RINGS OF DIFFERENTIAL OPERATORS ON CLASSICAL RINGS OF INVARIANTS

by

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ABSTRACT

We consider rings of differential operators over the classical rings of invariants, in the sense of Weyl [**We**]. Thus, let $\overline{\mathcal{X}}_k$ be one of the following varieties: *(CASE A)* all complex $p \times q$ matrices of rank $\leq k$; *(CASE B)* all symmetric $n \times n$ matrices of rank $\leq k$; *(CASE C)* all antisymmetric $n \times n$ matrices of rank $\leq 2k$. We prove that the ring of differential operators $\mathcal{D}(\overline{\mathcal{X}}_k) = \mathcal{D}(\mathcal{O}(\overline{\mathcal{X}}_k))$ defined on the ring of regular functions $\mathcal{O}(\overline{\mathcal{X}}_k)$ is a simple, finitely generated, Noetherian domain.

Assume further that $\overline{\mathcal{X}}_k$ is singular (which is the only interesting case). Then the result is proved by showing that $\mathcal{D}(\overline{\mathcal{X}}_k)$ is a factor ring of an enveloping algebra $U(\mathfrak{g})$. Here $\mathfrak{g} = \mathfrak{gl}(p+q)$, $\mathfrak{sp}(2n)$ and $\mathfrak{so}(2n)$ in the Cases A, B and C, respectively.

Finally, let $SO(k)$ act in the natural way on the ring $\mathbb{C}[X]$ of complex polynomials in kn variables. Then we prove that $\mathcal{D}(\mathbb{C}[X]^{SO(k)})$ has a similarly pleasant structure and, at least for $k \leq n$, is a finitely generated $U(\mathfrak{sp}(2n))$-module.

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CONTENTS

INTRODUCTION

0.1. Given a commutative $\mathbb{C}$-algebra R, the ring $\mathcal{D}(R)$ of $\mathbb{C}$-linear differential operators on R is defined, inductively, as follows. Let $\mathcal{D}_0(R) = Hom_R(R,R) \cong R$. For $m > 0$ define the set of differential operators of order $\leq m$ to be

$$\mathcal{D}_m(R) = \{\theta \in End_{\mathbb{C}}(R) : [\theta, a] \in \mathcal{D}_{m-1}(R) \text{ for all } a \in R\}.$$

Then $\mathcal{D}(R) = \bigcup_{m=0}^{\infty} \mathcal{D}_m(R)$, with multiplication defined by composition of functions. In order to avoid confusion with multiplication inside $\mathcal{D}(R)$, the action of $\theta \in \mathcal{D}(R)$ as a differential operator on $r \in R$ will always be written $\theta * r$. Basic facts about $\mathcal{D}(R)$ can be found, for example, in [**Sw**] or [**MR,** Chapter XV].

Given a quasi-affine algebraic variety $\mathcal{Y}$, with regular functions $\mathcal{O}(\mathcal{Y})$, write $\mathcal{D}(\mathcal{Y})$ for $\mathcal{D}(\mathcal{O}(\mathcal{Y}))$. One aspect of $\mathcal{D}(\mathcal{Y})$ is of particular relevance here. If $\mathcal{Y}$ is affine, non-singular and irreducible, it is well known that $\mathcal{D}(\mathcal{Y})$ has a particularly pleasant structure being, in particular, a simple Noetherian domain that is finitely generated as a $\mathbb{C}$-algebra (see, for example [**MR,** Chapter XV, §§1.20, 3.7 and 5.6] or [**SmSt,** §1.4]). In contrast, when $\mathcal{Y}$ has singularities, $\mathcal{D}(\mathcal{Y})$ need not be pleasant. For example, if $\mathcal{Y}$ is the cubic cone $x_1^3 + x_2^3 + x_3^3 = 0$ in complex 3-space, then $\mathcal{D}(\mathcal{Y})$ is neither simple nor Noetherian nor finitely generated, and even has an infinite ascending chain of two-sided ideals (see [**BGG**]).

0.2. The aim of this paper is to study the rings of differential operators on classical rings of invariants. Indeed, even though the varieties will often be singular, we will prove that the corresponding rings of differential operators will always be simple Noetherian domains, generated by the "obvious" differential operators of order ≤ 2.

Following Weyl [**We**] (but see also [**DP**]) we will consider 3 main classes of rings of invariants. To describe them, let $M_{p,q}(\mathbb{C})$ denote the space of $p \times q$ complex matrices, and fix $k \geq 1$.

(CASE A) Given $p \geq q \geq 1$, let $G' = GL(k) = GL(k,\mathbb{C})$ act on $\mathcal{X} = M_{p,k}(\mathbb{C}) \times M_{k,q}(\mathbb{C})$ by $g \cdot (\xi, \eta) = (\xi g^{-1}, g\eta)$ for $g \in GL(k)$ and $(\xi,\eta) \in \mathcal{X}$. Then $GL(k)$ acts on $\mathcal{O}(\mathcal{X})$ and the ring of invariants $\mathcal{O}(\mathcal{X})^{GL(k)}$ is described by the two Fundamental Theorems of Invariant Theory (see [**DP**] or (II, Theorem 2.3)). For the

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purposes of this introduction we merely note that $\mathcal{O}(\mathcal{X})^{GL(k)} \cong \mathcal{O}(\overline{\mathcal{X}}_k)$, where $\overline{\mathcal{X}}_k$ is the variety of $p \times q$ matrices of rank $\leq k$.

(CASE B) Given $n \geq 1$, let $G' = O(k) = O(k, \mathbb{C})$ act on $\mathcal{X} = M_{k,n}(\mathbb{C})$ by $g \cdot \xi = g\xi$ for $g \in O(k)$ and $\xi \in \mathcal{X}$. In this case $\mathcal{O}(\mathcal{X})^{O(k)} \cong \mathcal{O}(\overline{\mathcal{X}}_k)$, where $\overline{\mathcal{X}}_k$ is the variety of symmetric $n \times n$ matrices of rank $\leq k$ (see **[DP]** or (II, Theorem 3.3)).

(CASE C) Given $n \geq 1$, let $G' = Sp(2k) = Sp(2k, \mathbb{C})$ act on $\mathcal{X} = M_{2k,n}(\mathbb{C})$ by $g \cdot \xi = g\xi$ for $g \in Sp(2k)$ and $\xi \in \mathcal{X}$. Then $\mathcal{O}(\mathcal{X})^{Sp(2k)} \cong \mathcal{O}(\overline{\mathcal{X}}_k)$ where, now, $\overline{\mathcal{X}}_k$ is defined to be the variety of all antisymmetric $n \times n$ matrices of rank $\leq 2k$ (see **[DP]** or (II, Theorem 4.3)).

One may, of course, also consider the rings of invariants under the action of $SO(k)$ and $SL(k)$, but we defer comment on these cases until later in the introduction.

0.3. It follows immediately from (0.2) that $\mathcal{O}(\overline{\mathcal{X}}_k)$ is a polynomial ring if (and only if) $k \geq q$ in Case A, $k \geq n$ in Case B and $2k \geq n-1$ in Case C. Thus in these cases $\mathcal{D}(\overline{\mathcal{X}}_k)$ is nothing more than the Weyl algebra,

$$A_m(\mathbb{C}) = \mathbb{C}[x_1, \ldots, x_m, \partial/\partial x_1, \ldots, \partial/\partial x_m]$$

of an appropriate index m. The aim of this paper is to study $\mathcal{D}(\overline{\mathcal{X}}_k)$ for the remaining values of k. We therefore define k to be *sufficiently small* if $1 \leq k \leq q-1$ in Case A, $1 \leq k \leq n-1$ in Case B and $2 \leq 2k \leq n-2$ in Case C. We remark that, with a little work, one can prove that k *is sufficiently small if and only if* $\overline{\mathcal{X}}_k$ *is singular.*

The method we use to study $\mathcal{D}(\overline{\mathcal{X}}_k)$ is Howe's notion of a classical reductive dual pair in a symplectic group **[Ho1, Ho2]** (see Chapter I for the basic definitions and results). This provides, by means of the metaplectic representation, a natural map $\psi : U(\mathfrak{g}) \to \mathcal{D}(\overline{\mathcal{X}}_k)$. Here $\mathfrak{g}$ is the Lie algebra $\mathfrak{g} = \mathfrak{gl}(p+q)$ in Case A, $\mathfrak{g} = \mathfrak{sp}(2n)$ in Case B and $\mathfrak{g} = \mathfrak{so}(2n)$ in Case C, while $U(\mathfrak{g})$ denotes the enveloping algebra of $\mathfrak{g}$.

We can now state the main result of this paper.

THEOREM. *Suppose that k is sufficiently small, and that $\overline{\mathcal{X}}_k$ is defined as in (0.2). Then $J(k) = ker(\psi)$ is a completely prime, maximal ideal of $U(\mathfrak{g})$. Moreover, ψ induces an isomorphism*

$$U(\mathfrak{g})/J(k) \xrightarrow{\sim} \mathcal{D}(\overline{\mathcal{X}}_k).$$

This result is obtained by combining the main results of Chapters III and IV (see (III, Theorem 1.10) and (IV, Theorem 1.3), respectively).

0.4. Combining Theorem 0.3 with a non-commutative version of the First Fundamental Theorem of Invariant Theory ([**HoI**, Theorems 2 and 7]) gives:

COROLLARY. *$\mathcal{D}(\overline{\mathcal{X}}_k)$ is a simple Noetherian domain, generated by differential operators of degree ≤ 2. Moreover, these generators are the "obvious" elements of $\mathcal{D}(\overline{\mathcal{X}}_k)$ – see (IV, 1.9) for the precise statement.*

We remark that $\psi : U(\mathfrak{g}) \to \mathcal{D}(\overline{\mathcal{X}}_k)$ is still defined when k is not sufficiently small. However, in this case $\mathcal{D}(\overline{\mathcal{X}}_k)$ is a Weyl algebra, and hence by (IV, Remark 1.5) ψ cannot be surjective. Indeed, in this case $\mathcal{D}(\overline{\mathcal{X}}_k)$ will not even be finitely generated as a $U(\mathfrak{g})$-module. This provides the following, rather curious dichotomy: *The map ψ is nice (that is, surjective) if and only if $\overline{\mathcal{X}}_k$ is bad (that is, singular).*

0.5. In the course of the proof of Theorem 0.3 we prove a number of results about the Lie algebra $\mathfrak{g}$, of which the most significant are the following. For simplicity, we state these results here under the assumption that k is sufficiently small, although we do in fact prove analogous results for all values of k.

0.5.1. $J(k) = Ann(L(\lambda_k + \rho))$, where $L(\lambda_k + \rho)$ is a simple highest weight module, whose highest weight λ_k can be explicitly described. Under the isomorphism of Theorem 0.3, $L(\lambda_k + \rho)$ becomes the standard $\mathcal{D}(\overline{\mathcal{X}}_k)$-module $\mathcal{O}(\overline{\mathcal{X}}_k)$. (See (II, 2.7, 3.7 and 4.7)).

0.5.2. Write $\widetilde{k} = k$ in Cases A and B, but $\widetilde{k} = 2k$ in Case C. Let $\mathbf{O}_k$ be the nilpotent orbit $\{\xi \in \mathfrak{g} : \xi^2 = 0 \text{ and } \operatorname{rank}\xi = \widetilde{k}\}$, with Zariski closure $\overline{\mathbf{O}}_k$. Then the associated variety $\mathcal{V}(J(k))$ of the ideal $J(k)$ is equal to $\overline{\mathbf{O}}_k$ (see (II, 6.4)).

0.5.3. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the usual triangular decomposition of $\mathfrak{g}$. Then $\overline{\mathcal{X}}_k$ is an irreducible component of $\overline{\mathbf{O}_k \cap \mathfrak{n}^+}$ (see (II, Proposition 6.3)).

0.6. Let $G' = O(k)$ act on $\mathcal{X} = M_{k,n}(\mathbb{C})$ as in Case B of (0.2). This induces an action of $SO(k)$ on $\mathcal{X}$ and hence on $\mathcal{O}(\mathcal{X})$. We will also study the ring of differential operators $\mathcal{D}(A)$, for $A = \mathcal{O}(\mathcal{X})^{SO(k)}$. The natural way to approach this ring is to note that $\mathbb{Z}/2\mathbb{Z} \cong O(k)/SO(k)$ acts on both A and $\mathcal{D}(A)$ and so one should try and relate $\mathcal{D}(A)$ both to $\mathcal{D}(A)^{\mathbb{Z}/2\mathbb{Z}}$ and to $\mathcal{D}(A^{\mathbb{Z}/2\mathbb{Z}}) = \mathcal{D}(\mathcal{O}(\mathcal{X})^{O(k)})$. In this way we obtain:

THEOREM. *(V, Theorems 2.6 and 3.11). Let $A = \mathcal{O}(\mathcal{X})^{SO(k)}$. If $k \leq n$ then $\mathcal{D}(A)$ is a simple Noetherian ring, and is finitely generated as a module over the subring $R = U(\mathfrak{sp}(2n))/J(k)$. Moreover $\mathcal{D}(A)^{\mathbb{Z}/2\mathbb{Z}} = R$.*

Some remarks on this theorem are in order. First, note that it covers all the interesting values of k. For, the ring A is non-regular if and only if $k \leq n$. Moreover, for $k > n$ the group $\mathbb{Z}/2\mathbb{Z}$ acts trivially on A and so this case is already covered by the earlier results concerning $O(k)$ invariants. If $k < n$, then Theorem 0.3 shows that $R = \mathcal{D}(\overline{\mathcal{X}}_k) = \mathcal{D}(A^{\mathbb{Z}/2\mathbb{Z}})$ and so the above theorem implies that $\mathcal{D}(A)^{\mathbb{Z}/2\mathbb{Z}} = \mathcal{D}(A^{\mathbb{Z}/2\mathbb{Z}})$. The case $k = n$ is rather curious. For, (0.4) now implies that $R \neq \mathcal{D}(\overline{\mathcal{X}}_k)$ and one can even show that $\mathcal{D}(\overline{\mathcal{X}}_k)$ cannot be finitely generated as an R-module (see (IV, Remark 1.5)). Equivalently, $\mathcal{D}(A^{\mathbb{Z}/2\mathbb{Z}})$ is infinitely generated as a module over the subring $\mathcal{D}(A)^{\mathbb{Z}/2\mathbb{Z}}$.

It is natural to ask whether one can extend these results to cover the rings of invariants under the action of $SL(k)$. Unfortunately the methods of this paper will not apply to this case since there is no obvious enveloping algebra that can be used to generate all, or most of, $\mathcal{D}(\mathcal{O}(\mathcal{X})^{SL(k)})$.

0.7. There is an alternative way of viewing Theorems 0.3 and 0.6. Given a group K acting on a variety $\mathcal{Y}$ then this induces a natural action of K on $\mathcal{D}(\mathcal{Y})$. Moreover, one always has a map

$$\varphi : \mathcal{D}(\mathcal{Y})^K \longrightarrow \mathcal{D}(\mathcal{O}(\mathcal{Y})^K)$$

obtained by restriction of differential operators. Now, Howe in **[Ho1]** actually provides a map ω from $U(\mathfrak{g})$ *onto* $\mathcal{D}(\mathcal{X})^{G'}$. The map ψ of (0.3) is then simply $\psi = \varphi\omega$. Thus an equivalent formulation of Theorem 0.3 is that φ is surjective if (and only if) k is sufficiently small. Similarly, in the set-up described in (0.6) one can show that

$$\varphi' : \mathcal{D}(\mathcal{X})^{SO(k)} \longrightarrow \mathcal{D}(\mathcal{O}(\mathcal{X})^{SO(k)})$$

is surjective if and only if $k \leq n$. Consequently, at least when k is small enough, these results may be regarded as a non-commutative analogue of one of the basic results of classical invariant theory – that $\mathcal{O}(\mathcal{X})^{G'} = \mathcal{O}(\overline{\mathcal{X}}_k)$. However, in contrast to the commutative case, φ will not be injective (see (IV, Lemma 1.7)).

0.8. We next give a brief outline of the proofs of the main results and of the organisation of the paper. As was remarked earlier, one of the basic ideas behind the proof of Theorem 0.3 is to use Howe's work on reductive dual pairs from **[Ho1]**. Since Howe's paper has not been published, we give a brief survey of the relevant material in Chapter I. In Chapter II, we then give the more detailed computations that we will need concerning this material. In particular, in Chapter II we prove the results mentioned in (0.5). We remark that many of these results, at least for Cases A and B, can be found in the literature, notably in [**KV**]. We have included them here, both for the reader's convenience and because most of the details will be needed elsewhere in the paper.

0.9. The main step in the proof of Theorem 0.3 is given in Chapter III. This proves that (i) the rings $R = \psi(U(\mathfrak{g}))$ and $\mathcal{D} = \mathcal{D}(\overline{\mathcal{X}}_k)$ have the same full quotient ring $Q(R)$ and (ii) $\mathcal{D}$ is a finitely generated R-module. This is, essentially, equivalent to showing that $\mathcal{D}$ is the ring of $\mathfrak{k}$-finite vectors $L(L(\lambda_k+\rho),\, L(\lambda_k+\rho))$. To prove this, we adopt the approach used in [**LSS**]. Thus, the key point is to prove that $GKdim_R\mathcal{D}/R \leq GKdim\, R - 2$,where $GKdim$ stands for Gelfand-Kirillov dimension. This is done by computing the dimensions of certain associated varieties. One may then apply Gabber's Lemma [**Le2**] to conclude that $\mathcal{D}$ is a finitely generated R-module. (In the present situation, Gabber's Lemma implies that *there exists a unique, maximal, finitely generated submodule M of $Q(R)$ satisfying $GKdim\, M/R \leq GKdim\, R - 2$.*) Since [**Le2**] will not be published, we include a proof of this result, in its full generality, in an appendix to this paper. We remark that, after this research was completed, Joseph found another completely different method of proving that $\mathcal{D} = L(L(\lambda_k+\rho),\, L(\lambda_k+\rho))$ (see [**Jo3**]).

The equality $\mathcal{D} = L(L(\lambda_k+\rho),\, L(\lambda_k+\rho))$ is remarkably stable under translation. Indeed, let E be a finite dimensional $U(\mathfrak{g})$-module, and N any direct summand of the $U(\mathfrak{g})$-module $E \otimes_{\mathbb{C}} L(\lambda_k+\rho)$. Then $L(N,\, N) \cong \mathcal{D}^{U(\mathfrak{r}^-)}(N)$, the ring of "twisted differential operators" on N, regarded as a $U(\mathfrak{r}^-)$-module. Here $\mathfrak{r}^-$ is a certain abelian sub-Lie algebra of $\mathfrak{n}^-$, related to $\overline{\mathcal{X}}_k$. See (III, §3) for the full details.

0.10. In order to complete the proof of Theorem 0.3 it suffices to show that $J(k)$ is a maximal ideal of $U(\mathfrak{g})$. This is proved in Chapter IV and follows from the fact that, since λ_k is known and $J(k) = ann\, L(\lambda_k+\rho)$ (see (0.5)), one can use the results of Barbasch-Vogan, Joseph, et al to determine whether $J(k)$ is maximal.

Finally, the results on $SO(k)$ invariants are proved in Chapter V. For any k it is not difficult to show, in the notation of (0.6), that

$$R \;=\; \psi(U(\mathfrak{sp}(2n))) \;\subseteq\; \mathcal{D}(A)^{\mathbb{Z}/2\mathbb{Z}} \;\subseteq\; \mathcal{D} \;=\; \mathcal{D}(\overline{\mathcal{X}}_k).$$

Thus, when $k < n$, Theorem 0.3 forces $R = \mathcal{D}(A)^{\mathbb{Z}/2\mathbb{Z}} = \mathcal{D}$, which is a major part of Theorem 0.6. However, if $k = n$ then $\mathcal{D}$ is not even finitely generated as an R-module. Instead, some explicit calculations are needed to prove that $R = \mathcal{D}(A)^{\mathbb{Z}/2\mathbb{Z}}$ (see (V, Theorem 3.11)).

0.11. The results obtained in this paper have some connection with unipotent representations of the classical groups in question. For example, the algebra $\mathcal{D}(A)$ introduced in (0.6) is naturally a Harish-Chandra bimodule for the pair $(\mathfrak{sp}(2n), Sp(2n))$ and, as such, decomposes as a direct sum $\mathcal{D}(A) = R \oplus \mathcal{D}_-$, where $R = U(\mathfrak{g})/J(k)$ and $\mathcal{D}_-$ are irreducible Harish-Chandra bimodules. At least when k is odd, with $k < n$, these two modules can be viewed as concrete realisations (in terms of differential operators) of the two unipotent representations attached to the orbit $\mathbf{O}_k$ in [**BV2**, Theorem III]. For more details see (V, §5).

Again in relation to unipotent representations, we remark that the orbits $\mathbf{O}_k$ considered here are complex analogues of the orbits $G' \cdot \mu_k$ of [**Ad2**, pp.144-5]. Similarly, the representations $\pi(\mu)$ and $\pi(\mu)^{\pm}$ introduced in [**Ad2,** ibid] and [**Ad1**, Definition 4.6] correspond in the orthogonal case to the Harish-Chandra bimodules R and $\mathcal{D}_-$.

0.12. The results of this paper may also be viewed as part of a programme that aims to attach a completely prime, primitive ideal J to some of the nilpotent orbits $\mathbf{O}$ in $\mathfrak{g}$ and to give the structure of a commutative ring to certain highest weight modules $L(\lambda)$. In this paper, $J = J(k)$, $\mathbf{O} = \mathbf{O}_k$ and $L(\lambda) = L(\lambda_k + \rho) = \mathcal{O}(\overline{\mathcal{X}}_k)$. The results we have obtained also continue the idea, begun in [**LS**] and [**LSS**] ,of realizing the corresponding primitive factor domain $U(\mathfrak{g})/J$ as an algebra (or subalgebra) of the ring of differential operators on some irreducible component of $\overline{\mathbf{O} \cap \mathfrak{n}^+}$. By (0.5.3), $\overline{\mathcal{X}}_k$ is such a component. The results of [**LSS**] for Lie algebras of type A_n, C_n and D_n are just the case $k = 1$ of Theorem 0.3. However, [**LSS**] also proves a corresponding result for Lie algebras of type B_n, E_6 and E_7, and it would be interesting to know if Theorem 0.3 could be extended to cover these cases. Recently, Goncharov has also asserted (without proof) that ψ is surjective in the case $k = 1$ (see [**Go**]).

0.13. Finally, consider $\mathcal{D}(\mathcal{Z})$, where $\mathcal{Z}$ is an irreducible affine algebraic variety. As noted in (0.1) for the cubic cone, it can happen that $\mathcal{D}(\mathcal{Z})$ has almost no pleasant properties. Nevertheless we have now found a number of examples where this algebra has a nice structure; for example, when:

$$\begin{array}{lll} \mathcal{O}(\mathcal{Z}) &= \mathcal{O}(\mathcal{X})^{G'} & \text{is as in (0.2); see Corollary 0.4,} \\ \mathcal{O}(\mathcal{Z}) &= \mathcal{O}(\mathcal{X})^{SO(k)} & \text{see Theorem 0.6,} \\ \mathcal{O}(\mathcal{Z}) &= \mathcal{O}(\mathbb{C}^n)^{G} & \text{for } G \text{ finite; see [Le1],} \\ \mathcal{Z} & \text{is a quadratic cone in } \mathbb{C}^n\text{; see [LSS].} & \end{array}$$

It would be interesting to know for what other varieties $\mathcal{Z}$ the ring of differential operators is pleasant. In each of the cases mentioned above, $\mathcal{Z}$ has rational singularities, but unfortunately this condition is insufficient by itself to ensure that $\mathcal{D}(\mathcal{Z})$ is pleasant.

This is illustrated by the following example. Let

$$R = \mathbb{C}[x_1, x_2, x_3]/(x_1^3 + x_2^3 + x_3^3) = \mathcal{O}(\mathcal{X})$$

be the coordinate ring of the cubic cone. Then $< \sigma >= \mathbb{Z}/3\mathbb{Z}$ acts on R by the rule $\sigma : (x_1, x_2, x_3) \to (\omega^2 x_1, \omega x_2, \omega x_3)$, where ω is the cube root of unity. By [**Wa,** §4.6], $R^{\mathbb{Z}/3\mathbb{Z}}$ *is a* 2*-dimensional, normal ring with an isolated rational singularity at the origin*. Moreover, (x_1, x_2, x_3) is the unique prime ideal of R that is fixed by $\mathbb{Z}/3\mathbb{Z}$. Using the techniques of, for example [**Le1**], it follows easily that $\mathcal{D}(R)^{\mathbb{Z}/3\mathbb{Z}} \cong \mathcal{D}(R^{\mathbb{Z}/3\mathbb{Z}})$. Thus, combining [**BGG**] with standard results on fixed rings (see, in particular [**Mo,** Corollaries 1.12 and 2.6]), one obtains that $\mathcal{D}(R^{\mathbb{Z}/3\mathbb{Z}})$ *is neither Noetherian nor simple*. Further details may be found in [**Le3**].

This raises the following question:

0.13.1. Suppose that $\mathcal{Z}$ has rational singularities. Then what other hypotheses are required to ensure that the algebra $\mathcal{D}(\mathcal{Z})$ is finitely generated or simple or Noetherian?

Some related questions are given in [**CS**]. As a particular special case of (0.13.1) and as a generalisation of the commutative theory:

0.13.2. Let G be a reductive, algebraic group acting linearly on $\mathbb{C}^n$ and set $\mathcal{O}(\mathbb{C}^n)^G = \mathcal{O}(\mathcal{Z})$. Then does $\mathcal{D}(\mathcal{Z})$ satisfy the properties listed in (0.13.1)?

Finally:

0.13.3. When is $\mathcal{O}(\mathcal{Z})$ a simple $\mathcal{D}(\mathcal{Z})$-module? For example, this is the case if $\mathcal{Z}$ satisfies the hypotheses of (0.13.2) (see I, Proposition 3.5).

It is easy to see that the simplicity of $\mathcal{D}(\mathcal{Z})$ forces ${}_{\mathcal{D}(\mathcal{Z})}\mathcal{O}(\mathcal{Z})$ to be simple. However, the converse is false. An example is given by the subring

$$\mathcal{O}(\mathcal{Z}) = \mathbb{C} + x\mathbb{C}[x, y] + y^2\mathbb{C}[x, y]$$

of the polynomial ring $\mathbb{C}[x, y]$. This example was found jointly with M. Chamarie. The details are left to the reader.

0.14. Much of this research was conducted while the second author was visiting the University of Paris VI in November 1986, and he would like to thank the department there for its hospitality and financial support. This work was first reported at the conference in honour of I.N. Herstein in Chicago in March 1987.

INDEX OF NOTATION

Throughout this paper we will use the Lie algebra notation from **[B1]** while **[Di]** and **[Ja]** will form the basic references for enveloping algebras. The following notation will be assumed without comment. Unless otherwise stated, $\mathfrak{g}$ will denote a semi-simple, complex Lie algebra, with a Cartan subalgebra $\mathfrak{h}$ and corresponding triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. The Weyl group of $\mathfrak{g}$ will be denoted by W and the weight vector in $\mathfrak{g}$ corresponding to a root $\alpha \in \mathfrak{h}^*$ will be written X_α. Write ρ for the half sum of the positive roots and let $M(\lambda+\rho)$ denote the Verma module with highest weight λ and unique irreducible factor $L(\lambda+\rho)$. Write $J(\lambda+\rho)$ for the annihilator $ann_{U(\mathfrak{g})}(L(\lambda+\rho))$. Unless otherwise specified, all tensor products and vector spaces will be over $\mathbb{C}$ while all algebras will be $\mathbb{C}$-algebras. Given a vector space V, denote $Hom_{\mathbb{C}}(V, \mathbb{C})$ by V^*.

For more specialised notation the reader may refer to the following index. Many pieces of notation are given in Sections (II, §2), (II, §3) and (II, §4) for Cases A, B and C respectively. In order to combine these cases, we will use the notation (II, m.3), for example, to denote (II, 2.3), (II, 3.3) and (II, 4.3) in Cases A, B and C respectively.

(**0.1**) $\mathcal{D}(R)$, $\mathcal{D}_m(R)$, $\mathcal{D}(\mathcal{X})$

(**0.2**) $M_{p,q}(\mathbb{C})$

(**0.3**) $J(k)$, sufficiently small

(**I, 1.1**) $U^\sim$, $\Gamma = Sp(U^\sim)$, $<\ ,\ >^\sim$, classical reductive dual pair (G, G')

(**I, 1.2**) $U^\sim = U \oplus U^*$, $gr^n\mathcal{D}(U^*) = \Omega_n/\Omega_{n-1}$, $S_n(U)$, $S^n(U)$, metaplectic representation ω

(**I, 1.5**) φ, ψ

(**I, 3.1**) $\mathfrak{sp}^{(i,j)}$, $\mathfrak{g}^{(i,j)}$, $\mathfrak{p}^+$

(**II, 1.2**) λ_k, $J(k)$ (see also (**II, m.6**))

(**II, m.1**) V, E, F, k, p, q, n

(**II, m.3**) $\mathcal{X}_k$, $\overline{\mathcal{X}}_k$, $I(m)$, $g \cdot X$

(**II, 4.3**) I_k, J_k

(**II, m.4**) M, $\mathfrak{m}$, $\mathfrak{r}^+$, $\mathfrak{r}^-$, $\mathcal{I}(\mathcal{Y})$

(**II, 3.4**) $Sym_n(\mathbb{C})$

(**II, 4.4**) $Alt_n(\mathbb{C})$

(**II, 5.1**) π_+

(**II, 5.2**) $\widetilde{k}$, $\overline{I}(j)$, Killing form κ

(**II, 6.1**) $\mathcal{V}(J_k) = \overline{\mathbf{O}}_k$

(**III, 1.2**) $R = \psi(U(\mathfrak{g}))$, $\mathcal{C}(p)$, $\mathcal{D}(\mathcal{X})_p$

(**III, 1.3**) $Der A$

(**III, 1.6**) V_k

(**III**, **1**.**9**) $\mathfrak{k}$, module of $\mathfrak{k}$-finite vectors $L(M, N)$

(**III**, **2**.**1**) W_k, π_-

(**III**, **2**.**3**) R^-, $P = MR^-$

(**III**, **2**.**5**) J_r, K_r

(**IV**) (most Lie algebra notation may be found in (IV, 2.2 and 2.3))

(**IV**, **3**.**4**) left tableau $A(w)$

(**V**, **1**.**1**) T, $\{\pm\} = \{1, \sigma\}$

(**V**, **1**.**4**) $\mathcal{D}_{det}$

(**V**, **1**.**8**) $\mathcal{Y}$, $\mathcal{Y}_t$, $\mathcal{Z}$, $\mathcal{Z}_t$, π

(**V**, **1**.**9**) $\widetilde{Z}_{ij}$

(**V**, **2**.**2**) R, R', ψ'

(**V**, **2**.**3**) S, $\mathcal{D}$

CHAPTER I. REDUCTIVE DUAL PAIRS AND THE HOWE CORRESPONDENCE

One of the basic strategies behind the proof of Theorem 0.3 of the introduction is to use the machinery of Howe's papers [**Ho1**] and [**Ho2**]. Since these papers are unpublished, we collect in this chapter the notation and general results from these papers that we will need. In Chapter II we will give the more detailed analysis that will be required in the proofs of our main results. Thus, for example and in the notation of the introduction, it is shown in [**Ho1**] that $\mathcal{O}(\overline{\mathcal{X}}_k) = L(\lambda_k + \rho)$ is a simple highest weight module, and hence that $J(k)$ is a primitive ideal. But it needs the more explicit computations of Chapter II to find λ_k and the associated variety of $J(k)$.

1. Reductive Dual Pairs.

1.1. Let $U^\sim$ be a complex vector space, equipped with a symplectic form $< \ , \ >^\sim$. Write Γ for the group $Sp(U^\sim, < \ , \ >^\sim)$ and suppose that G and G' are two reductive subgroups of Γ. Then (G, G') is called a *reductive dual pair* if G and G' are mutual commutators in Γ. The pair is called a *classical reductive dual pair* if G and G' are also classical Lie groups. Set $\mathfrak{sp}(U^\sim) = Lie(\Gamma)$. In general, closed subgroups of Γ will be denoted by capital roman letters while the corresponding Lie subalgebras of $\mathfrak{sp}(U^\sim)$ will be denoted by the same letter in lower case German script. Thus, write $\mathfrak{g} = Lie\, G$ and $\mathfrak{g}' = Lie\, G'$. Note that $\mathfrak{g}$ is the centraliser of $\mathfrak{g}'$ in $\mathfrak{sp}(U^\sim)$, and vice-versa.

The classical reductive dual pairs (G, G') in Γ have been classified in [**Ho1**] and all arise from the following construction. One may write $U^\sim = E \otimes V$, where E and V are equipped with bilinear forms such that $< \ , \ >^\sim$ may be deduced from these forms. Then G and G' are appropriate subgroups of $GL(E)$ and $GL(V)$, respectively. The precise subgroups of $GL(E)$ and $GL(V)$ that occur will be described in Chapter II, but that level of detail is unnecessary for the development of this chapter. The action of a group H on a space V will always be written as $h \cdot v$ for $h \in H$ and $v \in V$.

1.2. In the cases that interest us, $U^\sim$ admits a polarisation $U^\sim = U \oplus U^*$ such that $G' \subseteq GL(U)$. Thus, as a subgroup of Γ, G' will act on U^* via the contragradient representation; $(g \cdot u^*)(u) = u^*(g^{-1} \cdot u)$, for $u \in U$, $u^* \in U^*$ and $g \in G'$. We will always identify the ring of regular functions $\mathcal{O}(U^*)$ with the symmetric algebra $S(U)$ via $< \ , \ >^\sim$. Thus $u \in U$ corresponds to $< u, \ >^\sim \in \mathcal{O}(U^*)$. This in turn identifies $\mathcal{D}(U^*) = \mathcal{D}(\mathcal{O}(U^*))$ with $\mathcal{D}(S(U))$ and hence with $S(U) \otimes S(U^*)$ as left $S(U)$-modules.

Here an element $u^* \in U^*$ is viewed as a derivation on $S(U)$ by the rule $v \mapsto u^*(v)$ for $v \in S(U)$. As will be seen shortly, this naturally gives rise to an action of Γ (and hence of G') on $\mathcal{D}(U^*)$ and $\mathcal{O}(U^*)$.

First, however, we need some more notation. Filter $\mathcal{D}(U^*)$ by total degree; thus set $\Omega_0 = \mathbb{C}$, $\Omega_1 = (U \oplus U^*) \oplus \Omega_0$ and for $m \geq 2$ let $\Omega_m = \Omega_{m-1}\Omega_1$. Write $gr^n(\mathcal{D}(U^*)) = \Omega_n/\Omega_{n-1}$ (where $\Omega_{-1} = 0$). Note that

$$gr\mathcal{D}(U^*) \;=\; \bigoplus_{n\geq 0} gr^n\mathcal{D}(U^*) \;\cong\; S(U^\sim).$$

Similarly for a vector space W, write $S_m(W) = \{f \in S(W) : deg f \leq m\}$ and $S^m(W) = S_m(W)/S_{m-1}(W)$. In particular,

$$gr^2\mathcal{D}(U^*) \;\cong\; S^2(U^\sim).$$

Moreover, the bracket $[P, Q] = PQ - QP$ on $\mathcal{D}(U^*)$ induces the structure of a Lie algebra, denoted by $\mathfrak{sp}$, on $gr^2\mathcal{D}(U^*)$ and there is an isomorphism $\omega : \mathfrak{sp}(U^\sim) \to \mathfrak{sp}$. In fact we may even identify $\mathfrak{sp}(U^\sim)$ with a Lie subalgebra of $\mathcal{D}(U^*)$. For, the set of *anticommutators* $ab + ba$ of elements $a, b \in S^1(U^\sim) = U \oplus U^*$ forms a Lie subalgebra $\mathfrak{a}$ inside $\mathcal{D}(U^*)$. The projection from $\mathcal{D}(U^*)$ onto $\mathcal{D}(U^*)/\Omega_1$ induces an isomorphism between $\mathfrak{a}$ and $\mathfrak{sp}$. Thus we will henceforth identify $\mathfrak{sp} = S^2(U^\sim)$ with $\mathfrak{a}$ and regard ω as a map from $\mathfrak{sp}(U^\sim)$ into $\mathcal{D}(U^*)$. This is called the *metaplectic representation* of $\mathfrak{sp}(U^\sim)$. (All of this is described in [**Ho1**, Theorems 4 and 5], but it is straightforward to write down $\omega(\mathfrak{sp}(U^\sim))$, as we will do in the next section.)

1.3. Now consider the actions of the Lie groups. First, the adjoint action of $\omega(\mathfrak{sp}(U^\sim))$ on $\mathcal{D}(U^*)$ integrates to give an action of Γ on $\mathcal{D}(U^*)$ as algebra automorphisms. Equivalently,

$$\xi \cdot P \;=\; [\omega(\xi), P] \qquad \text{for } \xi \in \mathfrak{sp}(U^\sim),\;\; P \in \mathcal{D}(U^*)$$

and the map ω is Γ-equivariant, where Γ is given the adjoint action on $U(\mathfrak{sp}(U^\sim))$. By [**Ho1**, Theorem 5], this action of Γ on $\mathcal{D}(U^*)$ is just the natural extension of the given action of Γ on $U^\sim = S^1(U^\sim)$.

We will need to view the action of G' on $\mathcal{O}(U^*)$ and $\mathcal{D}(U^*)$ in a number of different ways and we should emphasise that they are all the same. First, G' acts on $\mathcal{D}(U^*)$ by restricting the Γ-action. Of course, this is the natural extension of the action of G' on $U^\sim = U \oplus U^*$ obtained from the inclusion $G' \subset GL(U)$. Secondly, the contragradient representation of G' on U^* gives an action of G' on $\mathcal{O}(U^*)$ by $(g \cdot \theta)(u^*) = \theta(g^{-1} \cdot u^*)$, for $g \in G', \theta \in \mathcal{O}(U^*)$ and $u \in U^*$. When $\mathcal{O}(U^*)$ and $S(U)$ are identified by means of

$< \ , \ >^{\sim}$ this becomes the natural action of G' on $S(U)$ extending that on U. Finally, one also has the abstract action of G' on $\mathcal{D}(U^*)$ given by

$$(g \cdot P) * \theta \;=\; g \cdot (P * (g^{-1} \cdot \theta)) \qquad \text{for } g \in G',\, P \in \mathcal{D}(U^*) \text{ and } \theta \in \mathcal{O}(U^*). \tag{1.3.1}$$

Let us check, for example, that this is the same action as the one we began with. Thus, let $P \in U^*$ (identified with derivations on $S(U)$) and $\theta \in U$. Then

$$(g \cdot P) * \theta \;=\; g \cdot (P * (g^{-1} \cdot \theta)) \;=\; g \cdot (P(g^{-1} \cdot \theta)) \;=\; P(g^{-1} \cdot \theta),$$

where the final equality comes from the fact that $P(g^{-1} \cdot \theta) \in \mathbb{C}$. Thus the actions do indeed coincide.

1.4. Set $\mathcal{D}(U^*)^{G'} = \{P \in \mathcal{D}(U^*) : g \cdot P = P \text{ for all } g \in G'\}$, and define $\mathcal{O}(U^*)^{G'}$ similarly. By universality, the Lie algebra homomorphism ω defined in (1.2) extends to a ring homomorphism $\omega : U(\mathfrak{sp}(U^{\sim})) \to \mathcal{D}(U^*)$. The starting point of our investigation is:

THEOREM. [**Ho1,** Theorem 7] $\mathcal{D}(U^*)^{G'} = \omega(U(\mathfrak{g}))$.

REMARKS. With respect to the notation in [**Ho1**], we have interchanged the roles of G and G', as it will be $\mathfrak{g}$ rather than $\mathfrak{g}'$ that will be our main interest.

We will not give a proof of this theorem here, since that would involve a fair amount of extra notation and invariant theory. However, the idea behind the proof is fairly easy. The isomorphism $S(U^{\sim}) \cong gr\mathcal{D}(U^*)$ is G'-equivariant and so, by classical invariant theory, $(gr\mathcal{D}(U^*))^{G'}$ is generated by the degree 2 invariants. Moreover, under the identification $gr^2\mathcal{D}(U^*) \cong \mathfrak{sp}(U^{\sim})$ given in (1.2), the degree 2 invariants are simply $\mathfrak{g}$. The theorem follows from these observations by means of a straightforward induction.

