INVARIANT DIFFERENTIAL OPERATORS AND AN HOMOMORPHISM OF HARISH-CHANDRA

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1. Introduction and Applications

Let $\mathfrak{g}$ be a reductive complex Lie algebra, with adjoint group G, Cartan subalgebra $\mathfrak{h}$ and Weyl group W. Then G acts naturally on the algebra of polynomial functions $\mathcal{O}(\mathfrak{g})$ and hence on the ring of differential operators with polynomial coefficients, $\mathcal{D}(\mathfrak{g})$. Similarly, W acts on $\mathfrak{h}$ and hence on $\mathcal{D}(\mathfrak{h})$. In [HC2], Harish-Chandra defined an algebra homomorphism $\delta : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h})^W$. Recently, Wallach proved that, *if $\mathfrak{g}$ has no factors of type E_6, E_7 or E_8, then this map δ is surjective* [Wa, Theorem 3.1]. The significance of Wallach's result is that it enables him to give an easy proof of an important theorem of Harish-Chandra about invariant distributions and to give an elegant new approach to the Springer correspondence.

The main aim of this paper is to give an elementary proof of [Wa, Theorem 3.1] that also works for all reductive Lie algebras. Set

$$\mathcal{I} = \{D \in \mathcal{D}(\mathfrak{g})^G : D(p) = 0 \text{ for all } p \in \mathcal{O}(\mathfrak{g})^G\}.$$

Then, we prove:

Theorem 1. *The sequence* $0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{D}(\mathfrak{g})^G \xrightarrow{\delta} \mathcal{D}(\mathfrak{h})^W \longrightarrow 0$ *is exact.*

This theorem follows immediately from an abstract result that describes the fixed rings, under a finite group action, of rings of differential operators (see Theorem 5 and the comments thereafter). Before discussing that result, however, we wish to explain the significance of Theorem 1, for which we need some notation.

Fix a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ and assume that Ω is a fixed, open, completely invariant subset of $\mathfrak{g}_0$. Let $\mathcal{N}$ denote the cone of nilpotent elements in $\mathfrak{g}_0$. Write $\mathcal{D}'_{\mathcal{N}}(\Omega)^{G_0}$ for the set of distributions on Ω that are invariant under the action of the adjoint group G_0 of $\mathfrak{g}_0$ and supported in $\mathcal{N}$. One of the key ideas in [Wa] is to relate invariant distributions to Weyl group representations. While Wallach is able to make this translation without use of Theorem 1, the result becomes simpler and more natural with that result to hand. This is summarized by the following result, which is obtained by combining Theorem 1 with [Wa, Theorems 2.7, 6.1 and Proposition 5.5]:

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Theorem 2. *Assume that $\mathfrak{g}_0$ is semisimple and let $T \in \mathcal{D}'_{\mathcal{N}}(\Omega)^{G_0}$. Then, $\mathcal{D}(\mathfrak{g})^G T$ is killed by $\mathcal{I}$. Thus, by Theorem 1 we may regard $\mathcal{D}(\mathfrak{g})^G T$ as a module over $\mathcal{D}(\mathfrak{h})^W$. As such, $\mathcal{D}(\mathfrak{g})^G T \in C'_W$, the category of all finitely generated $\mathcal{D}(\mathfrak{h})^W$-modules on which $S(\mathfrak{h}^*)^W_+$ acts locally nilpotently. Moreover, C'_W is equivalent to the category of finite dimensional representations of W. Under this equivalence, $\mathcal{D}'_{\mathcal{N}}(\mathfrak{g}_0)^{G_0}$ becomes a finite W-module.*

By taking T to be the orbital integral corresponding to a nilpotent element X of a semisimple Lie algebra $\mathfrak{g}$, Wallach associates, in much this way, an irreducible representation of W to the G-orbit of X (see [Wa, Theorem 6.9]). This leads to his construction of the Springer correspondence. Wallach's proof can be further simplified since, by Theorem 1 and [Wa, Corollary 1.4], the categories $\mathcal{C}_W$ and $\check{\mathcal{C}}_W$ of [Wa, Section 2] are equal. Thus, [Wa, Theorem 2.4] and [Wa, Theorem 2.6] are now trivial consequences of [Wa, Theorem 1.4], respectively [Wa, Theorem 1.6], while [Wa, Lemma 2.5] is vacuously true.