1.5. The inclusion $\mathcal{O}(U^*)^{G'} \subset \mathcal{O}(U^*)$ induces a homomorphism

$$\varphi : \; \mathcal{D}(U^*)^{G'} \;\to\; \mathcal{D}(\mathcal{O}(U^*)^{G'}),$$

and hence, by Theorem 1.4, a homomorphism $\psi = \varphi\omega$ from $U(\mathfrak{g})$ to $\mathcal{D}(\mathcal{O}(U^*)^{G'})$. This raises the natural question as to the image and kernel of ψ, and much of this paper is devoted to answering it. For, it will be the case that U^* is the variety $\mathcal{X}$ of (0.2) of the introduction, while G' is the group $GL(k)$, $O(k)$ or $Sp(2k)$ that acts upon it. Similarly, $\mathfrak{g}$ is the Lie algebra defined in (0.3).

2. Formulae for the metaplectic representation.

2.1. As was remarked in (1.2), the Lie algebra $\mathfrak{sp}(U^\sim, <\ ,\ >^\sim)$ can be identified under the metaplectic representation ω with a subalgebra $\mathfrak{sp}$ of $\mathcal{D}(U^*)$ .It will be useful to have an explicit description of $\mathfrak{sp}$ in terms of the standard basis for $\mathcal{D}(U^*)$, and we give such a description in this section.

2.2. Fix a symplectic basis $\{u_1, \ldots, u_m, u_{-1}, \ldots, u_{-m}\}$ of $U^\sim$; thus $U^* = \bigoplus \mathbb{C}u_{-i}$. Since $U^* \cong \mathbb{C}^m$, $\mathcal{D}(U^*)$ is the m^{th} Weyl algebra, which we write here as

$$\mathcal{D}(U^*) \quad = \quad \mathbb{C}[q_1, \ldots, q_m, p_1, \ldots, p_m],$$

where $p_i = \partial/\partial q_i$. Under the identification $\mathcal{D}(U^*) \cong S(U) \otimes S(U^*)$, q_i corresponds to u_i and p_i to $-u_{-i}$. Now consider $\mathfrak{sp}$. In terms of the given symplectic basis, $\mathfrak{sp}$ may be written as the algebra of $2m \times 2m$ matrices

$$\mathfrak{sp}(2m) \quad = \quad \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} : \ A \in \mathfrak{gl}(m),\ B = {}^tB,\ C = {}^tC \right\}. \tag{2.2.1}$$

This then identifies the Cartan subalgebra $\mathfrak{h}^\sim$ of $\mathfrak{sp}(2m)$ with the diagonal matrices. It is convenient to write the matrix units in $\mathfrak{gl}(2m)$ as $\{E^\sim_{ij}\}$. It is immediate that the set of elements

$$\{H_i^\sim \quad = \quad -E^\sim_{i,i} + E^\sim_{m+i,m+i}\}$$

then forms a basis for $\mathfrak{h}^\sim$. Moreover, $\mathfrak{h}^\sim$ acts on $U^\sim$ by sending u_i to $-u_i$ but u_{-i} to u_{-i}.

Note that our basis $\{H_i^\sim\}$ of $\mathfrak{h}^\sim$ is the negative of the usual one. The reason for this non-standard choice is that elsewhere it will allow us to be consistent with other, equally standard notation. In particular, let $\mathfrak{sp} = \mathfrak{n}^+ \oplus \mathfrak{h}^\sim \oplus \mathfrak{n}^-$ be a triangular decomposition of $\mathfrak{sp}$. Then in terms of the matrix decomposition (2.2.1), $\mathfrak{n}^+$ contains the *lower* block triangular matrices $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$. We want this to happen because it implies that $S^2(U^*) \subseteq \mathfrak{n}^+$. This in turn means that, for example, in (3.1) we will be able to work with highest weight modules rather than lowest weight modules.

2.3. As remarked in (1.2), $\mathfrak{sp}$ is spanned by the set of anticommutators of elements of $S^1(U^\sim)$ inside $\mathcal{D}(U^*)$. It follows easily from this that our choice of Cartan subalgebra provides the following basis for $\mathfrak{sp}$. Here we take a dual basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$ of $\{H_1^\sim, \ldots, H_n^\sim\}$ and use the root space decomposition of $\mathfrak{sp}$ given in [**B1,** Planche III, p.254].

Then the basis elements are as follows:

$$
\begin{array}{ccccccccc}
 & & H_i^{\sim} & = & -\,q_ip_i-\frac{1}{2} & \text{for } 1\le i\le m; & & & \\
X_{-(\varepsilon_i+\varepsilon_j)} & = & -q_iq_j & \text{and} & X_{(\varepsilon_i+\varepsilon_j)} & = & p_ip_j & \text{for} & 1\le i<j\le m;\\
X_{-(\varepsilon_i-\varepsilon_j)} & = & -q_ip_j & \text{and} & X_{(\varepsilon_i-\varepsilon_j)} & = & -q_jp_i & \text{for} & 1\le i<j\le m;\\
X_{-2\varepsilon_i} & = & -\frac{1}{2}q_i^2 & \text{and} & X_{2\varepsilon_i} & = & \frac{1}{2}p_i^2 & \text{for} & 1\le i\le m.
\end{array}
$$

3. Preliminary results on the structure of $\mathbf{S(U)^{G'}}$ and $\mathbf{ker}(\psi)$.

3.1. For the moment, consider $\mathfrak{sp}$ as $S^2(U^\sim) \cong \Omega_2/\Omega_1$ (notation (1.2)). Then $\mathfrak{sp}$ decomposes;

$$\mathfrak{sp} \;=\; \mathfrak{sp}^{(0,2)} \oplus \mathfrak{sp}^{(1,1)} \oplus \mathfrak{sp}^{(2,0)},$$

where $\mathfrak{sp}^{(0,2)} = S^2(U^*)$, $\mathfrak{sp}^{(1,1)} = U \otimes U^*$ and $\mathfrak{sp}^{(2,0)} = S^2(U)$. In particular, $\mathfrak{sp}^{(1,1)} \oplus \mathfrak{sp}^{(0,2)}$ is a parabolic subalgebra of $\mathfrak{sp}$ with abelian radical $\mathfrak{sp}^{(0,2)}$ and Levi factor $\mathfrak{sp}^{(1,1)} \cong \mathfrak{gl}(U)$. (Equivalently, this is just the parabolic decomposition arising from (2.2.1). Thus, for example, $\mathfrak{sp}^{(0,2)}$ is the set of block lower triangular matrices $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$ in (2.2.1).) If we now identify $\mathfrak{sp}$ with a subalgebra of $\mathcal{D}(U^*)$, as in (2.3), then we get the same decomposition except that $\mathfrak{sp}^{(1,1)}$ must be identified with the set of anticommutators $\{ab + ba : a \in U,\, b \in U^*\}$.

By construction, $G' \subseteq GL(U)$ and so $\mathfrak{g}' \cong \omega(\mathfrak{g}') \subseteq \mathfrak{sp}^{(1,1)}$ (notation (1.2)). On the other hand, $\omega(\mathfrak{g})$ decomposes;

$$\omega(\mathfrak{g}) \;=\; \mathfrak{g}^{(0,2)} \oplus \mathfrak{g}^{(1,1)} \oplus \mathfrak{g}^{(2,0)}, \qquad \text{for} \quad \mathfrak{g}^{(i,j)} = \mathfrak{g} \cap \mathfrak{sp}^{(i,j)}.$$

(This decomposition of $\omega(\mathfrak{g})$ is implicit in **[Ho2]**, but can be proved directly as follows. By definition, $\omega(\mathfrak{g}) = \{M \in \mathfrak{sp} : G' \cdot M = M\}$. Since $\mathfrak{sp} = \mathfrak{sp}^{(0,2)} \oplus \mathfrak{sp}^{(1,1)} \oplus \mathfrak{sp}^{(2,0)}$ is an G'-module decomposition, it follows that $M \in \omega(\mathfrak{g})$ if and only if each $M_{(i,j)} \in \mathfrak{g}^{(i,j)}$, where $M_{(i,j)}$ denotes the projection of M into $\mathfrak{sp}^{(i,j)}$. Therefore, $\omega(\mathfrak{g}) = \mathfrak{g}^{(0,2)} \oplus \mathfrak{g}^{(1,1)} \oplus \mathfrak{g}^{(2,0)}$; as required. An explicit description of this decomposition will be given in (II, §§2,3 and 4).) Once again, $\mathfrak{p}^+ = \mathfrak{g}^{(1,1)} \oplus \mathfrak{g}^{(0,2)}$ is a parabolic subalgebra of $\omega(\mathfrak{g})$, with Levi factor $\mathfrak{g}^{(1,1)}$ and abelian radical $\mathfrak{g}^{(0,2)}$.

3.2. Since (G, G') is a dual pair, it is clear that the action of $\mathfrak{sp}$ as differential operators on $\mathcal{O}(U^*) = S(U)$ restricts to give an action of $\mathfrak{g}$ on $S(U)^{G'}$. (This is another way of viewing the map ψ of (1.5).) Similarly, $S^2(U)^{G'} = \mathfrak{g}^{(2,0)}$. Thus the Fundamental Theorem of Invariant Theory implies that $\mathbb{C}[\,\mathfrak{g}^{(2,0)}]$, the subalgebra of $S(U)$ generated by $\mathfrak{g}^{(2,0)}$, is precisely $S(U)^{G'}$. Moreover, one has:

PROPOSITION. *The $\mathfrak{g}$-module $S(U)^{G'}$ is a simple highest weight module. Furthermore, the constant function $1 \in S(U)^{G'}$ is a highest weight vector.*

PROOF: This is contained within **[Ho1,** Theorems 8, 9] and **[Ho2,** Theorem 3.9]. However, as it is fairly easy to give a direct proof, we will do so here. The fact that 1 is a highest weight vector follows directly from the next remark. The simplicity of $S(U)^{G'}$ will be proved in (3.5) after we have explored some of the consequences of the proposition.

REMARK. In fact $S(U)^{G'}$ is even a quotient of a generalised Verma module. For, the description of $\mathfrak{sp} \subset \mathcal{D}(U^*)$ in (2.3) shows that

$$\mathfrak{p}^+ * 1 \ \subseteq\ (\mathfrak{sp}^{(1,1)} \oplus \mathfrak{sp}^{(0,2)}) * 1 \ \subseteq\ \mathbb{C} * 1.$$

Thus $S(U)^{G'} \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^+)} V$ for some 1-dimensional $U(\mathfrak{p}^+)$-module V. We will be describing $\mathfrak{p}^+$ and $S(U)^{G'}$ in more detail in the next chapter.

3.3. COROLLARY. *(Notation 1.4) The ideal $J(k) = ker(\psi)$ is a completely prime, primitive ideal of $U(\mathfrak{g})$.*

PROOF: Since $\psi(U(\mathfrak{g})) \subset \mathcal{D}(S(U)^{G'})$ is a domain (see [**MR,** Chapter XV, Theorem 5.5]), certainly $J(k)$ is completely prime. On the other hand, $J(k)$ is, by definition, the annihilator in $U(\mathfrak{g})$ of $S(U)^{G'}$. Thus the proposition implies that $J(k)$ is primitive.

3.4. REMARKS. (i) In the cases that interest us, $J(k)$ is a maximal ideal. This will be proved in Chapter IV.

(ii) Note that the centre of $\mathfrak{g}$ acts by scalar multiplication on $S(U)^{G'}$, and so one may equally well identify $Im(\psi)$ with $U([\mathfrak{g}, \mathfrak{g}])/Ker\,\psi'$, where ψ' is the restriction of ψ to $[\mathfrak{g}, \mathfrak{g}]$. When $\mathfrak{g} = \mathfrak{gl}(n)$ (which is the only case where this is relevant) the centre of $\mathfrak{g}$ will act trivially on $S(U)^{G'}$. This is proved in (III, Remark 2.7).

3.5. We actually prove the following more general version of Proposition 3.2. Observe that, combined with Theorem 1.4, this next result does imply Proposition 3.2.

PROPOSITION. *Let K be a reductive, algebraic group and $K \to GL(V)$ be a finite dimensional representation. Thus this induces an action of K on $\mathcal{O}(V)$ and on $\mathcal{D}(V)$, as in (1.3). Then $\mathcal{O}(V)^K$ is a simple $\mathcal{D}(V)^K$-module.*

PROOF: While this result is more general than Proposition 3.2, the proof is essentially that given [**Ho1**]. Let $K^\wedge$ denote the isomorphism classes of finite dimensional representations of K. Since the action of K on $\mathcal{D}(V)$ is locally finite, we may decompose $\mathcal{D}(V)$ as $\mathcal{D}(V) = \mathcal{D}(V)^K \oplus \mathcal{D}(V)_K$, where $\mathcal{D}(V)_K = \bigoplus\{\mathcal{D}(V)_\chi : \chi \in K^\wedge \setminus \{1\}\}$ and $\mathcal{D}(V)_\chi$ is the isotypic component of type χ.

Let $f \in \mathcal{O}(V)^K$, with $f \neq 0$. Certainly, there exists $D \in \mathcal{D}(V)$ such that $D * f = 1$, and in order to prove the proposition we need to find $D' \in \mathcal{D}(V)^K$ such that $D' * f = 1$. Write $D = D^\sharp + D_\sharp$ for $D^\sharp \in \mathcal{D}(V)^K$ and $D_\sharp \in \mathcal{D}(V)_K$. We aim to prove that $D^\sharp * f = 1$. Let F be the K-submodule of $\mathcal{D}(V)_K$ generated by $D_\sharp$ and set

$F_e = \mathbb{C} \cdot D_\sharp * f \subseteq \mathcal{O}(V)$. Suppose that $F_e \neq 0$. Then we make F_e into a 1-dimensional K-module by giving it the trivial K-action.

Observe that, if $k \in K$, then

$$(kD) * f \ = \ k(D * k^{-1} f) \ = \ k(D * f) \ = \ k \cdot 1 \ = \ 1 \ = \ D * f \ = \ D_\sharp * f + D^\sharp * f.$$

Similarly,

$$(kD) * f \ = \ (kD_\sharp + D^\sharp) * f \ = \ (kD_\sharp) * f + D^\sharp * f.$$

Subtracting gives $(kD_\sharp) * f \ = \ D_\sharp * f$. But this implies that we can define a non-zero K-module map $F \to F_e$ by $P \mapsto P * f$, contradicting the fact that $F \subseteq \mathcal{D}(V)_K$. This contradiction implies that $D_\sharp * f = 0$ and hence that $D^\sharp * f = 1$; as required.

CHAPTER II. CLASSICAL REDUCTIVE DUAL PAIRS: EXPLICIT CALCULATIONS

1. Introduction.

1.1. In this chapter we look in detail at the classical reductive dual pairs $(G, G') \subset Sp(U^{\sim}))$. These have been classified in **[Ho1]** and there are 3 possibilities:

$$\begin{array}{llll} \text{(Case A)} & GL(p+q) \times GL(k) & \subset & Sp(2(kp+kq)), \\ \text{(Case B)} & Sp(2n) \times O(k) & \subset & Sp(2nk), \\ \text{(Case C)} & O(2n) \times Sp(2k) & \subset & Sp(4nk). \end{array}$$

The apparent symmetry between Cases B and C will be lost as soon as one writes $U^{\sim} = U \oplus U^*$, as in (I, 1.2).

The aim of this chapter is to study these cases in sufficient detail to prove the results stated in (0.5) of the introduction. The idea behind (say) the calculation of the highest weight λ_k of $S(U)^{G'}$ as a module over $\mathfrak{g} = Lie(G)$ is easy enough. From the explicit descriptions of the dual pair (G, G') one easily computes the image of $\mathfrak{g}$ under the metaplectic representation. Since the highest weight vector v of $S(U)^{G'}$ is $v = 1$ (see (I, Proposition 3.2)), the weight of this vector can then be read off. This particular computation, together with most of the other basic material for Cases A, B and C is given in Sections 2, 3 and 4, respectively. Section 5, among other things, provides notation that unifies the 3 cases, while Section 6 determines the associated variety $\mathcal{V}(J(k))$ of $J(k) = ker(\psi)$ (notation (I, 1.5)).

1.2. Throughout the chapter, we will build on the general results of Chapter I, and so the notation of that chapter will be used throughout. In particular, the polarisation $U^{\sim} = U \oplus U^*$, the groups $\Gamma = Sp(U^{\sim})$, G and G' with their Lie algebras $\mathfrak{sp}(U^{\sim})$, $\mathfrak{g}$ and $\mathfrak{g}'$ will be as described in (I, 1.1). However, to simplify the notation we identify $\mathfrak{g}$ and $\mathfrak{sp}(U^{\sim})$ with their images in $\mathcal{D}(U^*)$ under the metaplectic representation ω (notation (I, 1.2)). Here $\mathfrak{g} = \mathfrak{gl}(p+q)$, $\mathfrak{g} = \mathfrak{sp}(2n)$ and $\mathfrak{g} = \mathfrak{so}(2n)$ in Cases A, B and C, respectively. Notice that the algebra $\mathfrak{g}$ is independent of k, although ω and the highest weight module $S(U)^{G'} = \mathcal{O}(U^*)^{G'}$ do depend upon k. Thus for each k we will write $S(U)^{G'} = L(\lambda_k + \rho)$ and $J(k) = J(\lambda_k + \rho) = ann\, L(\lambda_k + \rho)$. By (I, 3.3),

$$J(k) \;=\; ker\{\psi : U(\mathfrak{g}) \to \mathcal{D}(S(U)^{G'})\}.$$

We remark that, in Case A, $J(k)$ depends on both p and q rather than just on $p+q$.

2. Description of Case A : $\mathbf{GL(p+q)\times GL(k)}$.

2.1. In this section we describe the basic material concerning Case A. In particular we compute the highest weight λ_k of $S(U)^{G'} = L(\lambda_k+\rho)$.

Fix integers k, p and q, all greater than zero, and let V, E and F be complex vector spaces of dimensions k, p and q, respectively. Set

$$U \;=\; (V\otimes E)\oplus(V^*\otimes F^*)$$

and identify U^* with $(V^*\otimes E^*)\oplus(V\otimes F)$. Write $U^\sim = U\oplus U^*$. Then $U^\sim$ has a natural symplectic form $<\ ,\ >^\sim$ given by

$$< u+u^*,\, v+v^* >^\sim \;=\; v^*(u)-u^*(v) \qquad \text{for } u,v\in U \text{ and } u^*,v^*\in U^*.$$

It is convenient to fix dual bases $\{v_1,\dots,v_k\}$ and $\{v_1^*,\dots,v_m^*\}$ of V and V^*, and similarly for E and F. Thus an equivalent definition of $<\ ,\ >^\sim$ is that

$$< v_i\otimes e_j,\, v_\ell^*\otimes e_m^* >^\sim \;=\; \delta_{i\ell}\delta_{jm} \quad\text{and}\quad < v_i^*\otimes f_j^*,\, v_\ell\otimes f_m >^\sim \;=\; \delta_{i\ell}\delta_{jm}, \qquad (2.1.1)$$

with all other products being zero.

Note that the above construction is symmetric in E and F^* and so, without loss of generality, we will always assume that $p\ge q$.

It is proved in **[Ho1,** §3] that the pair (G,G'), for $G=GL(E\oplus F)=GL(p+q)$ and $G'=GL(V)=GL(k)$, forms a reductive dual pair in $Sp(U^\sim)$. Note, in particular, that G' embeds in $GL(U)$ under its natural linear action on V, as is required by (I, 1.2).

2.2. It is useful to reinterpret (2.1) in terms of matrices. The specific choice of bases for $V,\dots,F^*$ gives identifications

$$V^*\otimes E^* \;\cong\; Hom(V,E^*) \;\cong\; M_{p,k}(\mathbb{C})$$

and $V\otimes F\cong Hom(F^*,V)\cong M_{k,q}(\mathbb{C})$. Thus $U^*\cong M_{p,k}(\mathbb{C})\times M_{k,q}(\mathbb{C})$. By tracing the action of G' through these isomorphisms one finds that G' acts on U^* by the rule

$$g\cdot(a,b) \;=\; (ag^{-1},gb) \qquad \text{for } g\in G', a\in M_{p,k}(\mathbb{C}) \text{ and } b\in M_{k,q}(\mathbb{C}).$$

(The inverse arises because G' acts contragradiently on $Hom(V,\_)$.)

Let $X=(x_{ij})$ and $Y=(y_{ij})$ be generic matrices of size $p\times k$ and $k\times q$, respectively. Here x_{ij} and y_{ij} will be identified with the coordinate functions on $e_i^*\otimes v_j^*\in M_{p,k}(\mathbb{C})$

and $v_i \otimes f_j \in M_{k,q}(\mathbb{C})$, respectively. Thus

$$\begin{aligned}\mathcal{O}(U^*) &= \mathbb{C}[\,x_{ij}, y_{\ell m}\ ; 1 \le i \le p,\ \ 1 \le j, \ell \le k,\ \ 1 \le m \le q] \\ &= \mathbb{C}[\,X, Y].\end{aligned}$$

Identify $\mathcal{O}(U^*)$ with $S(U)$, as in (I, 1.2). Then it is useful to note that (2.1.1) implies that, for all i and j, $x_{ij} = e_i \otimes v_j$ and $y_{ij} = v_i^* \otimes f_j^*$ as elements of $U \subset S(U)$.

2.3. By the remarks of (I, 1.3), the action of G' on U^* given in (2.2) uniquely defines an action of G' on $\mathcal{O}(U^*) = S(U)$. If $g \in G'$, then this is given by $g \cdot x_{ij} = (Xg)_{ij}$, the $(i,j)^{th}$ coefficient of the product matrix Xg. Similarly, $g \cdot y_{ij} = (g^{-1}Y)_{ij}$. It is natural to abbreviate these actions as $g \cdot X = Xg$ and $g \cdot Y = g^{-1}Y$. Given this notation, we can state the following results from classical invariant theory (see [**We**] or [**DP,** Theorems 3.1 and 3.4]).

THEOREM. **(i)** *$\mathcal{O}(U^*)^{G'} = \mathbb{C}[X, Y]^{G'} = \mathbb{C}[\,XY]$, where $\mathbb{C}[\,XY]$ denotes the ring generated by all coefficients of the product matrix XY.*

(ii) *If $Z = (z_{ij})$ is a generic $q \times p$ matrix and $0 \le m$ is an integer, let $I(m)$ be the ideal of $\mathbb{C}[Z]$ generated by all $(m+1) \times (m+1)$ minors of Z. Then $\mathcal{O}(U^*)^{G'} \cong \mathbb{C}[Z]/I(k)$.*

(iii) *For $0 \le m$, let $\overline{\mathcal{X}}_m$ denote the subvariety $\{\xi \in M_{q,p}(\mathbb{C}) : rk\,\xi \le m\}$ of $M_{q,p}(\mathbb{C})$. Then $\mathcal{O}(U^*)^{G'} \cong \mathbb{C}[Z]/I(k) = \mathcal{O}(\overline{\mathcal{X}}_k)$.*

The reason for the bar in $\overline{\mathcal{X}}_k$ is that, at least for $m \le q$, $\overline{\mathcal{X}}_k$ is the Zariski closure of $\mathcal{X}_k = \{\xi \in M_{q,p}(\mathbb{C}) : rk\ \xi = m\}$.

Eventually we will wish to identify $S(U)^{G'}$ with $\mathcal{O}(\overline{\mathcal{X}}_k)$ but since this identification needs a little care (see (5.2) below), at this stage it is more convenient to regard them as distinct objects.

2.4. If $k \ge q$ then it is immediate from (2.3) that $\overline{\mathcal{X}}_k = \overline{\mathcal{X}}_q = M_{q,p}(\mathbb{C})$ and so $\mathcal{O}(\overline{\mathcal{X}}_k)$ is a polynomial ring. As was explained in (0.3) and (0.4) this case is of limited interest to us. Thus we will mainly be concerned with the case when k is *sufficiently small*; that is $1 \le k < q$. When k is sufficiently small, $\overline{\mathcal{X}}_k$ is singular, with singular subvariety, *Sing* $\overline{\mathcal{X}}_k = \overline{\mathcal{X}}_{k-1}$. This is a well-known fact for which we do not know a good reference, although it can be obtained from [**Br,** §3]. However it is very easy to prove directly, as follows. Let *Sing* $\overline{\mathcal{X}}_k$ have defining ideal $I \subset \mathbb{C}[Z]$. Then the Jacobian criterion [**Ku,** Theorem 1.15, p.171] implies that, as a radical ideal, I is generated by the $\{\partial/\partial z_{ij}(f)\}$ where f runs through the generators of $I(k)$; that is, f runs through the $(k+1)\times(k+1)$ minors of Z. Thus I is generated by the $k\times k$ minors of Z and $Sing\ \overline{\mathcal{X}}_k = \overline{\mathcal{X}}_{k-1}$. Therefore, if $\overline{I}(k-1)$ denotes the image of $I(k-1)$ in $\mathcal{O}(\overline{\mathcal{X}}_k) = \mathbb{C}[Z]/I(k)$, then $Sing\ \overline{\mathcal{X}}_k$ has defining ideal $\mathcal{I}(Sing\ \overline{\mathcal{X}}_k) = \overline{I}(k-1)$.

Consider $M = GL(E)\times GL(F)$, identified in the natural way with a subgroup of $G = GL(E\oplus F)$. Then M acts on $M_{q,p}(\mathbb{C})$ by

$$(g_1, g_2)\cdot\xi \quad = \quad g_2\xi g_1^{-1} \text{ for } (g_1,g_2)\in M \text{ and } \xi\in M_{q,p}(\mathbb{C}).$$

Note that the orbits in $M_{q,p}(\mathbb{C})$ under this action are the $\mathcal{X}_r$ for $0\le r\le q$. This is, of course, just the statement that any matrix is equivalent to a diagonal matrix.

2.5. It is easy to identify the parabolic subalgebra $\mathfrak{p}^+\subset\mathfrak{g}=\mathfrak{gl}(E\oplus F)$ considered in (I, 3.1). For, by the comments of that section,

$$\mathfrak{m} \quad = \quad \mathfrak{g}\cap\mathfrak{sp}^{(1,1)} \quad = \quad \mathfrak{g}\cap\mathfrak{gl}(U) \quad = \quad \mathfrak{gl}(E)\times\mathfrak{gl}(F).$$

Note that $\mathfrak{m} = Lie\ M$, so the notation is consistent with (2.4). Next, consider $\mathfrak{r}^+ = Hom(E,F)\subset\mathfrak{g}$. Since $Hom(E,F)\cong Hom(F^*,E^*)$, the algebra $\mathfrak{r}^+$ acts on $U^\sim$ by sending $V\otimes E$ to $V\otimes F$ and $V^*\otimes F^*$ to $V^*\otimes E^*$. Thus, by (2.1), $\mathfrak{r}^+$ maps U to U^*. Therefore, under the metaplectic representation, $\mathfrak{r}^+$ identifies with a subalgebra of $\mathfrak{sp}^{(0,2)} = S^2(U^*)$. Similarly, $\mathfrak{r}^- = Hom(F,E)\subset\mathfrak{sp}^{(2,0)}$. Thus $\mathfrak{g}$ does indeed decompose as $\mathfrak{g} = \mathfrak{r}^+\oplus\mathfrak{m}\oplus\mathfrak{r}^-$ where $\mathfrak{r}^+ = \mathfrak{g}^{(0,2)}$, $\mathfrak{m} = \mathfrak{g}^{(1,1)}$, and $\mathfrak{r}^- = \mathfrak{g}^{(2,0)}$, in the notation of (I, 3.1).

In terms of matrices we may therefore write

$$\mathfrak{g} \quad = \quad \left\{\begin{pmatrix} A & B\\ C & D\end{pmatrix} :\ A\in\mathfrak{gl}(p),\ B\in M_{p,q}(\mathbb{C}),\ C\in M_{q,p}(\mathbb{C}) \text{ and } D\in\mathfrak{gl}(q)\right\}.$$

Under this representation, $\mathfrak{m}$, $\mathfrak{r}^+$ and $\mathfrak{r}^-$ may be identified with the subalgebras of matrices of the form

$$\mathfrak{m} \quad = \quad \left\{\begin{pmatrix} A & 0\\ 0 & D\end{pmatrix}\right\}, \quad \mathfrak{r}^+ \quad = \quad \left\{\begin{pmatrix} 0 & 0\\ C & 0\end{pmatrix}\right\} \quad \text{and} \quad \mathfrak{r}^- \quad = \quad \left\{\begin{pmatrix} 0 & B\\ 0 & 0\end{pmatrix}\right\}.$$

This accords with the matrix decomposition of $\mathfrak{sp}$ given in (I, 2.2).

Eventually, we will want to identify $\overline{\mathcal{X}}_k$ with a subvariety of $\mathfrak{r}^+$, but as this can be done simultaneously for all 3 cases, it will be deferred until (5.2).

2.6. We next want to compute the image under the metaplectic representation ω of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ as this will allow us to calculate the highest weight λ_k of $S(U)^{G'}$ as a $\mathfrak{g}$-module. This has two stages. First, identify $\mathfrak{h}$ inside the Cartan subalgebra $\mathfrak{h}^{\sim}$ of $\mathfrak{sp}(2m) = \mathfrak{sp}(U^{\sim})$. Then use the explicit description of ω in (I, 2.3) to write down $\omega(\mathfrak{h})$.

Compare the notation of this section with that of (I, §2). In particular, the basis of U is

$$\begin{aligned}\{u_1, ..., u_m\} &= \{x_{ij} = e_i \otimes v_j \ : \ 1 \le i \le p,\ 1 \le j \le k\} \\ &\qquad \cup \{y_{ij} = v_i^* \otimes f_j^* \ : \ 1 \le i \le k,\ 1 \le j \le q\}.\end{aligned}$$

Now consider the action of $\mathfrak{m} = \mathfrak{gl}(E) \times \mathfrak{gl}(F)$ on these elements. Since $\mathfrak{gl}(E)$ acts on $E \otimes V$ via its natural left action on E, the matrix unit $E_{ij} \in \mathfrak{gl}(E)$ acts on these elements by

$$E_{ij} \cdot x_{ab} = \delta_{ja} x_{ib} \quad \text{and} \quad E_{ij} \cdot y_{ab} = 0.$$

However, $GL(F)$ acts on $V^* \otimes F^*$ via the contragradient representation. Thus the matrix unit $F_{ij} \in \mathfrak{gl}(F)$ acts by the rules

$$F_{ij} \cdot x_{ab} = 0 \quad \text{and} \quad F_{ij} \cdot y_{ab} = -\delta_{ib} y_{aj}.$$

In the notation of (I, 2.2), therefore, E_{ii} is the sum of k of the elements $(E^{\sim}_{jj} - E^{\sim}_{j+m,j+m}) = -H^{\sim}_j$, and similarly for the F_{ii}. Now, (I, 2.3) says that, under the metaplectic representation,

$$\begin{aligned}\{H^{\sim}_j\} &= \{-q_\ell p_\ell - \tfrac{1}{2} :\ 1 \le \ell \le m = nk\} \\ &= \{-x_{st}\partial/\partial x_{st} - \tfrac{1}{2},\ \ -y_{uv}\partial/\partial y_{uv} - \tfrac{1}{2} :\ 1 \le s \le p,\ 1 \le t, u \le k,\ 1 \le v \le q\}.\end{aligned}$$

Thus $E_{ii} \in \mathfrak{gl}(E)$ maps to the element

$$\Big(\sum_{j=1}^{k} x_{ij}\partial/\partial x_{ij}\Big) + \tfrac{1}{2}k \ \in \ \mathcal{D}(U^*),$$

while $F_{ii} \in \mathfrak{gl}(F)$ maps to the element $-\big(\sum_{j=1}^{k} y_{ji}\partial/\partial y_{ji}\big) - \frac{1}{2}k$.

The matrix representation of $\mathfrak{g}$ in (2.5) ensures that the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is spanned by $\{E_{11}, \dots, E_{pp}, F_{11}, \dots, F_{qq}\}$. Thus we take as a basis for the Cartan subalgebra $\mathfrak{h}_1$ of $\mathfrak{sl}(p+q) = [\mathfrak{g}, \mathfrak{g}]$ the following elements:

$$\begin{aligned} t_i &= E_{i+1,i+1} - E_{i,i}\ ; \quad 1 \le i \le p-1, \\ t_p &= F_{1,1} - E_{p,p} \\ t_{p+j} &= F_{j+1,j+1} - F_{j,j}\ ; \quad 1 \le j \le q-1. \end{aligned}$$

We emphasise that this choice of basis is dictated by the triangular decomposition of $\mathfrak{g}$ given in (2.5). By the comments of the last paragraph, the action of the t_i on the vector

$1 \in S(U)^{G'}$ is given by

$$t_i * 1 \;=\; 0 \text{ for } i \neq p \quad \text{but} \quad t_p * 1 \;=\; -k.$$

2.7. Let $\varpi_1, ..., \varpi_{p+q}$ be the fundamental weights in $\mathfrak{h}_1^*$, as defined for example in [**B1,** Planche I, p.250]. Then $\varpi_p = t_p^*$ and so the final equation of (2.6) shows that the vector $1 \in S(U)^{G'}$ has weight $\lambda_k = -k\varpi_p$. Combined with (I, Proposition 3.2) this proves:

PROPOSITION. *Consider* $S(U)^{G'} = L(\lambda_k + \rho)$ *as a module over* $\mathfrak{sl}(p+q) = [\mathfrak{g}, \mathfrak{g}]$. *Then* $S(U)^{G'}$ *has highest weight* $\lambda_k = -k\varpi_p$.

REMARK. The centre of $\mathfrak{g}$ is spanned by the element $\eta = \sum E_{ii} + \sum F_{jj}$. By (2.6) η acts trivially on $1 \in S(U)^{G'}$. Thus the centre of $\mathfrak{g}$ acts trivially on all of $S(U)^{G'}$ and it makes little difference whether one regards $S(U)^{G'}$ as a module over $\mathfrak{g}$ or $[\mathfrak{g}, \mathfrak{g}]$.

3. Description of Case B : $Sp(2n) \times O(k)$.

3.1. Here we repeat the calculations of Section 2 for the case of $Sp(2n) \times O(k)$. Inevitably, a number of the results will be exactly analogous to the corresponding result of Section 2, and in such cases some of the details will be left to the reader.

In this section V will denote a k-dimensional vector space equipped with a non-degenerate symmetric bilinear form $(\ ,\ )$ while E will be a $2n$-dimensional space with a symplectic form denoted by $<\ ,\ >$. (In this case the auxilary vector space F is unnecessary because one uses $(\ ,\ )$ to identify V and V^*, by equating $v \in V$ with $(v,\ ) \in V^*$. This allows one to combine E and F.) Take a polarisation $E = L \oplus L^*$ and set $U = V \otimes L$. Then the identification of V with V^* means that U^* can be identified with $V \otimes L^*$, and so we may take $U^\sim = V \otimes E = U \oplus U^*$. On this occasion $<\ ,\ >^\sim$ is just the form $(\ ,\ ) \otimes <\ ,\ >$.

Let $G' = O(V)$ and $G = Sp(E)$. Then it follows from [**Ho1,** §3] that (G, G') is a reductive dual pair inside $Sp(U^\sim)$. Moreover, G' acts linearly on U via the natural embedding $G' \subset GL(V) \subset GL(U)$.

3.2. Pick an orthonormal basis $\{v_1, \dots, v_k\}$ for V and a symplectic basis $\{e_1, \dots, e_n, e_{-1}, \dots, e_{-n}\}$ for E such that $L = \bigoplus \mathbb{C}e_i$ and $L^* = \bigoplus \mathbb{C}e_{-i}$. This provides the matrix representation $U^* = V \otimes L^* \cong Hom(L, V) \cong M_{k,n}(\mathbb{C})$. The induced action of $g \in G'$ on $\xi \in M_{k,n}(\mathbb{C})$ is simply $g \cdot \xi = g\xi$, where g is identified with a matrix in $O(k) \subset GL(k)$ by means of the basis $\{v_i\}$ of V. Write $\mathcal{O}(U^*) = \mathbb{C}[X]$ where $X = (x_{ij})$ is a generic $k \times n$ matrix and x_{ij} is the coordinate function on $v_i \otimes e_{-j}$. When (I, 1.2) is used to identify $\mathcal{O}(U^*)$ with $S(U)$, this implies that x_{ij} identifies with $v_i \otimes e_j \in U \subset S(U)$.

3.3. In the notation of (2.3), the induced action of $g \in G'$ on $\mathbb{C}[X] = \mathcal{O}(U^*)$ is given by $g \cdot x_{ij} = (g^{-1}X)_{ij}$, which we again write as $g \cdot X = g^{-1}X$. The following theorem therefore follows from classical invariant theory (see [**We**] or [**DP,** Theorems 5.6 and 5.7]).

THEOREM. **(i)** $\mathcal{O}(U^*)^{G'} = \mathbb{C}[X]^{G'} = \mathbb{C}[{}^tXX]$, *the ring generated by all coefficients of the product matrix* ${}^tXX$.

(ii) *Let* $Z = (z_{ij})$ *be a generic, symmetric* $n \times n$ *matrix and, for any* $m \geq 0$ *, let* $I(m)$ *denote the ideal of* $\mathbb{C}[Z]$ *generated by all* $(m+1) \times (m+1)$ *minors of* Z *. Then* $\mathcal{O}(U^*)^{G'} \cong \mathbb{C}[Z]/I(k)$.

(iii) *For* $m \geq 0$ *, let* $\overline{\mathcal{X}}_m$ *denote the subvariety of* $M_n(\mathbb{C})$ *consisting of all symmetric* $n \times n$ *matrices of rank* $\leq m$ *. Then* $\mathbb{C}[Z]/I(k) = \mathcal{O}(\overline{\mathcal{X}}_k)$.

The reason for the bar in $\overline{\mathcal{X}}_m$ is that, if $m \leq n$, then $\overline{\mathcal{X}}_m$ is the Zariski closure of the set $\mathcal{X}_m = \{\text{all symmetric } n \times n \text{ matrices of rank } = m\}$.

3.4. Note that $O(\overline{\mathcal{X}}_k)$ is a polynomial ring when $k \geq n$. Thus we will usually be interested in the case when k is *sufficiently small;* that is $1 \leq k < n$. In this case $\overline{\mathcal{X}}_k$ is singular with $Sing\, \overline{\mathcal{X}}_k = \overline{\mathcal{X}}_{k-1}$. Once again this is part of the folklore. It can either be proved by using [**Br,** §3] or by using the Jacobian criterion, as in (2.4). (In the latter case the following observation is useful. Given a generic, symmetric $t \times t$ matrix $W = (w_{ij})$, then $\partial/\partial w_{ii}(det\, W) = \widetilde{W}_{ii}$ but $\partial/\partial w_{ij}(det\, W) = \pm 2\widetilde{W}_{ij}$ for $i \neq j$. Here $\widetilde{W}_{ij}$ is the $(i,j)^{th}$ minor of W. See for example [**Mi,** Ex.22, p.193].) Thus $\mathcal{I}(Sing\, \overline{\mathcal{X}}_k) = \overline{I}(k-1)$, where $\overline{I}(k-1)$ is the image of $I(k-1)$ in $\mathcal{O}(\overline{\mathcal{X}}_k) = \mathbb{C}[Z]/I(k)$.

If $G = Sp(2n)$ is written matricially in terms of the basis $\{e_1, \dots, e_{-n}\}$ of E, then $M = GL(L)$ is identified with the matrix subgroup

$$M \;\cong\; \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & {}^t\alpha^{-1} \end{pmatrix} : \alpha \in GL(n) \right\} \;\subseteq\; G.$$

Therefore M acts on the variety $Sym_n(\mathbb{C})$ of all symmetric $n \times n$ matrices by the action $g \cdot \xi = g\xi({}^tg)$ for $g \in M$ and $\xi \in Sym_n(\mathbb{C})$. Since any symmetric matrix is congruent to a diagonal matrix, the M-orbits in $Sym_n(\mathbb{C})$ are precisely the $\mathcal{X}_r$ for $0 \leq r \leq n$.