In [Wa], Wallach also gives a relatively easy proof of a fundamental result of Harish-Chandra. Using Theorem 1, one can further simplify his argument. Indeed, by combining Theorem 1 with the first two paragraphs of the proof of [Wa, Theorem 5.4] one obtains an easy proof of the following result:

Theorem 3. (see [HC3, Theorem 5]) $\mathcal{I} = \{D \in \mathcal{D}(\mathfrak{g})^G : D\mathcal{D}'(\Omega)^{G_0} = 0\}$.

Moreover, the argument given at the end of [Wa, Section 5] shows that this, in turn, easily implies a second result from [HC3]:

Theorem 4. *Let $\mathfrak{g}'$ denote the set of all generic elements of $\mathfrak{g}$. Suppose that $T \in \mathcal{D}'(\Omega)^{G_0}$ satisfies $\dim S(\mathfrak{g})^G T < \infty$ and $T_{|\Omega\cap\mathfrak{g}'} = 0$. Then $T = 0$.*

We end the introduction by discussing the proof of Theorem 1. This is based on a structure theorem for rings of differential operators fixed by an arbitrary finite group, that is of independent interest. Let V be an ℓ-dimensional vector space over a field k of characteristic zero and $W \subset GL(V)$ a finite group. Let $\mathcal{O}(V) \cong k[x_1, \dots, x_\ell]$ denote the ring of polynomial functions on V and write $\mathcal{D}(V) = \left\{\sum f_i(\partial/\partial x_1)^{i_1} \cdots (\partial/\partial x_\ell)^{i_\ell} : f_i \in \mathcal{O}(V)\right\}$ for the ring of k-linear differential operators on $\mathcal{O}(V)$. We may identify the symmetric algebra $S(V^*)$ with $\mathcal{O}(V)$ and the algebra $S(V)$ with the ring of constant coefficient differential operators on V. This provides a natural identification of left $S(V^*)$-modules: $\mathcal{D}(V) = S(V^*) \otimes_k S(V)$ (see, for example, [HC2, Section 3, Corollary 4]). The diagonal action of W on $S(V^*) \otimes_k S(V)$ now identifies with the usual action of W on $D(V)$ given by $g(\theta)(f) = g \cdot (\theta(g^{-1} \cdot f))$, for $g \in W$, $\theta \in \mathcal{D}(V)$ and $f \in \mathcal{O}(V)$ (see, for example, [HC2, pp.234-5]).

Now consider $\mathcal{D}(V)^W$. Two obvious subspaces are the W-invariant polynomial functions, $\mathcal{O}(V)^W = S(V^*)^W$, and the W-invariant, constant coefficient differential operators, $S(V)^W$. Remarkably, these suffice to generate $\mathcal{D}(V)^W$:

Theorem 5. *Let $\mathcal{B}$ be the k-subalgebra of $\mathcal{D}(V)$ generated by $S(V^*)^W$ and $S(V)^W$. Then, $\mathcal{B} = \mathcal{D}(V)^W$.*

If W is a Weyl group with no factors of type E_6, E_7 or E_8, then this theorem is proved in [Wa], by means of a case by case analysis. The relevance of Theorem 5 is that, as is shown in [Wa], it has Theorem 1 as an immediate corollary. The argument is as follows:

Proof of Theorem 1 from Theorem 5. Inside $\mathcal{D}(\mathfrak{g})^G$ one has the natural subrings $S(\mathfrak{g}^*)^G$ and $S(\mathfrak{g})^G$. Let $\mathcal{A}$ be the subalgebra $\mathbb{C}\langle S(\mathfrak{g}^*)^G, S(\mathfrak{g})^G\rangle$ that they generate. Then the restriction of Harish-Chandra's homomorphism δ to $S(\mathfrak{g})^G$ is just the Chevalley isomorphism $S(\mathfrak{g})^G \xrightarrow{\sim} S(\mathfrak{h})^W$. Similarly, the restriction of δ to $S(\mathfrak{g}^*)^G$ is the Chevalley isomorphism $S(\mathfrak{g}^*)^G \xrightarrow{\sim} S(\mathfrak{h}^*)^W$ (see [HC1, Theorem 1]). Thus, with $V = \mathfrak{h}$ and W the Weyl group, Theorem 5 implies that δ maps $\mathcal{A}$ surjectively onto $\mathcal{B} = \mathbb{C}\langle S(\mathfrak{h}^*)^W, S(\mathfrak{h})^W\rangle = \mathcal{D}(\mathfrak{h})^W$. Since $\mathcal{I} = \operatorname{Ker}(\delta)$ (see [HC2, Theorem 1]) this completes the proof of Theorem 1, modulo proving Theorem 5. Note that this argument also shows that $\mathcal{D}(\mathfrak{g})^G = \mathcal{A} + \mathcal{I}$. $\square$