3.5. In terms of the basis $\{e_1, \dots, e_{-n}\}$ of E, the Lie algebra $\mathfrak{g} = \mathfrak{sp}(E)$ identifies with the set of $2n \times 2n$ matrices of the form

$$\mathfrak{g} \;=\; \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \;:\; A \in \mathfrak{gl}(n),\, B = {}^tB \text{ and } C = {}^tC \right\}. \tag{3.5.1}$$

This makes it easy to identify the parabolic subalgebra $\mathfrak{p}^+ \subset \mathfrak{g}$ (notation (I, 3.1)). For, as in (2.5), $\mathfrak{m} \;=\; \mathfrak{g} \cap \mathfrak{sp}^{(1,1)} \;=\; \mathfrak{g} \cap \mathfrak{gl}(U) \;=\; \mathfrak{gl}(L)$. Thus, in terms of (3.5.1),

$$\mathfrak{m} \;=\; \left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix} \;:\; A \in \mathfrak{gl}(n) \right\}.$$

Once again, $\mathfrak{m} = Lie\, M$, and so this notation does accord with (3.4). In a similar manner, we write $\mathfrak{r}^+$ for the subalgebra of $\mathfrak{g}$ obtained by putting $A = B = 0$ and $\mathfrak{r}^-$ for the subalgebra given by taking $A = C = 0$. Then certainly $\mathfrak{g} = \mathfrak{r}^- \oplus \mathfrak{m} \oplus \mathfrak{r}^+$. As in (2.5) it is clear that $\mathfrak{r}^- \subset \mathfrak{sp}^{(2,0)}$ and $\mathfrak{r}^+ \subset \mathfrak{sp}^{(0,2)}$, and hence that $\mathfrak{r}^- = \mathfrak{g}^{(2,0)}$ and $\mathfrak{r}^+ = \mathfrak{g}^{(0,2)}$, in the notation of (I, 3.1). This substantiates the claim made in (I, 3.1) that $\mathfrak{g} = \mathfrak{g}^{(0,2)} \oplus \mathfrak{g}^{(1,1)} \oplus \mathfrak{g}^{(2,0)}$.

3.6. The matrix representation of $\mathfrak{g}$ in (3.5) suggests that we choose

$$\mathfrak{h} \;=\; \left\{\begin{pmatrix} h & 0 \\ 0 & -h \end{pmatrix} \;:\; h \text{ a diagonal matrix in } \mathfrak{gl}(L)\right\}$$

for the Cartan subalgebra of $\mathfrak{g}$. Given the parabolic decomposition of $\mathfrak{g}$ in (3.5), we choose $\{t_i \;=\; -E_{i,i}+E_{n+i,n+i} : 1 \leq i \leq n\}$ as the basis for $\mathfrak{h}$. We now mimic (2.6) in order to compute the image of the t_ℓ under the metaplectic representation. Since $G = Sp(E)$ acts on $U = L\otimes V$ via its natural action on E, t_ℓ acts on $v_i \otimes e_j = x_{ij}$ by the rule $t_\ell \cdot x_{ij} \;=\; -\delta_{j\ell}v_i\otimes e_\ell \;=\; -\delta_{j\ell}x_{i\ell}$. But by (3.2),

$$\mathcal{D}(U^*) \;=\; \mathbb{C}[x_{ij},\ \partial/\partial x_{ij} \;:\; 1\leq i\leq k,\ 1\leq j\leq n].$$

Thus, in terms of the explicit basis for $\mathfrak{sp}\subset\mathcal{D}(U^*)$ given in (I, 2.3), this implies that

$$t_\ell \;=\; -\Big(\sum_{i=1}^{k} x_{i\ell}\partial/\partial x_{i\ell}\Big) - \tfrac{1}{2}k \;\in\; \mathcal{D}(U^*), \qquad \text{for } 1\leq \ell\leq n.$$

3.7. Let $\varpi_1,...,\varpi_n$ be the fundamental weights for $\mathfrak{g}$ as defined, for example, in [**B1,** Planche III, p.254]. Then by (3.6), $t_\ell * 1 = -\frac{1}{2}k$ for $1\leq\ell\leq n$ and so $1\in S(U)^{G'}$ is a vector of weight $-\frac{1}{2}k\varpi_n$. Combined with (I, Proposition 3.2) this proves

PROPOSITION. *The highest weight of* $S(U)^{G'} = L(\lambda_k+\rho)$ *as a module over* $\mathfrak{g} = \mathfrak{sp}(2n)$ *is* $\lambda_k = -\frac{1}{2}k\varpi_n$.

REMARK. The value of λ_k can also be obtained by setting $\lambda = (0,\ldots,0)$ in [**KV,** Chapter II, Theorem 7.2] and our proof is very similar to this special case of the proof in [**KV**]. Similarly, the value of λ_k in Case A follows from [**KV,** Chapter III, Theorem 7.2]. However, as most of the results leading up to our calculation of λ_k will be needed later in the paper, the direct computation of λ_k has necessitated very little extra work.

4. Description of Case C : $\mathbf{O(2n) \times Sp(2k)}$.

4.1. The construction here is very similar to that of Case B, except that the roles of Sp and O are interchanged. Thus V now denotes a $2k$-dimensional vector space equipped with a symplectic form $<\ ,\ >$, while E is a $2n$-dimensional vector space with a non-degenerate, symmetric form $(\ ,\ )$. As in (3.1) we identify $v \in V$ with $< v,\ \ > \in V^*$. Thus, put $E = L \oplus L^*$ and $U = V \otimes L$, identify U^* with $V \otimes L^*$ and write $U^\sim = U \oplus U^* = V \otimes E$. Then $<\ ,\ >^\sim = <\ ,\ > \otimes (\ ,\ )$ is a symplectic form on $U^\sim$. If $G = O(E)$ and $G' = Sp(V)$ then (G, G') is again a classical reductive dual pair in $Sp(U^\sim)$ (see **[Ho1, §3]**).

4.2. Fix a symplectic basis $\{v_1, \ldots, v_k, v_{-1}, \ldots, v_{-k}\}$ for V and dual bases $\{e_1, \ldots, e_n\}$ and $\{e_{-1}, \ldots, e_{-n}\}$ for L and L^*, respectively. Thus

$$(e_i, e_j) = \delta_{-i,j} \ , \quad < v_i, v_j > = \delta_{-i,j} \quad \text{but} \quad < v_{-i}, v_j > = -\delta_{i,j}$$

for the appropriate indices i and j, with i positive. This provides identifications

$$U^* = V \otimes L^* \cong Hom(L, V) \cong M_{2k,n}(\mathbb{C})$$

and $U \cong Hom(L^*, V)$. Identify $\mathcal{O}(U^*)$ with $S(U)$, as in (I, 1.2). Next, let $X = (x_{ij})$ be a generic $2k \times n$ matrix and write $\mathcal{O}(U^*) = \mathbb{C}[X]$ by identifying x_{ij} with $v_i \otimes e_j \in S(U) = \mathcal{O}(U^*)$. On this occasion, this means that x_{ij} is the coordinate function on $v_{-i} \otimes e_{-j}$ if $i > 0$ but is the coordinate function on $-v_{-i} \otimes e_{-j}$ if $i < 0$.

Identify $g \in G' = Sp(V)$ with a matrix via its action on the $\{v_i\}$. Then g acts on $\xi \in M_{2k,n}(\mathbb{C})$ by the rule $g \cdot \xi = g\xi$ and so it acts on $\mathbb{C}[X]$ by the rule $g \cdot X = g^{-1}X$ (notation (2.2)).

4.3. If I_k is the $k \times k$ identity matrix, then J_k will denote the matrix:

$$J_k = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix} \in M_{2k}(\mathbb{C}).$$

THEOREM. **(i)** $\mathcal{O}(U^*)^{G'} = \mathbb{C}[X]^{G'} = \mathbb{C}[{}^tXJ_kX]$.

(ii) *Let $Z = (z_{ij})$ be a generic, antisymmetric $n \times n$ matrix. For $m \geq 0$, let $I(m)$ denote the ideal of $\mathbb{C}[Z]$ generated by Pfaffians of the principal minors of Z of size $(2m+2) \times (2m+2)$. Then $\mathcal{O}(U^*)^{G'} \cong \mathbb{C}[Z]/I(k)$.*

(iii) *Let $\overline{\mathcal{X}}_m$ denote the variety of alternating $n \times n$ matrices of rank $\leq 2m$. Then $\mathcal{O}(U^*)^{G'} \cong \mathbb{C}[Z]/I(k) = \mathcal{O}(\overline{\mathcal{X}}_k)$.*

For a proof of this theorem, see **[We]** or **[DP,** Theorems 6.6 and 6.7]. If $2m \le n$, then $\overline{\mathcal{X}}_m$ is the Zariski closure of $\mathcal{X}_m = \{\text{alternating } n \times n \text{ matrices of rank } = 2m\}$.

4.4. It is easy to see that $\mathcal{O}(\overline{\mathcal{X}}_k)$ is a polynomial ring when $2k \ge n-1$, and so we will usually be concerned with the case when k is *sufficiently small;* that is, $2 \le 2k < n-1$. In this case $\overline{\mathcal{X}}_k$ is singular, with $Sing\ \overline{\mathcal{X}}_k = \overline{\mathcal{X}}_{k-1}$ and $\mathcal{I}(Sing\ \overline{\mathcal{X}}_k) = \overline{I}(k-1)$ (see **[Kl,** Proposition 3.2]).

If $G = O(2n)$ is written as a matrix group in terms of the basis $\{e_1, \dots, e_{-n}\}$ of E, then $M = GL(L)$ again identifies with matrices of the form

$$M \;\cong\; \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & {}^t\alpha^{-1} \end{pmatrix} \;:\; \alpha \in GL(n) \right\}.$$

Thus M acts on the variety $Alt_n(\mathbb{C})$ of alternating $n \times n$ matrices by the rule $g \cdot \xi = g\xi({}^tg)$ for $g \in M$ and $\xi \in Alt_n(\mathbb{C})$. Any alternating, $n \times n$ matrix of rank $2r$ is congruent to $\begin{pmatrix} J_r & 0 \\ 0 & 0 \end{pmatrix}$, and so the M-orbits in $Alt_n(\mathbb{C})$ are exactly the $\mathcal{X}_r$ for $0 \le r \le \frac{1}{2}n$.

4.5. In terms of the given basis $\{e_1, \dots, e_{-n}\}$ of E, the Lie algebra $\mathfrak{g} = \mathfrak{so}(E)$ identifies with the set of $2n \times 2n$ matrices of the form

$$\mathfrak{g} \;=\; \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \;: A \in \mathfrak{gl}(n),\;\; B = -{}^tB \text{ and } C = -{}^tC \right\}.$$

Inside $\mathfrak{g}$ take $\mathfrak{m}$, $\mathfrak{r}^+$ and $\mathfrak{r}^-$ to be the subalgebras consisting of matrices of the form

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \right\},$$

respectively. Thus $\mathfrak{m} \cong \mathfrak{gl}(L)$ and $\mathfrak{g} = \mathfrak{r}^- \oplus \mathfrak{m} \oplus \mathfrak{r}^+$. Just as in (2.5) one readily checks that

$$\mathbf{m} \;=\; \mathfrak{g} \cap \mathfrak{gl}(U) \;=\; \mathfrak{g} \cap \mathfrak{sp}^{(1,1)} \;=\; \mathfrak{g}^{(1,1)}$$

while $\mathfrak{r}^+ = \mathfrak{g} \cap \mathfrak{sp}^{(0,2)} = \mathfrak{g}^{(0,2)}$ and $\mathfrak{r}^- = \mathfrak{g} \cap \mathfrak{sp}^{(2,0)} = \mathfrak{g}^{(2,0)}$. Thus $\mathfrak{g} = \mathfrak{g}^{(2,0)} \oplus \mathfrak{g}^{(1,1)} \oplus \mathfrak{g}^{(0,2)}$ as claimed in (I, 3.1).

4.6. Finally, we want to calculate the highest weight of $S(U)^{G'} = L(\lambda_k + \rho)$, and this is almost identical to that in Case B. As in (3.6), the matrix representation of $\mathfrak{g}$ means that we take the Cartan subalgebra $\mathfrak{h}$ to have basis $\{t_i = -E_{i,i} + E_{n+i,n+i} \;: 1 \le i \le n\}$.

Thus $\mathfrak{h}$ acts on $S(U)$ by $t_\ell * x_{ij} = -\delta_{j\ell}x_{i\ell}$. Under the metaplectic representation (see (I, 2.3)) this implies that

$$t_\ell \quad = \quad \Big(\sum_{i=1}^{2k} x_{i\ell}\partial/\partial x_{i\ell}\Big) - k \quad \in \quad \mathcal{D}(U^*).$$

(Note that the sum now has $2k$ terms since V is $2k$-dimensional.)

4.7. Let $\overline{\omega}_1, ..., \overline{\omega}_n$ be the fundamental weights for $\mathfrak{g}$ (see, for example, [**B1,** Planche IV, p.256]). Then $t_\ell * 1 = -k$ for $1 \leq \ell \leq n$ and so the vector $1 \in S(U)^{G'}$ has weight $-2k\overline{\omega}_n$. Combined with (I, Proposition 3.2) this proves:

PROPOSITION. *The highest weight of* $S(U)^{G'}$ *as a module over* $\mathfrak{g} = \mathfrak{so}(2n)$ *is* $\lambda_k = -2k\overline{\omega}_n$.

5. Comments and Notation.

5.1. For the rest of the paper we will keep the common notation of the last 3 sections and simply refer to Cases A, B and C if we need to distinguish between them. In particular, the numbers k, $p \geq q$ and n will retain their meanings from those sections, as will the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ and variety $\overline{\mathcal{X}}_k$, etc. In fact, in many of the subsequent results we will be able to consider all 3 cases simultaneously but in order to achieve this we need to introduce some unifying notation. To save repetition, we will use the number (m.a) to refer to sections (2.a), (3.a) and (4.a), simultaneously.

The most important notation concerns the parabolic decomposition of $\mathfrak{g}$ given in (m.5). This will be denoted by

$$\mathfrak{g} \quad = \quad \left\{ z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \ A \in \mathfrak{gl}(p),\, B \in M_{p,q}(\mathbb{C}),\, C \in M_{q,p}(\mathbb{C}) \text{ and } D \in \mathfrak{gl}(q) \right\},$$

and any $z \in \mathfrak{g}$ will be written in this form. In Cases B and C one therefore has the following additional (and usually implicit) assumptions:

$$\text{(Case B)} \quad p = q = n, \quad D = -{}^tA, \quad B = {}^tB\,, \quad C = {}^tC\,;$$

$$\text{(Case C)} \quad p = q = n, \quad D = -{}^tA, \quad B = -{}^tB, \quad C = -{}^tC.$$

The subalgebra $\mathfrak{r}^+$ of $\mathfrak{g}$ is then identified with the set of all matrices of the form $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$. The subalgebras $\mathfrak{m}$ and $\mathfrak{r}^-$ are given in a similar manner. The projection of $\mathfrak{g}$ onto $\mathfrak{r}^+$ along $\mathfrak{p}^- = \mathfrak{m} \oplus \mathfrak{r}^-$ is simply the map

$$\pi_+ \ : \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}.$$

5.2. As was remarked in (m.5), it will be convenient to regard $\mathcal{X}_k$ as a subvariety of $\mathfrak{r}^+$, and we will show next how to do this. This will also permit us to identify $\mathcal{O}(\overline{\mathcal{X}}_k)$ with $S(U)^{G'}$ in an M-equivariant manner, a fact that will be important in Chapter III.

Let $\mathcal{R}$ be the variety $M_{q,p}(\mathbb{C})$, $Sym_n(\mathbb{C})$ or $Alt_n(\mathbb{C})$ in Cases A, B or C, respectively. The group M acts on $\mathcal{R}$ by the rule described in (m.4) and on $\mathfrak{r}^+$ via the adjoint action. This implies that the map

$$\eta \ : \quad \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \quad \longmapsto \quad C$$

is an M-equivariant isomorphism between $\mathfrak{r}^+$ and $\mathcal{R}$. Henceforth, this will be used to identify $\mathcal{O}(\mathfrak{r}^+)$ with $\mathcal{O}(\mathcal{R}) = \mathbb{C}[Z]$.

Now consider the variety $\overline{\mathcal{X}}_k$. Given an integer d, set

$$\widetilde{d} \;=\; d \text{ in Cases } A \text{ and } B \quad \text{but} \quad \widetilde{d} \;=\; 2d \text{ in Case } C. \tag{5.2.1}$$

Then η^{-1} restricts to give an isomorphism of M-varieties:

$$\overline{\mathcal{X}}_k \;=\; \{C\in\mathcal{R} \;: rk\, C\le \widetilde{k}\} \;\cong\; \left\{\begin{pmatrix}0 & 0\\ C & 0\end{pmatrix}\in\mathfrak{g} \;: rk\, C\le\widetilde{k}\right\} \;\subseteq\; \mathfrak{r}^+.$$

A similar description is available for $\mathcal{X}_k$ – just replace "$\le \widetilde{k}$" by "$= \widetilde{k}$". In future we will always regard $\overline{\mathcal{X}}_k$ and $\mathcal{X}_k$ as subvarieties of $\mathfrak{r}^+$ in this manner. Observe that, with $I(j)$ defined by (m.3), this identification implies that the co-morphism of the embedding $\overline{\mathcal{X}}_k\subset\mathfrak{r}^+$ is simply the map

$$\mathcal{O}(\mathfrak{r}^+)=\mathbb{C}[Z] \;\longrightarrow\; \mathbb{C}[Z]/I(k)=\mathcal{O}(\overline{\mathcal{X}}_k).$$

For any $j\le k$, write $\overline{I}(j)$ for the image of $I(j)$ in $\mathcal{O}(\overline{X}_k)$. Thus $\overline{\mathcal{X}}_j$ is the variety of zeros of $\overline{I}(j)$ in $\overline{\mathcal{X}}_k$.

We next wish to consider $\mathfrak{r}^-$ and the map $\psi' = \psi\mid_{U(\mathfrak{r}^-)}$. Since $\mathfrak{r}^-$ is abelian, $U(\mathfrak{r}^-)=S(\mathfrak{r}^-)$. By (I, 3.1), the metaplectic representation ω maps $\mathfrak{r}^-$ onto $\mathfrak{g}^{(2,0)} = S^2(U)^{G'}$. Recall from (I, 1.5) that φ restricts to the identity map on $S(U)^{G'}$ and so by (I, 3.1) ω induces the map ψ' from $U(\mathfrak{r}^-)$ onto $S(U)^{G'}$. On the other hand, the Killing form on $\mathfrak{g}$ induces an M-equivariant isomorphism $\kappa : U(\mathfrak{r}^-)\;\xrightarrow{\sim}\;\mathcal{O}(\mathfrak{r}^+)\cong\mathbb{C}[Z]$ and we will use κ to identify $U(\mathfrak{r}^-)$ with $\mathcal{O}(\mathfrak{r}^+)$. In particular, for any j, the image of $I(j)$ under κ^{-1} will again be denoted by $I(j)$.

We claim that $Ker(\psi')=I(k)$. Recall from (I, 1.3) that $\omega : U(\mathfrak{sp}(U^\sim))\to\mathcal{D}(U^*)$ is Γ-equivariant. Since $M\subset G\subset\Gamma$, M acts on $S(U)^{G'}\subset\mathcal{O}(U^*)$ and the map $\psi' : U(\mathfrak{r}^-)\to S(U)^{G'}$ is therefore M-equivariant. Thus $Ker(\psi')$ is an M-equivariant prime ideal in $U(\mathfrak{r}^-)$. Moreover, its variety of zeros $\mathcal{V}(Ker(\psi'))$ satisfies

$$dim\;\mathcal{V}(ker(\psi')) \;=\; Kdim\; S(U)^{G'} \;=\; dim\,\overline{\mathcal{X}}_k,$$

by Theorem (m.3). But (m.4) shows that the *only* M-stable, closed (and irreducible) sub-varieties of $\mathfrak{r}^+$ are the $\overline{\mathcal{X}}_j$. Thus $\mathcal{V}(Ker(\psi'))=\overline{\mathcal{X}}_k$ and $Ker(\psi')=I(k)$. In summary:

LEMMA. *There is an M-equivariant identification $U(\mathfrak{r}^-)=\mathcal{O}(\mathfrak{r}^+)=\mathbb{C}[Z]$. By applying ψ' one obtains an M-equivariant isomorphism*

$$\chi:\;\mathcal{O}(\overline{\mathcal{X}}_k) \;=\; U(\mathfrak{r}^-)/I(k) \;\xrightarrow{\sim}\; S(U)^{G'}.$$

In particular, $I(k) \;=\; Ker(\psi') \;=\; Ker(\psi)\cap U(\mathfrak{r}^-) \;=\; J(k)\cap U(\mathfrak{r}^-)$.

REMARK. It would be natural at this point to simply identify $\mathcal{O}(\overline{\mathcal{X}}_k)$ and $S(U)^{G'}$ by means of χ. However, we will not do so until Paragraph 1.4 of Chapter III since the more detailed nature of χ will be needed there. Thus for the moment let the isomorphism $\mathcal{D}(\overline{\mathcal{X}}_k) \xrightarrow{\sim} \mathcal{D}(S(U)^{G'})$ induced by χ again be denoted by χ and write ψ_1 for the map

$$\psi_1 : \ U(\mathfrak{g}) \ \xrightarrow{\psi} \ D(S(U)^{G'}) \ \xrightarrow{\chi^{-1}} \ \mathcal{D}(\overline{\mathcal{X}}_k).$$

5.3. The following facts about $\overline{\mathcal{X}}_k$ will prove useful. Given a point y in a variety $\mathcal{Y}$, write $\mathcal{O}(\mathcal{Y})_y$ for the local ring of functions regular at y.

LEMMA. **(i)** *The variety $\overline{\mathcal{X}}_k$ is normal, Cohen-Macaulay, and even has rational singularities.*

(ii) *The dimension of $\overline{\mathcal{X}}_k$ is given by the formulae*

$$\begin{array}{llcll} (\textit{Case A}) & \dim \overline{\mathcal{X}}_k & = & k(p+q-k) & \text{for} \quad 1 \le k \le q \le p, \\ (\textit{Case B}) & \dim \overline{\mathcal{X}}_k & = & nk - \frac{1}{2}k(k-1) & \text{for} \quad 1 \le k \le n, \\ (\textit{Case C}) & \dim \overline{\mathcal{X}}_k & = & 2nk - k(2k+1) & \text{for} \quad 2 \le 2k \le n. \end{array}$$

(iii) *Lower the upper bound for k by 1; that is assume that $1 \le k < q$ in Case A, $1 \le k < n$ in Case B and $2 \le 2k < n$ in Case C. Then $\dim \overline{\mathcal{X}}_k \ge \dim \overline{\mathcal{X}}_{k-1} + 2$ and $\mathcal{O}(\overline{\mathcal{X}}_k) = \mathcal{O}(\mathcal{X}_k)$.*

(iv) *For any positive value of k and any $p \in \mathcal{X}_k$, one has $\mathcal{O}(\overline{\mathcal{X}}_k)_p = \mathcal{O}(\mathcal{X}_k)_p$.*

REMARK. If $\widetilde{k}$ is large; that is, if $\widetilde{k} \ge q$ (where $q = n$ in Case B and C), then $\mathcal{O}(\overline{\mathcal{X}}_k) = \mathcal{O}(\overline{\mathcal{X}}_q) = \mathbb{C}[Z]$. Thus the lemma may be regarded as computing $\dim \overline{\mathcal{X}}_k$ for all values of k. Similarly, if $\widetilde{k} > q$ then $\mathcal{X}_k$ is empty and so part (iii) is vacuously true.

PROOF: (i) Since $\mathcal{O}(\overline{\mathcal{X}}_k)$ is the fixed ring under a linear action of G', this follows from [**Bo**].

(ii) This is well-known; see for example [**Ea,** Theorem 2], [**Jz,** Theorem 2.3] and [**Kl,** Proposition 3.1], for Cases A, B and C respectively. Alternatively, it is fairly easy to prove the formulae directly using the fact that, by (m.4), $\mathcal{X}_k$ is an M-orbit. To do this, one makes a judicious choice of $x \in \mathcal{X}_k$ and uses the formula

$$\dim \overline{\mathcal{X}}_k \ = \ \dim \mathcal{X}_k \ = \ \dim \mathfrak{m} - \dim \mathfrak{s},$$

where $\mathfrak{m} = \mathrm{Lie}\, M$ and $\mathfrak{s} = \{s \in \mathfrak{m} : s \cdot x = 0\}$ (see [**Hu,** Theorem 13.2, p.88]). The details are left to the reader.

(iii) The dimension inequality is immediate from part (i). Also, $\mathcal{X}_k = \overline{\mathcal{X}}_k \setminus \overline{\mathcal{X}}_{k-1}$ and $\overline{\mathcal{X}}_{k-1}$ is a closed subset of $\overline{\mathcal{X}}_k$. Thus [**Gr,** Lemma 1, p.239] or [**KP2,** Lemma 9.1] may be applied to give the identity $\mathcal{O}(\overline{\mathcal{X}}_k) = \mathcal{O}(\mathcal{X}_k)$.

(iv) We use the notation of the remark. Thus, if $\widetilde{k} > q$ then the remark shows that the result is vacuously true. Conversely, if $\widetilde{k} \leq q$ then $\mathcal{X}_k$ is open in $\overline{\mathcal{X}}_k$ and the result is immediate.

5.4. In Case C the construction of U requires that the vector space E has even dimension (see (4.1)). This, in turn, means that it is only for m even that we have obtained a non-trivial representation for $\mathfrak{so}(m)$ as differential operators. It would be interesting to find an analogous construction for $\mathfrak{g}_1 = \mathfrak{so}(2n+1)$.Of course, one could try to use the construction given in Case B, but with $\mathfrak{g}_1 = \mathfrak{g}'$. Unfortunately, the lack of symmetry in that construction (and in particular the fact that $G' \subseteq GL(U)$) means that this is unlikely to produce a map from $U(\mathfrak{g}_1)$ onto a ring of differential operators.

There is, however, one factor ring of $U(\mathfrak{g}_1)$, for $\mathfrak{g}_1 = \mathfrak{so}(2n{+}1)$, that can be written as $\mathcal{D}(\mathcal{Y})$ for a singular variety $\mathcal{Y}$. This uses Goncharov's construction of the Joseph ideal and is given in [**LSS**]. The variety $\mathcal{Y}$ in question is the set of isotropic vectors in $\mathbb{C}^{2n-1}$ and really corresponds to the case $k = 1$ of the present paper.

6. The Associated Variety of $\mathbf{J(k) = Ker(\psi)}$.

6.1. Recall that

$$J(k) \;=\; ann_{U(\mathfrak{g})}\mathcal{O}(\overline{\mathcal{X}}_k) \;=\; ker\{\psi_1 : U(\mathfrak{g}) \to \mathcal{D}(\overline{\mathcal{X}}_k)\}$$

(see Remark 5.2). The aim of this section is to prove the results quoted in (0.5.2) and (0.5.3) of the introduction. However in this section, unlike (0.5), we will not be assuming that k is sufficiently small. Let J be a primitive ideal of $U(\mathfrak{g})$, with associated graded ideal $gr(J) \subset S(\mathfrak{g}) = gr(U(\mathfrak{g}))$. Then, by **[Jo2,** Theorem 3.10], the associated variety of zeros $\mathcal{V}(J) \subset \mathfrak{g}$ (where we have identified $\mathfrak{g}^*$ with $\mathfrak{g}$ under the Killing form) is the closure of a nilpotent orbit in $\mathfrak{g}$. If $J = J(k)$, we denote this orbit by $\mathbf{O}_k$. Thus the aim of this section is to identify $\mathbf{O}_k$ and to prove that $\overline{\mathcal{X}}_k$ is an irreducible component of $\overline{\mathbf{O}_k \cap \mathfrak{n}^+}$.

6.2. We first need to show that $\overline{\mathcal{X}}_k \subseteq \overline{\mathbf{O}_k \cap \mathfrak{n}^+}$. There are two possible proofs of this; first, by explicitly describing $\overline{\mathcal{X}}_k$ and $\mathbf{O}_k$, and secondly by using the identifications of Section 5. We will give the former proof in (6.4), since the description of $\mathbf{O}_k$ will be needed elsewhere. However, since it is easy and natural, we will give the latter proof here.

Recall from (5.2) that $\mathcal{O}(\overline{\mathcal{X}}_k) \cong U(\mathfrak{r}^-)/I(k)$ as a $U(\mathfrak{r}^-)$-module. Alternatively, as a $U(\mathfrak{g})$-module $\mathcal{O}(\overline{\mathcal{X}}_k)$ is a simple highest weight module, $L(\lambda_k + \rho)$, with highest weight vector 1 (see (I, Propopsition 3.2)). Thus, write $L(\lambda_k+\rho) = U(\mathfrak{g})/A$, where $A = ann(1)$. Provide $U(\mathfrak{g})$ and $U(\mathfrak{r}^-)$ with their natural filtrations. Now, (I, Remark 3.2) shows that $L(\lambda_k + \rho)$ is a factor of a generalised Verma module, induced from a 1-dimensional module over $\mathfrak{p}^+ = \mathfrak{m} \oplus \mathfrak{r}^+$. Therefore the isomorphism $U(\mathfrak{g})/A \cong U(\mathfrak{r}^-)/I(k)$ is a filtered morphism. Moreover, by its definition, $I(k)$ is a determinental ideal and so is homogeneous. Thus, by passing to the associated graded objects, one obtains maps

$$S(\mathfrak{g})/gr(J(k)) \;\twoheadrightarrow\; S(\mathfrak{g})/gr(A) \;\cong\; S(\mathfrak{r}^-)/gr(I(k)) \;=\; S(\mathfrak{r}^-)/I(k).$$

But by (5.2), the variety of zeros of $I(k)$ in $\mathfrak{r}^+$ is just $\overline{\mathcal{X}}_k$. Thus the displayed equation must be the co-morphism of the map $\overline{\mathcal{X}}_k \;\cong\; \mathcal{V}(gr\,A) \;\hookrightarrow\; \mathcal{V}(J(k))$. Of course, since $L(\lambda_k + \rho)$ is induced from a 1-dimensional $\mathfrak{p}^+$-module, $gr\,A$ contains $\mathfrak{p}^+S(\mathfrak{g})$. Thus the isomorphism $\overline{\mathcal{X}}_k \cong \mathcal{V}(gr\,A)$ is just the map induced from the inclusion $\mathfrak{r}^+ \subset \mathfrak{g}$. Therefore, as subvarieties of $\mathfrak{g}$, we have shown that $\overline{\mathcal{X}}_k \subseteq \mathcal{V}(J(k)) \cap \mathfrak{r}^+$. In particular:

LEMMA. *The variety* $\overline{\mathbf{O}_k \cap \mathfrak{n}^+}$ *contains* $\overline{\mathcal{X}}_k$.

6.3. The observations of (6.2) can be extended to prove

PROPOSITION. *The variety $\mathcal{V}(J(k))$ is the closure of a nilpotent orbit $\mathbf{O}_k$ such that $\overline{\mathbf{O}_k \cap \mathfrak{n}^+}$ contains $\overline{\mathcal{X}}_k$ as an irreducible component. Moreover,*

$$dim\,\overline{\mathcal{X}}_k \;=\; dim\,\overline{\mathbf{O}_k \cap \mathfrak{n}^+} \;=\; \tfrac{1}{2}dim\,\overline{\mathbf{O}}_k.$$

PROOF: By Lemma 6.2, $\overline{\mathcal{X}}_k \subseteq \overline{\mathbf{O}_k \cap \mathfrak{n}^+}$. By the Spaltenstein-Steinberg equality, [**Jo1,** §3.1], the variety $\overline{\mathbf{O}_k \cap \mathfrak{n}^+}$ is equidimensional of dimension precisely $\frac{1}{2}dim\,\overline{\mathbf{O}}_k$. Thus, in order to prove the proposition, it suffices to show that $dim\,\overline{\mathcal{X}}_k = \frac{1}{2}dim\,\overline{\mathbf{O}}_k$. Let $GKdim$ stand for Gelfand-Kirillov dimension. Then, by [**Ja,** Satz 10.9],

$$dim\,\overline{\mathbf{O}}_k \;=\; GKdim\,U(\mathfrak{g})/J(k) \;=\; 2GKdim\,L(\lambda_k+\rho).$$

Now, by (6.2) the isomorphism $L(\lambda_k+\rho) \cong U(\mathfrak{g})/A \cong S(\mathfrak{r}^-)/I(k)$ is a filtered morphism. Therefore,

$$Gkdim_{U(\mathfrak{g})}L(\lambda_k+\rho) \;=\; GKdim_{S(\mathfrak{r}^-)}S(\mathfrak{r}^-)/I(k) \;=\; GKdim\,\mathcal{O}(\overline{\mathcal{X}}_k).$$

Thus $dim\,\overline{\mathbf{O}}_k = 2dim\,\overline{\mathcal{X}}_k$; as required.

6.4. Since the nilpotent orbits in $\mathfrak{g}$ have been classified, Proposition 6.3 makes it easy to find $\mathbf{O}_k$. The following result provides the method.

COROLLARY. *Suppose that Ω is a nilpotent orbit in $\mathfrak{g}$ such that $\mathcal{X}_k \subset \Omega$. Then $\Omega = \mathbf{O}_k$.*

PROOF: Suppose that $\Omega \neq \mathbf{O}_k$. Then, as orbits are disjoint, $\mathcal{X}_k \cap \mathbf{O}_k = \emptyset$. Now, $\mathbf{O}_k$ is open in $\overline{\mathbf{O}}_k$ (see, for example, the proof of [**Hu,** Proposition 8.3, p.60]) and hence $\mathbf{O}_k \cap \mathfrak{n}^+$ is open in $\overline{\mathbf{O}_k \cap \mathfrak{n}^+}$. But, by Proposition 6.3, $\mathcal{X}_k \subseteq \overline{\mathbf{O}_k \cap \mathfrak{n}^+}$ and so both $\mathcal{X}_k$ and $\overline{\mathcal{X}}_k$ are contained in the closed set $\Delta = \overline{\mathbf{O}_k \cap \mathfrak{n}^+} \setminus (\mathbf{O}_k \cap \mathfrak{n}^+)$. But by [**Hu,** Proposition 8.3, p.60]), Δ is a union of M-orbits of dimension strictly less than $dim\,\overline{\mathbf{O}_k \cap \mathfrak{n}^+}$. Thus

$$dim\,\overline{\mathcal{X}}_k \;\leq\; dim\,\Delta \;<\; dim\,\overline{\mathbf{O}_k \cap \mathfrak{n}^+};$$

contradicting Proposition 6.3.

6.5. We may now apply Corollary 6.4 to find $\mathbf{O}_k$. In each case, the orbit is uniquely determined by two criteria; by (5.2) the matrices in $\mathcal{X}_k$ have square zero and rank equal to $\widetilde{k}$. In the statement of these results, we will restrict the values of k, just as we did in Lemma 5.4. However, using Remark 5.4, these results do describe $\mathbf{O}_k$ for *all* positive values of k.

6.5.1. (Case A, $1 \le k \le q \le p.$) By [**KP1,** §1.1], the nilpotent orbits in $\mathfrak{gl}(p+q)$ are classified by partitions of $r = p+q$ in the following manner. Given an orbit Δ, pick $D \in \Delta$ and let D have Jordan normal form D', with Jordan blocks of size $d_1 \ge d_2 \ge \dots \ge d_t$. Then Δ corresponds to the partition $(d_1, \dots, d_t)$. Let $(2^x, 1^y)$ denote the partition $(2, \dots, 2, 1, \dots, 1)$, where 2 appears x times and 1 appears y times. Then the orbit Ω that we wish to consider corresponds to $(2^k, 1^{r-2k})$. Equivalently,

$$\Omega \;=\; \big\{z \in \mathfrak{gl}(p+q) : \;\; rk\, z \;=\; k \text{ and } z^2 \;=\; 0\big\},$$

with $\overline{\Omega} = \{z \in \mathfrak{gl}(p+q) : rk\, z \le k \text{ and } z^2 = 0\}$. The identification of $\mathcal{X}_k$ in (5.2) ensures that $\mathcal{X}_k \subset \Omega$. Thus $\Omega = \mathbf{O}_k$, by Corollary 6.4.

6.5.2. (Case B, $1 \le k \le n.$) By [**KP2,** §2.2], the nilpotent orbits in $\mathfrak{sp}(2n)$ are classified by partitions $(d_1 \;\ge\; d_2 \;\ge \dots \ge\; d_t)$ of $2n$ such that an even number of the (d_i)'s are odd (use the same algorithm as in (6.5.1)). Take Ω to be the orbit corresponding to $(2^k, 1^{2n-4k})$. Thus

$$\Omega \;=\; \big\{z \in \mathfrak{sp}(2n) : rk\, z \;=\; k \text{ and } z^2 \;=\; 0\big\}.$$

Once again $\mathcal{X}_k \subset \Omega$ and so $\Omega = \mathbf{O}_k$ by Corollary 6.4.

6.5.3. (Case C, $2 \le 2k \le n.$) Here, the nilpotent orbits in $\mathfrak{so}(2n)$ under the action of the group $O(2n)$ are classified by partitions $(d_1 \;\ge\; d_2 \dots \ge\; d_t)$ of $2n$ where, now, an even number of the d_i are even (see [**KP2,** §2.2], again). Thus, let Ω correspond to the partition $(2^{2k}, 1^{2n-4k})$ or, equivalently,

$$\Omega \;=\; \big\{z \in \mathfrak{so}(2n) : rk\, z \;=\; 2k \text{ and } z^2 \;=\; 0\big\}.$$

Of course, we need to consider $SO(2n)$-orbits. However, by [**KP2,** Proposition 2.3], Ω will be an $SO(2n)$-orbit, provided that $2k < n$. If $2k = n$ then the same result shows that $\Omega = \Omega' \cup \Omega''$ for two $SO(2n)$-orbits Ω' and Ω''. Thus, by (5.2) and Corollary 6.4, $\Omega = \mathbf{O}_k$ if $2k < n$. If $2k = n$ then $\mathcal{X}_k$ is an M-orbit (see (4.4)) and so is contained in $SO(2n)$-orbit. Thus by (5.2) and (6.4) again, $\Omega' = \mathbf{O}_k$ (after possibly re-ordering) when $2k = n$.

6.6. REMARK. It is not only in (6.5) that the case $2k = n$ of Case C is exceptional. For, at least among the values of k considered in (6.5), it is only in this case that $J(k) = ker(\psi)$ will not be a maximal ideal (see (IV, 5.4)). Of course, since this value of k is not sufficiently small, this is not relevant to the main purpose of this paper (which is the surjectivity of ψ), but it does suggest that the case $2k = n$ deserves further study.

6.7. Using the notation of (5.1) and (5.2.1), the orbit $\mathbf{O}_k$ may in each of the three cases be written as

$$\mathbf{O}_k \;=\; \{z \in \mathfrak{g} : rk\; z \;=\; \widetilde{k} \text{ and } z^2 \;=\; 0\}.$$

The closure $\overline{\mathbf{O}}_k$ of $\mathbf{O}_k$ may be obtained by replacing "$rk\, z = \widetilde{k}$" by "$rk\, z \leq \widetilde{k}$". Naturally, we assume here that k lies in the appropriate range; that is $1 \leq k \leq q \leq p$ in Case A, $1 \leq k \leq n$ in Case B and $2 \leq 2k \leq n$ in Case C. Note that, under the identifications of (5.2), $\mathbf{O}_k \cap \mathfrak{r}^+ = \mathcal{X}_k$. Further useful facts about $\mathbf{O}_k$ are contained in the next lemma.

LEMMA. *Assume that k lies in the range given above. Then*

(i) *The dimension of $\overline{\mathbf{O}}_k$ is given by the formulae*

$$\dim \overline{\mathbf{O}}_k \;=\; \begin{cases} 2k(p+q-k) & \textit{in Case A} \\ 2nk - k(k-1) & \textit{in Case B} \\ 4nk - 2k(2k+1) & \textit{in Case C.} \end{cases}$$

(ii) *The variety $\overline{\mathbf{O}}_k$ is normal, and even has rational singularities. Moreover, $\dim \overline{\mathbf{O}}_k - \dim \overline{\mathbf{O}}_{k-1} \geq 2$ and $\mathcal{O}(\overline{\mathbf{O}}_k) = \mathcal{O}(\mathbf{O}_k)$.*

PROOF: (i) For Case A, the formula for $\dim \overline{\mathbf{O}}_k$ follows from [**KP1,** Proposition 1.3], while in Cases B and C it follows from [**KP2,** Proposition 2.4] (for which the notation is given in [**KP2,** §§1.1, 2.1 and 2.2]). Alternatively, since $\dim \overline{\mathbf{O}}_k = 2\dim \overline{\mathcal{X}}_k$, this also follows from Lemma 5.3 (or vice-versa).

(ii) By [**He,** Theorem, p.108 and Criterion 2, p.109], $\overline{\mathbf{O}}_k$ is a normal variety with rational singularities. The dimension inequality follows from the classical fact that nilpotent orbits in $\mathfrak{g}$ have even dimension. Finally, the equality $\mathcal{O}(\overline{\mathbf{O}}_k) = \mathcal{O}(\mathbf{O}_k)$ follows, for example, from [**BK,** Lemma 3.7].

CHAPTER III. DIFFERENTIAL OPERATORS ON CLASSICAL RINGS OF INVARIANTS

1. Reduction of the main theorem.

1.1. The aim of this chapter is to prove the main step in the proof of Theorem 0.3 of the introduction. Thus, for k sufficiently small and in the notation of (II, Remark 5.2), we will show that (i) the rings $R = \psi_1(U(\mathfrak{g}))$ and $\mathcal{D}(\overline{\mathcal{X}}_k)$ have the same full quotient ring and (ii) $\mathcal{D}(\overline{\mathcal{X}}_k)$ is a finitely generated R-module. The main result of Chapter IV will be that R is a simple ring. It follows very easily from these results that $R = \mathcal{D}(\overline{\mathcal{X}}_k)$, as is required by Theorem 0.3.

Most of the results of this chapter will actually hold for one more value of k than is allowed by k being sufficiently small. Thus unless otherwise stated, in this chapter we will assume that:

$$\begin{aligned} &(\text{Case A}) \qquad 1 \le k \le q \le p, \\ &(\text{Case B}) \qquad 1 \le k \le n, \\ &(\text{Case C}) \qquad 2 \le 2k \le n-1. \end{aligned} \tag{1.1.1}$$

We will continue to use the notation developed in the last two chapters; thus $\mathfrak{g}$ is the Lie algebra $\mathfrak{gl}(p+q)$ in Case A, $\mathfrak{sp}(2n)$ in Case B and $\mathfrak{so}(2n)$ in Case C, with the parabolic decomposition $\mathfrak{g} = \mathfrak{r}^- \oplus \mathfrak{m} \oplus \mathfrak{r}^+$ described in (II, 2.5, 3.5 and 4.5). Further, ψ is defined in (I, 1.5), with $J(k) = ker(\psi)$, while $\mathcal{X}_k \subset \mathbf{O}_k$ are described in (II, 5.2 and 6.4). Most of the other notation that we will use, and in particular the maps χ, ψ_1, and ψ', are defined in (II, 5.1 and 5.2).

1.2. We next give an outline of the proof of the main theorem, but to do so some comments on localisation are required. Since $\mathfrak{r}^-$ is abelian and the non-zero elements of $U(\mathfrak{r}^-)$ act ad-nilpotently on $U(\mathfrak{g})$, it follows from [**KL,** Lemma 4.7] that any multiplicatively closed subset of $\psi_1(U(\mathfrak{r}^-))$ is an Ore set in $R = \psi_1(U(\mathfrak{g}))$.Recall that $\psi_1(U(\mathfrak{r}^-)) = \mathcal{O}(\overline{\mathcal{X}}_k)$. Thus, given $p \in \overline{\mathcal{X}}_k$, with corresponding maximal ideal $\mathcal{I}(p) \subset \mathcal{O}(\overline{\mathcal{X}}_k)$, set $\mathcal{C}(p) = \mathcal{O}(\overline{\mathcal{X}}_k) \setminus \mathcal{I}(p)$ and write $R_p = R_{\mathcal{C}(p)}$ for the local ring at p. Similarly, one may form

$$\mathcal{D}(\overline{\mathcal{X}}_k)_p \;=\; \mathcal{D}(\mathcal{O}(\overline{\mathcal{X}}_k)_p) \;\cong\; \mathcal{D}(\overline{\mathcal{X}}_k)_{\mathcal{C}(p)} \;\cong\; \mathcal{O}(\overline{\mathcal{X}}_k)_p \otimes_{\mathcal{O}(\overline{\mathcal{X}}_k)} \mathcal{D}(\overline{\mathcal{X}}_k)$$

(see for example [**Le1,** §1.2, p.160] or [**SmSt,** §1.3d]). Of course, $R_p \subseteq \mathcal{D}(\overline{\mathcal{X}}_k)_p$ for each $p \in \overline{\mathcal{X}}_k$, and so Theorem 0.3 would follow easily if one could prove that $\mathcal{D}(\overline{\mathcal{X}}_k)_p = R_p$ for all $p \in \overline{\mathcal{X}}_k$. If $p \in \mathcal{X}_k$ then this is easy to prove; basically because $\mathcal{D}(\overline{\mathcal{X}}_k)_p$ is then a regular ring and hence is generated by $\mathcal{O}(\overline{\mathcal{X}}_k)_p$ and its derivations (see (1.3) and (1.4), below). If $p \in \overline{\mathcal{X}}_k \setminus \mathcal{X}_k$, then $\mathcal{O}(\overline{\mathcal{X}}_k)_p$ is (usually) no longer regular and its derivations are harder to identify. Thus it is not clear how to show directly that $R_p = \mathcal{D}(\overline{\mathcal{X}}_k)_p$. Instead, a more roundabout approach is taken (see (1.6) and (1.7)). This reduces the proof of the main theorem of this chapter to that of computing the dimension of a certain subvariety of $\mathbf{O}_k$. This computation really forms the heart of the proof and is delayed until Section 2.

1.3. In this paragraph, and in (1.4), k will be arbitrary. Let M be the subgroup of G defined in (II, 2.4, 3.4 and 4.4), for Cases A, B and C respectively. Then, by those sections, M acts linearly on $\mathfrak{r}^+$ with orbits being precisely the $\mathcal{X}_t$. The differential of this action gives a homomorphism of Lie algebras

$$\alpha' : \ \mathfrak{m} \ \longrightarrow \ \ Der\ \mathcal{O}(\mathfrak{r}^+),$$

where $Der\, A$ denotes the module of $\mathbb{C}$-linear derivations on a ring A. Thus, by restriction, α' induces a map $\alpha : \mathfrak{m} \to \ Der\ \mathcal{O}(\overline{\mathcal{X}}_k)$.

LEMMA. *For each $p \in \mathcal{X}_k$, $\alpha(\mathfrak{m})$ generates $Der\ \mathcal{O}(\overline{\mathcal{X}}_k)_p$ as an $\mathcal{O}(\overline{\mathcal{X}}_k)_p$-module. Thus $\mathcal{D}(\overline{\mathcal{X}}_k)_p$ is generated as an algebra by $\mathcal{O}(\overline{\mathcal{X}}_k)_p$ and $\alpha(\mathfrak{m})$.*

PROOF: This is, essentially, [**LSS,** Lemma 2.6], but since it is short we include a proof here. By (II, Lemma 5.3(iv)), $\mathcal{O}(\overline{\mathcal{X}}_k)_p = \mathcal{O}(\mathcal{X}_k)_p$, for any $p \in \mathcal{X}_k$. Since $\mathcal{X}_k$ is an M-orbit, $\mathcal{X}_k \cong M/Stab_M(p)$, where $Stab_M(p) = \{m \in M : m \cdot p = p\}$. Thus the tangent space $T_p\mathcal{X}_k$ of $\mathcal{X}_k$ at p satisfies

$$T_p\mathcal{X}_k \ \cong \ T_p\{M/Stab_M(p)\} \ \cong \ \mathfrak{m}/Lie(Stab_M(p)).$$

But, by definition, $T_p\mathcal{X}_k = \mathcal{O}(\mathcal{X}_k)_p/\mathcal{I}(p)_p \otimes Der(\mathcal{O}(\mathcal{X}_k)_p)$. Thus, by Nakayama's Lemma, $\alpha(\mathfrak{m})$ generates $Der\ \mathcal{O}(\mathcal{X}_k)_p = Der\ \mathcal{O}(\overline{\mathcal{X}}_k)_p$. Since $\mathcal{O}(\overline{\mathcal{X}}_k)_p$ is a regular ring, the final assertion of the lemma follows from the fact that, for any regular $\mathbb{C}$-algebra A, $\mathcal{D}(A)$ is generated by A and its derivations (see [**MR,** Chapter XV, Corollary 5.6]).

1.4. Unfortunately, $\alpha(\mathfrak{m})$ and $\psi_1(\mathfrak{m})$ are not quite equal, and so a little more work has to be done in order to prove that R_p and $\mathcal{D}(\overline{\mathcal{X}}_k)_p$ are equal for $p \in \mathcal{X}_k$. To prove this, we will need to examine the difference between $\alpha(\mathfrak{m})$ and $\psi_1(\mathfrak{m})$. Thus in this paragraph (but no others) we will have to be a little careful about the distinction between $\mathcal{O}(\overline{\mathcal{X}}_k)$ and $S(U)^{G'}$, and we will therefore make extensive use of the comments made in (II, 5.2).

By (I, 3.1) the metaplectic representation ω maps $\mathfrak{m}$ to $\omega(\mathfrak{m}) = \mathfrak{g}^{(1,1)} \subset \mathfrak{sp}^{(1,1)}$. By (I, 2.3 and 3.1) $\mathfrak{sp}^{(1,1)} \subset \mathbb{C} \oplus Der\ S(U)$ and hence, in the notation of (I, 1.5),

$$\psi(\mathfrak{m}) \;=\; \varphi\omega(\mathfrak{m}) \;\subset\; \mathbb{C} \oplus Der\ S(U)^{G'}.$$

Thus, given $\xi \in \mathfrak{m}$, write $\psi(\xi) = d_\xi + c_\xi$ for some $d_\xi \in Der\ S(U)^{G'}$ and $c_\xi \in \mathbb{C}$. (If ξ is in the Cartan subalgebra of $\mathfrak{sp}$, then the formulae in (I, 2.3) show that $c_\xi \neq 0$. Thus $\psi(\xi)$ cannot be equal to the derivation $\alpha(\xi)$.)

Recall that, by (I, 3.2), $\psi(U(\mathfrak{r}^-)) = \omega(U(\mathfrak{r}^-)) = S(U)^{G'}$. Therefore, given $\xi \in \mathfrak{m}$ and $y \in \mathfrak{r}^-$, one has

$$\psi([\xi, y]) \;=\; [\psi(\xi), \psi(y)] \;=\; [d_\xi + c_\xi, \psi(y)] \;=\; d_\xi(\psi(y)). \tag{1.4.1}$$

There is another way of viewing this equation. Since $M \subset G \subset \Gamma$, M acts on $S(U)^{G'}$ and the differential of this action is just the restriction of the adjoint representation of $\omega(\mathfrak{sp}(U^\sim))$ (see (I, 1.3)). Denote this action of $\mathfrak{m}$ by β, and let $\xi \in \mathfrak{m}$ and $y \in \mathfrak{r}^-$. Since (I, 1.5) implies that $\psi(y) = \omega(y)$, one has

$$\beta(\xi)(\psi(y)) \;=\; [\omega(\xi), \omega(y)] \;=\; d_\xi(\psi(y)).$$

Thus $d_\xi = \beta(\xi)$.

Finally, consider the map $\alpha : \mathfrak{m} \to Der\ \mathcal{O}(\overline{\mathcal{X}}_k)$. Recall that the isomorphism $\chi : \mathcal{O}(\overline{\mathcal{X}}_k) \xrightarrow{\sim} S(U)^{G'}$ is M-equivariant (see (II, 5.2)). Moreover, by construction, the differential of the M-action on the two sides of this equation are the actions of $\mathfrak{m}$ given by α and β, respectively. Thus $\alpha(\xi) = \chi^{-1}(d_\xi)$. Combined with Lemma 1.3, this gives

PROPOSITION. *(k arbitrary) Write $R = \psi_1(U(\mathfrak{g})) \subseteq \mathcal{D}(\overline{\mathcal{X}}_k)$. Then $R_p = \mathcal{D}(\overline{\mathcal{X}}_k)_p$ for all $p \in \mathcal{X}_k$.*

REMARK. From now on $\mathcal{O}(\overline{\mathcal{X}}_k)$ and $S(U)^{G'}$ will be identified through χ. Thus we will write ψ rather than $\psi_1 = \chi^{-1}\psi$ for the homomorphism from $U(\mathfrak{g})$ to $\mathcal{D}(\overline{\mathcal{X}}_k)$ (see [**We,** footnote, p.289]).

1.5. COROLLARY. *Assume that k satisfies (1.1.1). Let $d \in \mathcal{D}(\overline{\mathcal{X}}_k)$ and set $K = \{a \in \mathcal{O}(\overline{\mathcal{X}}_k) : ad \in R\}$. Then, for some $i \geq 0$, $K \supseteq \overline{I}(k-1)^i$.*

PROOF: Recall from (II, 5.2) that $\overline{I}(k-1)$ is the ideal of $\mathcal{O}(\overline{\mathcal{X}}_k)$ defining $\overline{\mathcal{X}}_{k-1}$. Since $\overline{\mathcal{X}}_{k-1} = \overline{\mathcal{X}}_k \backslash \mathcal{X}_k$, Proposition 1.4 may be rephrased as saying that, for all $p \in \mathcal{X}_k$, there exists $a \in \mathcal{C}(p)$ such that $a \in K$. This is equivalent to the statement of the corollary.

1.6. The significance of Corollary 1.5 is that, in proving the main result of this chapter, we need only worry about the left ideal $R.\overline{I}(k-1)$ of $R = \psi(U(\mathfrak{g}))$. This is most easily dealt with in terms of the associated variety $\overline{\mathbf{O}}_k = \mathcal{V}(J(k))$ of $J(k)$, for which we adopt the notation of (II, 5.2 and 6.6). Thus

$$\begin{aligned} \mathbf{O}_k &= \left\{ z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{g} : z^2 = 0 \text{ and } rk\ z = \widetilde{k} \right\} \\ \supset \quad \mathcal{X}_k &= \left\{ z = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \in \mathfrak{g} : rk\ C = \widetilde{k} \right\}. \end{aligned}$$

Define $V_k = \{ z \in \mathbf{O}_k : \pi_+(z) \in \overline{\mathcal{X}}_{k-1} \}$, where π_+ is defined in (II, 5.1). Equivalently,

$$V_k = \left\{ z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{O}_k : \ C = \pi_+(z) \text{ has rank } < \widetilde{k} \right\}.$$

Observe that $\overline{V}_k = \{ z \in \overline{\mathbf{O}}_k : \pi_+(z) \in \overline{\mathcal{X}}_{k-1} \}$.

KEY LEMMA. *Assume that k satisfies (1.1.1). Then*

$$dim\ \mathbf{O}_k - dim\ V_k \geq \begin{cases} q-k+1 & \textit{(Case A)} \\ n-k+1 & \textit{(Case B)} \\ n-2k & \textit{(Case C)} \end{cases}.$$

In particular, if k is sufficiently small (notation (0.3)), then $dim\ \mathbf{O}_k - dim\ V_k \geq 2$.

The proof of this lemma will be delayed until the next section, but as we will show here, the main theorem is a fairly easy consequence of it.

1.7. Recall from (II, 5.2) that $\mathcal{O}(\overline{\mathcal{X}}_k) = \psi(U(\mathfrak{r}^-)) \subset R$. Thus the set K defined in (1.5) is contained in R.

LEMMA. *If k is sufficiently small, then $GKdim\ R/RK \leq GKdim\ R - 2$.*

Proof: By Corollary 1.5, $K \supseteq \overline{I}(k-1)^i$ for some i. Thus, by [**KL,** Theorem 7.7], it suffices to prove that $GKdim\ R/R.\overline{I}(k-1) \leq GKdim\ R - 2$. In order to prove this, we reinterpret the problem in terms of associated varieties, as this will allow us to apply the Key Lemma. Recall from (II, 5.2) that $\overline{I}(k-1)$ is the image under $\psi' = \psi \mid_{U(\mathfrak{r}^-)}$ of the ideal $I(k-1)$ of $U(\mathfrak{r}^-)$. Since $J(k) = ker(\psi)$ this implies that

$$R/R.\overline{I}(k-1) \;\cong\; U(\mathfrak{g})/\{J(k) + U(\mathfrak{g})I(k-1)\}.$$

Moreover, if $U(\mathfrak{g})$ and $U(\mathfrak{r}^-)$ are given their natural filtrations, then $I(k-1)$ is a determinental ideal (see (II, 2.3, 3.3, and 4.3)) and hence is homogeneous. Thus, by passing to associated graded objects one obtains

$$gr(R/R.\overline{I}(k-1)) \;\cong\; S(\mathfrak{g})/\{gr(J(k)) + S(\mathfrak{g})I(k-1)\}$$

and $gr\,R \cong S(\mathfrak{g})/gr(J(k))$. By definition, the associated variety of $J(k)$ is $\overline{\mathbf{O}}_k$ while the variety of zeros of $I(k-1)$ in $\mathfrak{r}^+$ is exactly $\overline{\mathcal{X}}_{k-1}$ (see (II, 5.2)). Thus the associated variety of $\{gr(J(k)) + S(\mathfrak{g})I(k-1)\}$ is

$$\{z \in \overline{\mathbf{O}}_k : \pi_+(z) \in \overline{\mathcal{X}}_{k-1}\} \;=\; \overline{V}_k.$$

Therefore $GKdim(R/R.\overline{I}(k-1)) = dim\,\overline{V}_k$ while, by definition, $GKdim\ R = dim\,\overline{\mathbf{O}}_k$.

Thus, in order to prove the lemma, we need only prove that $dim\,\overline{\mathbf{O}}_k \geq 2 + dim\,\overline{V}_k$. Let W be an irreducible component of $\overline{V}_k$. Then one of two things can happen. First, W could be contained in $\overline{\mathbf{O}}_{k-1} = \overline{\mathbf{O}}_k \setminus \mathbf{O}_k$. In this case (II, Lemma 6.6) implies that

$$dim\,W \;\leq\; dim\,\overline{\mathbf{O}}_{k-1} \;\leq\; dim\,\overline{\mathbf{O}}_k\, - 2.$$

Alternatively, $W \cap \mathbf{O}_k \neq \emptyset$. In this case, since $W \cap \mathbf{O}_k$ is an open subspace of the irreducible variety W, one has $dim\,W = dim(W \cap \mathbf{O}_k)$. But $W \cap \mathbf{O}_k \subseteq V_k$, and so Lemma 1.6 implies that

$$dim\,W \;\leq\; dim\,\mathbf{O}_k\, - 2 \;=\; dim\,\overline{\mathbf{O}}_k\, - 2;$$

as required.

1.8. We will need the following result from [**Le2**]. Since this will not appear in print, a proof will be given in an appendix to this paper. Recall that a finitely generated left $U(\mathfrak{g})$-module M is called *d-homogeneous* if $GKdim\ M = GKdim\ N = d$ for all non-zero submodules N of M.

GABBER'S LEMMA. *Let $\mathfrak{g}$ be any finite dimensional, complex Lie algebra. Let M be a finitely generated, d-homogeneous left $U(\mathfrak{g})$-module, for some integer d, and suppose that E is an essential extension of M (we do not assume that E is a finitely generated $U(\mathfrak{g})$-module). Then the set of left $U(\mathfrak{g})$-modules*

$$\mathcal{S} \;=\; \big\{M' : M \subseteq M' \subseteq E \text{ with } M' \text{ finitely generated and } GKdim\ M'/M \leq d-2\big\}$$

contains a unique maximal element.

Consequently, if $GKdim\ E/M \leq d-2$, then E is a finitely generated $U(\mathfrak{g})$-module.

REMARK. In every case in which the lemma is used, M will be a domain and so will automatically be homogeneous.

1.9. Write $\mathfrak{k} = \{(x,-x) : x \in \mathfrak{g}\}$ for the diagonal copy of $\mathfrak{g}$ in $\mathfrak{g}\times\mathfrak{g}$. If M and N are left $U(\mathfrak{g})$-modules, then the natural $\mathfrak{g}$-bimodule structure on $Hom_{\mathbb{C}}(M,N)$ induces a left $U(\mathfrak{k})$-module structure on it. Define the corresponding *module of $\mathfrak{k}$-finite vectors* to be

$$L(M,N) \;=\; \big\{\theta \in Hom_{\mathbb{C}}(M,N) : dim_{\mathbb{C}} U(\mathfrak{k})\theta < \infty\big\}.$$

The basic properties of $L(M,N)$ can be found, for example, in [**Ja,** §§6.8 and 6.9]. Since $\mathfrak{n}^-$ acts nilpotently on any finite dimensional $\mathfrak{g}$-module, certainly $\mathfrak{n}^-$, and hence $\mathfrak{r}^-$ act ad-nilpotently on $L(M,N)$. On the other hand, $\mathcal{D}(\overline{\mathcal{X}}_k)$ is *defined* to be the set of elements $x \in Hom_{\mathbb{C}}\big(\mathcal{O}(\overline{\mathcal{X}}_k),\, \mathcal{O}(\overline{\mathcal{X}}_k)\big)$ on which $U(\mathfrak{r}^-) = \mathcal{O}(\mathfrak{r}^+)$ acts ad-nilpotently. Thus we have proved:

LEMMA. *As a $U(\mathfrak{g})$-module, $\mathcal{O}(\overline{\mathcal{X}}_k)$ satisfies $\mathcal{D}(\overline{\mathcal{X}}_k) \;\supseteq\; L\big(\mathcal{O}(\overline{\mathcal{X}}_k),\, \mathcal{O}(\overline{\mathcal{X}}_k)\big)$.*

1.10. It is now easy to prove the main result of this chapter.

THEOREM. *Assume that k is sufficiently small; that is, $1 \leq k < q \leq p$ in Case A, $1 \leq k < n$ in Case B and $2 \leq 2k < n-1$ in Case C. Write $R = \psi(U(\mathfrak{g})) \subseteq \mathcal{D}(\overline{\mathcal{X}}_k)$. Then:*

(i) *$\mathcal{D}(\overline{\mathcal{X}}_k)$ is finitely generated as a left or right R-module and has the same full quotient ring as R.*

(ii) $\mathcal{D}(\overline{\mathcal{X}}_k) = L\big(\mathcal{O}(\overline{\mathcal{X}}_k),\ \mathcal{O}(\overline{\mathcal{X}}_k)\big)$.

PROOF: By Proposition 1.4, R and $\mathcal{D}(\overline{\mathcal{X}}_k)$ have the same quotient division ring and so, in particular, $\mathcal{D}(\overline{\mathcal{X}}_k)$ is an essential extension of R as a module over either R or $U(\mathfrak{g})$. Let $d \in \mathcal{D}(\overline{\mathcal{X}}_k)$, and set $K = \{a \in \mathcal{O}(\overline{\mathcal{X}}_k) : ad \in R\}$. Then, by Lemma 1.7,

$$GKdim(R + Rd/R) \ \leq \ GKdim\ R/RK \ \leq \ GKdim\ R - 2.$$

Thus $GKdim_R \mathcal{D}(\overline{\mathcal{X}}_k)/R \leq GKdim\ R - 2$ and Lemma 1.8 implies that $\mathcal{D}(\overline{\mathcal{X}}_k)$ is a finitely generated left R-module.

By (I, Proposition 3.2) $\mathcal{O}(\overline{\mathcal{X}}_k)$ is a simple $U(\mathfrak{g})$-module. Also, by Lemma 1.9 and Proposition 1.4,

$$R \ \subseteq \ L\left(\mathcal{O}(\overline{\mathcal{X}}_k), \mathcal{O}(\overline{\mathcal{X}}_k)\right) \ \subseteq \ \mathcal{D}(\overline{\mathcal{X}}_k)$$

and all 3 rings have the same full quotient ring. Combined with the result of the last paragraph, [**JS,** Theorem 2.9] therefore implies that $\mathcal{D}(\overline{\mathcal{X}}_k) = L(\mathcal{O}(\overline{\mathcal{X}}_k), \mathcal{O}(\overline{\mathcal{X}}_k))$. Finally, this implies that $\mathcal{D}(\overline{\mathcal{X}}_k)$ is also finitely generated as a right R-module (see, for example [**JS,** §2.4]).

2. Dimensions of associated varieties.

2.1. The aim of this section is to prove the Key Lemma 1.6 and so throughout the section we will assume that k satisfies (1.1.1). We will consistently use the notation set out in (II, 5.1, 5.2 and 6.6) as on many occasions this will allow us to prove results for Cases A, B and C simultaneously.

The proof of (1.6) has 3 steps:

2.1.1 The variety V_k is stable under the adjoint action of a parabolic subgroup $P = MR^-$ of G.

2.1.2 Let π_- be the projection of $\mathfrak{g}$ onto $\mathfrak{r}^-$ along $\mathfrak{m}\oplus\mathfrak{r}^+$ (notation (II, 5.2)). Thus π_- maps $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{g}$ onto B, in the notation of (II, 5.1). Set

$$W_k = \{z \in V_k : rk\ \pi_-(z) = \widetilde{k}\}.$$

Then for all $z \in V_k$, $\quad P \cdot z \cap W_k \neq \emptyset$. Equivalently, $dim\ V_k = dim\ W_k$.

2.1.3 Finally, compute $dim\ W_k$.

2.2. At the risk of being pedantic here, but in order to avoid worries later on, we make some remarks about block matrix decompositions. We have already been writing $z \in \mathfrak{g}$ as a block 2×2 matrix (and will shortly be writing it as a 4×4 matrix). By a slight abuse of notation we will continue to use standard matrix notation to describe these entries. Thus if a matrix N is written in block matrix form

$$M \quad = \quad \begin{array}{c} \\ d \\ e \\ \\ \end{array} \!\!\begin{array}{c} \begin{array}{cccc} a & b & c & \end{array} \\ \begin{pmatrix} N_{11} & N_{12} & N_{13} & \dots \\ N_{21} & \dots & & \\ \vdots & & & \end{pmatrix} \end{array}$$

then we call N_{11} the (1,1) entry and $(N_{11}\, N_{12} \dots)$ the first row, etc. The integers a, b, ... indicate the sizes of the blocks, but will usually be omitted as the sizes will be clear from the context. Note that much of elementary matrix algebra carries over to this situation. For example, if N_{1t} is invertible then the elementary operation of replacing $Row\ r$ by $Row\ r - (N_{rt}N_{1t}^{-1}) \times Row\ 1$ is a well defined operation, in the sense that the various sums and products of the N_{ij} that one is required to make in this operation are automatically defined.

2.3. We will also write subgroups of G in block matrix form in a manner analogous to (II, 2.4). Thus the subgroup M defined in (II, 2.4, 3.4, and 4.4) will be written

$$M = \left\{ \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} : \ g_1 \in GL(p), \ \ g_2 \in GL(q) \right\}$$

with the tacit assumption that $p = q = n$ and $g_1 = g$, $g_2 = {}^t g^{-1}$ in Cases B and C. Similarly, define

$$R^- = \left\{ \begin{pmatrix} I_p & T \\ 0 & I_q \end{pmatrix} : \ T \in M_{pq}(\mathbb{C}) \right\},$$

where again $p = q = n$ in Cases B and C, but now we also assume that $T \in Sym_n(\mathbb{C})$ in Case B and $T \in Alt_n(\mathbb{C})$ in Case C.

Define $P = MR^-$. Of course, since $P \subset G$, P acts on $\mathfrak{g}$ by conjugation. Thus $g \cdot A = gAg^{-1}$ for $g \in P$ and $A \in \mathfrak{g}$. Let π_+ be as defined in (II, 5.1).

2.4. LEMMA. *The variety V_k is stable under the action of P. Indeed, for any $b \le k-1$, the subvariety $\mathbf{O}_k \cap \{(\pi_+)^{-1}(\mathcal{X}_b)\}$ is stable under the action of P.*

PROOF: Since $\overline{\mathcal{X}}_{k-1} = \mathcal{X}_0 \cup \mathcal{X}_1 \cup ... \cup \mathcal{X}_{k-1}$, it suffices to prove that $\mathbf{O}_k \cap (\pi_+)^{-1}(\mathcal{X}_b)$ is stable under P. This is achieved by the simple method of multiplying together the appropriate matrices. However, since the explicit form of $\gamma \cdot z$ for $\gamma \in P$ and $z \in \mathbf{O}_k \cap (\pi_+)^{-1}(\mathcal{X}_b)$ will be needed later, we will give the details. Note that, as $\mathbf{O}_k$ is a nilpotent orbit, the fact that $\gamma \cdot z \in \mathbf{O}_k$ is automatic.

Thus, set $z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{O}_k \cap (\pi_+)^{-1}(\mathcal{X}_b)$, and let

$$\gamma_1 = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \in M \quad \text{and} \quad \gamma_2 = \begin{pmatrix} I_p & T \\ 0 & I_q \end{pmatrix} \in R^-.$$

In Case A we obtain

$$\gamma_1 \cdot z = \begin{pmatrix} g_1 A g_1^{-1} & g_1 B g_2^{-1} \\ g_2 C g_1^{-1} & g_2 D g_2^{-1} \end{pmatrix} \quad \text{and} \quad \gamma_2 \cdot z = \begin{pmatrix} A + TC & E \\ C & D - TC \end{pmatrix}, \tag{2.4.1}$$

where $E = B + TD - AT - TCT$.

In Cases B and C, write $g_1 = g$ and hence $g_2 = {}^t g^{-1}$. Thus

$$\gamma_1 \cdot z = \begin{pmatrix} gAg^{-1} & gB({}^t g) \\ ({}^t g^{-1})Cg^{-1} & ({}^t g^{-1})D({}^t g) \end{pmatrix} \quad \text{and} \quad \gamma_2 \cdot z = \begin{pmatrix} A + TC & E \\ C & D - TC \end{pmatrix}, \tag{2.4.2}$$

where, again, $E = B + TD - AT - TCT$. Here, of course, $D = -{}^t A$.

The condition $z \in \pi_+^{-1}(\mathcal{X}_b)$ is just the statement that $z = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $rk\, C = b$. Clearly, in each case the (2,1) block of $\gamma_i \cdot z$ does have *rank* b; as required.

2.5. We will use (2.4) frequently since, at least if one is willing to move around the P-orbit $P \cdot z$ of an element $z \in V_k$, one can obtain strong conditions on the entries of z. In particular, the structure of $\gamma_1 \cdot z$ is just general enough to "diagonalise" B. To make this more precise, we need some more notation. For an integer r, recall that J_r is the $2r \times 2r$ matrix $\begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$. Define

$$K_r \quad = \quad \begin{cases} I_r & \text{in Cases A and B} \\ \sqrt{-1}\, J_r & \text{in Case C.} \end{cases}$$

The point behind this definition is that, in each case, $K_r^{-1} = K_r$. Recall from (II, 5.2) that, for an integer c, we have defined $\widetilde{c} = c$ in Cases A and B but $\widetilde{c} = 2c$ in Case C.

Recall that the M-orbits in $\mathfrak{r}^+$ (or $\mathfrak{r}^-$) are the $\mathcal{X}_t$ (see (II, 2.4, 3.4 and 4.4)). Thus we have the following immediate consequence of the description of $\gamma_1 \cdot z$ given in (2.4.1.) and (2.4.2).

COROLLARY. *Let $z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{g}$ and set $\widetilde{r} = rk\, B$. Then there exists $\gamma_1 \in M$ such that*

$$w \quad = \quad \gamma_1 \cdot z \quad = \quad \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \quad \textit{where} \quad B' \quad = \quad \begin{pmatrix} K_r & 0 \\ 0 & 0 \end{pmatrix}.$$

A parallel result holds for the block matrix C.

2.6. Since $V_k \subset \mathbf{O}_k$, any $w \in V_k$ satisfies $w^2 = 0$ (see (II, 6.6)). In particular, if w has the form described by Corollary 2.5, then this places severe restrictions on its entries.

LEMMA. *Let $w = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in V_k$ and suppose that $B' = \begin{pmatrix} K_c & 0 \\ 0 & 0 \end{pmatrix}$ for some c. Then:*

(i) *The entries of w have the following block matrix form. (In each case A_1 is a $\widetilde{c} \times \widetilde{c}$ matrix, which uniquely determines the sizes of the remaining blocks.)*

$$A' \quad = \quad \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix} \qquad\qquad B' \quad = \quad \begin{pmatrix} K_c & 0 \\ 0 & 0 \end{pmatrix}$$

$$C' = \begin{pmatrix} -K_cA_1^2 & -K_c(A_1A_2 + A_2A_4) \\ D_3K_cA_1 - D_4D_3K_c & C_4 \end{pmatrix}$$

and

$$D' = \begin{pmatrix} -K_cA_1K_c & 0 \\ D_3 & D_4 \end{pmatrix}.$$

Furthermore, $A_4^2 = 0$ *.*

(ii) *Moreover,* ${}^tA_1 = A_1$ *in Case B while* ${}^tA_1 = -J_cA_1J_c$ *in Case C.*

PROOF: **(i)** Write

$$w = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} A_1 & A_2 & K_c & 0 \\ A_3 & A_4 & 0 & 0 \\ C_1 & C_2 & D_1 & D_2 \\ C_3 & C_4 & D_3 & D_4 \end{pmatrix}$$

where, again, A_1 , C_1 and D_1 are $\widetilde{c} \times \widetilde{c}$ matrices. Now compute entries of $w^2 = 0$, using the fact that $K_c^{-1} = K_c$. The computation is left to the reader, with the advice that one should compute all the conditions on the A_i and D_j before considering the C_ℓ .

(ii) In Cases B and C one has the extra requirement that $D' = -{}^tA'$. Thus from the $(1, 1)$ entry of D' one obtains $-{}^tA_1 = -K_cA_1K_c$; as required.

2.7. Recall that, if $z \in V_k$, then $rk\ z = \widetilde{k}$.

COROLLARY. *Let* $z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in V_k$ *and pick* $\gamma \in M$ *such that*

$$w = \gamma \cdot z = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \quad \textit{with} \quad B' = \begin{pmatrix} K_c & 0 \\ 0 & 0 \end{pmatrix}.$$

Here $\widetilde{c} = rk\,B$*. Write* w *as a block* 4×4 *matrix as in Lemma 2.4. Then:*

(i) $\widetilde{c} = \widetilde{k}$ *if and only if the following conditions hold:*

$$A_4 = 0, \qquad D_4 = 0 \qquad \text{and} \qquad C_4 - D_3K_cA_2 = 0.$$

(ii) *Thus if* $\widetilde{c} = \widetilde{k}$*, then* w *has the block matrix form*

$$w = \begin{pmatrix} A_1 & A_2 & K_k & 0 \\ 0 & 0 & 0 & 0 \\ -K_kA_1^2 & -K_kA_1A_2 & -K_kA_1K_k & 0 \\ D_3K_kA_1 & D_3K_kA_2 & D_3 & 0 \end{pmatrix}.$$

PROOF: **(i)** Write out w as a block 4×4 matrix, using the entries given by Lemma 2.6. If $A_4 \neq 0$ then the first two rows of w are

$$\begin{matrix} A_1 & A_2 & K_c & 0 \\ 0 & A_4 & 0 & 0 \end{matrix}$$

and so

$$\widetilde{k} \;=\; rk\ z \;=\; rk\ w \;\geq\; rk\ K_c + rk\ A_4 \;>\; \widetilde{c}.$$

Similarly, if $D_4 \neq 0$ then $rk\ z > \widetilde{c}$. Thus we may assume that $A_4 = D_4 = 0$. Thus w now has the form:

$$w \;=\; \begin{pmatrix} A_1 & A_2 & K_c & 0 \\ 0 & 0 & 0 & 0 \\ -K_cA_1^2 & -K_cA_1A_2 & -K_cA_1K_c & 0 \\ D_3K_cA_1 & C_4 & D_3 & 0 \end{pmatrix}.$$

Now add $(K_cA_1)\times Row\ 1$ to $Row\ 3$ and $(-D_3K_c)\times Row\ 1$ to $Row\ 4$. Then we obtain the new matrix

$$w' \;=\; \begin{pmatrix} A_1 & A_2 & K_c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & C_4 - D_3K_cA_2 & 0 & 0 \end{pmatrix}.$$

But $\widetilde{k} = rk\ z = rk\ w'$. Thus $\widetilde{k} = \widetilde{c}$ if and only if $C_4 - D_3K_cA_2 = 0$. This proves part (i), and part (ii) is then an immediate consequence.

2.8. One is lead to consider the subspace

$$W_k \;=\; \left\{ z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in V_k :\ \ rk\ B = \widetilde{k} \right\}.$$

For, let $z \in W_k$ and suppose that $w = \gamma_1 \cdot z$ is chosen by Corollary 2.5. Then Corollary 2.7 says that most of the entries of w are fixed and this makes it relatively easy to determine the dimension of W_k. Moreover, as the next result shows, the passage from V_k to W_k involves no serious loss of generality. This forms the most significant step in the proof of the Key Lemma.

PROPOSITION. *Assume that (1.1.1) holds. Then, for all* $z \in V_k$ *there exists* $\gamma \in P = MR^-$ *such that* $w = \gamma \cdot z \in W_k$.

Proof: Among elements of $P \cdot z$ choose one, say $z = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, again, such that $\widetilde{c} = rk\, B$ is as large as possible. We assume that $\widetilde{c} < \widetilde{k}$ and aim for a contradiction. Observe that, by (2.4.1) and (2.4.2), the rank of B is unaffected if we replace z by $w = \gamma_1 \cdot z$ for any $\gamma_1 \in M$. By Corollary 2.5 we may therefore assume that

$$B = \begin{pmatrix} K_c & 0 \\ 0 & 0 \end{pmatrix}.$$

By Lemma 2.6, write

$$z \;=\; \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;=\; \begin{pmatrix} A_1 & A_2 & K_c & 0 \\ 0 & A_4 & 0 & 0 \\ C_1 & C_2 & D_1 & 0 \\ C_3 & C_4 & D_3 & D_4 \end{pmatrix}$$

where, as before, A_1, C_1 ,and D_1 are $\widetilde{c} \times \widetilde{c}$ matrices. Only later will we need the more precise descriptions of the C_i and D_j given in (2.6). We next wish to conjugate z by a matrix $\gamma \in R^-$. By (2.3) γ has the form

$$\gamma \;=\; \begin{pmatrix} I_p & T \\ 0 & I_q \end{pmatrix}$$

(with the appropriate extra conditions in Cases B and C). We will take T to have the form

$$T \;=\; \begin{array}{c c} & \begin{array}{cc} \widetilde{c} & q-\widetilde{c} \end{array} \\ \begin{array}{c} \widetilde{c} \\ p-\widetilde{c} \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & T' \end{pmatrix} \end{array}.$$

Here T' is, for the moment an arbitrary $(p-\widetilde{c}) \times (q-\widetilde{c})$ matrix (although it has to be symmetric or alternating in Cases B or C respectively). We emphasise that such a non-trivial decomposition is possible. For, by assumption and (1.1.1), $\widetilde{c} < \widetilde{k} \le min(p,q)$, and so T' is at least 1×1.

Equations (2.4.1) and (2.4.2) combine to show that

$$\gamma \cdot z \;=\; \begin{pmatrix} A + TC & E \\ C & D - TC \end{pmatrix},$$

where $E = B + TD - AT - TCT$. Since the matrices A_1, C_1 and D_1 are $\widetilde{c} \times \widetilde{c}$ blocks and T has the zero $\widetilde{c} \times \widetilde{c}$ matrix in its $(1,1)$ position, computing E gives

$$E \;=\; \begin{pmatrix} K_c & -A_2 T' \\ T'D_3 & T'D_4 - A_4 T' - T'C_4 T' \end{pmatrix}.$$

Recall that $K_c^{-1} = K_c$. Thus we may replace *Row* 2 of E by *Row* $2-(T'D_3K_c)\times$*Row* 1. This gives the new matrix

$$E' \;=\; \begin{pmatrix} K_c & -A_2T' \\ 0 & F \end{pmatrix},$$

where $F \;=\; T'D_4 - A_4T' - T'(C_4 + D_3K_cA_2)T'$.

The maximality of $\widetilde{c} = rk\ B$ implies that $\widetilde{c} \geq rk\ E = rk\ E'$. Thus $F = 0$. To complete the proof we therefore need to show that this in turn implies that $A_4 = 0$, $D_4 = 0$ and $C_4 - D_3K_cA_2 = 0$. For, Corollary 2.7(i) will then force $\widetilde{c} = \widetilde{k}$; as required. As yet, T' is (essentially) an arbitrary matrix, and it is by varying T' that we will show that these matrices are zero.