2. The Proof of Theorem Five

We will continue to use the identifications described in the introduction; in particular we will always make the identification $S(V^*) \otimes_k S(V) = \mathcal{D}(V)$. Let $S(V) = \bigoplus_{i\geq 0} S^i(V)$ be the usual graded structure of $S(V)$ and write $\{S(V)_n = \bigoplus_{i\leq n} S^i(V)\}$ for the corresponding filtration of $S(V)$. This induces a filtered structure $\mathcal{D}(V) = \bigcup \mathcal{D}(V)_n$ on $\mathcal{D}(V)$ by defining $\mathcal{D}(V)_n = S(V^*) \otimes_k S(V)_n$. Since $S(V)$ has been identified with the constant coefficient differential operators, this filtration is nothing more than the filtration of $D(V)$ by degree of differential operators and so, in particular, is a filtration of $\mathcal{D}(V)$ as a k-algebra. Given any subalgebra, or even subspace, $R \subseteq \mathcal{D}(V)$, we will always filter R by the induced filtration: $\{R_n = R \cap \mathcal{D}(V)_n\}$. The associated graded object $\operatorname{gr}(R)$ is defined to be $\operatorname{gr}(R) = \bigoplus \operatorname{gr}_n R$, where $\operatorname{gr}_n R = R_n/R_{n-1}$. Observe that

$$\operatorname{gr}_n \mathcal{D}(V) \;=\; \big(S(V^*) \otimes S(V)_n\big)\big/\big(S(V^*) \otimes S(V)_{n-1}\big) \;\cong\; S(V^*) \otimes S^n(V). \qquad (\dagger)$$

Thus, $\operatorname{gr}(\mathcal{D}(V))$ is isomorphic to the *commutative ring* $S(V^*) \otimes S(V)$. Moreover, for any subalgebra $R \subseteq \mathcal{D}(V)$, the associated graded ring $\operatorname{gr}(R)$ is isomorphic to

$$\bigoplus\, (R \cap \mathcal{D}(V)_n + \mathcal{D}(V)_{n-1})/\,\mathcal{D}(V)_{n-1} \;\subseteq\; \operatorname{gr}(\mathcal{D}(V)).$$

Lemma 6. *Let $W \subset GL(V)$ be a finite group and write $\mathcal{B} = k\langle S(V^*)^W, S(V)^W\rangle$ for the subalgebra of $\mathcal{D}(V)^W$ generated by $S(V^*)^W$ and $S(V)^W$. Then, $\mathcal{B}$ is a Noetherian domain and $\mathcal{D}(V)^W$ is finitely generated as both a left and a right $\mathcal{B}$-module.*

Proof. Observe that $S(V^*)\otimes_k S(V) = \bigoplus S(V^*)\otimes_k S^n(V)$ is a decomposition of W-modules. Thus, $(\dagger)$ is also an isomorphism of W-modules. Indeed, the identification $\mathcal{D}(V) = S(V^*) \otimes S(V)$ gives an action of $W \times W$ on $\mathcal{D}(V)$ by $(w_1, w_2) \circ (s_1 \otimes s_2) = w_1(s_1) \otimes w_2(s_2)$. Although this is not an action of $W \times W$ by algebra automorphisms, $(\dagger)$ is still an isomorphism of $W \times W$ modules. Now consider $\mathcal{B}$ and its natural k-subspace $L = S(V^*)^W \otimes S(V)^W$. Then $(\dagger)$ ensures that $\operatorname{gr}(L) = S(V^*)^W \otimes S(V)^W$ as a subspace of $\operatorname{gr}(\mathcal{D}(V)) = S(V^*) \otimes S(V)$. Thus, $\operatorname{gr}(L)$ is a ring; indeed $\operatorname{gr}(L) = \{S(V^*) \otimes S(V)\}^{W\times W}$. In particular, $\operatorname{gr}(L)$ is a Noetherian domain and $\operatorname{gr}(\mathcal{D}(V))$ is a finitely generated $\operatorname{gr}(L)$-module.

Clearly, $\operatorname{gr}(L) \subseteq \operatorname{gr}(\mathcal{B}) \subseteq \operatorname{gr}(\mathcal{D}(V)^W) \subseteq \operatorname{gr}(\mathcal{D}(V))$. Thus, $\operatorname{gr}(\mathcal{B})$ is Noetherian and so, by [MR, Lemma 7.6.11], $\mathcal{B}$ is a Noetherian ring. Next, as $\mathcal{B}$ is a filtered subalgebra of $\mathcal{D}(V)^W$, $\mathcal{D}(V)^W$ is a filtered (left or right) $\mathcal{B}$-module and its associated graded module is just $\operatorname{gr}(\mathcal{D}(V)^W)$. But, by the last paragraph, $\operatorname{gr}(\mathcal{D}(V)^W)$ is a finitely generated (right or left) $\mathrm{gr}(\mathcal{B})$-module. Thus, by [MR, Lemma 7.6.11], again, $\mathcal{D}(V)^W$ is finitely generated as both a left and a right $\mathcal{B}$-module. $\square$

Let C be a commutative ring that is a localisation of a finitely generated k-algebra. The ring of k-linear differential operators $\mathcal{D}(C)$ is defined to be $\mathcal{D}(C) = \bigcup_{n\geq 0}\mathcal{D}(C)_n$, where $\mathcal{D}(C)_0 = C$ and, inductively,

$$\mathcal{D}(C)_n = \{\theta \in \mathrm{End}_k(C) : \theta c - c\theta \in \mathcal{D}(C)_{n-1} \text{ for all } c \in C\}.$$

The ring $\mathcal{D}(C)$ is filtered by the $\mathcal{D}(C)_n$ and, when $C = S(V^*)$, this does agree with the earlier definition. Now assume that F is a finitely generated field extension of k of transcendence degree ℓ and pick a transcendence basis, say $\{u_1, \dots, u_\ell\}$, for F/k. Then, by [MR, Corollaries 15.1.12(iv) and 15.2.5] the derivations $\partial_i = \frac{\partial}{\partial u_i}$ extend to derivations on F and $\mathcal{D}(F) \cong F\langle \partial_1, \dots, \partial_\ell\rangle$. Given $I = (i_1, \dots, i_\ell) \in \mathbb{N}^\ell$, let $\partial^I = \partial_1^{i_1}\cdots\partial_\ell^{i_\ell}$. Then, as a left F-module, $\mathcal{D}(F)$ has basis $\{\partial^I : I \in \mathbb{N}^\ell\}$ ([MR, Proposition 15.3.2]). If $I = (i_1, \dots, i_\ell) \in \mathbb{N}^\ell$, write $|I| = \sum_j i_j$. We totally order $\mathbb{N}^\ell$ by $I = (i_1, \dots, i_\ell) > J = (j_1, \dots, j_\ell)$ if either $|I| > |J|$ or $|I| = |J|$ and there exists $1 \leq m \leq \ell$ such that $i_n = j_n$ for $n < m$ but $i_m > j_m$. Given a k-algebra R and $r \in R$, let $\mathrm{ad}(r) \in \mathrm{End}_k(R)$ be defined by $\mathrm{ad}(r)(s) = [r, s] = rs - sr$, for $s \in R$.

Lemma 7. *Keep the above notation. Then:*

(i) *If $J = (j_1, \dots, j_\ell) \in \mathbb{N}^\ell$, define $ad(u)^J = ad(u_1)^{j_1}\cdots ad(u_\ell)^{j_\ell}$. Then, $ad(u)^J(\partial^I) = \lambda\partial^{I-J}$, for some $\lambda \in \mathbb{Z}\setminus\{0\}$. By convention, if $I < J$, then $\partial^{I-J} = 0$.*

(ii) *Let $f \in \mathcal{D}(F)\setminus F$ and write $f = f_I\partial^I + \sum_{J<I} f_J\partial^J$, with $f_I \neq 0$. Suppose that $I = (i_1, \dots, i_\ell)$, with $i_r > 0$ and set $I' = (i_1, \dots, i_r - 1, \dots, i_\ell)$. Then, $\partial_r + f_{r+1}\partial_{r+1} + \cdots + f_\ell\partial_\ell + f_0 \in F\cdot ad(u)^{I'}(f)$, for some $f_j \in F$.*

Proof. (i) By construction, $\mathrm{ad}(u_j)(\partial_i) = -\delta_{ij}$ and

$$\mathrm{ad}(u_j)(\partial^I) = -(i_j)\partial_1^{i_1}\cdots\partial_j^{i_j-1}\cdots\partial_\ell^{i_\ell}.$$

The result now follows from the obvious induction.