We now have to consider the various possibilities separately. Suppose first that $A_4 \neq 0$. Let E_{ij} be the 2×2 matrix with a 1 in the $(i,j)^{th}$ position and zeros elsewhere. Now, as $A_4 \neq 0$ but $A_4^2 = 0$ (see Lemma 2.6), A_4 must be at least 2×2. Thus there exist matrices g and H such that A_4 has associated Jordan canonical form

$$gA_4g^{-1} \;=\; \begin{pmatrix} E_{12} & 0 \\ 0 & H \end{pmatrix}.$$

Since A_4 is a $(p-\widetilde{c})\times(p-\widetilde{c})$ matrix, $p-\widetilde{c}\geq 2$ and H is a $(p-\widetilde{c}-2)\times(p-\widetilde{c}-2)$ matrix. (Here and elsewhere $0\times r$ matrices are held to be vacuous. Thus if $p-\widetilde{c}=2$ then the above description means that $gA_4g^{-1} = E_{12}$.)

Now consider the 3 cases separately. In Case A, T' is a $(p-c)\times(q-c)$ matrix. Since, by hypothesis, $q > c$, this implies that T' is at least 2×1 and we may take

$$T' \;=\; g^{-1}\begin{pmatrix} E'_{21} & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{with } E'_{21} \text{ the } 2\times 1 \text{ matrix } \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let Y be the $(p-c)\times(p-c)$ matrix $Y = \begin{pmatrix} E_{11} & 0 \\ 0 & 0 \end{pmatrix} g$. Observe that $YT' = 0$. Thus

$$YF \;=\; -\,YA_4T' \;=\; \begin{pmatrix} E_{11} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} E_{12} & 0 \\ 0 & H \end{pmatrix}\begin{pmatrix} E'_{21} & 0 \\ 0 & 0 \end{pmatrix} \;\neq\; 0.$$

This contradicts the fact that $F = 0$ and hence forces $A_4 = 0$. By a similar argument (essentially the transpose of the one above) one obtains $D_4 = 0$.

Now consider Case B. Here T' must be symmetric, and therefore square, and so a different choice of T' is required. However, as now $p = q = n$ and $p-\widetilde{c}\geq 2$, the submatrix T' is at least 2×2. Thus we may pick

$$T' \;=\; g^{-1}\begin{pmatrix} E_{22} & 0 \\ 0 & 0 \end{pmatrix}({}^tg^{-1}),$$

which is symmetric. As before, we take $Y = \begin{pmatrix} E_{11} & 0 \\ 0 & 0 \end{pmatrix} g$. Just as in Case A, $YT' = 0$ and so $YF = -YA_4T' \neq 0$. This contradicts the fact that $F = 0$ and forces $A_4 = 0$.

In Case C, the matrix T' needs to be antisymmetric. In this case (1.1.1) implies that $n > \widetilde{k} = 2k$. Thus, by assumption, $n > 2k > 2c$ and so $p - \widetilde{c} = n - 2c \geq 3$. Thus T' is at least 3×3. (We need T' to be this large, since there are no non-invertible, non-zero antisymmetric 2×2 matrices.) Now, let J_1' and E_{31}' be the 3×3 matrices

$$J_1' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \text{and} \qquad E_{31}' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Set $T' = g^{-1} \begin{pmatrix} J_1' & 0 \\ 0 & 0 \end{pmatrix} ({}^t g^{-1})$, which is antisymmetric, and $Y = \begin{pmatrix} E_{11} & 0 \\ 0 & 0 \end{pmatrix} g$. Then $YT' = 0$ and so

$$YF = -YA_4T' = \begin{pmatrix} E_{31}' & 0 \\ 0 & 0 \end{pmatrix} ({}^t g^{-1}) \neq 0,$$

a contradiction. Thus in each case we have shown that $A_4 = D_4 = 0$.

It remains to consider $X = C_4 - D_3K_cA_2$, which we assume to be non-zero. Note that now $0 = F = -T'XT'$. Thus Case B is particularly easy; we take T' to be the (symmetric) matrix $I_{n-\tilde{c}}$ in order to obtain the desired contradiction.

Now consider Case C. Thus T' (and hence X) are $(n-2c) \times (n-2c)$ matrices. If n is even then $T' = J_r$ for, $r = \frac{1}{2}(n - 2c)$, will give the required contradiction. However, for n odd we need to be a little more careful. Note that

$$X = C_4 - D_3K_cA_2 = C_4 - {}^tA_2K_cA_2$$

is antisymmetric. Thus, for some $g \in GL(n - 2c)$, one has $gX({}^t g) = \begin{pmatrix} J_r & 0 \\ 0 & 0 \end{pmatrix}$, where $r = rk\ X$. Thus if we set

$$T' = {}^t g \begin{pmatrix} J_r & 0 \\ 0 & 0 \end{pmatrix} g,$$

then

$$F = -T'XT' = {}^t g \begin{pmatrix} J_r & 0 \\ 0 & 0 \end{pmatrix} g \neq 0;$$

a contradiction. Thus $X = 0$, as required.

It remains to consider Case A. As we remarked earlier, T' is an $t \times s$ matrix where $t = p - c \geq s = q - c \geq 1$. Thus X is an $s \times t$ matrix. Pick invertible matrices g_1 and g_2 of the appropriate size such that $g_1Xg_2 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where $r = rk\,X$.

Therefore, if $T' = g_2 \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} g_1$, then

$$F \;=\; T'XT' \;=\; g_2 \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} g_1 \;\neq\; 0;$$

giving the required contradiction.

2.9. Given a quasi-affine variety $\mathcal{X}$ and a point $x \in \mathcal{X}$, recall that the local ring of $\mathcal{X}$ at x is written $\mathcal{O}(\mathcal{X})_x$. Let $dim_x\mathcal{X}$ denote the dimension of $\mathcal{O}(\mathcal{X})_x$.

COROLLARY. *Assume that (1.1.1) holds. Then* $dim\ V_k = dim\ W_k$.

PROOF: Note that W_k is an open subspace of V_k , defined for example by the condition that at least one of the $\widetilde{k} \times \widetilde{k}$ minors in the (1,2) block is non-zero. Thus $dim_\xi W_k = dim_\xi V_k$ for all $\xi \in W_k$. However, given $\xi \in V_k$, Proposition 2.8 shows that there exists $\gamma \in P$ such that $\gamma \cdot \xi \in W_k$. Since $dim_{\gamma\cdot\xi} V_k = dim_\xi V_k$, this implies that $dim\ V_k = dim\ W_k$.

2.10. There is one, further, restriction on the entries of w that holds for $w \in W_k$.

LEMMA. *Let* $z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in V_k$. *Then* $min\{rk\,A,\ rk\,D\} < \widetilde{k}$. *Of course, in Cases B and C, one has* $D = -{}^tA$ *and so this implies that* $rk\ A < \widetilde{k}$.

PROOF: Suppose that $rk\ C = \widetilde{c}$. By the definition of V_k , $\widetilde{c} < \widetilde{k} = rk\,z$. By Corollary 2.5 there exists $\gamma \in M$ such that $w = \gamma \cdot x = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ where, now, $C' = \begin{pmatrix} K_c & 0 \\ 0 & 0 \end{pmatrix}$. This gives the block matrix decomposition

$$w \;=\; \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \;=\; \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \\ K_c & 0 & D_1 & D_2 \\ 0 & 0 & D_3 & D_4 \end{pmatrix}$$

where A_1 , B_1 and D_1 are $\widetilde{c} \times \widetilde{c}$ matrices and the sizes of the other blocks are defined accordingly. Since $w \in V_k \subset \mathbf{O}_k$, we have $w^2 = 0$ (see (1.6)). Thus by multiplying the third row of w by the second column one finds that $K_c A_2 = 0$ and hence $A_2 = 0$. Similarly, $D_3 = 0$. But this implies that

$$\widetilde{k} \;=\; rk\ w \;\geq\; rk\ K_c + rk\ A_4 + rk\ D_4.$$

This certainly forces $\widetilde{k} - \widetilde{c} > min\{rk\,A_4, rk\,D_4\}$, which in turn implies that $\widetilde{k} > min\{rk\,A', rk\,D'\}$. Since $\gamma \in M$, (2.5.1) and (2.5.2) imply that $rk\ A' = rk\ A$ and $rk\ D' = rk\ D$. Thus $\widetilde{k} > min\{rk\,A, rk\,D\}$, as required.

2.11. In order to complete the computation of $dim\ W_k$ we will use a result of Eisenbud from [**Ei**], but to state this we need some notation. Let $H = M_{w,v}(\mathbb{C})$ for some integers $w \leq v$ and let F be a subspace of H of dimension m. Set

$$F^{\perp} \ = \ \big\{\theta \in H^* : \theta(f) = 0 \quad \text{for all} \quad f \in F\big\}.$$

For any integer u write $H_u = \{h \in H : rk\, h \leq u\}$ and $F_u = F \cap H_u$. Similarly, we identify H^* with $M_{v,w}(\mathbb{C})$ and define $(F^{\perp})_u = \{f \in F^{\perp} : rk\, f \leq u\}$. Write

$$\delta(F) \ = \ dim\ (F^{\perp})_1 - (w-1).$$

Then the following result is a particular case of [**Ei,** Theorem 2.3].

PROPOSITION. *With the above notation,*

$$dim\ F_{w-1} \ \leq \ m - (v - (w-1)) \ + \ max\{0,\ \delta(F)\}.$$

2.12. We will apply Proposition 2.11 to compute the dimension of the variety of matrices of the form described by Corollary 2.7(ii). This amounts to proving the following result. We have omitted Case A as it will turn out to be a trivial consequence of (II, 5.3).

COROLLARY. **(i)** *Assume that* $n \geq k$ *and set*

$$\mathcal{L}_B \ = \ \big\{ A \ = \ (A_1\ A_2) \ \in \ M_{k,n}(\mathbb{C}) :\ A_1 \ \in \ Sym_k(\mathbb{C}) \ \text{ and } \ rk\, A \ < \ k\big\}.$$

Then $dim\ \mathcal{L}_B \ \leq \ nk - \frac{1}{2}k(k-1) - (n-k+1)$.

(ii) *Assume that* $n \geq 2k$ *and set*

$$\mathcal{L}'_C \ = \ \{A \ = \ (A_1\ A_2) \ \in \ M_{2k,n}(\mathbb{C}) :\ A_1 \ \in \ Alt_{2k}(\mathbb{C}) \ \text{and} \ rk\, A \ < \ 2k\}.$$

Then $dim\ \mathcal{L}'_C \ \leq \ 2nk - k(2k+1) - (n-2k)$.

PROOF: (i) We apply Proposition 2.11 with $v = n \geq w = k$, while $H = M_{k,n}(\mathbb{C})$ and

$$F \ = \ \big\{A = (A_1\ A_2) \ \in \ H :\ \ A_1 \in Sym_n(\mathbb{C})\big\}.$$

Thus

$$m \ = \ dim\ F \ = \ \tfrac{1}{2}k(k+1) + k(n-k) \ = \ nk - \tfrac{1}{2}k(k-1).$$

It is an elementary exercise to prove that $Sym_k(\mathbb{C})^{\perp} = Alt_k(\mathbb{C})$ as subspaces of $M_{k,k}(\mathbb{C})$. Thus

$$F^{\perp} \;=\; \left\{ B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \in M_{n,k}(\mathbb{C}) : \;\; B_1 \in Alt_k(\mathbb{C}) \text{ and } B_2 = 0 \right\}.$$

In particular, $(F^{\perp})_1 = 0$ and $\delta(F) \leq 0$. Therefore, by Proposition 2.11,

$$dim\ \mathcal{L}'_B \;\leq\; m - (n-k-1)),$$

as required.

(ii) In this case, take $v = n \geq w = 2k$, while $H = M_{2k,n}(\mathbb{C})$ and

$$F \;=\; \{A = (A_1\ A_2) \;\in\; H : \; A_1 \in Alt_{2k}(\mathbb{C})\}.$$

Thus $m \;=\; dim\ F \;=\; 2nk - k(2k+1)$. As before, $Alt_{2k}(\mathbb{C})^{\perp} = Sym_{2k}(\mathbb{C})$ and, therefore,

$$F^{\perp} \;=\; \left\{ B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \in M_{n,2k}(\mathbb{C}) : \;\; B_1 \in Sym_{2k}(\mathbb{C}) \text{ and } B_2 = 0 \right\}.$$

Thus $(F^{\perp})_1 \cong \{B_1 \in Sym_{2k}(\mathbb{C}) : rk\ B_1 \leq 1\}$. In this case, $dim(F^{\perp})_1 = 2k$ (use (II, Lemma 5.3)) and $\delta(F) = 1$. Therefore $dim\ \mathcal{L}'_C \leq m - (n - (2k-1)) + 1$, by Proposition 2.11.

2.13. REMARK. The proof of Corollary 2.12, though short, does require the use of non-trivial geometric machinery through [**Ei,** Theorem 2.3]. It should be remarked that one can fairly easily give a direct proof of (2.12). For, if one knows that $\mathcal{L}_B$ and $\mathcal{L}'_C$ are irreducible then it is an easy exercise to compute their dimensions (and these equal the bounds given above). In order to prove irreducibility, the main point is to show that $\mathcal{L}_B$, respectively $\mathcal{L}'_C$, is the closure of an irreducible quasi-affine variety $\mathcal{L}$. In the two cases $\mathcal{L}$ is defined by

$$\mathcal{L} \;=\; \begin{cases} \{A = (A_1\ A_2) \;\in\; \mathcal{L}_B : \; rk\ A_1 = k-1\} & \text{in Case B} \\ \{A = (A_1\ A_2) \;\in\; \mathcal{L}'_C : \; rk\ A_1 = 2k-2\} & \text{in Case C.} \end{cases}$$

The proof of these observations is left to the interested reader.

Let F be defined as in (2.11). If $\delta(F) \leq 0$, then [**Ei,** Theorem 2.3] also proves that F_{w-1} is irreducible. Combined with the computations in (2.12), this gives another proof of the irreducibility of $\mathcal{L}_B$. However, the proof of Corollary 2.12 shows that $\delta(F) = 1$ in Case C. Thus [**Ei,** Theorem 2.3] cannot be used to prove that $\mathcal{L}'_C$ is irreducible.

2.14. When we apply Corollary 2.12(ii), it will be useful to have another description of the variety $\mathcal{L}'_C$.

LEMMA. *$\mathcal{L}'_C \cong \mathcal{L}_C$, where*

$$\mathcal{L}_C = \big\{A = (A_1\ A_2) \in M_{2k,n}(\mathbb{C}) : \ A_1 \in M_{2k,2k}(\mathbb{C}),\ {}^tA_1 = -J_k A_1 J_k \text{ and } rk\ A < 2k\big\}.$$

PROOF: Since ${}^tJ_k = J_k^{-1} = -J_k$, the equation ${}^tA_1 = -J_kA_1J_k$ may be rewritten as ${}^t(J_kA_1) = -J_kA_1$. Since J_k is invertible the required isomorphism from $\mathcal{L}_C$ to $\mathcal{L}'_C$ is therefore given by

$$(A_1\ A_2) \longmapsto J_k(A_1\ A_2) = (J_kA_1\ J_kA_2).$$

2.15. The following classical result will prove useful. The proof is left to the reader.

LEMMA. *Suppose that $\theta : \mathcal{Z} \to \mathcal{Y}$ is a morphism of quasi-affine varieties. Then*

$$dim\ \mathcal{Z} \leq dim\ \mathcal{Y} + sup\{dim\ \theta^{-1}(y) : y \in \mathcal{Y}\}.$$

2.16. We can now combine the earlier results of this section to prove the Key Lemma 1.6.

THEOREM. **(i)** *Assume that (1.1.1) holds. Then*

$$dim\ V_k = dim\ W_k \leq \begin{cases} 2k(p+q-k) - (q+1-k) & \text{in Case A} \\ 2nk - k(k-1) - (n-k+1) & \text{in Case B} \\ 2(2nk - k(2k+1)) - (n-2k) & \text{in Case C.} \end{cases}$$

(ii) *In particular, if k is sufficiently small, then*

$$dim\ V_k = dim\ W_k \leq dim\ \mathbf{O}_k - 2.$$

REMARKS. This completes the proof of Lemma 1.6 and hence of Theorem 1.10. Suppose that k is not sufficiently small, but does satisfy (1.1.1). Thus $k = q,\ n$ or $\frac{1}{2}(n-1)$ in Cases A, B or C, respectively. Then the given bound on $dim\ W_k$ only provides the inequality $dim\ V_k \leq dim\ \mathbf{O}_k - 1$. In fact this must be an equality. For, in each case (IV, Remark 1.5), below, implies that Theorem 0.3 fails for this value of k. But it is only at the present step that the proof of Theorem 0.3 will fail. Alternatively, one can fairly easily show that the given inequality for $dim\ W_k$ is actually an equality whenever k satisfies (1.1.1).

PROOF: Recall that $dim\ V_k = dim\ W_k$ by Corollary 2.9. By (II, Lemma 6.6) we have

$$dim\ \mathbf{O}_k \;=\; \begin{cases} 2k(p+q-k) & \text{in Case A} \\ 2nk - k(k-1) & \text{in Case B} \\ 2(2nk - k(2k+1)) & \text{in Case C.} \end{cases}$$

Thus part (ii) of the theorem and the second paragraph of the remark both follow from part (i) of the theorem.

It remains to bound $dim\ W_k$. We will first simplify the problem by applying Lemma 2.15. Recall, from the definitions of V_k in (1.6) and W_k in (2.8), that the projection

$$\pi_- : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;\longmapsto\; B$$

maps W_k onto $\mathcal{X}_k$. Furthermore, dimension does not change along an orbit, and the action of the group M, given in (2.3), commutes with that of π_-. Thus Corollary 2.5 implies that

$$sup\{dim\ (\pi_-)^{-1}(y) : y \in \mathcal{X}_k\} \;=\; dim\ (\pi_-)^{-1}(y_0) \quad \text{for any} \quad y_0 \in \mathcal{X}_k.$$

Obviously, we will take $y_0 = \begin{pmatrix} K_k & 0 \\ 0 & 0 \end{pmatrix}$. Thus Lemma 2.15, with $\mathcal{Z} = W_k$, $\mathcal{Y} = \mathcal{X}_k$ and $\theta = \pi_-$, implies that

$$dim\ W_k \;\leq\; dim\ \mathcal{X}_k + dim\ (\pi_-)^{-1}(y_0).$$

The value of $dim\ \mathcal{X}_k$ is given in (II, Lemma 5.3), and equals $\frac{1}{2} dim\ \mathbf{O}_k$. Thus in order to complete the proof of the theorem it suffices to prove:

2.17. SUBLEMMA. *If* $y_0 = \begin{pmatrix} K_k & 0 \\ 0 & 0 \end{pmatrix}$ *then*

$$dim\ (\pi_-)^{-1}(y_0) \;\leq\; \begin{cases} k(p+q-k) - (q+1-k) & \textit{in Case A} \\ nk - \frac{1}{2}k(k-1) - (n-k+1) & \textit{in Case B} \\ 2nk - k(2k+1) - (n-2k) & \textit{in Case C.} \end{cases}$$

PROOF: At this stage it seems easiest to consider the three cases separately. Thus, assume that we are in Case A. By Corollary 2.7(ii) and using the notation of that result, just 3 of the entries of $w \in (\pi_-)^{-1}(y_0)$ completely determine w; these being A_1, A_2 and D_3. Here, $A_1 \in M_{k,k}(\mathbb{C})$, $A_2 \in M_{k,p-k}(\mathbb{C})$ and $D_3 \in M_{q-k,k}(\mathbb{C})$. Furthermore, as $K_k = I_k$, Lemma 2.10 implies that

$$max\left\{dim\ (A_1\ A_2),\ \ dim\ \begin{pmatrix} -A_1 \\ D_3 \end{pmatrix}\right\} \ < \ k.$$

Thus by ignoring the determined entries of w, Corollary 2.7(ii) implies that $(\pi_-)^{-1}(y_0) \cong \mathcal{L}_1 \cup \mathcal{L}_2$, where

$$\mathcal{L}_1 \ = \ \left\{\begin{pmatrix} A_1 & A_2 \\ D_3 & 0 \end{pmatrix} :\ rk\ (A_1\ A_2) < k\right\}$$

and

$$\mathcal{L}_2 \ = \ \left\{\begin{pmatrix} -A_1 & A_2 \\ D_3 & 0 \end{pmatrix} :\ rk\ \begin{pmatrix} -A_1 \\ D_3 \end{pmatrix} < k\right\}.$$

Certainly

$$dim\ \mathcal{L}_1 \ \leq \ dim\ \big\{A = (A_1\ A_2) \in M_{k,p}(\mathbb{C}) : rk\ A < k\big\} \ + \ dim\ M_{q-k,k}(\mathbb{C}).$$

By (II, Lemma 5.3) this implies that

$$\begin{aligned} dim\ \mathcal{L}_1 \ &\leq \ \big\{(k-1)(p+q) - (k+1)\big\} + (q-k)k \\ &= \ k(p+q-k) - (q+1-k). \end{aligned}$$

Similarly, $dim\ \mathcal{L}_2 \ \leq \ k(p+q-k) - (p+1-k)$. Since $p \geq q$ this suffices to prove (2.17) in Case A.

Now repeat the above proof in Case B. Here, Lemma 2.6(ii) implies that one has the extra conditions $D_3 = -{}^tA_1$ and $A_1 = {}^tA_1$ in the formula for $w \in (\pi_-)^{-1}(y_0)$ given in Corollary 2.7(ii). Thus by applying Corollary 2.7(ii) and Lemma 2.10, and ignoring determined entries of w, one now obtains that

$$(\pi_-)^{-1}(y_0) \ \cong \ \mathcal{L}_B \ = \ \big\{(A_1\ A_2) \in M_{k,n}(\mathbb{C}) : A_1 \in Sym_k(\mathbb{C}) \text{ and } rk\ A < k\big\}.$$

Thus (2.17) follows immediately from Corollary 2.12(i).

Finally, consider Case C. Now $K_k = \sqrt{-1}\, J_k$ and so Lemma 2.6 implies that $D_3 = -{}^tA_1$ and $A_1 = -J_k({}^tA_1)J_k$. Thus the argument used above implies that

$$\begin{aligned} (\pi_-)^{-1}(y_0) \ \cong \ \mathcal{L}_C \ = \ \big\{(A_1\ A_2) \in M_{2k,n}(\mathbb{C}) :\ & A_1 = -J_k({}^tA_1)J_k \in M_{2k,2k}(\mathbb{C}) \\ & \text{and } rk\ A < 2k\big\}. \end{aligned}$$

Thus (2.17) follows from Lemma 2.14 and Corollary 2.12(ii).

This completes the proof of the theorem.

2.18. In Chapter V we will require the following variant of Case B of Theorem 2.16. Fix

$$\begin{aligned}\mathbf{O}_n \;&=\; \left\{ z = \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} : \;\; B,\, C \in Sym_n(\mathbb{C}),\;\; z^2 = 0 \text{ and } rk\ z = n \right\} \\ &\supset\;\; V_n \;=\; \{ z \in \mathbf{O}_n : \;\; rk\ C < n \} \\ &\supset\;\; U_n \;=\; \{ z \in \mathbf{O}_n : \;\; rk\ C < n-1 \}.\end{aligned}$$

PROPOSITION. *With the above notation, dim* $U_n \;\leq\; dim\ \mathbf{O}_n - 2$.

Proof: If $n = 1$, then U_n is defined to be empty, in which case the Proposition follows from (II, Lemma 6.6). Thus, assume that $n \geq 2$. As this proof is similar to that of the earlier results of this section, some of the details are left to the reader.

It is clear that U_n is stable under the action of the group P (see Lemma 2.4.) Thus by Proposition 2.8, for any $z \in U_n$, there exists $\gamma \in P$ such that

$$\gamma \cdot z \;\in\; \widetilde{W}_n \;=\; W_n \cap U_n \;=\; \left\{ y = \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \in U_n : rk\ B = n \right\}.$$

The proof of Corollary 2.9 therefore implies that $dim\ U_n = dim\ \widetilde{W}_n$. Furthermore, given $z \in \widetilde{W}_n$, there exists, by Corollary 2.5, some $\gamma \in M$ such that

$$w \;=\; \gamma \cdot z \;=\; \begin{pmatrix} A' & I_n \\ C' & -{}^tA' \end{pmatrix}.$$

By Lemma 2.10, the submatrix A' has rank $< n$. Since $w \in U_n \subseteq \mathbf{O}_n$ one has $w^2 = 0$. This forces $C' = -(A')^2$ and $A' = {}^t(A')$. In particular, from the definition of U_n, one has $rk(A')^2 \leq n-2$. Thus, just as in the first part of the proof of Theorem 2.16, this implies that

$$dim\ \widetilde{W}_n \;\leq\; dim\{B \in Sym_n(\mathbb{C}) : rk\ B = n\} \;+\; dim\ \mathcal{L},$$

where $\mathcal{L}$ is the variety

$$\mathcal{L} \;=\; \{A \in Sym_n(\mathbb{C}) : \;\; rk\ A \leq n-1 \text{ and } rk\ A^2 \leq n-2\}.$$

Now $\mathcal{L} \subseteq \mathcal{M} = \{A \in Sym_n(\mathbb{C}) : rk\ A \leq n-1\}$. By (II, Lemma 5.3), $\mathcal{M}$ is an irreducible variety with $dim\ \mathcal{M} = \frac{1}{2}n(n+1) - 1$. Since

$$\begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix} \notin \mathcal{L},$$

clearly $\mathcal{L}$ is a proper, closed subvariety of $\mathcal{M}$. Thus $dim\ \mathcal{L} \leq dim\ \mathcal{M} - 1$. Since $dim\ Sym_n(\mathbb{C}) = \frac{1}{2}n(n+1)$, we conclude that

$$dim\ U_n \;=\; dim\ \widetilde{W}_n \;\leq\; n(n+1) - 2.$$

But $dim\ \mathbf{O}_n = n(n+1)$, by (II, 6.6). Thus $dim\ U_n \leq dim\ \mathbf{O}_n - 2$, as required.

2.19. COROLLARY. *Set* $R = U(\mathfrak{sp}(2n))/J(n)$. *Let* $I(n-2)$ *be the ideal defining*

$$\overline{\mathcal{X}}_{n-2} = \big\{C \in Sym_n(\mathbb{C}) : rk\ C \leq n-2\big\}$$

inside $\mathcal{O}(\overline{\mathcal{X}}_n) = \mathcal{O}(Sym_n(\mathbb{C}))$. *Then*

$$GKdim\ R/R.I(n-2)^i \;\leq\; GKdim\ R - 2 \quad \text{for all} \quad i \geq 1.$$

PROOF: Use the proof of Lemma 1.7, but with Proposition 2.18 replacing the Key Lemma 1.6.

3. Twisted Differential Operators.

3.1. Let $L = L(\lambda_k + \rho) = \mathcal{O}(\overline{\mathcal{X}}_k)$. Then Theorem 1.10 shows that $L(L, L) = \mathcal{D}(L)$, and this suggests that one should compare $L(L, L)$ with $\mathcal{D}(L)$ under translation. More precisely, let M be a direct summand of $L \otimes_{\mathbb{C}} E$, for some finite dimensional $\mathfrak{g}$-module E. In this section we will prove that $L(M, M) = \mathcal{D}^{U(\mathfrak{r}^-)}(M)$; the ring of "twisted differential operators" on M regarded as a $U(\mathfrak{r}^-)$-module. This ring is defined just as was $\mathcal{D}(R)$ in (0.1). Formally, if M is a module over a commutative $\mathbb{C}$-algebra R, define $\mathcal{D}_0^R(M) = End_R(M)$ and, for each $i > 0$, set

$$\mathcal{D}_i^R(M) \;=\; \big\{\theta \in End_{\mathbb{C}}(M) : \;[\theta, a] \in \mathcal{D}_{i-1}^R(M) \quad \text{for all} \quad a \in R\big\}.$$

Then $\mathcal{D}^R(M) = \bigcup \mathcal{D}_i^R(M)$. The basic properties of $\mathcal{D}^R(M)$ are similar to those of $\mathcal{D}(R)$ and can be found, for example, in **[MR]** or **[SmSt]**. We emphasise that, as L is a factor ring of $U(\mathfrak{r}^-)$, one has $\mathcal{D}(\overline{\mathcal{X}}_k) = \mathcal{D}^{U(\mathfrak{r}^-)}(L)$.

3.2. We fix the following notation. First, all tensor products and endomorphism rings will be defined over $\mathbb{C}$, unless otherwise indicated. Let $\mathfrak{g}$ be a semi-simple, complex Lie algebra and $\mathfrak{a}$ an abelian subalgebra of $\mathfrak{n}^-$. Let L be any finitely generated $\mathfrak{g}$-module. Then L will be viewed as a $U(\mathfrak{a})$-module by restriction. We remark that in these circumstances one always has the inclusion

$$L(L, L) \;\subseteq\; \mathcal{D}^{U(\mathfrak{a})}(L). \tag{3.2.1}$$

The proof of this is identical to that of Lemma 1.9.

Next, let E be a finite dimensional $\mathfrak{g}$-module and write $F = E \oplus \mathbb{C}_0$ where $\mathbb{C}_0$ denotes the trivial $\mathfrak{g}$-module. Regard $L \otimes F$ as a $\mathfrak{g}$-module, and hence as an $\mathfrak{a}$-module, via the usual diagonal action;

$$g \cdot (a \otimes b) \;=\; (ga) \otimes b + a \otimes (gb) \qquad \text{for } g \in \mathfrak{g},\; a \in L \text{ and } b \in F.$$

3.3. PROPOSITION. *(Notation 3.2). Suppose that* $L(L, L) = \mathcal{D}^{U(\mathfrak{a})}(L)$. *Then*

$$L(L \otimes F, L \otimes F) \;=\; \mathcal{D}^{U(\mathfrak{a})}(L \otimes F).$$

PROOF: Identify $End(L \otimes F)$ with $M_n(\mathbb{C}) \otimes End(L)$, where $n = dim\, F$. This induces the standard decomposition

$$L(L \otimes F, L \otimes F) \;=\; M_n(\mathbb{C}) \otimes L(L, L)$$

(see, for example, **[Ja,** §6.8.2]). Also, by (3.2.1) we have the inclusion

$$L(L\otimes F,\, L\otimes F) \;\subseteq\; \mathcal{D}^{U(\mathfrak{a})}(L\otimes F).$$

Thus there exists the following chain of embeddings:

$$\begin{aligned} M_n(\mathbb{C})\otimes\mathcal{D}^{U(\mathfrak{a})}(L) \;&=\; M_n(\mathbb{C})\otimes L(L,\, L) \;=\; L(L\otimes F,\, L\otimes F) \\ &\subseteq\; \mathcal{D}^{U(\mathfrak{a})}(L\otimes F) \;\subseteq\; M_n(\mathbb{C})\otimes End(L). \end{aligned} \tag{3.3.1}$$

Fix a basis $\{e_1, \ldots, e_{n-1}\}$ of E and $\{e_n\}$ of $\mathbb{C}_0$ and let $\{e_{ij}\}$ be the corresponding matrix units in $M_n(\mathbb{C}) \cong End(F)$. In particular, e_{nn} is the projection of F onto $\mathbb{C}_0$ along E. The reason for working with F rather than E is that, as elements of $End_{\mathbb{C}}(L\otimes F)$, e_{nn} commutes with $\mathfrak{a}$. To see this, note that

$$e_{nn} \;=\; e_{nn}\otimes 1 \;\in\; M_n(\mathbb{C})\otimes L(L,\, L) \;\subseteq\; End(F)\otimes End(L).$$

On the other hand, $\mathfrak{a}$ is embedded in $End(F)\otimes End(L)$ via the diagonal map; $a \to a\otimes 1 + 1\otimes a$, for $a\in\mathfrak{a}$. Thus

$$[a,\, e_{nn}] \;=\; [a\otimes 1 + 1\otimes a,\; e_{nn}\otimes 1] \;=\; ae_{nn}\otimes 1 - e_{nn}a\otimes 1 + e_{nn}\otimes a - e_{nn}\otimes a.$$

Since $\mathfrak{a}$ acts trivially on $\mathbb{C}_0$, it annihilates the projection e_{nn} from either side. Thus $[\mathfrak{a},\, e_{nn}] = 0$.

Let $\theta\in\mathcal{D}^{U(\mathfrak{a})}(L\otimes F)$ and write $\theta = \sum e_{ij}\theta_{ij}$ for the appropriate $\theta_{ij}\in End(L)$. Certainly each $e_{ij}\in M_n(\mathbb{C})\subseteq L(L\otimes F,\, L\otimes F)$. Thus in order to prove the proposition it suffices, by (3.3.1), to show that $\theta_{ij}\in\mathcal{D}^{U(\mathfrak{a})}(L)$ for each i and j. Now each e_{uv} commutes with every θ_{xy}. Therefore, for any i and j,

$$e_{nn}\theta_{ij} \;=\; e_{ni}\theta e_{jn} \;\in\; \mathcal{D}^{U(\mathfrak{a})}(L\otimes F).$$

Thus we need only prove the following inductive statement:

Suppose that $\varphi\in End(L)$ is such that $e_{nn}\varphi\in\mathcal{D}_m^{U(\mathfrak{a})}(L\otimes F)$ for some integer m. Then $\varphi\in\mathcal{D}_m^{U(\mathfrak{a})}(L)$. (3.3.2)

If we define $\mathcal{D}_{-1}^{U(\mathfrak{a})}(L) = 0$ then (3.3.2) does hold for $m=-1$, and this begins the induction. Now suppose that $e_{nn}\varphi\in\mathcal{D}_m^{U(\mathfrak{a})}(L\otimes F)$, for some $m\geq 0$. For any $a\in\mathfrak{a}$ one has $[a,\, e_{nn}]=0$ and so

$$e_{nn}[\varphi,\, a] \;=\; [e_{nn}\varphi,\, a] \;\in\; \mathcal{D}_{m-1}^{U(\mathfrak{a})}(L\otimes F).$$

By induction on m, one obtains that $[\varphi, a] \in \mathcal{D}^{U(\mathfrak{a})}_{m-1}(L)$. Since $a \in \mathfrak{a}$ was arbitrary, this implies that $\varphi \in \mathcal{D}^{U(\mathfrak{a})}_{m}(L)$; as required.

3.4. COROLLARY. *(Notation 3.2) Suppose that* $L(L, L) = \mathcal{D}^{U(\mathfrak{a})}(L)$ *and that* M *is a direct summand of* $L \otimes E$, *as a* $U(\mathfrak{g})$*-module. Then* $L(M, M) = \mathcal{D}^{U(\mathfrak{a})}(M)$.

PROOF: Clearly M is also a direct summand of $L \otimes F$ where, as in (3.2), $F = E \oplus \mathbb{C}_0$. Let $\eta \in End_{U(\mathfrak{g})}(L \otimes F)$ be the corresponding projection $L \otimes F \to M$. Then certainly $\eta \in L(L \otimes F, L \otimes F)$ and we claim that

$$\eta L(L \otimes F, L \otimes F)\eta \;=\; L(M, M). \tag{3.4.1}$$

To see this, note that $\eta L(L \otimes F, L \otimes F)\eta \subseteq L(M, M)$, by composition of functions (see [**Ja,** §6.8.6]). For the other inclusion, suppose that $\theta \in L(M, M)$. Let $\varphi \in End(L \otimes F)$ be defined by $\varphi(x, y) = (\theta(x), 0)$ for $x \in M$ and y in the complement of M. Then it is routine to check that $\varphi \in L(L \otimes F, L \otimes F)$. Since $\eta\varphi\eta = \theta$, this suffices to prove (3.4.1).

As $\eta \in End_{U(\mathfrak{g})}(L \otimes F) \subseteq End_{U(\mathfrak{a})}(L \otimes F)$ a similar argument shows that

$$\eta \mathcal{D}^{U(\mathfrak{a})}(L \otimes F)\eta \;=\; \mathcal{D}^{U(\mathfrak{a})}(M).$$

Now apply the proposition.

3.5. We can, in particular, apply Corollary 3.4 to $L(\lambda_k + \rho)$.

COROLLARY. *Let* $L = L(\lambda_k + \rho)$, *for* k *sufficiently small. Let* E *be a finite dimensional* $U(\mathfrak{g})$*-module and* M *a direct summand of* $L \otimes E$. *Then:*

(i) M *is a highest weight module that is finitely generated as a* $U(\mathfrak{r}^-)$*-module.*

(ii) $L(M, M) = \mathcal{D}^{U(\mathfrak{r}^-)}(M)$.

PROOF: Since L is a highest weight module and is a factor ring of $U(\mathfrak{r}^-)$, part (i) is clear. Part (ii) is obtained by combining Corollary 3.4 with Theorem 1.10.

3.6. Although it is getting rather far from our frame of reference, there exist several generalisations of Corollary 3.4. Let us mention just one, suggested by recent work of Joseph [**Jo3**]. Suppose, now, that $\mathfrak{a}$ is any sub-Lie algebra of $\mathfrak{n}^-$. Given an $U(\mathfrak{a})$-module M, let $A_{U(\mathfrak{a})}(M)$ denote the set of $\mathbb{C}$-endomorphisms of M on which $\mathfrak{a}$ acts ad-nilpotently. Of course, if $\mathfrak{a}$ were abelian, this would simply be the definition of $\mathcal{D}^{U(\mathfrak{a})}(M)$ given in (3.1).

With minor alterations, (3.3) and (3.4) can be used to prove that *Corollary* 3.4 *still holds for any subalgebra* $\mathfrak{a}$ *of* $\mathfrak{n}^-$,*provided only that one replaces* $\mathcal{D}^{U(\mathfrak{a})}(\ )$ *by* $A_{U(\mathfrak{a})}(\ )$. We remark that one does require that $\mathfrak{a} \subseteq \mathfrak{n}^-$ for this to hold. For, if $\mathfrak{a} \not\subseteq \mathfrak{n}^-$, then $\mathfrak{a}$ need not act nilpotently on finite dimensional $\mathfrak{g}$-modules and so there would be no reason why (3.2.1) should hold.

CHAPTER IV. THE MAXIMALITY OF $\mathbf{J(k)}$ AND THE SIMPLICITY OF $\mathcal{D}(\overline{\mathcal{X}}_{\mathbf{k}})$

1. Introduction and consequences of the maximality of J(k) .

1.1. We will continue to use the notation and conventions established earlier and described, for example, in (III, 1.1). Recall from Chapter II that in each of the Cases A, B and C we have constructed a map $\psi : U(\mathfrak{g}) \to \mathcal{D}(\overline{\mathcal{X}}_k)$. Moreover, when k is sufficiently small we have proved in (III, Theorem 1.10) that (i) the rings $R = U(\mathfrak{g})$ and $\mathcal{D}(\overline{\mathcal{X}}_k)$ have the same full quotient ring and (ii) $\mathcal{D}(\overline{\mathcal{X}}_k)$ is a finitely generated left and right R-module. The aim of this chapter is to complete the proof of Theorem 0.3 of the Introduction by showing that $J(k) = ker\,\psi$ is a maximal ideal of $U(\mathfrak{g})$ and (as is an easy consequence) that ψ is actually onto.

In fact we show that $J(k)$ is maximal under the slightly weaker assumption given in (III, 1.1.1).

THEOREM. *The ideal $J(k)$ is maximal in the following cases:*

$$\begin{cases} 1 \le k \le q \le p & \textit{in Case A} \\ 1 \le k \le n & \textit{in Case B} \\ 2 \le 2k \le n-1 & \textit{in Case C.} \end{cases}$$

1.2. The proof of Theorem 1.1 will take up most of this chapter. Various consequences of this theorem, in particular the simplicity of $\mathcal{D}(\overline{\mathcal{X}}_k)$, will be given later in this section. In Section 2 we will outline the various possible approaches to the proof of such a theorem and we give the proof for the 3 cases in the following 3 sections. In each case, the proof is purely Lie-theoretic and follows from the fact that we have already computed the highest weight λ_k of $L(\lambda_k + \rho) = \mathcal{O}(\overline{\mathcal{X}}_k)$ as a $U(\mathfrak{g})$-module.

Thus, on the one hand, the results of this chapter can be used as an introduction to and illustration of the way in which the theory of primitive ideals, as developed by Joseph and Barbash-Vogan, et al, can be used to answer specific questions about the ideals of $U(\mathfrak{g})$. Of course the length of the chapter also indicates that the combinatorics involved are non-trivial.

On the other hand, it would be more satisfactory if one could find a more intuitive proof of the maximality of $J(k)$. For example, using Lemma 1.3(ii) below, Theorem 1.1

would follow if one could prove that $\mathcal{D}(\overline{\mathcal{X}}_k)$ were simple. We remark that the maximality of $J(k)$ is not simply a consequence of the fact that the map ψ is obtained via the metaplectic representation. Indeed, if $2k = n$ in Case C, then $J(k)$ is *not* maximal (see (5.4), below).