(ii) Let $\mathbf{1}_r = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^\ell$, where the 1 occurs in the r^{th} position. By part (i), $\mathrm{ad}(u)^{I'}(f) = \mu f_I\partial_r + \sum_{K<\mathbf{1}_r} g_K\partial^K$, for some non-zero integer μ and some $g_K \in F$. This is equivalent to the assertion of the lemma. $\square$

Suppose that $K = k(u_1, \dots, u_\ell)$ is a field of rational functions and let R be the subalgebra of $\mathcal{D}(K)$ generated by K and one operator $\partial_1^{i_1}\cdots\partial_\ell^{i_\ell}$, for which $i_j > 0$, for each j. Then an easy exercise using Lemma 7 shows that $R = \mathcal{D}(K)$. The same basic idea leads to the following curious result.

Lemma 8. *Let F be a finitely generated field extension of k of transcendence degree ℓ. Let $R \subseteq \mathcal{D}(F)$ be a subalgebra such that $F \subseteq R$ and give R the induced filtration $\{R_n = R \cap \mathcal{D}(F)_n\}$. Assume that*

$$\limsup_{n\to\infty}\{\log_n(\dim_F gr_n(R))\} > \ell - 2. \tag{8.1}$$

Then, $R = \mathcal{D}(F)$.

Remark. By [KL, Theorem 4.5] the condition (8.1) holds, in particular, if $\mathrm{gr}(R)$ contains a finitely generated, graded F-subalgebra of Krull dimension ℓ.

Proof. Let $d_i = (\partial_i + F)/F$ denote the image of ∂_i in $\mathrm{gr}(\mathcal{D}(F))$. By [MR, Proposition 15.3.2], $\mathrm{gr}(\mathcal{D}(F)) \cong F[d_1, \dots, d_\ell]$, the (commutative) polynomial ring in the d_j. In particular, $\mathrm{gr}_n(\mathcal{D}(F)) \cong F[d_1, \dots, d_\ell]^n$, the vector space of homogeneous polynomials of degree n. Now, $F[d_2, \dots, d_\ell]^n$ has dimension $O(n^{\ell-2})$. Thus, by (8.1), there exists n such that, for any $1 \le r \le \ell$, one has $\dim \mathrm{gr}_n R > \dim F[d_1, \dots, d_{r-1}, d_{r+1}, \dots, d_\ell]^n$.

For n chosen as in the last paragraph, pick an F-basis $\{t_1, \dots, t_\nu\}$ for $\mathrm{gr}_n R$. As $F \subseteq R$, we may choose this basis such that, for $1 \le \lambda \le \nu$,

$$t_\lambda = d^{I(\lambda)} + \sum \left\{ f_{J(\lambda)} d^{J(\lambda)} : J(\lambda) < I(\lambda) \text{ and } f_{J(\lambda)} \in F \right\}$$

and

$$\text{if } \lambda \neq \mu \text{ then } d^{I(\lambda)} \neq d^{I(\mu)}.$$

Now pick $1 \le r \le \ell$. By the choice of n, and the fact that $\nu = \dim_F(\mathrm{gr}_n R)$, there exists $1 \le \lambda \le \nu$ such that $d^{I(\lambda)} \notin F[d_1, \dots, d_{r-1}, d_{r+1}, \dots, d_\ell]$. For this λ, pick $f \in R_n$ such that $(f + R_{n-1})/R_{n-1} = t_\lambda$. Then $f = \partial^{I(\lambda)} + \sum_{J<I(\lambda)} f_J \partial^J$ and $I(\lambda) = (i_1, \dots, i_\ell)$ with $i_r \neq 0$. Since each $u_j \in F \subset R$, we may apply Lemma 7(ii) to conclude that there exist $f_j = f_{jr} \in F$ such that

$$\Delta_r = \partial_r + f_{r+1}\partial_{r+1} + \cdots + f_\ell \partial_\ell \ \in \ R.$$

But $\mathcal{D}(F) = F\langle \partial_1, \dots, \partial_\ell \rangle = F\langle \Delta_1, \dots, \Delta_\ell \rangle$. Thus, $R = \mathcal{D}(F)$. □

The next result is proved in [LS, Lemma IV.1.3] but, since it is crucial to this paper, we will reprove it here. In the case of Weyl groups, it is the only ingredient of our proof of Theorem 5 that is not implicit in [Wa].

Lemma 9. *Let $R \subseteq S$ be Noetherian domains, with the same division ring of fractions. Assume that S is a simple ring and that S is finitely generated as a left and a right R-module. Then, $R = S$.*

Proof. We may write $S = \sum_{i=1}^m s_i R$, for some $s_i \in S$. As $S \subset Q(R)$, the quotient ring of R, we may write each s_i over a common denominator; say $s_i = c^{-1} r_i$, for some $c, r_i \in R$. Thus, $c \in \ell\text{-ann}_R(S) = \{r \in R : rS \subseteq R\}$ and so $N = \ell\text{-ann}_R(S) \neq 0$. Observe that N is a right ideal of S. Similarly, as S is a finitely generated left R-module, $M = r\text{-ann}_R(S) \neq 0$ and M is a left ideal of S. Thus, as S is a domain, MN is a non-zero two-sided ideal of S. Since S is a simple, $S = MN \subseteq R$. □

Finally, we may combine these lemmas to prove Theorem 5 of introduction:

Theorem 5. *Let $\mathcal{B}$ be the k-subalgebra of $\mathcal{D}(V)$ generated by $S(V^*)^W$ and $S(V)^W$. Then, $\mathcal{B} = \mathcal{D}(V)^W$.*

Proof. Recall that $\mathcal{O}(V) = S(V^*)$ and that $\ell = \dim_k V$. Let $\mathcal{C} = \mathcal{O}(V)^W \setminus \{0\}$. By the definition of differential operators, $\mathcal{C}$ is an ad-nilpotent set of elements of $\mathcal{D}(V)$ and hence of the subalgebras $\mathcal{B}$ and $\mathcal{D}(V)^W$. Thus, by [KL, Theorem 4.9], $\mathcal{C}$ is a (left and right) Ore set in each of these rings. Let F denote the field of fractions of $\mathcal{O}(V)^W$ and K the field of fractions of $\mathcal{O}(V)$. Since $\mathcal{O}(V)$ is a finite $\mathcal{O}(V)^W$-module, $\mathcal{O}(V)_\mathcal{C} = K$ and hence, by [MR, Theorem 15.5.5], $\mathcal{D}(V)_\mathcal{C} = \mathcal{D}(K)$. Similarly, $\mathcal{D}(\mathcal{O}(V)^W)_\mathcal{C} = \mathcal{D}(F)$. Next, pick ℓ algebraically independent elements $u_i \in \mathcal{O}(V)^W \subset F$ and set $\partial_i = \partial/\partial u_i$, as before. Then

$\mathcal{D}(F) = F\langle\partial_1, \ldots, \partial_\ell\rangle \subset \mathcal{D}(K) = K\langle\partial_1, \ldots, \partial_\ell\rangle$. We also have an homomorphism $\mathcal{D}(V)^W \to \mathcal{D}(\mathcal{O}(V)^W)$, given by restriction of differential operators. Since $\mathcal{D}(V)^W$ is simple (see, for example, [Mo, Theorem 2.15, p.32] or [Wa, Lemma 1.2]) this map must be an injection. Thus, there is a chain of inclusions

$$F = \mathcal{O}(V)^W_{\mathcal{C}} \subset \mathcal{B}_{\mathcal{C}} \subseteq \mathcal{D}(V)^W_{\mathcal{C}} \subseteq \mathcal{D}(\mathcal{O}(V)^W)_{\mathcal{C}} = \mathcal{D}(F) \subseteq \mathcal{D}(V)_{\mathcal{C}} = \mathcal{D}(K).$$