1.3. As we show next, (III, Theorem 1.10) and Theorem 1.1 combine to have Theorem 0.3 as an easy consequence.

LEMMA. *Let $A \subset B$ be Noetherian domains with the same quotient division ring. Then:*

(i) *If A is simple and B is a finitely generated left A-module, then $A = B$.*

(ii) *If B is simple and B is finitely generated as both a left and a right A-module then $A = B$.*

PROOF: **(i)** Write $B = \sum_1^r Ad_i$. By assumption, we may write $d_i = b_i c^{-1}$ for some non-zero b_i and $c \in A$. Thus $N = r\text{-}ann_A(B/A) \ni c$, and so N is a non-zero ideal of A. Since A is a simple ring, this forces $N = A$ and hence $A = B$.

(ii) In this case, both $N = r\text{-}ann_A(B/A)$ and $M = \ell\text{-}ann_A(B/A)$ are non-zero. Thus, NM is a non-zero ideal of B. The simplicity of B therefore implies that $B = NM = A$.

1.4. COROLLARY. *Suppose that k is sufficiently small; that is, $1 \leq k < q \leq p$ in Case A, $1 \leq k < n$ in Case B and $2 \leq 2k < n-1$ in Case C. Then $U(\mathfrak{g})/J(k) \cong \mathcal{D}(\overline{\mathcal{X}}_k)$. Moreover, $\mathcal{D}(\overline{\mathcal{X}}_k)$ is a simple Noetherian domain and is finitely generated as a $\mathbb{C}$-algebra.*

PROOF: By Theorem 1.1, $R = \psi(U(\mathfrak{g}))$ is a simple subring of the domain $\mathcal{D}(\overline{\mathcal{X}}_k)$. Lemma 1.3(i) and (III, Theorem 1.10(i)) can therefore be combined to show that $R = \mathcal{D}(\overline{\mathcal{X}}_k)$.

1.5. The following result, which complements Corollary 1.4, is a slight generalisation of [**LSS,** §3.9]. Recall that the m^{th} Weyl algebra $A_m(\mathbb{C})$ is the ring of differential operators on affine m-space.

LEMMA. *Suppose that $\mathfrak{g}$ is a semi-simple, finite dimensional complex Lie algebra and that $\theta : U(\mathfrak{g}) \to A_m(\mathbb{C})$ is a ring homomorphism, for some $m \geq 1$. Then θ is not surjective. Moreover, assume that $S = \theta(U(\mathfrak{g}))$ and $A_m(\mathbb{C})$ have the same full quotient ring. Then $A_m(\mathbb{C})$ cannot be finitely generated as a (left or right) S-module.*

REMARK. Suppose that k is not sufficiently small. Then by (II, 2.4, 3.4 and 4.4), $\mathcal{O}(\overline{\mathcal{X}}_k)$ is a polynomial ring and so $\mathcal{D}(\overline{\mathcal{X}}_k)$ *is* a Weyl algebra. Moreover, (III, Proposition 1.4) implies that $R = \psi(U(\mathfrak{g}))$ and $\mathcal{D}(\overline{\mathcal{X}}_k)$ have the same full quotient ring. Thus the lemma shows that: *If k is not sufficiently small, then $\mathcal{D}(\overline{\mathcal{X}}_k)$ cannot even be finitely generated as a $U(\mathfrak{g})$-module.*

PROOF: Since $\mathfrak{g}$ is semi-simple, S decomposes under the adjoint action of $\mathfrak{g}$ as a direct sum of simple modules. In particular, $S = \mathbb{C}\cdot 1 \oplus V$ for some $(ad\text{-}\mathfrak{g})-$ module V. Now, for any a, b and $c \in S$ one has $[ab, c] = [a, bc] + [b, ca]$. Thus

$$[S, S] = [\mathfrak{g}, S] = [\mathfrak{g}, V] \subseteq V.$$

Therefore, $1 \notin [S, S]$. In particular this implies that $S \neq A_m(\mathbb{C})$. Finally, if $A_m(\mathbb{C})$ were finitely generated as a one-sided $U(\mathfrak{g})$-module, then **[JS,** Theorem 2.9] and (III, Lemma 1.9) would combine to prove that $A_m(\mathbb{C})$ would be finitely generated as *both* a left and a right $U(\mathfrak{g})$-module. Since $A_m(\mathbb{C})$ is simple this contradicts Lemma 1.3(ii).

1.6. There is a second way to interpret Corollary 1.4. Recall that $\psi = \varphi\omega$, where $\omega : U(\mathfrak{g}) \to \mathcal{D}(\mathcal{X})^{G'}$ is the surjective homomorphism defined by the metaplectic representation and $\varphi : \mathcal{D}(\mathcal{X})^{G'} \to \mathcal{D}(\overline{\mathcal{X}}_k)$ is the restriction map. Thus Corollary 1.4 and Remark 1.5 combine to prove: *φ is nice (that is, surjective) if and only if $\mathcal{O}(\mathcal{X})^{G'}$ is nasty (that is, singular) if and only if k is sufficiently small.*

1.7. The fact that φ is surjective when k is sufficiently small may be regarded as a non-commutative analogue of the Second Fundamental Theorem of invariant theory – which in our notation is just the statement that the restriction map $\mathcal{O}(\mathcal{X})^{G'} \to \mathcal{O}(\overline{\mathcal{X}}_k)$ is an isomorphism. However, unlike the commutative case, ϕ will (almost never) be an isomorphism, since it will have a large kernel.

LEMMA. *Assume that k satisfies (III, 1.1.1) and let*

$$\Theta = GKdim\ \mathcal{D}(\mathcal{X})^{G'} - GKdim\ \mathcal{D}(\overline{\mathcal{X}}_k).$$

Then

$$\Theta = \tfrac{1}{2}\{GKdim\ \mathcal{D}(\mathcal{X}) - GKdim\ \mathcal{D}(\overline{\mathcal{X}}_k)\} = dim\ \mathcal{X} - dim\ \overline{\mathcal{X}}_k = dim\ G'.$$

Thus $\Theta = k^2$ in Case A, $\Theta = \frac{1}{2}k(k-1)$ in Case B and $\Theta = k(2k+1)$ in Case C.

PROOF: Since we do not wish to spend too much time on this somewhat peripheral result, we will only prove it in Case B, where a short cut is available. In this case the proof of (V, Lemma 3.7), below, can be used to show that the natural isomorphism of abelian groups

$$\alpha : \mathcal{D}(\mathcal{X}) \xrightarrow{\sim} \mathcal{O}(\mathcal{X})\otimes\mathcal{O}(\mathcal{X}^*) \cong \mathcal{O}(M_{k,2n}(\mathbb{C}))$$

induces an isomorphism

$$\alpha' : \mathcal{D}(\mathcal{X})^{G'} \xrightarrow{\sim} \mathcal{O}(M_{k,2n}(\mathbb{C}))^{G'} \tag{1.7.1}$$

where the action of G' on the right hand side is that induced from the natural action $g \cdot \xi = g\xi$ of $g \in G'$ on $\xi \in M_{k,2n}(\mathbb{C})$. Moreover, as α respects the natural finite dimensional filtration on both sides, this implies that

$$GKdim\ \mathcal{D}(\mathcal{X})^{G'} = GKdim\{\mathcal{O}(M_{k,2n}(\mathbb{C}))^{G'}\} = 2nk - \tfrac{1}{2}k(k-1),$$

where the final equality follows from (II, Lemma 5.3). But (II, 6.4.2) implies that

$$GKdim\ \mathcal{D}(\overline{\mathcal{X}}_k) = 2dim\ \overline{\mathcal{X}}_k = 2nk - k(k-1),$$

Thus this proves the lemma in Case B.

In Cases A and C a little more work needs to be done, as in these cases the action of G' on the right hand side of (1.7.1) is slightly twisted. The glitch comes in the proof of (V, Equation 3.7.1), where one uses the fact that ${}^tg = g^{-1}$ holds in $O(k)$ in order to show that G has the natural action on the right hand side of (1.7.1).

1.8. In **[Mu1]** and **[Mu2]** Musson examines the rings of invariants obtained from the action of a torus T on $\mathbb{C}^n$. In particular, in **[Mu1]** he has obtained necessary and sufficient conditions for the map $\varphi : \mathcal{D}(\mathbb{C}^n)^T \to \mathcal{D}(\mathcal{O}(\mathbb{C}^n)^T)$ to be surjective. In **[Mu2]** he examines the problem of whether $\mathcal{D}(\mathbb{C}^n)^T$ is a factor ring of a Lie algebra. The results he obtains for $T = \mathbb{C}^*$ are, essentially, the same as we have obtained in Case A with $k = 1$.

1.9. It is fairly easy to use Corollary 1.4 to write down an explicit set of generators for $\mathcal{D}(\overline{\mathcal{X}}_k)$. To do this, recall that the generators of $\mathfrak{sp}(2m)$ under the metaplectic representation ω are described explicitly in (I, §2). Next, rather as we did for the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ in (II, 2.6, 3.6 and 4.6), it is easy to compute the image of $\mathfrak{g}$ under ω. Indeed, Howe's Theorem (I, Theorem 1.4) can be regarded as saying that $\omega(\mathfrak{g})$ consists of the "obvious" invariant elements in $\mathcal{D}(\mathcal{X})^{G'}$. Finally, as $\psi = \varphi\omega$, this gives the generators for $\mathcal{D}(\overline{\mathcal{X}}_k)$.

In this manner one obtains the following sets of generators for $\mathcal{D}(\overline{\mathcal{X}}_k)$. In each case we assume that k is sufficiently small (so that Corollary 1.4 can be applied) and we freely use the notation of (II, §5). For a Lie algebra $\mathfrak{a}$, the set $\Psi(\mathfrak{a})$ denotes the image under ψ of the standard basis for $\mathfrak{a}$, except that one ignores the various scalars coming from the metaplectic representation.

In Case A, the generators are:

$$\left\{\sum_{j=1}^{k} x_{uj}y_{jv} : 1 \leq u \leq p,\ 1 \leq v \leq q\right\} \quad = \quad \Psi(\mathfrak{r}^+) \quad = \quad \text{generators of } \mathcal{O}(\overline{\mathcal{X}}_k)$$

and

$$\left\{\sum_{j=1}^{k} \partial^2/\partial x_{uj}\partial y_{jv} : 1 \leq u \leq p,\ 1 \leq v \leq q\right\} \quad = \quad \Psi(\mathfrak{r}^-)$$

and

$$\left\{\begin{array}{l} \displaystyle\sum_{j=1}^{k} x_{wj}\partial/\partial x_{uj} : 1 \leq u, w \leq p \quad \text{and} \\[2ex] \displaystyle\sum_{j=1}^{k} y_{jw}\partial/\partial y_{jv} : 1 \leq v, w \leq p \end{array}\right\} \quad = \quad \Psi(\mathfrak{m}).$$

In Case B the generators are:

$$\left\{\sum_{j=1}^{k} x_{ju}x_{jv} : 1 \leq u \leq v \leq n\right\} \quad = \quad \Psi(\mathfrak{r}^+) \quad = \quad \text{generators of } \mathcal{O}(\overline{\mathcal{X}}_k)$$

and

$$\left\{\sum_{j=1}^{k} \partial^2/\partial x_{ju}\partial x_{jv} : 1 \leq u \leq v \leq n\right\} \quad = \quad \Psi(\mathfrak{r}^-)$$

and

$$\left\{\sum_{j=1}^{k} x_{jw}\partial/\partial x_{ju} : 1 \leq u, w \leq n\right\} \quad = \quad \Psi(\mathfrak{m}).$$

In Case C the generators are:

$$\left\{\sum_{j=1}^{k}(x_{j,a}x_{j+k,b}-x_{j,b}x_{j+k,a}) : 1\leq a<b\leq n\right\} \quad = \quad \Psi(\mathfrak{r}^{+}) \quad = \quad \text{generators of } \mathcal{O}(\overline{\mathcal{X}}_k)$$

and

$$\left\{\sum_{j=1}^{k}(\partial^2/\partial x_{j,a}\partial x_{j+k,b}-\partial^2/\partial x_{j,b}\partial x_{j+k,a}) : 1\leq a<b\leq n\right\} \quad = \quad \Psi(\mathfrak{r}^{-})$$

and

$$\left\{\sum_{j=1}^{2k}(x_{ja}\partial/\partial x_{jb}) : 1\leq a,b\leq n\right\} \quad = \quad \Psi(\mathfrak{m}).$$

1.10. There is a second way of describing the generators of $\mathcal{D}(\overline{\mathcal{X}}_k)$. For, the second fundamental theorem of invariant theory allows one to identify $\mathcal{O}(\overline{\mathcal{X}}_k)=\mathbb{C}[Z]/I(k)$. Here, $Z=(z_{ij})$ is an appropriate generic matrix and both it and $I(k)$ are given explicitly in (II, Theorems 2.3(ii), 3.3(ii) and 4.3(ii)) for Cases A, B and C, respectively. One may then write $\mathcal{D}(\overline{\mathcal{X}}_k)$ in terms of the z_{ij} and $\partial/\partial z_{uv}$ in the following way. Write $\mathbb{C}[Z]=\mathcal{O}(\mathcal{Z})$ for the appropriate affine space $\mathcal{Z}$ and set $L(k)=I(k)\mathcal{D}(\mathcal{Z})$. The *idealizer* $\mathbb{I}(L(k))$ is defined to be

$$\mathbb{I}(L(k)) \quad = \quad \big\{\theta\in\mathcal{D}(\mathcal{Z}) :\ \theta L(k)\ \subseteq\ L(k)\big\}.$$

The point of this definition is that [**SmSt,** Proposition 1.6] implies that $\mathcal{D}(\overline{\mathcal{X}}_k)\cong\mathbb{I}(L(k))/L(k)$.

It is now easy enough in principle (although the practice is rather tedious) to compute the inverse images in $\mathbb{I}(L(k))$ of the generators of $\mathcal{D}(\overline{\mathcal{X}}_k)$. Moreover, except in a few low dimensional cases, the degree of elements of $L(k)$ is large in comparison with that of the generators of $\mathcal{D}(\overline{\mathcal{X}}_k)$. Thus, each generator of $\mathcal{D}(\overline{\mathcal{X}}_k)$ has a unique inverse image of lowest total degree back in $\mathbb{I}(L(k))$. We can now describe the generators of $\mathcal{D}(\overline{\mathcal{X}}_k)$ in terms of the z_{ij} and $\partial/\partial z_{uv}$. However we will only give these generators in Case A (this should at least give the reader an idea of their structure. In the other two cases the generators have a similar but considerably messier form). As in (1.9), we assume that k is sufficiently small and we will not distinguish between elements of $\mathcal{D}(\mathcal{Z})$ and their images in $\mathcal{D}(\overline{\mathcal{X}}_k)$. In each case an element, say $\alpha(u,v)\in\Psi(\mathfrak{r}^{-})$, is equal to the element with the same indices, say $\beta(u,v)\in\Psi(\mathfrak{r}^{-})$ given in (1.9).

In Case A, the generators are:

$$\left\{ z_{uv} \;=\; \sum_{j=1}^{k} x_{uj} y_{jv} : 1 \le u \le p,\ 1 \le v \le q \right\} \;=\; \Psi(\mathfrak{r}^+) \;=\; \text{generators of } \mathcal{O}(\overline{\mathcal{X}}_k)$$

and

$$\left\{ \sum_{i=1}^{p} \sum_{j=1}^{q} z_{ij} \partial^2/\partial z_{uj} \partial z_{iv} \;+\; k\partial/\partial z_{uv} : 1 \le u \le p,\ 1 \le v \le q \right\} \;=\; \Psi(\mathfrak{r}^-)$$

and

$$\left\{ \begin{array}{ll} \displaystyle\sum_{j=1}^{q} z_{wj} \partial/\partial z_{uj} : 1 \le u, w \le p & \text{and} \\ \\ \displaystyle\sum_{j=1}^{p} z_{jw} \partial/\partial z_{jv} : 1 \le v, w \le p & \end{array} \right\} \;=\; \Psi(\mathfrak{m}).$$

In the case $k = 1$ Goncharov in [**Go,** §**6**] has also written down the generators of $\mathcal{D}(\overline{\mathcal{X}}_k)$ in the above manner, but there seem to be some minor errors in his scalars.

2. Outline of the proof of the maximality of $\mathbf{J}(\mathbf{k})$.

2.1. Recall that, as a $U(\mathfrak{g})$-module, $\mathcal{O}(\overline{\mathcal{X}}_k) = L(\lambda_k + \rho)$ is a highest weight module for which the highest weight λ_k has been explicitly determined (see (II, 2.7, 3.7 and 4.7)). In this chapter we will show that, under the hypotheses of Theorem 1.1, this forces $J(k) = ann\ L(\lambda_k + \rho)$ to be a maximal ideal. There are several methods available in the literature for proving this and we outline the various techniques in this section. It will be quickest to use different methods in the different cases.

While the results used in this chapter are due to various people, most are available in [**Ja**] and we will reference that book as much as possible. Unfortunately our notation differs from Jantzen's since we shift our weights by ρ, the half sum of the positive roots. Thus our $M(\lambda+\rho)$ is his $M(\lambda)$ and our $J(w(\lambda+\rho))$, for $w \in W$, is his $I(w\cdot\lambda)$, where $w \cdot \lambda = w(\lambda+\rho) - \rho$; etc.

2.2. The following standard notation (over and above that given in the index of notation) will be used without comment throughout this chapter. We will usually write μ_k for $\lambda_k + \rho$.

Fix a Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{g}$ and write R for the set of roots of the pair $(\mathfrak{g}, \mathfrak{h})$. Fix a basis $B = \{\alpha_1, \dots, \alpha_\ell\}$ of R and let R^+ denote the set of positive roots. The co-root of $\alpha \in R$ will be denoted by H_α. Let $\overline{\omega}_1, \dots, \overline{\omega}_\ell$ be the fundamental weights and write $P(R) = \sum \mathbb{Z}\overline{\omega}_i$ for the lattice of (integral) weights. We emphasise that $\rho \in P(R)$ (see [**B1,** Proposition 29, p.168]). Similarly, write $Q(R) = \sum \mathbb{Z}\alpha_i$ for the lattice of roots. If $\Lambda = \lambda + P(R)$ is the coset of λ in $\mathfrak{h}^*/P(R)$ then $R_\Lambda = \{\alpha \in R : \lambda(H_\alpha) \in \mathbb{Z}\}$ will denote the corresponding root system, with restricted Weyl group

$$W_\Lambda \;=\; \big\{w \in W : w\lambda - \lambda \in Q(R)\big\}.$$

Fix a basis B_Λ for R_Λ such that $B_\Lambda \subset R^+$. If $w \in W_\Lambda$ then define

$$\tau(w) \;=\; \big\{\alpha \in B_\Lambda : w\alpha < 0\big\}.$$

Let $\nu \in \Lambda$. Then ν is *dominant* (respectively *dominant regular*) if $\nu(H_\alpha) \geq 0$ (respectively $\nu(H_\alpha) > 0$) for all $\alpha \in B_\Lambda$. (Note that this is, again, a shifted version of Jantzen's notation.) There will always exist $\pi \in W_\Lambda\lambda$ such that π is dominant (see [**Ja,** §2.5]), and, moreover, $\Lambda = \pi + P(R)$. Finally, given $\pi \in \Lambda$ dominant, set $B^o_\pi = \{\alpha \in B_\Lambda : \pi(H_\alpha) = 0\}$.

2.3. Given $\Lambda = \lambda + P(R)$, set $\mu = \lambda + \rho$ and pick $\pi \in W_\Lambda\mu$ such that π is dominant. Write

$$\zeta_\mu \;=\; \big\{J \in \mathit{Spec}\, U(\mathfrak{g}) : J \cap Z(\mathfrak{g}) = J(\mu) \cap Z(\mathfrak{g})\big\}.$$

Then, by Duflo's Theorem ([**Ja,** Satz 7.3]),

$$\zeta_\mu \;=\; \big\{J(w\mu) : w \in W\big\} \;=\; \big\{J(w\mu) : w \in W_\Lambda\big\} \;=\; \zeta_\pi.$$

Moreover, ζ_μ has a unique maximal element – viz $J(\pi)$ (see [**Ja,** Corollar 5.21]). Thus the aim of this chapter is to prove that $J(\mu_k) = J(\pi)$ where $\mu_k = \lambda_k + \rho$ and $\pi \in W_\Lambda(\lambda_k + \rho)$ is dominant.

This motivates the following definition. Let $\chi \in \Lambda$ be dominant regular and pick $w, w' \in W_\Lambda$. Then define w and w' to be in the same *left cell,* written $w \sim_\ell w'$, if $J(w\chi) = J(w'\chi)$. By [**Ja,** §14.15] this does not depend on the choice of χ. The advantage of this concept is that, as will be seen later, there exists a combinatorial procedure for determining whether $w \sim_\ell w'$. Moreover, while the dominant weight π will almost never be regular, the following translation principle allows one to pass to the regular case.

THEOREM. [**Ja,** Satz 5.8] *Let Λ, π and χ be as above. Then there exists an isomorphism of ordered sets*

$$T_*^\pi : \big\{J(w\chi) : w \in W_\Lambda \text{ and } B_\pi^o \subseteq \tau(w)\big\} \;\longrightarrow\; \zeta_\pi$$

given by $T_*^\pi\big(J(w\chi)\big) = J(w\pi\big)$.

2.4. This provides the first method for determining whether $J(\mu_k)$ is maximal:

Method (a). *Write* $\Lambda = \lambda_k + P(R)$, set $\mu_k = \lambda_k + \rho$ *and pick* $\pi \in W\mu_k$ *such that* π *is dominant. Next, find* $y, w \in W_\Lambda$ *such that*

$$w\pi \;=\; \pi \;=\; y^{-1}\mu_k \quad \text{and} \quad B_\pi^o \;\subseteq\; \tau(w) \cap \tau(y).$$

Then by (2.3), *in order to show that* $J(\mu_k)$ *is maximal, it suffices to show that* $w \sim_\ell y$. *In order to do this one uses a combinatorial procedure, called the Robinson-Schensted algorithm, to compute the standard left tableaux* $A(w)$ *and* $A(y)$. *Finally, at least for the classical Lie algebras, the equality* $A(w) = A(y)$ *implies that* $w \sim_\ell y$ *and hence that* $J(\mu_k)$ *is maximal* (see [**Ja,** §§16.13, 16.14 and 5.25]).

We have been deliberately vague in the final part of Method (a), since left tableaux are defined differently for the different Lie algebras. Thus the details for this step will be given later.

2.5. The computations involved in Method (a) can be reasonably tedious. However, there is a second method using special nilpotent orbits which, while not always available, can be much easier. Unfortunately, the various steps in this method will vary slightly from case to case and so at this stage we will merely indicate the basic steps and leave the details until later (see, in particular, Section 4).

In a classical Lie algebra $\mathfrak{g}$, the nilpotent orbits can be classified by symbols (see, for example, [**BV1,** p.165]), and the orbit is *special* if the corresponding symbol is special. Alternatively, by [**BV1,** §1.10], an orbit $\mathbf{O}$ is special if and only if $\overline{\mathbf{O}} = \mathcal{V}(J)$ is the associated variety of a primitive ideal J that has an integral central character. Next, let ${}^L\mathfrak{g}$ denote the *dual Lie algebra* of $\mathfrak{g}$, as for example defined in [**BV2,** Definition 1.12]. Then to any special nilpotent orbit $\mathbf{O}$ in $\mathfrak{g}$ one may associate via its symbol a (unique) dual nilpotent orbit ${}^L\mathbf{O}$ in ${}^L\mathfrak{g}$ (see [**BV2,** Corollary 3.25]). The orbit ${}^L\mathbf{O}$ is *even* if its Dynkin diagram is labelled by 0 and 2 ([**BV2,** Definition 2.11]). If the orbit ${}^L\mathbf{O}$ is even, then there exists a dominant integral weight $\lambda_{\mathbf{O}}$ in $\mathfrak{h}^*$ related to $\mathbf{O}$ (see [**BV2,** §5.4]). For typographical reasons we will write $\lambda(\mathbf{O})$ in place of $\lambda_{\mathbf{O}}$.

There exists an algorithm for each step in the above procedure, but it will be easier to describe these in the specific examples than in general. However, the point behind these definitions is the following result.

THEOREM. *Assume that $\mathbf{O}$ is a special nilpotent orbit with ${}^L\mathbf{O}$ even. Let $\mu \in \mathfrak{h}^*$ and suppose that $J(\mu) \in \zeta_{\lambda(\mathbf{O})}$ and that GKdim $U(\mathfrak{g})/J(\mu) =$ dim $\mathbf{O}$. Then $J(\mu) = J(\lambda(\mathbf{O}))$ and $J(\mu)$ is a maximal ideal of $U(\mathfrak{g})$.*

PROOF: Use the equivalence of parts (b) and (c) of [**BV2,** Definition 5.23] together with [**BV2,** Corollary 5.20(i)].

2.6. Recall that the nilpotent orbit $\mathbf{O}_k$ is defined by $\overline{\mathbf{O}}_k = \mathcal{V}(J(k))$, the associated variety of $J(k)$. Using Theorem 2.5 one obtains

METHOD (b). *Assume that $\mathbf{O}_k$ is special and that ${}^L\mathbf{O}_k$ is even. Then, by definition, μ_k satisfies GKdim $U(\mathfrak{g})/J(\mu_k) =$ dim $\mathbf{O}_k$. Assume, further, that $\mu_k \in W\lambda(\mathbf{O}_k)$. Then $J(\mu_k) \in \zeta_{\lambda(\mathbf{O}_k)}$ and so Theorem 2.5 implies that $J(\mu_k) = J(\lambda(\mathbf{O}_k))$. Therefore, $J(\mu_k)$ is maximal.*

2.7. Of the two methods given above, the second is shorter, but is unfortunately not always available. This is because either $\mathbf{O}_k$ will not be special (which happens in Case B when k is odd) or ${}^L\mathbf{O}_k$ will not be even (which can happen in Case A). The first method is always available. Thus we will use Method (b) in Case B (for k even) and Case C, while Method (a) will be used in Case A and Case B (for k odd).

Finally, the question of exactly when $\mathbf{O}_k$ is special in Case B will be of interest in Chapter V, where we consider the ring of $SO(k)$-invariants as a $U(\mathfrak{sp}(2n))$-module (see (V, 5.5)).

3. The maximality of J(k) in Case A.

3.1. In this section we will use Method (a) of (2.4) in order to prove that $J(k)$ is maximal in Case A. Thus, fix $\mathfrak{g} = \mathfrak{sl}(n)$ where $n = p+q$ and assume that $1 \leq k \leq q \leq p$. We will also fix the following notation. Let $\mathfrak{h} = \{\sum_1^n \lambda_i H_i : \sum \lambda_i = 0\}$ be the usual Cartan subalgebra of $\mathfrak{g}$ and write $\{\epsilon_i = H_i^*\}$ for the dual basis of $\{H_i\}$. Then the roots of $\mathfrak{g}$ are given by $R = \{\pm(\epsilon_i - \epsilon_j)\}$ with the standard basis

$$B \;=\; \{\alpha_1 = \epsilon_1 - \epsilon_2, \ldots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n\}.$$

A weight $\lambda \in \mathfrak{h}^*$ will be identified with an element $\sum \lambda^i \epsilon_i$ of $\sum \mathbb{C}\epsilon_i$ and written $\lambda = (\lambda^1, \ldots, \lambda^n)$.

We remark that, while B is the standard basis for R, it is the negative of the basis given in (II, 2.6). Thus by (II, 2.7) the weight λ_k is now the element

$$\lambda_k \;=\; \big(-k/2, \ldots, -k/2,\; k/2, \ldots, k/2\big), \tag{3.1.1}$$

where there are p negative and q positive entries. As before, set $\mu_k = \lambda_k + \rho$.

3.2. It follows from [**B1,** Planche I, p.251] that

$$\rho \;=\; \tfrac{1}{2}\big(n-1,\; n-3, \ldots, -n-1\big)$$

and an easy computation shows that

$$\mu_k \;=\; \big(a_1, \ldots, a_{p-k},\; b_1, \ldots, b_k,\; b_1, \ldots, b_k,\; c_1, \ldots, c_{q-k}\big)$$

where $a_i = \frac{1}{2}(n-k-1)+1-i$ and $b_i = -\frac{1}{2}(n-k-1)+q-i$ and $c_i = -\frac{1}{2}(n-k-1)+q-k-i$ for each i.

The coroots are $\{H_{\alpha_i} = H_i - H_{i+1} : 1 \leq i \leq n-1\}$ and hence (3.1.1) implies that λ_k is integral; that is, $\lambda_k^i - \lambda_k^{i+1} = \lambda(H_{\alpha_i}) \in \mathbb{Z}$ for each i. In particular,

$$\Lambda \;=\; \lambda_k + P(R) \;=\; P(R).$$

3.3 We can now begin to describe the elements required of Method (a). The Weyl group W of $\mathfrak{sl}(n)$ is just the symmetric group S_n acting on $\mathfrak{h}^*$ by permutation of the ϵ_i. In the description of μ_k in (3.2) note that the entries are descending integers, with the one exception that $b_k < b_1$ at the p^{th} entry. Therefore

$$\pi \;=\; \big(a_1, \ldots, a_{p-k},\; b_1,\; b_1,\; b_2,\; b_2, \ldots, b_k,\; b_k,\; c_1, \ldots, c_{q-k}\big)$$

is dominant and belongs to $W\mu_k$. Clearly

$$B^o_\pi \;=\; \{\alpha_{p-k+1}, \,\ldots\, , \alpha_{p+k-1}\}.$$

We next want to find $w, y \in W$ such that $w\pi = \pi = y^{-1}\mu_k$ and $B^o_\pi \subseteq \tau(w) \cap \tau(y)$. For w take the element that swops each pair of b_i's in the description of π and fixes the other entries. Thus

$$w \;=\; \left(I \;\middle|\; \begin{matrix} (p-k+1) & (p-k+2) & \ldots & (p+k-1) & (p+k) \\ (p-k+2) & (p-k+1) & \ldots & (p+k) & (p+k-1) \end{matrix} \;\middle|\; I \right).$$

(Here and elsewhere the I will imply that w fixes the appropriate number of entries at the beginning and end.) Trivially, $w\pi = \pi$ and $B^o_\pi = \tau(w)$. For future reference we note that $w = w^{-1}$.

In order that $B^o_\pi \subseteq \tau(y)$ we choose for y the permutation that sends the "first b_i" in π to the "second b_i" in μ_k but fixes the a_u and c_v. Thus

$$y \;=\; \left(I \;\middle|\; \begin{matrix} (p-k+1) & (p-k+2) & (p-k+3) & (p-k+4) & \ldots \\ (p+1) & (p-k+1) & (p+2) & (p-k+2) & \ldots \end{matrix} \;\middle|\; I \right).$$

and

$$y^{-1} \;=\; \left(I \;\middle|\; \begin{matrix} (p-k+1) & (p-k+2) & \ldots & (p+1) & (p+2) & \ldots \\ (p-k+2) & (p-k+4) & \ldots & (p-k+1) & (p-k+3) & \ldots \end{matrix} \;\middle|\; I \right).$$

Once again it is clear that $y\pi = \mu_k$ and that $B^o_\pi \subseteq \tau(y)$.

3.4. It remains to compute the left tableaux

$$A(w) \;=\; A\left(w^{-1}(1),\, w^{-1}(2),\, \ldots\, , w^{-1}(n)\right)$$

and $A(y)$. The algorithm for this is given, for example, in **[Ja,** §5.23] but in outline is as follows. A *left tableau* is an array of numbers (ξ_{ij}) arranged in rows $(\xi_{i1}, \xi_{i2}, \ldots, \xi_{i,n(i)})$ such that $\xi_{ij} \le \xi_{i,j+1}$ and $n(i) \ge n(i+1)$ for each i and j. A tableau is associated to a permutation σ as follows. Suppose that $\sigma^{-1}(1),\ \sigma^{-1}(2), \ldots, \sigma^{-1}(r)$ have been arranged in a tableau. Then $\sigma^{-1}(r+1)$ will either replace the smallest element larger than itself in the first row or, if that is not applicable, will be added to the end of the first row. The displaced element will be added to the second row using the same rule. Using the fact that $w^{-1} = w$, this procedure can be used to prove that

$$A(w) \;=\; A(y) \;=$$

1	2	$\cdots$	k	$\cdots$	$p-k+1$	$p-k+3$	$\cdots$
$p-k+2$	$p-k+4$	$\cdots$	$p+k$				

By **[Ja,** Satz 5.25] this implies that $w \sim_\ell y$. Finally, by Method (a), one obtains

$$J(\pi) \;=\; J(w\pi) \;=\; J(y\pi) \;=\; J(\mu_k),$$

and so $J(\mu_k)$ is, indeed, maximal. Thus we have proved:

PROPOSITION. *(Case A) If $1 \le k \le q \le p$ then $J(k)$ is maximal.*

4. The Maximality of J(k) in Case B.

4.1. In this section $\mathfrak{g}$ will always denote the Lie algebra $\mathfrak{sp}(2n)$ and we assume that $1 \le k \le n$. As was remarked at the end of Section 2, in proving the maximality of $J(k)$ we aim to use Method (b) whenever possible. Thus in this section we will show that, for k even, the orbit $\mathbf{O}_k$ is special and Method (b) can be applied. Unfortunately when k is odd we will have to use Method (a).

For the reader's convenience, we begin with some standard notation and results about $\mathfrak{g}$ as described, for example, in [**B1,** Planche III, p.254]. Once again, write $\mathfrak{h} = \sum_1^n \mathbb{C}H_i$ and let $\{\varepsilon_i = H_i^*\}$ be a dual basis of $\{H_i\}$. Then the roots of $\mathfrak{g}$ are $R = \{\pm(\varepsilon_i \pm \varepsilon_j),\ 2\varepsilon_i\}$ for which we fix the basis

$$B \;=\; \big\{\alpha_i = \varepsilon_i - \varepsilon_{i+1} : 1 \le i \le n-1, \quad \alpha_n = 2\varepsilon_n\big\}.$$

The coroots are $\{H_{\alpha_i} = H_i - H_{i+1} : 1 \le i \le n-1,\ H_{\alpha_n} = H_n\}$. As before, an element $\lambda \in \mathfrak{h}^*$ will be written $\lambda = \sum_1^n \lambda^i \varepsilon_i = (\lambda^1, \dots, \lambda^n)$. In particular, $\rho = (n, n-1, \dots, 2, 1)$ and, by (II, 3.6), $\lambda_k = (-k/2, \dots, -k/2)$.

4.2. We first want to determine when $\mathbf{O}_k$ is special, which we do by means of the algorithm from [**BV1,** p.165] or [**Ja,** Chapter 16]. This requires the following definition. For a Lie algebra of type C_n, define a *symbol* Σ to be any $(2m+1)$-tuple of positive integers

$$\Sigma \;=\; \begin{pmatrix} \sigma \\ \tau \end{pmatrix} \;=\; \begin{pmatrix} \sigma_1 & \dots & \sigma_m & \sigma_{m+1} \\ \tau_1 & \dots & \tau_m & \end{pmatrix}$$

such that the following technical conditions are satisfied: $\sigma_i < \sigma_{i+1}$, $\tau_i < \tau_{i+1} \le \sigma_{i+2}$, $\sigma_i \le \tau_{i+1} + 1$ for each i and $\sum_1^n \sigma_i + \sum_1^n \tau_i = n + m^2$. Then Σ is *special* if $\sigma_i \le \tau_i \le \sigma_{i+1}$ for each i. From a symbol Σ one can obtain a partition of n as follows. Take the set $\{2\sigma_i,\ 2\tau_i + 1\}$ and re-order it with an increasing order; say $\{v_j : 1 \le j \le 2m+1\}$. Then $\{w_j = v_j - j + 1\}$ provides a partition of n in which each odd number occurs an even number of times. Given a nilpotent orbit $\mathbf{O}$ one can reverse the procedure by passing to its associated partition (as in (II, 6.6.2)) and then to the corresponding symbol, *Symb* $\mathbf{O}$ (up to an equivalence relation on the set of symbols, which we ignore, this is unique). Finally, $\mathbf{O}$ is *special* if *Symb* $\mathbf{O}$ is special.

We may now apply this technique to the orbit $\mathbf{O}_k$.

LEMMA. *The orbit $\mathbf{O}_k$ is special if and only if either (i) k is even or (ii) $k = n$ and k is odd.*

PROOF: If k is even, then it is easy to check that

$$Symb\ \mathbf{O}_k = \begin{pmatrix} 0 & 1 & \dots & (n-k+1)\widehat{} & \dots & n-\frac{1}{2}k & n-\frac{1}{2}k+1 \\ 1 & 2 & \dots & & \dots & n-\frac{1}{2}k & \end{pmatrix}$$

where, as usual, the hat $\widehat{}$ indicates that that term is to be deleted. This is a special symbol, and so $\mathbf{O}_k$ is special. If $k=n$ is odd, then

$$Symb\ \mathbf{O}_k = \begin{pmatrix} 1 & \dots & \frac{1}{2}(n-1) & \frac{1}{2}(n+1) \\ 1 & \dots & \frac{1}{2}(n-1) & \end{pmatrix}$$

which is again special. Finally, if k is odd with $k<n$, then

$$Symb\ \mathbf{O}_k = \begin{pmatrix} 1 & 2 & \dots & & \dots & n-\frac{1}{2}(k+1) & n-\frac{1}{2}(k-1) \\ 0 & 1 & \dots & (n-k)\widehat{} & \dots & n-\frac{1}{2}(k+1) & \end{pmatrix}$$

which is clearly not special.

4.3. In order to apply Method (b), we next need to find the element $\lambda(\mathbf{O}_k)$. The algorithm for this is as follows. Recall that the dual Lie algebra ${}^L\mathfrak{sp}(2n)$ equals $\mathfrak{so}(2n+1)$ (see for example [**BV2,** Definition 1.12]). Given the orbit $\mathbf{O}_k$ with symbol

$$\Sigma = \begin{pmatrix} \sigma \\ \tau \end{pmatrix} = \begin{pmatrix} \sigma_1 & \dots & \sigma_m & \sigma_{m+1} \\ \tau_1 & \dots & \tau_m & \end{pmatrix},$$

one first attaches a nilpotent orbit ${}^L\mathbf{O}_k$ in ${}^L\mathfrak{g}$ as follows. Let $t=\sigma_{m+1}$ and write $\Sigma' = \begin{pmatrix} \sigma' \\ \tau' \end{pmatrix}$ for the symbol whose first (respectively second) row consists of the integers j such that $0 \le j \le t$ and $t-j$ *does not* belong to the other row. Then ${}^L\mathbf{O}_k$ is defined by $\Sigma' = Symb\ {}^L\mathbf{O}_k$. (However, as one now has a Lie algebra ${}^L\mathfrak{g}$ of type B_n, the partition of n is obtained from Σ' by using $\{v_j\} = \{2\sigma'_i+1,\ 2\tau'_i\}$. For more details see [**Lu,** 4.5.5] or [**BV2,** Corollary 3.25].)

Next, pick an $\mathfrak{sl}(2)$-triplet $\{{}^LE, {}^LH, {}^LF\}$ in ${}^L\mathfrak{g}$ for which ${}^LE \in {}^L\mathbf{O}_k$ and such that ${}^LH \in {}^L\mathfrak{h} = \mathfrak{h}^*$ with ${}^LH$ dominant (see [**BV2,** §5.3]). By [**BV2,** Theorem 2.6] ${}^LH$ is uniquely determined by ${}^L\mathbf{O}_k$. Finally, $\lambda(\mathbf{O}_k) = \frac{1}{2}({}^LH)$ (see [**BV2,** §5.4]).