Notice that the various filtrations on these rings are compatible. To see this, filter $\mathcal{D}(K)$ by degree of differential operator. Then the original filtrations on $\mathcal{D}(V)$ and hence on $\mathcal{B}$ are just the restrictions of that on $\mathcal{D}(K)$; in particular, $\mathcal{B}_n = \mathcal{B}\cap\mathcal{D}(K)_n$. Similarly, for example by the choice of $\{\partial_j\}$, $\mathcal{D}(F)_n = \mathcal{D}(F) \cap \mathcal{D}(K)_n$. Hence, the filtrations on $\mathcal{B}_{\mathcal{C}}$ induced from $\mathcal{D}(F)$ and $\mathcal{B}$ do indeed coincide. Now, consider $\mathrm{gr}(\mathcal{B}_{\mathcal{C}})$. Since $\mathcal{B} \supset S(V)^W$, certainly $\mathrm{gr}(\mathcal{B}_{\mathcal{C}}) \supseteq F \otimes_k S(V)^W$. Since $F \otimes_k S(V)$ is a finitely generated $F \otimes_k S(V)^W$-module, $\mathrm{Kdim}(F \otimes_k S(V)^W) = \ell$. Thus, by Lemma 8 and the remark thereafter, $\mathcal{B}_{\mathcal{C}} = \mathcal{D}(F)$. Therefore, $\mathcal{B}_{\mathcal{C}} = \mathcal{D}(V)^W_{\mathcal{C}}$ and so $\mathcal{B}$ and $\mathcal{D}(V)^W$ have the same quotient division ring. By Lemma 8, both $\mathcal{B}$ and $\mathcal{D}(V)^W$ are Noetherian domains and $\mathcal{D}(V)^W$ is finitely generated as a left or right $\mathcal{B}$-module. Thus, as $\mathcal{D}(V)^W$ is a simple ring, all the hypotheses of Lemma 9 are satisfied and $\mathcal{B} = \mathcal{D}(V)^W$. □

Wallach's version of Theorem 5 for Weyl groups with no factors of type E_n is obtained as a corollary of a theorem about Poisson algebras and we end the paper by discussing the connections between his results and the techniques of this paper. Let $\{\ ,\ \}$ denote the usual Poisson bracket on $S(V^*) \otimes S(V)$ and let $W \subset GL(V)$ be a finite group. Following [Wa], the pair (V, W) is called *good* if, whenever C is a subalgebra of $S(V^*) \otimes_k S(V)$ that contains $S(V^*)^W \otimes S(V)^W$ and is closed under $\{\ ,\ \}$, then $C \supseteq (S(V^*) \otimes S(V))^W$. In [Wa], Wallach proves that (V, W) is good in the case when W is a Weyl group with no factors of type E_m and he shows that, whenever (V, W) is good, the conclusion of Theorem 5 will also hold.

This raises the natural question of whether (V, W) is good for any finite group W. However, it is easy to construct counterexamples, even in the case where W is generated by pseudo-reflections. For example, let V be 1-dimensional complex vector space and $W = \langle\sigma\rangle \cong \mathbb{Z}_3$ acting by $\sigma(v) = \omega v$, where ω is a primitive third root of unity. We may identify $S(V^*) \otimes S(V) = S(V^* \times V) = \mathbb{C}[x, \xi]$, which we grade by total degree in x and ξ. It is easy to check that $R = (S(V^*) \otimes S(V))^W = \mathbb{C}[x^3, x\xi, \xi^3]$. Let R' be the graded subalgebra $\mathbb{C} + \bigoplus_{n\geq 3} R_n$ of R. Since $\{R_n, R_m\} \subseteq R_{n+m-2}$, certainly R' is closed under the Poisson bracket. But, $x\xi \in R \setminus R'$ and so the pair (V, W) is not good.

One of the key points in the proof of Theorem 5 is the fact that $\mathcal{D}(V)^W$ is simple. The analogous result for Poisson algebras is false. To see this, let $W \subset GL(V)$ be any finite group such that $V^W = \{0\}$. Give $S(V^*) \otimes S(V)$ the natural grading by total degree. Then, $C = (S(V^*) \otimes S(V))^W$ will contain no homogeneous element of degree one. Hence, the augmentation ideal $\bigoplus_{n\geq 1} C_n = \bigoplus_{n\geq 2} C_n$ will be a non-trivial Poisson ideal. It is therefore unlikely that the techniques of this paper could be used to prove that (V, W) is good when W is a Weyl group.

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