Now apply this procedure to $\mathbf{O}_k$. If k is even then using Lemma 4.2 one obtains

$$Symb\ {}^L\mathbf{O}_k = \begin{pmatrix} 0 & n-\frac{1}{2}k+1 \\ \frac{1}{2}k & \end{pmatrix}$$

and hence that ${}^L\mathbf{O}_k$ has partition $(d_1,\ d_2,\ d_3) = (2n-k+1,\ k-1,\ 1)$. Next, one uses the algorithm from [**SpSt,** Chapter 4, §2.32] in order to find ${}^LH$. In our notation, this proceeds as follows. Take the numbers

$$\{t_{n(i,j)} \;=\; 1-d_i+2j \;:\; 1\le i\le 3 \text{ and } 0\le j\le d_{i-1}\}$$

and reorder them as $\{t_1 \ge \ldots \ge t_{2n+1}\}$. Then, using the notation of (4.1),

$${}^L H \;=\; (t_1, \ldots, t_n) \;\in\; {}^L\mathfrak{h} \;=\; \mathfrak{h}^*.$$

In this manner one obtains that

$${}^L H \;=\; \big(2n-k,\ 2n-k-2, \ldots, k+2,\ k,\ k-2,\ k-2, \ldots, 2,\ 2,\ 0\big).$$

Observe that, as k is even, ${}^L H(H_\alpha)$ is even for each co-root $H_\alpha \in \mathfrak{h}$ (as described in (4.1)). Equivalently, ${}^L\mathbf{O}_k$ is *even* (see [**BV2,** Definition 2.7]).

In summary, we have proved the first part of the following lemma. The proof of the second part of the lemma is left to the reader as it is similar and will not be needed subsequently.

LEMMA. *Assume that $\mathbf{O}_k$ is special. Then ${}^L\mathbf{O}_k$ is even and the weight $\lambda(\mathbf{O}_k)$ is given by:*

(i) *If k is even then*

$$\lambda(\mathbf{O}_k) \;=\; \big(n-\tfrac12 k,\ n-\tfrac12 k-1, \ldots, \tfrac12 k,\ \tfrac12 k-1,\ \tfrac12 k-1, \ldots, 1,\ 1,\ 0\big).$$

(ii) *If $k=n$ is odd then* $\lambda(\mathbf{O}_k) \;=\; \tfrac12\big(n-1,\ n-1, \ldots, 1,\ 1,\ 0\big)$.

4.4. COROLLARY. *(Case B) If k is even and $1\le k\le n$ then $J(k)$ is a maximal ideal.*

PROOF: By (4.1), $\mu_k \;=\; \big(n-\tfrac12 k,\ n-\tfrac12 k-1,\ \ldots, -(\tfrac12 k-1)\big)$. It remains to show that $\mu_k \in W\lambda(\mathbf{O}_k)$. But by [**B1,** Planche III, p.255] the Weyl group W in type C_n is a semidirect product of S_n and $(\mathbb{Z}/2\mathbb{Z})^n$. Here S_n acts by permuting the ε_i's while, if γ_i is the generator of the i^{th} copy of $\mathbb{Z}/2\mathbb{Z}$, then $\gamma_i(\varepsilon_j) = (-1)^{\delta_{ij}}\varepsilon_j$. Clearly, therefore, $\mu_k \in W\lambda(\mathbf{O}_k)$. Thus, by Lemma 4.3, the conditions of Method (b) of (2.6) are met and $J(\mu_k) = J(\lambda(\mathbf{O}_k))$ is indeed maximal.

REMARK. Unfortunately Method (c) cannot be applied in the case when $k=n$ is odd. For, in this case $\lambda(\mathbf{O}_k)$ is integral (see 4.3) whereas μ_k is not. Thus $\mu_k \notin W\lambda(\mathbf{O}_k)$.

4.5. We assume from now on that k is odd and we will use Method (a) in order to show that $J(k)$ is maximal. Since the principle is the same as that in Case A, although the details are messier, we will leave a number of the computations to the reader.

From the description of μ_k in (4.4) one observes that, for each i and j,

$$\mu_k(H_{2\varepsilon_i}) = n - \tfrac{1}{2}k - i + 1 \ \notin \ \mathbb{Z} \qquad \text{but} \qquad \mu_k(H_{\varepsilon_i \pm \varepsilon_j}) = \mu(H_i) \pm \mu(H_j) \ \in \ \mathbb{Z}.$$

Thus $R_\Lambda \ = \ \bigl\{ \pm(\varepsilon_i \pm \varepsilon_j) : 1 \le i < j \le n \bigr\}$. This is a root system of type D_n and we fix a basis

$$B_\Lambda \ = \ \bigl\{\beta_1 = \varepsilon_1 - \varepsilon_2, \dots, \beta_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \quad \beta_n = \varepsilon_{n-1} + \varepsilon_n\bigr\}.$$

One can easily check that μ_k is dominant when $k = 1$ or $k = 3$ and so in these cases $J(\mu_k)$ is maximal by (2.3). Thus we may assume that $k \ge 5$. By **[B1,** Planche IV, p.257], W_Λ is the semidirect product of S_n and $(\mathbb{Z}/2\mathbb{Z})^{n-1}$. As a subgroup of W it is generated by S_n and all *even* products of the γ_i (notation 4.4).

4.6. (k odd) Write $p = \frac{1}{2}(k-1) \in \mathbb{N}$ and note that

$$\begin{aligned} \mu_k \ &= \ \bigl(n - \tfrac{1}{2}k,\ n - \tfrac{1}{2}k - 1, \dots, -(\tfrac{1}{2}k - 1)\bigr) \\ &= \ \bigl(a_1, \dots, , a_{n-k+1},\ b_1, \dots, b_p,\ -b_p, \dots, -b_1\bigr) \end{aligned}$$

for appropriate positive half integers a_i and b_j satisfying $a_i > a_{i+1} > b_j > b_{j+1}$ for each i and j. Set

$$\pi \ = \ \bigl(a_1, \dots, a_{n-k+1},\ b_1,\ b_1, \dots, b_{p-1},\ b_{p-1},\ b_p,\ (-1)^p b_p\bigr).$$

We claim that there exist $w,\ y \in W_\Lambda$ such that $w\pi = \pi = y^{-1}\mu_k$. (It should be remarked that the $(-1)^p$ occurs in the definition of π because only even products of γ_j's are permitted in W_Λ.) The elements w and y are chosen in a similar manner (and for similar reasons) to that given in (3.3) for Case A. Thus, w swops each pair of b_i's in π while y maps the "first b_i" in π to the "second b_i" in μ_k. But, of course, one must also now insert the appropriate number of minus signs. More formally,

$$w \ = \ w^{-1} \ = \ \left(I \ \middle| \ \begin{matrix} (n-k+2) & (n-k+3) & \dots & (n-1) & n \\ (n-k+3) & (n-k+2) & \dots & n & (n-1) \end{matrix} \right) \gamma_n^p \gamma_{n-1}^p$$

and

$y^{-1} \;=$

$$\left(I \;\middle|\; \begin{matrix} (n-k+2) & (n-k+3) & \dots & (n-p) & (n-p+1) & & \dots & \\ (n-k+3) & (n-k+5) & \dots & n & (n-1) & (n-3) & \dots & (n-k+2) \end{matrix} \right) \gamma$$

where $\gamma \;=\; \gamma_n \gamma_{n-1} \dots \gamma_{n-p+1} \gamma_{n-p}^{p}$. It is readily checked that $w,\ y \in W_\Lambda$ and satisfy $w\pi = \pi = y^{-1}\mu_k$. It is also left to the reader to show that $\tau(w) \cap \tau(y) \supseteq B_\pi^o$. In this case, $B_\pi^o \;=\; \{\beta_{n-k+2},\ \beta_{n-k+4},\ \dots, \beta_{n-\nu}\}$, where $\nu = -1$ if p is even but $\nu = 0$ if p is odd.

4.7. In order to apply Method (a) of (2.4) it remains to prove that $w \sim_\ell y$. As in Case A, to do this one uses the standard tableaux $A(y)$ and $A(w)$, although the algorithm is now a little more involved. Let $z \in W_\Lambda$.Following **[Ja,** §16.12], define $\widetilde{z}$ to be the ordered set:

$$\widetilde{z} \;=\; \{z^{-1}(1)^+, \dots, z^{-1}(n)^+,\ z^{-1}(-n)^+, \dots, z^{-1}(-1)^+\},$$

where $z(-j) = -z(j)$ and $j^+ = j$ but $(-j)^+ = 2n+1-j$ for $1 \le j \le n$. Then $\widetilde{z}$ is a permutation of $\{1,\ \dots, 2n\}$ and the Robinson-Schensted algorithm, as described in (3.4), associates to $\widetilde{z}$ a standard tableaux $A(z) = A(\widetilde{z})$. Applying this procedure to w and y gives

$A(w) \;=\; A(y) \;=$

1	2	$\cdots$		$\cdots$	$2n$
$n-2p+2$	$n-2p+4$	$\cdots$	$n+2p$		

when p is even, but

$A(w) \;=\; A(y) \;=$

1	2	$\cdots$						$\cdots$	$2n$
$n-2p+2$	$n-2p+4$	$\cdots$	$n-2$	$n+1$	$n+2$	$\cdots$	$n+2p$		

when p is odd. Finally, by **[Ja,** §16.14(i)], the equality $A(w) = A(y)$ implies that $w \sim_\ell y$. By Method (a) from (2.4), this implies that $J(\mu_k) = J(\pi)$ is maximal. Thus we have completed the proof of

PROPOSITION. *(Case B) If* $1 \le k \le n$ *and* k *is odd then* $J(k)$ *is maximal.*

5. The maximality of J(k) in Case C.

5.1. In this case we assume that $\mathfrak{g} = \mathfrak{so}(2n)$ and that $2 \le 2k \le n-1$. As will be shown, $\mathbf{O}_k$ is always special and so we may use Method (b) of (2.6) in order to prove that $J(k)$ is maximal. The calculations are very similar to those used when k was even in Case B, and so many of the details will be left to the reader. Unfortunately the definitions and algorithms – indeed even the definition of a special symbol – differ slightly from those in Case B. Rather than write them out, again, we will refer the reader to [**Ja,** §16.10], [**BV1**] and [**Lu**] for the formal definitions.

5.2. As before, write $\mathfrak{h} = \sum_1^n \mathbb{C}H_i$, set $\varepsilon_i = H_i^*$ and denote a weight $\lambda = \sum \lambda^i \varepsilon_i \in \mathfrak{h}^*$ by $\lambda = (\lambda^1, \ldots, \lambda^n)$.

LEMMA. *The orbit $\mathbf{O}_k$ is special, ${}^L\mathbf{O}_k$ is even and the weight $\lambda(\mathbf{O}_k)$ is given by*

$$\lambda(\mathbf{O}_k) \;=\; \big(n-k-1,\ n-k-2, \ldots, k+1,\ k,\ k, \ldots, 1,\ 1,\ 0\big).$$

PROOF: From the algorithm for Lie algebras of type D_n given in [**Ja,** §16.10] or [**BV1,** p.165] one deduces that $\mathbf{O}_k$ has the symbol

$$Symb\ \mathbf{O}_k \;=\; \begin{pmatrix} 0 & 1 & \ldots & \widehat{(n-2k)} & \ldots & (n-k) \\ 1 & 2 & \ldots & & \ldots & (n-k) \end{pmatrix}.$$

By [**BV1**], again, this is a special symbol. By [**Lu,** p.89], $Symb\ \mathbf{O}_k$ is non-degenerate ("nicht symmetrisch" in [**Ja**]). From [**Lu,** §§4.6.8 and 4.5.5] one obtains

$$Symb\ {}^L\mathbf{O}_k \;=\; \begin{pmatrix} n-k \\ n \end{pmatrix}.$$

Recall that ${}^L\mathfrak{so}(2n) = \mathfrak{so}(2n)$. Thus by [**BV1,** p.165], again, ${}^L\mathbf{O}_k$ corresponds to the partition $(2n-2k-1,\ 2k+1)$. Using the algorithm from [**SpSt,** Chapter 4, §2.3.2], one can show that

$${}^LH \;=\; \big(2n-2k-2,\ 2n-2k-4, \ldots, 2k+2,\ 2k,\ 2k, \ldots, 2,\ 2,\ 0\big)$$

Thus, certainly $\lambda(\mathbf{O}_k) = \frac{1}{2}({}^LH)$ has the desired form. Finally, by [**B1,**Planche IV, p.256] the coroots are

$$\big\{H_{\alpha_i} = H_i - H_{i+1} : 1 \le i \le n-1 \quad \text{and} \quad H_{\alpha_n} = H_n + H_{n-1}\big\}.$$

Clearly each ${}^LH(H_{\alpha_i})$ is even and so by [**BV2,** Definition 2.7], ${}^L\mathbf{O}_k$ is even.

5.3. In order to apply Method (b) it remains to show that $\mu_k \in W\lambda(\mathbf{O}_k)$. Now the Weyl group W for a Lie algebra of type D_n was described in (4.5), and we use the same notation. By combining (II, 4.6) with [**B1,** Planche IV, p.257] one obtains

$$\mu_k \;=\; \lambda_k + \rho \;=\; \big(n-k-1,\ n-k-2,\ \dots,k,\ k-1,\ \dots,-(k-1),\ -k\big).$$

Thus by (5.2) $\mu_k = w_2 w_1 \lambda(\mathbf{O}_k)$, where

$$w_1 \;=\; \left(I \;\middle|\; \begin{matrix} (n-2k) & (n-2k+1) & & \dots & & \\ (n-2k) & n & & (n-2k+1) & (n-1) & \dots \end{matrix}\right)$$

and $w_2 \;=\; (\gamma_{n-k})^k \gamma_{n-k+1} \dots \gamma_n$. (Here, the term $(\gamma_{n-k})^k$ is included to ensure that w_2 has an even number of terms. Of course, since the $(n-k)^{th}$ entry of μ_k is zero, it matters not whether one multiplies it by (-1).) This implies that $\mu_k \in W\lambda(\mathbf{O}_k)$.

Combining these observations with (5.2) and Method (b) of (2.6) yields

PROPOSITION. *(Case C) If* $2 \le 2k \le n-1$ *then* $J(k)$ *is maximal.*

5.4. If $2k = n$, in Case C, then it turns out that $J(k)$ *is not maximal*. We will not give details here but merely note that the method of proof is rather similar to Method (a) of (2.4). Indeed, one must find w and y satisfying the conditions of Method (a), except that now one requires that $w \not\sim_\ell y$. Once again there exists a combinatorial procedure for determining this (see [**Ja,** Chapter 16] and [**Lu,** Chapter 5]).

CHAPTER V. DIFFERENTIAL OPERATORS ON THE RING OF SO(k) INVARIANTS

1. Introduction and background.

1.1. Fix $\mathcal{X} = M_{k,n}(\mathbb{C})$ and $T = \mathcal{O}(\mathcal{X})$, for some integers k and n. In this chapter we aim to mimic the results of Chapter III by describing $\mathcal{D}(T^{SO(k)})$. Here the action of $SO(k)$ is the natural one; $g \cdot \xi = g\xi$ for $g \in SO(k)$ and $\xi \in \mathcal{X}$, and we emphasise that this is just the restriction to $SO(k)$ of the action of $O(k)$ on $\mathcal{X}$ that was given in (II, §3). Since $O(k)/SO(k) \cong \mathbb{Z}/2\mathbb{Z}$ – which we will always write as $\{\pm\}$, with elements 1 and σ – one can expect that differential operators over $T^{SO(k)}$ and $T^{O(k)}$ will be closely related, and this is usually the case. In particular, if $k < n$ then we will prove the following results (see Theorem 2.6):

(i) *$\mathcal{D}(T^{SO(k)})$ is a simple Noetherian ring, and is finitely generated as a module over the subring $R = U(\mathfrak{sp}(2n))/J(k)$.*

(ii) $\mathcal{D}(T^{SO(k)})^{\{\pm\}} \;=\; \mathcal{D}(T^{O(k)}) \;=\; R$.

(iii) *The natural map $\mathcal{D}(T)^{SO(k)} \to \mathcal{D}(T^{SO(k)})$ is surjective.*

1.2. If $k > n$ then the ring of $SO(k)$ invariants gives nothing of interest (see Lemma 2.4), but the case $k = n$ is rather curious. For, (1.1(i)) and (1.1(iii)) do still hold for $k = n$. In contrast the analogue of (1.1(ii)) cannot now hold, basically because $R = U(\mathfrak{sp}(2n))/J(k)$ will now have infinite index in $\mathcal{D}(T^{O(k)})$ (see (IV, Remark 1.5)). However it is still true that $\mathcal{D}(T^{SO(k)})^{\{\pm\}} = R$ (see Theorem 3.11). We remark that, by Lemma 1.9 below, the ring $T^{SO(k)}$ is singular if and only if $2 \le k \le n$. Thus, just as in (IV, 1.6), one has the curious dichotomy that *the natural map $\mathcal{D}(T)^{SO(k)} \to \mathcal{D}(T^{SO(k)})$ is surjective if and only if $T^{SO(k)}$ is singular* . (Here we have to exclude the case $k = 1$, but as $SO(1) = 1$, this case is bound to be exceptional.)

1.3. In general one should not expect $\mathcal{D}(T^{SO(k)})$ to be isomorphic to a factor ring of the enveloping algebra $U(\mathfrak{g})$ for $\mathfrak{g}$ semi-simple. However, when $k = 2$, one has the coincidence that $SO(2) = GL(1)$ and, moreover, the action of $SO(2)$ on $\mathcal{X}$ identifies with the action of $GL(1)$ on $M_{n,1}(\mathbb{C}) \times M_{1,n}(\mathbb{C})$ as defined in (II, §2). As we show in Section 4, this implies that, for any $n \ge 2$, the group $\{\pm\}$ acts on the ring

$$U(\mathfrak{sl}(2n))/J(-\overline{\omega}_n + \rho) \;\cong\; \mathcal{D}(T^{GL(1)})$$

(notation (II, 2.7)), and has the ring of invariants

$$U(\mathfrak{sl}(2n))/J(-\overline{\omega}_n+\rho)^{\{\pm\}} \;\cong\; U(\mathfrak{sp}(2n))/J(-\overline{\omega}_n+\rho).$$

(Of course, $-\overline{\omega}_n$ is a different fundamental weight on the two sides of this equation – see (II, 2.7 and 3.7).)

1.4. Assume that $k \leq n$. As a module over $R = U(\mathfrak{sp}(2n))/J(k)$ the ring $\mathcal{D} = \mathcal{D}(T^{SO(k)})$ splits as a direct sum of two simple R-bimodules; $\mathcal{D} = R \oplus \mathcal{D}_-$ where $\mathcal{D}_- = \{D \in \mathcal{D} : \sigma \cdot D = -D\}$. Further, $\mathcal{D}_-$ is naturally a Harish-Chandra $(\mathfrak{g}, K)$-module, where $K = G = Sp(2n)$ and $\mathfrak{g} = \mathfrak{sp}(2n)$. This is related to (a special case of) some conjectures of Vogan, and is discussed in Section 5.

1.5. The results of this chapter are well illustrated by, and trivially proven in the case $k = 1 \leq n$. Here $O(1) = \{\pm\} \supset SO(1) = \{1\}$ and so

$$\mathcal{D}(T) \;=\; \mathcal{D}(T^{SO(1)}) \;=\; \mathcal{D}(T)^{SO(1)} \;\cong\; A_n \;=\; \mathbb{C}[p_1, \dots, q_n],$$

the n^{th} Weyl algebra. The group $O(1)$ acts on A_n by $\sigma \cdot p_i = -p_i$ and $\sigma \cdot q_j = -q_j$ for each i and j. It follows easily that

$$\mathcal{D}(T)^{O(1)} \;=\; \mathbb{C}[\, p_ip_j,\ p_iq_j,\ q_iq_j \;\; : 1 \leq i, j \leq n].$$

Moreover, by Theorem 0.3, this ring equals $U = U(\mathfrak{sp}(2n))/J(1)$. It follows from **[LSS]** that $J(1)$ is the Joseph ideal. The remaining results stated in (1.1) and (1.2) follow trivially. In this case the module $\mathcal{D}_-$ of (1.4) is just $\mathcal{D}_- = \sum Up_i + \sum Uq_i$.

1.6. For the remainder of this section we consider the classical ring of invariants $T^{SO(k)}$ and describe those results and notation that we will need, most of which come from **[We]**. Thus, fix $\mathcal{X} = M_{k,n}(\mathbb{C})$ and write $T = \mathcal{O}(\mathcal{X}) = \mathbb{C}[X]$, where $X = (x_{ij})$ is a generic $k \times n$ matrix. As in (II, §3),

$$T^{O(k)} \;=\; \mathbb{C}[\,({}^tX)X] \;\cong\; \mathbb{C}[Z]/I(k),$$

where Z is the generic, symmetric $n \times n$ matrix, $Z = (z_{uv})$. The isomorphism is given by mapping z_{uv}, to $({}^tX)_uX_v$, where X_r denotes the r^{th} column of X.

1.7. A similar, but far more complicated description is available for $T^{SO(k)}$. Fortunately, the only case in which we need an explicit description of this ring is when $k = n$, and in this case $T^{SO(k)}$ has a fairly pleasant form. Thus the following facts, easily derived from **[We,** p.77], are sufficient for our purposes.

LEMMA. **(i)** *If* $k > n$ *then* $T^{O(k)} = T^{SO(k)} = \mathbb{C}[Z]$ *is a polynomial ring in* $\frac{1}{2}n(n+1)$ *variables.*

(ii) *If* $1 \leq k \leq n$ *then* $T^{SO(k)} \supsetneq T^{O(k)}$ *but* $T^{SO(k)}$ *is a finite* $T^{O(k)}$*-module.*

(iii) *If* $k = n$ *then*

$$\begin{aligned} T^{SO(n)} &\cong \mathbb{C}[y,\ z_{\ell m} : 1 \leq \ell \leq m \leq n]/\big(y^2 - det(z_{\ell m})\big) \\ &\supsetneq T^{O(n)} = \mathbb{C}[Z]. \end{aligned}$$

Under this isomorphism y *maps to* $\det X$. *Note that* $T^{O(n)}$ *is a polynomial ring.*

(iv) *The group* $O(k)/SO(k) = \{\pm\} = \{1,\ \sigma\}$ *acts on* $T^{SO(k)}$ *with fixed ring* $T^{O(k)}$. *If* $k = n$ *then the action is given by* $\sigma \cdot y = -y$.

1.8. When $k = n$, it will be convenient to reinterpret (1.7) geometrically. Define

$$\mathcal{Y} = \overline{\mathcal{X}}_n = Spec\, T^{O(n)} = Sym_n(\mathbb{C})$$

and

$$\mathcal{Z} = Spec\, T^{SO(n)} = \left\{ \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} : A \in Sym_n(\mathbb{C}),\ a^2 = det\ A \right\}.$$

Thus the co-morphism of the inclusion $T^{O(n)} \hookrightarrow T^{SO(n)}$ is just the projection

$$\pi : \mathcal{Z} \twoheadrightarrow \mathcal{Z}/\{\pm\} = \mathcal{Y}$$

and is defined by $\begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} \longmapsto A.$ Write

$$\mathcal{Y}_t = \{A \in \mathcal{Y} : rk\ A = t\} \qquad \text{and} \qquad \mathcal{Z}_t = \left\{ \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} \in \mathcal{Z} : rk\ A = t \right\},$$

for any $t \leq n$, but with the convention that $\mathcal{Y}_t = \mathcal{Z}_t = \emptyset$ for $t < 0$. (Note that $\mathcal{Y}_t$ is the variety $\mathcal{X}_t$ of (II, §3).) Obviously $\pi(\mathcal{Z}_t) = \mathcal{Y}_t$ and this map is a bijection if and only if $t \leq n-1$.

1.9. Since the only relation among the generators of $T^{SO(n)}$ is the equation $y^2 - det(z_{\ell m}) = 0$ it is easy to compute $Sing\ \mathcal{Z}$.

LEMMA. *If* $n = k \geq 2$, *then* $Sing\ \mathcal{Z} = \overline{\mathcal{Z}}_{n-2} = \mathcal{Z}_0 \cup \ldots \cup \mathcal{Z}_{n-2}$.

Proof: Set $V = \mathbb{C}[y,\ z_{\ell m} : 1 \leq \ell \leq m \leq n]$. Observe that $\overline{\mathcal{Z}}_{n-2}$ is isomorphic to $\{A \in Sym_n(\mathbb{C}) : rk\ A \leq n-2\}$. Thus it follows from (III, Theorem 3.3) that the defining ideal $\mathcal{I}(\overline{\mathcal{Z}}_{n-2})$ of $\overline{\mathcal{Z}}_{n-2}$ in V is generated by y and the $(n-1) \times (n-1)$ minors of the matrix $Z = (z_{\ell m})$. On the other hand, $\mathcal{O}(\mathcal{Z}) = V/(y^2 - det(z_{\ell m}))$. Thus, the Jacobian criterion [**Ku,** Theorem 1.15, p.171] shows that $\mathcal{I}(Sing\ \mathcal{Z})$ is generated by $2y = \partial/\partial y\{y^2 - det(z_{\ell m})\}$ and the elements

$$\big\{H_{ij} \;=\; \partial/\partial z_{ij}\big(y^2 - det(z_{\ell m})\big) :\ 1 \leq i \leq j \leq n\big\}.$$

Let $\widetilde{Z}_{ij}$ denote the $(i,j)^{th}$ minor of the matrix $Z = (z_{\ell m})$. Then an easy computation using partial differentiation proves that $H_{ij} = -(-1)^{i+j}\widetilde{Z}_{ij}$ if $i = j$ but $H_{ij} = -2(-1)^{i+j}\widetilde{Z}_{ij}$ if $i \neq j$ (see [**Mi,** Ex.22, p.193]). This proves the lemma.

1.10. $(k = n)$ Write $\mathbb{Z}_2$ for the multiplicative group $\{1,\ -1\}$. Define an action of $GL(n) \times \mathbb{Z}_2$ on $\mathcal{Z}$ by

$$(g,\ \varepsilon)\cdot\begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} \;=\; \begin{pmatrix} gA(^tg) & 0 \\ 0 & \varepsilon(det\ g)a \end{pmatrix},$$

for $\begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} \in \mathcal{Z},\ g \in G$ and $\varepsilon \in \mathbb{Z}_2$. Since A is symmetric, it is readily checked that

$$\mathcal{Z}_n \;=\; (GL(n) \times \mathbb{Z}_2)\cdot\begin{pmatrix} I_n & 0 \\ 0 & 1 \end{pmatrix},$$

while

$$\mathcal{Z}_t \;=\; (GL(n) \times \mathbb{Z}_2)\cdot\begin{pmatrix} I_t & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for} \quad 0 \leq t \leq n-1.$$

Note that, if $\mathcal{Y}$ is given its classical $GL(n)$-action and the trivial $\mathbb{Z}_2$-action, then π is $(GL(n) \times \mathbb{Z}_2)$-equivariant.

2. Differential operators on $\mathcal{O}(\mathcal{X})^{\mathbf{SO(k)}}$ for $\mathbf{k \neq n}$.

2.1. We retain the notation of Section 1. In this section we make some fairly easy observations about $\mathcal{D}(T^{SO(k)})$ and $\mathcal{D}(T)^{SO(k)}$ for arbitrary k. However, these are sufficient to prove the results stated in (1.1) for $k < n$, basically because we already know so much about $\mathcal{D}(T^{O(k)})$ and $\mathcal{D}(T)^{O(k)}$ for these values of k.

2.2. Recall from (I, 1.5) that there are homomorphisms

$$\psi : U(\mathfrak{sp}(2n)) \xrightarrow{\omega} \mathcal{D}(T)^{O(k)} \xrightarrow{\varphi} \mathcal{D}(T^{O(k)}).$$

Write $J(k) = ker(\psi)$ and $R = Im(\psi) \subseteq \mathcal{D}(T^{O(k)})$. Similarly there are natural maps

$$\psi' : U(\mathfrak{sp}(2n)) \xrightarrow{\omega} \mathcal{D}(T)^{O(k)} \xhookrightarrow{i'} \mathcal{D}(T)^{SO(k)} \xrightarrow{\varphi'} \mathcal{D}(T^{SO(k)}).$$

Set $R' = Im(\psi') \subseteq \mathcal{D}(T^{SO(k)})$. Our first aim is to relate these objects.

LEMMA. **(i)** *The identity* $T^{O(k)} = (T^{SO(k)})^{\{\pm\}}$ *induces an embedding*

$$\mathcal{D}(T^{SO(k)})^{\{\pm\}} \hookrightarrow \mathcal{D}(T^{O(k)}).$$

(ii) *Under this embedding,* R *identifies with* R'.

PROOF: **(i)** Certainly the identity $T^{O(k)} = (T^{SO(k)})^{\{\pm\}}$ induces a map

$$\alpha : \mathcal{D}(T^{SO(k)})^{\{\pm\}} \longrightarrow \mathcal{D}(T^{O(k)}).$$

The proof of **[Le1,** Théorème 5(b)] may be used unaltered to show that α is injective.

(ii) Let $d \in \mathcal{D}(T)$. Then the map $\varphi' i'$ is simply defined by

$$\varphi' i'(d) = d \mid_{T^{SO(k)}}.$$

Similarly, $\varphi(d) = d \mid_{T^{O(k)}}$. Thus φ factors through $\varphi' i'$ and hence the map ψ factors through ψ'.

On the other hand, if $d \in \mathcal{D}(T)^{O(k)}$, then the action of $\varphi' i'(d)$ on $T^{SO(k)}$ is that induced from its action on T. Since d is invariant under $O(k)$, this implies that $\varphi' i'(d)$ is invariant under $\{\pm\} = O(k)/SO(k)$. Thus

$$R' = Im(\psi') \subseteq \mathcal{D}(T^{SO(k)})^{\{\pm\}}.$$

Therefore, by part (i), $R' \subseteq R = Im(\psi)$. Combined with the observations of the last paragraph, this forces $Im(\psi) = Im(\psi')$.

2.3. *From now onwards, identify* R' *with* R *by means of the embedding of Lemma* 2.2(i). *Similarly, write*

$$S \;=\; \varphi'(\mathcal{D}(T)^{SO(k)}) \;\subseteq\; \mathcal{D} \;=\; \mathcal{D}(T^{SO(k)})$$

and note that $R = R' \subseteq S$. Thus we have the following commutative diagram:

$$\begin{tikzcd}
& U(\mathfrak{sp}(2n)) \arrow[d, two heads] & & \\
& \mathcal{D}(T)^{O(k)} \arrow[r, hook] \arrow[d, two heads] & \mathcal{D}(T)^{SO(k)} \arrow[d, two heads] & \\
Im\,\varphi \;= & R \arrow[r, hook] \arrow[d, hook] & S & = Im\,\varphi' \\
& \mathcal{D}^{\{\pm\}} \arrow[r, hook] \arrow[d, hook] & \mathcal{D} & = \mathcal{D}(T^{SO(k)}) \\
& \mathcal{D}(T^{O(k)}) & &
\end{tikzcd}$$

2.4. We are now ready to describe $\mathcal{D}(T^{SO(k)})$ for $k \neq n$. The case $k > n$ is a triviality.

LEMMA. *Assume that* $k > n$. *Then*

$$\mathcal{D}(T^{O(k)}) \;=\; \mathcal{D}(T^{SO(k)}) \;=\; \mathcal{D}(T^{SO(k)})^{\{\pm\}} \;\cong\; A_m,$$

the Weyl algebra of index $m = \frac{1}{2}n(n+1)$. *Moreover,* $\mathcal{D}(T^{SO(k)}) \neq R$.

PROOF: By Lemma 1.7(i), $\mathcal{D}(T^{O(k)}) = \mathcal{D}(T^{SO(k)}) \cong A_m$. Since $\{\pm\}$ acts trivially on $T^{SO(k)}$, it also acts trivially on $\mathcal{D}(T^{SO(k)})$ (see (I, 1.3)). Finally, $R \neq \mathcal{D}(T^{SO(k)})$ by (IV, Remark 1.5).

2.5. (Notation 2.3) Note that, by construction, $T^{SO(k)} \subseteq S$. Since elements of $T^{SO(k)}$ act ad-nilpotently on $\mathcal{D}$, they do so on S. Therefore any multiplicatively closed subset of $T^{SO(k)}$ is automatically an Ore set in both $\mathcal{D}$ and S (see [**KL,** Lemma 4.7]). Thus the next lemma is meaningful, and follows immediately from [**MR,** Chapter XV, Corollary 5.6].

LEMMA. *Let $\mathcal{C}$ be a multiplicatively closed subset of $T^{SO(k)}$ and assume that $(T^{SO(k)})_{\mathcal{C}}$ is a regular ring. Then $S_{\mathcal{C}} = \mathcal{D}_{\mathcal{C}}$ if (and only if) $Der(T^{SO(k)}) \subset S_{\mathcal{C}}$.*

2.6. THEOREM. *(Notation 2.3) Assume that $k < n$. Then:*

(i) $Im(\psi) \;=\; \mathcal{D}(T^{O(k)}) \;=\; \mathcal{D}(T^{SO(k)})^{\{\pm\}}$.

(ii) $\mathcal{D}(T^{SO(k)})$ *is a simple, Noetherian domain, finitely generated as a left or right $U(\mathfrak{sp}(2n))$-module.*

(iii) *The natural map $\varphi' : \mathcal{D}(T)^{SO(k)} \;\to\; \mathcal{D}(T^{SO(k)})$ is surjective.*

PROOF: By (IV, Corollary 1.4), $\mathcal{D}(T^{O(k)}) = R$ and this ring is simple. Thus part (i) is immediate from (2.3). Combined with [**Mo,** Corollary 5.8, p.79 & Theorem 7.11, p.116], part (i) implies part (ii).

Since $R \subseteq S \subseteq \mathcal{D}$ in the notation of (2.3), part (ii) implies that $\mathcal{D}$ is a finite S-module (on either side). Suppose, further, that S and $\mathcal{D}$ have the same quotient division ring. Then (IV, Lemma 1.3(ii)) implies that $S = \mathcal{D}$.

It therefore remains to show that S and $\mathcal{D}$ have the same quotient ring. Consider the co-morphism of $T^{O(k)} \subset T^{SO(k)}$; that is, the map

$$\pi : \mathcal{Z} \;=\; Spec\, T^{SO(k)} \;\twoheadrightarrow\; \mathcal{Y} \;=\; \mathcal{Z}/\{\pm\} \;=\; Spec\, T^{O(k)}.$$

Since π is surjective, it is certainly dominant (which means that $\pi(\mathcal{Z})$ is dense in $\mathcal{Y}$). Also, $\mathcal{Z}$ and $\mathcal{Y}$ are both irreducible. Thus, by [**Ha,** Lemma 10.5, p.271], the set of points at which π is étale is a non-empty, open (and hence dense) subset of $\mathcal{Z}$. Since the sets of non-singular points of $\mathcal{Y}$ and of $\mathcal{Z}$ are also open dense subsets, we may therefore pick $q \in \mathcal{Y}$ and $p \in \pi^{-1}(q)$ such that $\mathcal{Y}$ is non-singular at q, $\mathcal{Z}$ is non-singular at p and π is étale at p. Then by [**Le1,** Proposition 3, p.169],

$$Der(T^{SO(k)})_p \;\cong\; (T^{SO(k)})_p \otimes_{(S^{O(k)})_q} Der(S^{O(k)})_q.$$

But, as $k < n$, (IV, Corollary 1.4) implies that $Der\; T^{O(k)} \;\subset\; R \;\subseteq\; S$. Thus Lemma 2.5 implies that $S_p = \mathcal{D}_p$, which certainly ensures that S and $\mathcal{D}$ have the same full quotient ring.

3. Differential operators on $\mathcal{O}(\mathcal{X})^{\mathbf{SO(k)}}$, when k = n .

3.1. Retain the notation of the last two sections, notably (2.3), but assume now that $k = n$. Thus $T = \mathcal{O}(M_{n,n}(\mathbb{C}))$ on which $O(n)$ and $SO(n)$ have their natural action. In (IV, Remark 1.5), we showed that $\mathcal{D}(T^{O(n)})$ is not even finitely generated as a (left or right) module over $R = \varphi(\mathcal{D}(T)^{O(n)}) = \psi(U(\mathfrak{sp}(2n))$. In this section we show, in contrast, that $\mathcal{D}(T^{SO(n)})$ is a homomorphic image of $\mathcal{D}(T)^{SO(n)}$ and is a finitely generated $U(\mathfrak{sp}(2n))$-module. Indeed, if one deletes the reference to $\mathcal{D}(T^{O(k)})$, then all of Theorem 2.6 will still hold in this case.

3.2. The proof of these results for $k = n$ are considerably more complicated than those for $k < n$, simply because $\mathcal{D}(T^{O(n)})$ cannot be utilised effectively. However, many of the ideas used in the proof of Theorem 2.6 – and those of Chapter III – do still work here. In outline, the proof is as follows. The aim is to prove, in the notation of (2.3), that $S_p = \mathcal{D}_p$ holds for sufficiently many $p \in \mathcal{Z} = Spec\ T^{SO(n)}$ to conclude that $GKdim\ \mathcal{D}/S \leq GKdim\ S - 2$. Once one has this, Gabber's Lemma (III, Lemma 1.8) may be applied to show that $\mathcal{D} = S$. If $p \in \mathcal{Z}_n$ (notation, 1.10) then one can still "lift" the equality $\mathcal{D}_p = S_p$ from the results of Chapter III, but this will only give the bound

$$GKdim\ \mathcal{D}/S \ \leq \ GKdim\ S/S.\mathcal{I}(\overline{\mathcal{Z}}_{n-1}) \ = \ GKdim\ S - 1.$$

(This is more or less inevitable – (III, Remark 2.16) implies that, at least for $O(n)$-invariants, one obtains an equality at this point.) However $\mathcal{Z}_{n-1}$ also consists of non-singular points of $\mathcal{Z}$ and one can save the proof by showing that $S_p = \mathcal{D}_p$ also holds when $p \in \mathcal{Z}_{n-1}$. At this point of the proof one has to do some fairly explicit computations since the "orbit trick" of (III, Lemma 1.3) cannot be applied to $\mathcal{Z}_{n-1}$.

3.3. We begin by considering S_p for $p \in \mathcal{Z}_n$. Recall that, by (2.3), $S \supset R = \psi(U(\mathfrak{sp}(2n))$.

PROPOSITION. *If $p \in \mathcal{Z}_n$ then $S_p = \mathcal{D}_p$. Indeed, if $q = \pi(p) \in \mathcal{Y} = Spec\ T^{O(n)}$, then*

$$Der(T^{SO(n)})_p \ = \ (T^{SO(n)})_p \otimes_{(T^{O(n)})_q} Der(T^{O(n)})_q. \tag{3.3.1}$$

PROOF: This is very similar to the last part of the proof of Theorem 2.6. By (1.10), $\mathcal{Z}_n$ is an orbit for the group $GL(n) \times \mathbb{Z}_2$ and π is $(GL(n) \times \mathbb{Z}_2)$-equivariant. Thus it suffices to prove the lemma for just one point $p \in \mathcal{Z}_n$.

Now, (II, Proposition 1.4) *does* prove that

$$R_q = \mathcal{D}(T^{O(n)})_q \quad \text{for all} \quad q \in \mathcal{Y}_n.$$

Furthermore, just as in the proof of Theorem 2.6, π is étale on a dense open subset V of $\mathcal{Z}_n$ and, of course, both $\mathcal{Z}_n$ and $\mathcal{Y}_n$ consist of non-singular points of $\mathcal{Z}$ and $\mathcal{Y}$ respectively. Thus by **[Le1,** Proposition 3, p.169], (3.3.1) does hold for all $p \in V$. Therefore it also holds for all $p \in \mathcal{Z}_n$. Finally, by Lemma 2.5, this in turn implies that $S_p = \mathcal{D}_p$.

3.4. Let $p \in \mathcal{Z}_n$ and put $q = \pi(p) \in \mathcal{Y}_n$. As usual, set

$$(T^{SO(n)})_q = T^{SO(n)} \otimes_{T^{O(n)}} (T^{O(n)})_q \quad \text{and} \quad S_q = S \otimes (T^{SO(n)})_q,$$

etc. We need to relate S_q and $\mathcal{D}_q$. Since $T^{SO(n)}$ is a finite $T^{O(n)}$-module, $(T^{SO(n)})_q$ is a semilocal ring, with maximal ideals corresponding to the points in $\pi^{-1}(q)$. It is immediate from (1.10) that $\pi^{-1}(q) \subset \mathcal{Z}_n$. In particular

$$S_q = \bigcap \{S_{p'} : p' \in \pi^{-1}(q)\},$$

and similarly for $\mathcal{D}_q$.

COROLLARY. **(i)** *Let $p \in \mathcal{Z}_n$ and $q = \pi(p)$. Then $S_q = \mathcal{D}_q$.*

(ii) *As a left or right R-module, $\mathcal{D}$ is an essential extension of S.*

PROOF: Part (i) is an immediate consequence of Proposition 3.3, combined with the comments given before the statement of the Corollary. In order to prove part (ii) one need only note that, given $d \in \mathcal{D} \setminus \{0\}$ then $d \in \mathcal{D}_q = S_q$ and so $d = c^{-1}b$ for some $c \in T^{O(n)} \subset R$ and $b \in S$. Thus $Rd \cap S \supseteq Rb$; as required.

3.5. If $p \in \mathcal{Z}_{n-1}$ then Lemma 1.9 shows that p is still a non-singular point of $\mathcal{Z}$, but the proof of Proposition 3.3 cannot be used to show that $S_p = \mathcal{D}_p$. This is for two reasons. First, $R_q \neq \mathcal{D}(T^{O(n)})_q$, where $q = \pi(p)$. For example, if $n = 1$ then $\mathcal{Y}_{n-1} = \mathcal{Y} \setminus \mathcal{Y}_n$ and so equality here would imply that $R = \mathcal{D}(T^{O(n)})$, contradicting (IV, Remark 1.5). Secondly, π need not be étale at p and so one may not be able to – and indeed cannot – obtain $Der(T^{SO(n)})_p$ from $Der(T^{O(n)})_q$. Thus in order to obtain $S_p = \mathcal{D}_p$ we will need to explicitly compute $Der(T^{SO(n)})_p$.

3.6. Recall from (IV, 1.9) that the generators of $\mathcal{D}(T)^{O(n)}$ are known; indeed in the notation of (1.6), $\mathcal{D}(T)^{O(n)}$ is generated by the elements

$$\begin{cases} z_{ij} &= \sum_{\ell=1}^{n} x_{\ell i}x_{\ell j}\ , \\ D_{ij} &= \sum_{\ell=1}^{n} x_{\ell i}\partial/\partial x_{\ell j} \qquad \text{for } 1 \leq i \leq j \leq n. \\ \sum_{\ell=1}^{n} \partial/\partial x_{\ell i}\partial/\partial x_{\ell j} \end{cases}$$

In order to keep our notation within reasonable bounds we will use the same symbols to denote elements of $\mathcal{D}(T)^{O(n)}$ (respectively $\mathcal{D}(T)^{SO(n)}$) and their images in $\mathcal{D}(T^{O(n)})$ (respectively $\mathcal{D}(T^{SO(n)})$). Thus the elements displayed above also denote the generators of $R = \varphi(U(\mathfrak{sp}(2n)) \subset \mathcal{D}(T^{O(n)})$.

Certainly the D_{ij} lie in $Der(T^{SO(n)})$, but there is also a second set of derivations that we will require. These are defined as follows. Given an $m \times m$ matrix $A = (a_{ij})$ write $\widetilde{A}_{uv}$ for the $(m-1) \times (m-1)$ minor corresponding to the element a_{uv}. Let Δ_{ij} be the derivation

$$\Delta_{ij} \;=\; \sum_{s=1}^{n}(-1)^{s+i}\widetilde{X}_{si}\partial/\partial x_{sj} \;\in\; Der(T).$$

This can be written formally as the determinant

$$\Delta_{ij} \;=\; (-1)^{n-i}\begin{vmatrix} x_{11} & \dots & \widehat{x}_{1i} & \dots & x_{1n} & \partial/\partial x_{1j} \\ \vdots & & \vdots & & \vdots & \vdots \\ x_{n1} & \dots & \widehat{x}_{ni} & \dots & x_{nn} & \partial/\partial x_{nj} \end{vmatrix},$$

where, as usual, $\widehat{x}_{ab}$ means that x_{ab} has been deleted from the matrix. However, in computing the determinant one must make sure that derivations are always written to the right of regular functions.

3.7. LEMMA. *(Notation (3.6)) Each Δ_{ij} belongs to $\mathcal{D}(T)^{SO(n)}$.*

PROOF: Unfortunately, the $SO(n)$ invariants of $\mathcal{D}(T)$ have, apparently, not been described in the literature and so for once we will have to explicitly describe the action of $SO(n)$ on the generators of $\mathcal{D}(T)$. Thus, write $T = \mathbb{C}[X]$, where $X = (x_{ij})$ is a generic $n \times n$ matrix,and let $\Xi = (\xi_{ij})$ be the $n \times n$ matrix with entries $\xi_{ij} = \partial/\partial x_{ij}$. Note that, as a subalgebra of $\mathcal{D}(T)$, the ring $\mathbb{C}[\Xi]$ is isomorphic to a commutative polynomial ring.

Consider the action of $O(n)$ on the ξ_{ij}. From the final displayed equation of (I, 1.3), this is defined by

$$(g \cdot \xi_{ij}) * x_{uv} \;=\; \xi_{ij} * (g^{-1} \cdot x_{uv})$$

for $g \in O(n)$ and any integers i, j, u, v. The action of $O(n)$ on $\mathbb{C}[X]$ is given explicitly in (II, 3.2). With respect to an orthonormal basis $\{v_i\}$ of V, one identifies $h \in O(n)$ with an orthogonal matrix and then defines $h \cdot x_{uv} = (h^{-1}X)_{uv}$ (notation, (II, 2.3)). Thus, writing g as the $n \times n$ matrix (g_{ab}) one obtains

$$(g \cdot \xi_{ij}) * x_{uv} \quad = \quad \xi_{ij} * (gX)_{uv} \quad = \quad \xi_{ij} * \sum g_{ub} x_{bv} \quad = \quad \delta_{jv} g_{ui}.$$

Equivalently, the induced action of $O(n)$ on $\mathbb{C}[\Xi]$ is given by

$$g \cdot \xi_{ij} \quad = \quad ({}^t g \Xi)_{ij} \quad = \quad (g^{-1}\Xi)_{ij} \quad \text{for any } g \in O(n). \tag{3.7.1}$$

Thus we have shown that the induced action of $O(n)$ on $\mathbb{C}[\Xi]$ is identical to that of $O(n)$ on $\mathbb{C}[X]$. Equivalently, the action of $SO(n)$ on the *commutative* ring $\mathbb{C}[X] \otimes_{\mathbb{C}} \mathbb{C}[\Xi]$ is that induced from the action of $SO(n)$, by left translation, on

$$M_{n,n}(\mathbb{C}) \times M_{n,n}(\mathbb{C}) \quad = \quad M_{n,2n}(\mathbb{C}).$$

Here, $\mathbb{C}[\Xi]$ is identified with the ring of regular functions on the second copy of $M_{n,n}(\mathbb{C})$. The invariants $\mathcal{O}(M_{n,2n}(\mathbb{C}))^{SO(n)}$ are described in **[We,** p.77**]** and include the element

$$\Delta'_{ij} \quad = \quad det[X_1, \; \ldots \;, \widehat{X}_i, \; \ldots \; X_n, \; \Xi_j],$$

where Ξ_j is the j^{th} column of Ξ.

Now return to $\mathcal{D}(T)$. At least as abelian groups, $\mathcal{D}(T) \cong \mathbb{C}[X] \otimes \mathbb{C}[\Xi]$. Thus the rings of invariants $\mathcal{D}(T)^{SO(n)}$ and $(\mathbb{C}[X] \otimes \mathbb{C}[\Xi])^{SO(n)}$ coincide; providing only that one is careful to always write the derivations $\xi_{ij} \in \mathcal{D}(T)$ to the right of the regular functions x_{uv}. Under this identification, Δ'_{ij} is exactly $(-1)^{i-n}\Delta_{ij}$; as required.

REMARK. The full set of generators of $\mathcal{O}(M_{n,2n}(\mathbb{C}))^{SO(n)}$ are given in **[We]** and so the above proof also enables one to write down a complete set of generators, both for $\mathcal{D}(T)^{SO(n)}$ and for $\mathcal{D}(T^{SO(n)})$. We will give this set at the end of the section.

3.8. By (3.6) and (3.7) the derivations D_{ij} and Δ_{ij} lie in $\mathcal{D}(T)^{SO(n)}$. We next want to give an alternative description of their images in $\mathcal{D}(T^{SO(n)})$, which we will again denote by D_{ij} and Δ_{ij}. Recall that, from Lemma 1.7,

$$T^{SO(n)} \;\; \cong \;\; \mathbb{C}[\, z_{\ell,m}, \; y : 1 \le \ell \le m \le n]/\big(y^2 - det(z_{\ell m})\big).$$

As before, set $Z = (z_{ab})$ and write $\widetilde{Z}_{ab}$ for the $(a,b)^{th}$ minor of Z. Write $\partial_{ab} = \partial/\partial z_{ab}$ and $\partial_y = \partial/\partial y$.

LEMMA. *As elements of Der* $T^{SO(n)}$, D_{ij} *and* Δ_{ij} *are given by the following formulae. Set*

$$D'_{ij} \;=\; 2z_{ij}\partial_{jj} \;+\; \sum_{a=1}^{j-1} z_{ia}\partial_{aj} \;+\; \sum_{b=j+1}^{n} z_{ib}\partial_{jb}.$$

Then

$$D_{ij} \;=\; \begin{cases} y\partial_y + D'_{ii} & \textit{if} \;\; i=j \\ D'_{ij} & \textit{if} \;\; 1 \le i < j \le n \end{cases}$$

and

$$\Delta_{ij} \;=\; \begin{cases} (-1)^{i+j}\widetilde{Z}_{ij}\partial_y \;+\; y\partial_{ij} & \textit{if} \;\; i<j \\ \widetilde{Z}_{ii}\partial_y \;+\; 2y\partial_{ii} & \textit{if} \;\; i=j. \end{cases}$$

PROOF: This follows from the obvious computations, and so we will just check one case here – that of Δ_{nn} – and leave the remaining cases to the reader. (We remark that the Δ_{ij} , for $(i,\ j) \neq (n,\ n)$, will not be needed subsequently.)

As elements of $T^{SO(n)}$, one has $z_{ab} = \sum_{s=1}^{n} x_{sa}x_{sb}$ and $y = det(x_{ab})$. Thus

$$\Delta_{nn} * z_{pq} \;=\; \left\{\sum_{t=1}^{n}(-1)^{t+n}\widetilde{X}_{tn}\partial/\partial x_{tn}\right\} * \sum_{s=1}^{n} x_{sp}x_{sq}$$

$$= \begin{cases} 2\sum_{t=1}^{n}(-1)^{t+n}\widetilde{X}_{tn}x_{tn} \;=\; 2y & \text{if} \;\; p=q=n \\ \sum_{t=1}^{n}(-1)^{t+n}\widetilde{X}_{tn}x_{tq} \;=\; 0 & \text{if} \;\; p=n\neq q \\ \sum_{t=1}^{n}(-1)^{t+n}\widetilde{X}_{tn}x_{tp} \;=\; 0 & \text{if} \;\; q=n\neq p \\ 0 & \text{otherwise.} \end{cases}$$

Similarly,

$$\begin{aligned} \Delta_{nn} * y \;&=\; \left\{\sum_{t=1}^{n}(-1)^{t+n}\widetilde{X}_{tn}\partial/\partial x_{tn}\right\} * det(X) \\ &=\; \sum_{t=1}^{n}(-1)^{t+n}\widetilde{X}_{tn}(-1)^{t+n}\widetilde{X}_{tn} \\ &=\; \sum_{t=1}^{n}\widetilde{X}_{tn}\widetilde{X}_{tn}. \end{aligned}$$

Given $n\times n$ matrices $A = BC$, recall that $\widetilde{A}_{uv} = \sum_{t=1}^{n} \widetilde{B}_{ut}\widetilde{C}_{tv}$ (see [**B2,** Ex 6, p.III.192]). Thus

$$\Delta_{nn} * y \;=\; {}^{t}\widetilde{X}_{nn}\widetilde{X}_{nn} \;=\; \widetilde{Z}_{nn}.$$

This implies that $\Delta_{nn} = 2y\partial_{nn} + \widetilde{Z}_{nn}\partial_y$; as required.

3.9. PROPOSITION. *(Notation 1.10, 2.3) Let* $p\in\mathcal{Z}_{n-1}$ *and* $q=\pi(p)\in\mathcal{Y}_{n-1}$. *Then* $S_p=\mathcal{D}_p$ *; indeed* $S_q=\mathcal{D}_q$.

PROOF: As π induces an isomorphism $\mathcal{Z}_{n-1}\xrightarrow{\sim}\mathcal{Y}_{n-1}$, one has $(T^{SO(n)})_p = (T^{SO(n)})_q$. Thus $S_p=S_q$ and $\mathcal{D}_p=\mathcal{D}_q$. Therefore it suffices to prove that $S_p=\mathcal{D}_p$.

Since $\mathcal{Z}$ is non-singular at p, Lemma 2.5 implies that we need only prove that $Der(T^{SO(n)})_p\subset S_p$. Furthermore, since

$$\mathcal{Z}_{n-1} \;=\; GL(n)\cdot\begin{pmatrix} I_{n-1} & 0\\ 0 & 0\end{pmatrix},$$

(see (1.10)), we need only prove this result for $p=\begin{pmatrix} I_{n-1} & 0\\ 0 & 0\end{pmatrix}$. Write $A=(T^{SO(n)})_p$. Thus the maximal ideal $\mathbf{m}(p)=\mathcal{I}(p)_p$ of A is generated by the set of elements

$$\Omega' \;=\; \big\{z_{ii}-1:\ 1\le i\le n-1,\ z_{nn},\ z_{\ell m}:\ 1\le \ell<m\le n,\ y\big\}.$$

Note that $\widetilde{Z}_{nn}\equiv(-1)\ mod\ \mathbf{m}(p)$ and so is invertible in A. Thus, from the equation

$$y^2 \;=\; det\ Z \;=\; \sum(-1)^{u+n}z_{un}\widetilde{Z}_{un},$$

one obtains that $\mathbf{m}(p)$ is generated by the set $\Omega=\Omega'\setminus\{z_{nn}\}$. But the cardinality of Ω equals $dim\,A$ and hence the elements of Ω form an r-sequence in $\mathbf{m}(p)$ (see [**Ma,** Theorem 31, p.108]). Consequently, a basis for the module of Kähler differentials $\Omega_{A/\mathbb{C}}$ is given by the differentials

$$\big\{dy,\ dz_{ij}:\ 1\le i\le j\le n,\ (i,\ j)\neq(n,\ n)\big\}$$

(see [**Ha,** Theorem 8.8, p.174]). Thus a base for $Der\,A$ is given by the duals of these elements.

At this point we should be a little careful about notation. This is because the usual notation for the dual of dy is ∂_y but it is definitely not the case that this derivation is the same as the ∂_y of (3.8). Thus let $\{\varepsilon_{ij},\ \varepsilon_y\}$ denote the duals of $\{dz_{ij},\ dy\}$. Thus, for example, ε_y is uniquely defined by the requirement that $\varepsilon_y\in Der\,A$ and satisfies

$$\varepsilon_y(y) \;=\; 1 \quad\text{but}\quad \varepsilon_y(z_{ij}) \;=\; 0 \quad\text{for}\quad 1\le i\le j\le n,\ (i,\ j) \;\neq\; (n,\ n).$$

Now, consider the matrix equation defining Δ_{nn} and the D_{ij}, for $1 \leq i \leq j \leq n$ but $(i,\ j) \neq (n,\ n)$, in terms of ε_y and the ε_{ij}. Set

$$\underline{\mathbf{D}} \ = \ {}^t(D_{11},\ D_{12},\ \dots\ ,D_{n-1,n},\ \Delta_{nn}) \quad \text{and} \quad \underline{\varepsilon} \ = \ {}^t(\varepsilon_{11},\ \dots\ ,\varepsilon_{n-1,n},\ \varepsilon_y).$$

Let N be the matrix with coefficients in A defined by $\underline{\mathbf{D}} = N\underline{\varepsilon}$. The coefficients of this matrix are determined by using the formulae in Lemma 3.8 and simply ignoring any occurence of ∂_{nn}. Moreover, since we only wish to show that N is invertible, we need only compute the entries of N modulo $\mathbf{m}(p)$. But, *mod* $\mathbf{m}(p)$,

$$D_{ij} \ \equiv \ \begin{cases} 2z_{jj}\partial_{jj} \ \equiv \ 2\varepsilon_{jj} & \text{if} \ \ i = j < n \\ \ z_{ii}\partial_{ij} \ \equiv \ \ \varepsilon_{ij} & \text{if} \ \ i < j, \end{cases}$$

and $\Delta_{nn} = (\widetilde{Z}_{nn})^{-1}\varepsilon_y$. Thus, modulo $\mathbf{m}(p)$, N has exactly one non-zero entry in each row and column and so N is invertible over A. Therefore,

$$Der\,A \ = \ \sum A\varepsilon_{ij} + A\varepsilon_y \ = \ \sum AD_{ij} + A\Delta_{nn} \ \subset \ S_p.$$

As remarked at the beginning of the proof, this suffices to prove the proposition.

3.10. REMARK. Compare (3.3) with (3.9). The former shows that $Der(T^{SO(n)})_p$ is induced from $Der\,T^{O(n)}$ for all $p \in \mathcal{Z}_n$. However it is implicit in (3.9) that $Der(T^{SO(n)})_p$ is *not* induced from $Der\,T^{O(n)}$ when $p \in \mathcal{Z}_{n-1}$. To see this, note that by (3.6), $Der\,T^{O(n)}$ is generated by the D_{ij}. Now consider the matrix N' defining the elements $\{D_{ij} : 1 \leq i \leq j \leq n\}$ in terms of the vector $\underline{\varepsilon}$. Working modulo $\mathbf{m}(p)$ as in (3.9), one obtains the same equations for the D_{ij} when $(i,\ j) \neq (n,\ n)$ but now

$$D_{nn} \ \equiv \ 2z_{nn}\partial_{nn} \ \equiv \ 0 \quad mod\,\mathbf{m}(p).$$

Thus N' is not invertible and ε_y cannot be obtained from the D_{ij}.

3.11. We are now ready to prove the main result of this chapter.

THEOREM. *Let $R \cong U(\mathfrak{sp}(2n))/J(n)$ be the image of $U(\mathfrak{sp}(2n))$ inside $\mathcal{D}(T^{O(n)})$. Then:*

(i) $R \ = \ D(T^{SO(n)})^{\{\pm\}} \ \neq \ \mathcal{D}(T^{O(n)})$.

(ii) *$\mathcal{D}(T^{SO(n)})$ is a simple Noetherian domain that is finitely generated as a (left or right) R-module.*

(iii) *The natural map $\varphi' : \mathcal{D}(T)^{SO(n)} \to \mathcal{D}(T^{SO(n)})$ is surjective.*

PROOF: We use the notation of (2.3). By [**Mo,** Corollary 5.8, p.79], $\mathcal{D}(T)^{SO(n)}$ is finitely generated as a $\mathcal{D}(T)^{O(n)}$-module, and so $S = \varphi'(\mathcal{D}(T)^{SO(n)})$ is a finitely generated module over $R = \varphi(\mathcal{D}(T)^{O(n)})$. Consider $\mathcal{D}/S$ as a R-module. Fix $d \in \mathcal{D}$ and let $K = \{a \in T^{O(n)} : ad \in S\}$. Pick $q \in \mathcal{Y}_n \cup \mathcal{Y}_{n-1}$ with corresponding maximal ideal $\mathcal{I}(q) \subset T^{O(n)}$. Then Corollary 3.4 and Proposition 3.9 imply that $K \cap \{T^{O(n)} \backslash \mathcal{I}(q)\} \neq \emptyset$. Thus, if $I(n-2)$ is the ideal in $\mathcal{O}(\mathcal{Y}) = T^{O(n)}$ defining $\overline{\mathcal{Y}}_{n-2}$, then we have shown that $K \supseteq I(n-2)^i$ for some $i \geq 1$. Thus, by (III, Corollary 2.19),

$$GKdim\ Rd + S/S \ \leq\ GKdim\ R/R.I(n-2)^i \ \leq\ GKdim\ R - 2.$$

By Corollary 3.4(ii) and Gabber's Lemma, (III, Lemma 1.8) this implies that $\mathcal{D}/S$, and hence $\mathcal{D}$, are finitely generated left R-modules.

By (2.3) there are inclusions $R \subseteq \mathcal{D}^{\{\pm\}} \subseteq \mathcal{D}$ and so $\mathcal{D}^{\{\pm\}}$ is a finitely generated left R-module. By (2.3) there are also inclusions

$$R \ \subseteq\ \mathcal{D}^{\{\pm\}} \ \subseteq\ \mathcal{D}(T^{O(n)})$$

and by (III, Proposition 1.4) these three rings have the same quotient division ring. Moreover, R is a simple ring (see (IV, Theorem 1.1)). Thus we may apply (IV, Lemma 1.3(i)), and conclude that $R = \mathcal{D}^{\{\pm\}}$. That $R \neq \mathcal{D}(T^{O(n)})$ is just (IV, Remark 1.5).

(ii) This follows from part (i) combined with [**Mo,** Corollary 5.8, p.79 and Theorem 7.11, p.116].

(iii) By Proposition 3.3, S and $\mathcal{D}$ certainly have the same quotient division ring. By part (i), $\mathcal{D}$ is finitely generated as a left or right S-module, while by part (ii), $\mathcal{D}$ is a simple ring. Thus (IV, Lemma 1.3(ii)) implies that $S = \mathcal{D}$.

3.12. Although the elements become rather messy, Theorems 2.6 and 3.11 can be used to obtain a set of generators for $\mathcal{D}(T^{SO(k)})$ when $k \leq n$. This is done as follows. Write $M_{k,2n}(\mathbb{C}) = M_{k,n}(\mathbb{C}) \times M_{k,n}(\mathbb{C})$, as in the proof of Lemma 3.7, and thereby identify

$$\begin{aligned} \mathcal{O}(M_{k,2n}(\mathbb{C})) \ &=\ \mathbb{C}[\, x_{ij},\ \xi_{ij},\ : 1 \leq i \leq k,\ 1 \leq j \leq n] \\ &=\ \mathbb{C}[X,\ \Xi] \ =\ \mathbb{C}[\Omega], \end{aligned}$$

where Ω is the $k \times 2n$ matrix $(X\ \Xi)$. Let $SO(k)$ act on $M_{k,2n}(\mathbb{C})$ by left translation. Then [**We,** p.77] implies that $\mathcal{O}(M_{k,2n}(\mathbb{C}))^{SO(n)}$ is generated by the elements

$${}^t\Omega_u \Omega_v \quad \text{for} \quad 1 \leq u \leq v \leq 2n$$

and

$$\chi_\alpha \ =\ det\,[\Omega_{\alpha_1},\ \ldots,\ \Omega_{\alpha_k}] \quad \text{for any} \quad \alpha \ =\ (1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_k \leq 2n).$$

Finally, in each ${}^t\Omega_u\Omega_v$ and χ_α, replace each ξ_{ij} by $\partial/\partial x_{ij}$. However, in formally expanding these elements, one must always keep the $\partial/\partial x_{ij}$ to the right of any x_{ab}. Then the argument used in Lemma 3.7 shows that the elements so obtained do actually generate $\mathcal{D}(T)^{SO(k)}$. Thus, by Theorems 2.6 and 3.11, their images generate $\mathcal{D}(T^{SO(k)})$.

In one respect, these generators are rather different from those obtained in (IV, 1.9) for invariants under the groups $GL(k)$, $O(k)$ and $Sp(2k)$. For, the generators described in (IV, 1.9) all have order ≤ 2 as differential operators. However the generators given above for $\mathcal{D}(T)^{SO(k)}$ include, in particular, the $k \times k$ determinant, $det(\partial/\partial x_{ij})$. This, of course, has order k as a differential operator.

3.13. There is one further class of classical rings of invariants, for which the generators and relations are given in [**We**], but which have not been considered in this paper. These are the invariants obtained from the action of $SL(k)$ on affine space. Unfortunately, it would seem that rather different methods from those of this paper are needed to describe $\mathcal{D}(\mathbb{C}[X]^{SL(k)})$. In particular, since $GL(k)/SL(k)$ is an infinite group, one cannot easily mimic the methods of this chapter, as these used the finiteness of $O(k)/SO(k)$ rather strongly.

4. On the identity SO(2) = GL(1).

4.1. The aim of this brief section is to note that the isomorphism of Lie groups $SO(2) \cong GL(1)$ gives an action of $\{\pm\} = O(2)/SO(2)$ on an appropriate factor ring of $U(\mathfrak{sl}(2n))$ and the fixed ring is then isomorphic to a factor of $U(\mathfrak{sp}(2n))$.

4.2. Consider the action of $O(2)$ on $\mathcal{X} = M_{2,n}(\mathbb{C})$ as given in (II, 3.2). This is obtained by writing $\mathcal{X} = V \otimes L^*$ where V is a 2-dimensional vector space equipped with a non-degenerate symmetric form and L^* is an n-dimensional space. Then the action of $O(2)$ on $\mathcal{X}$ is obtained by identifying $O(2)$ with $O(V)$. We have previously identified the action of $O(2)$ on $S = \mathcal{O}(\mathcal{X})$ by picking an orthonormal basis for V (see, for example, the proof of Lemma 3.7). However, now we wish to take a polarisation $V = K \oplus K^* = \mathbb{C}v \oplus \mathbb{C}v^*$ for V. This identifies $SO(2)$ with the diagonal matrices

$$\left\{ g \;=\; \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha \in \mathbb{C}^* \right\}$$

inside $GL(V)$. For such an element $g \in SO(2)$ one now has $g{\cdot}v = \alpha v$ and $g{\cdot}v^* = \alpha^{-1}v^*$. Thus the action of $SO(2)$ on $\mathcal{X} = V \otimes L^*$ is identical with the action of $GL(1)$ on $M_{n,1}(\mathbb{C}) \times M_{1,n}(\mathbb{C})$ given by $\alpha \cdot (a,\ b) = (a\alpha^{-1},\ \alpha b)$ for $(a,\ b) \in M_{n,1}(\mathbb{C}) \times M_{1,n}(\mathbb{C})$ and $\alpha \in GL(1) = \mathbb{C}^*$. Observe that this latter action is precisely the one considered in (II, 2.2).

4.3. Assume that $n \geq 2$ and let $O(2)$ act on $\mathcal{X} = M_{2,n}(\mathbb{C})$ as above. Write

$$R \;=\; Im\{\varphi : \mathcal{D}(T)^{O(2)} \;\longrightarrow\; \mathcal{D}(T^{O(2)})\}$$

and $S \;=\; Im\{\varphi' : \mathcal{D}(T)^{SO(2)} \;\longrightarrow\; \mathcal{D}(T^{SO(2)})\}$ as in (2.2) and (2.3). By (4.2) and (II, Proposition 2.7), $S \;=\; U(\mathfrak{sl}(2n))/J(-\overline{\omega}_n + \rho)$. Similarly, by (II, Proposition 3.7), $R \;=\; U(\mathfrak{sp}(2n))/J(-\overline{\omega}_n + \rho)$. Moreover, Theorems 2.6 and 3.11 combine to prove that $S^{\{\pm\}} = R$. Thus, by combining these results we obtain:

PROPOSITION. *For any $n \geq 2$ there exists an action of $\{\pm\} = \mathbb{Z}/2\mathbb{Z}$ on*

$$S \;=\; U(\mathfrak{sl}(2n))/J(-\overline{\omega}_n + \rho)$$

such that $S^{\{\pm\}} \;\cong\; U(\mathfrak{sp}(2n))/J(-\overline{\omega}_n + \rho).$

5. The structure of $\mathcal{D}(\mathcal{O}(\mathcal{X})^{\mathbf{SO(k)}})$ as a $\mathbf{U}(\mathfrak{sp}((\mathbf{2n}))$-module .

5.1. Throughout this section we assume that $k \leq n$ and we retain the notation of (1.1). The aim of this section is to study the structure of $\mathcal{D} = \mathcal{D}(T^{SO(k)})$ as a $U(\mathfrak{sp}(2n))$-module. We prove in particular that $\mathcal{D}$ is a Harish-Chandra $(\mathfrak{g}, K)$-bimodule, where $K = G = Sp(2n)$ and $\mathfrak{g} = Lie(G) = \mathfrak{sp}(2n)$. Moreover $\mathcal{D}$ splits as a direct sum of two simple $(\mathfrak{g}, K)$-modules; $\mathcal{D} = R \oplus \mathcal{D}_-$, for an appropriate module $\mathcal{D}_-$. This decomposition illustrates a conjecture of Vogan (see (5.5) below).

5.2. Set $R = Im\{\varphi : U(\mathfrak{sp}(2n)) \to \mathcal{D}(T^{O(k)})\}$ and recall from Theorems 2.6 and 3.11 that $\mathcal{D}^{\{\pm\}} = R$. This allows one to obtain detailed information about the structure of $\mathcal{D}$ as an R-module. An automorphism α of a ring A is called *inner* if there exists a unit $u \in A$ such that $\alpha(r) = uru^{-1}$ for all $r \in A$. If α is not inner then it is defined to be an *outer automorphism.* The reader is reminded that the elements of the group $\{\pm\}$ are denoted by 1 and σ.

PROPOSITION. **(i)** *σ is an outer automorphism of $\mathcal{D}$.*

(ii) *Set $\mathcal{D}_- = \{r \in \mathcal{D} : \sigma \cdot r = -r\}$. Then $\mathcal{D}$ decomposes; $\mathcal{D} = R \oplus \mathcal{D}_-$ as an R–bimodule.*

(iii) *$\mathcal{D}_-$ is projective and uniform as a left or right R–module, and is simple as an R–bimodule.*

PROOF: **(i)** Suppose that σ is inner – say $\sigma \cdot r = uru^{-1}$ for some $u \in \mathcal{D}$. Since the order of a differential operator is a degree function, every unit in $\mathcal{D}$ must have order zero. Thus $u \in T^{SO(k)}$. But this implies that σ acts trivially on $T^{SO(k)}$, contradicting Lemma 1.7.

(ii) If $d \in \mathcal{D}$ then

$$d \;=\; \tfrac{1}{2}(r + \sigma \cdot r) \;+\; \tfrac{1}{2}(r - \sigma \cdot r) \;\in\; R + \mathcal{D}_- .$$

Thus $\mathcal{D} = R \oplus \mathcal{D}_-$ as vector spaces and, clearly, $\mathcal{D}_-$ is an R–bimodule.

(iii) Since $R \subseteq \mathcal{D}(T^{O(k)})$, certainly R is a domain. For any $d \in \mathcal{D}_- \setminus \{0\}$ one has

$$\mathcal{D}_- \;\cong\; d\mathcal{D}_- \;=\; I \;\subseteq\; R.$$

Thus $\mathcal{D}_-$ is a finitely generated uniform (right) R-module. Suppose, next, that $\mathcal{D}_-$ is not simple as a R-bimodule. Then it has a proper factor bimodule; say $\overline{\mathcal{D}} = \sum_1^t Rr_i$, for some $r_i \in \overline{\mathcal{D}}$. Since $\mathcal{D}_- \cong I$, $\overline{\mathcal{D}} \cong I/J$ where I and J are non-zero right ideals of R. Since R is a Noetherian domain, the elements $r_i \in \overline{\mathcal{D}}$ therefore have a common right annihilator. Thus

$$r\text{-}ann_R(\overline{\mathcal{D}}) \;=\; \bigcap r\text{-}ann_R(r_i) \;\neq\; 0.$$

But $r\text{-}ann_R(\overline{\mathcal{D}})$ is then a non-zero ideal of the simple ring R (IV, Corollary 1.4), a contradiction. Thus $\mathcal{D}_-$ is a simple R-bimodule. Finally, $\mathcal{D}$ is a simple ring, by Theorems 2.6 and 3.11. Thus [**Mo,** Theorem 2.4, p.21] and part (i) combine to show that $\mathcal{D}_-$ is a projective R-module.

5.3. Recall from (II, 3.1) that $G = Sp(2n)$ and $G' = O(k)$ are mutual commutators in $Sp(U^\sim, <\ ,\ >^\sim)$. Thus the action of G on $\mathcal{D}(T) = \mathcal{D}(\mathcal{X})$ restricts to give an action on the fixed rings $\mathcal{D}(T)^{SO(k)}$ and $\mathcal{D}(T)^{O(k)}$. We wish to use this to induce an action of G on $\mathcal{D}$.

LEMMA. *The action of G on $\mathcal{D}(T)^{SO(k)}$ induces a locally finite action of G on $\mathcal{D}$ The differential of this action is just the adjoint action of $\mathfrak{g} = \mathfrak{sp}(2n)$, where $\mathfrak{g}$ is identified with its image in R. Moreover, the decomposition $\mathcal{D} = R \oplus \mathcal{D}_-$ is G-stable.*

PROOF: By Theorems 3.11 and 2.6, the map $\varphi' : \mathcal{D}(T)^{SO(k)} \to \mathcal{D}$ is surjective. Thus the action of G on $\mathcal{D}$ will be defined by

$$g \cdot \varphi'(d) \;=\; \varphi'(g \cdot d) \qquad \text{for} \quad g \in G \quad \text{and} \quad d \in \mathcal{D}(T)^{SO(k)}.$$

In order to prove that this is well-defined it suffices to show that the ideal

$$Ker(\varphi') \;=\; \left\{ P \in \mathcal{D}(T) : P * (T^{SO(k)}) = 0 \right\}$$

is G-stable. Now, the action of G on $\mathcal{D}(T)$ is induced from its action on $U^\sim \;=\; \mathcal{X} \oplus \mathcal{X}^*$, and so it is certainly a locally finite action. Thus, in order to show that $Ker(\varphi')$ is G-stable it suffices to show that $Ker(\varphi')$ is stable under the differentiation of this action. But by [**Ho1,** Theorem 5] this is simply the adjoint action of $\mathfrak{g}$, where $\mathfrak{g}$ is now identified with its image in $\mathcal{D}(T)^{O(k)}$ under the metaplectic representation ω. Thus let $P \in Ker(\varphi')$ and $X \in \mathfrak{g}$. Then

$$\begin{aligned} [X,\, P] * T^{SO(k)} \;&=\; X * (P * T^{SO(k)}) - P * (X * T^{SO(k)}) \\ &=\; -\, P * (X * T^{SO(k)}). \end{aligned}$$

But as $X \in \mathcal{D}(T)^{O(k)}$, one has $X * T^{SO(k)} \subseteq T^{SO(k)}$. Therefore $[X,\, P] * T^{SO(k)} = 0$. Thus $[X,\, P] \in Ker(\varphi')$ and the action of G on $\mathcal{D}$ is indeed well defined. That this action is locally finite and that its differential is the adjoint action of $\mathfrak{g}$ both follow immediately from the observation that these statements hold for the action of G on

$\mathcal{D}(T)^{SO(k)}$. Finally, as G and $O(k)$ are commutators in $Sp(U^{\sim})$, the action of G on $\mathcal{D}$ certainly commutes with that of $\{\pm\} = O(k)/SO(k)$. Thus the decomposition $\mathcal{D} = R \oplus \mathcal{D}_-$ is indeed G-stable.

5.4. COROLLARY-DEFINITION. *The ring $\mathcal{D}$ is a Harish-Chandra $(\mathfrak{g}, K)$-bimodule for the group $K = G$ and Lie algebra $\mathfrak{g} = Lie(G) = \mathfrak{sp}(2n)$. In other words:*

(i) *$\mathcal{D}$ is a $\mathfrak{g}$-bimodule that is finitely generated as a left or right $U(\mathfrak{g})$-module.*

(ii) *$\mathcal{D}$ is equipped with a locally finite action of the group K such that the differential of this action is the adjoint action of $\mathfrak{g}$.*

(iii) *$g \cdot (uv) = [Ad_g(u)](g \cdot v)$ for all $g \in K$, $u \in U(\mathfrak{g})$ and $v \in \mathcal{D}$.*

(iv) *Each isotypic component $\mathcal{D}_\alpha$, for $\alpha \in \widehat{K}$, is finite dimensional.*

PROOF: Parts (i), (ii) and (iii) follow immediately from (5.2) and (5.3). Since R is a simple factor ring of $U(\mathfrak{g})$, the centre of $U(\mathfrak{g})$ must act by scalars on R and hence on $\mathcal{D}$. Thus part (iv) follows from [**V1,** Corollary 5.4.16].

5.5. The results of (5.2) and (5.4) illustrate [**BV,** Conjecture 3.18] (which is true for $\mathfrak{g}$ classical) and [**V2,** Conjecture 1.26] in the case when $\mathfrak{g} = \mathfrak{sp}(2n)$. In order to describe these applications, we will use the notation from [**BV2**] that was introduced in Chapter IV.

Suppose, first, that k is even. Then the orbit $\mathbf{O}_k$ is special, ${}^L\mathbf{O}_k$ is even and $\mu_k = \lambda_k + \rho$ is conjugate to $\lambda(\mathbf{O}_k)$ (see (IV, 4.2, 4.3 and 4.4), respectively). To $\mathbf{O}_k$ (as to any special nilpotent orbit) one may associate a factor of the component group called the *Lusztig canonical quotient,* and written $\overline{A}(\mathbf{O}_k)$. This is defined in [**Lu,** Chapter 13] or [**BV2,** §4.4(a)], but all we need to know about it is that, in the present circumstances, $\overline{A}(\mathbf{O}_k) = \mathbb{Z}/2\mathbb{Z}$ (see [**Lu,** p.88]). Now, [**BV2,** Theorem III] gives a bijection between $\overline{A}(\mathbf{O}_k)$ and the set of unipotent representations attached to $\mathbf{O}_k$; that is, to the set of irreducible Harish-Chandra $(\mathfrak{g}, K)$-modules with central character $(\lambda(\mathbf{O}_k), \lambda(\mathbf{O}_k))$ and Gelfand-Kirillov dimension equal to *dim* $\mathbf{O}_k$. In the notation of that theorem, (5.2) and (5.4) imply that $X_{triv} = R$ and $X_{sign} = \mathcal{D}_-$ are the two corresponding unipotent representations. Furthermore, the ring extension $R \subset \mathcal{D}$ satisfies the properties required in [**BV3,** Conjecture 3.18].

Suppose, next, that $k = n$ is odd. By (IV, §4), again, $\mathbf{O}_k$ is special, ${}^L\mathbf{O}_k$ is even but now $\lambda(\mathbf{O}_k)$ is *not* conjugate to μ_k (see (IV, Remark 4.4)). In this case [**Lu,** p.88] implies that $\overline{A}(\mathbf{O}_k) = \{1\}$. Thus there is a unique unipotent representation attached to $\mathbf{O}_k$; namely $X_{triv} = U(\mathfrak{g})/J(\lambda(\mathbf{O}_k))$. Note that X_{triv} is not now equal to R.

Finally, suppose that $1 \leq k \leq n$, but otherwise that k is arbitrary. Of course, $\mathbf{O}_k$ need not be special (for $k < n$ odd – see (IV, Lemma 4.2)) but (5.2) and (5.4) are still true. Thus to $\mathbf{O}_k$ we have attached two irreducible Harish-Chandra modules, R and $\mathcal{D}_-$ and a ring extension $R \subset \mathcal{D}$. It is known that the component group of $G = Sp(2n)$ is $\mathbb{Z}/2\mathbb{Z}$ and by (II, Lemma 6.7) $\overline{\mathbf{O}}_k$ is a normal variety. Thus (5.2) and (5.4) should be viewed as an illustration of **[V2,** Conjecture 1.26**]**; in other words, the two extensions $R \subseteq R$ and $R \subseteq \mathcal{D}$ of R correspond to coverings of $\overline{\mathbf{O}}_k$.

APPENDIX. GABBER'S LEMMA

A.1. The aim of this appendix is to prove the result announced as (III, Lemma 1.8); that is,

GABBER'S LEMMA. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra (over any field k) and let M be a finitely generated, s-homogeneous left $U(\mathfrak{g})$-module, for some integer s. Let E be an essential extension of M and set*

$$\mathcal{S}_M = \big\{\text{finitely generated left } U(\mathfrak{g})\text{-modules } M' \text{ such that } M \subseteq M' \subseteq E \text{ and } GKdim(M'/M) \leq d-2\big\}.$$

Then $\mathcal{S}_M$ contains a unique maximal element.

This will be proved using the homological techniques developed in [**Bj,** Chapter 2], and so we will use that book as the basic reference for this appendix. As we remarked earlier, the lemma is proved in [**Le2**] but, as that paper will not be published, it seems appropriate to give a proof of it here. In fact since it seems to be a rather useful result, we will actually prove the lemma in a slightly more general setting than that given above.

A.2. It is perhaps worth remarking on the commutative case of Lemma A.1 (thus, when $\mathfrak{g}$ is abelian) where one is, in particular, proving the following result. Let R be a commutative domain of finite type over k, with quotient field Q, and write

$$R' = \bigcap\{R_p : ht(p) = 1\}.$$

Then R' is the unique, maximal R-module M such that $R \subseteq M \subseteq Q$ with $GKdim\ M/R \leq GKdim\ R - 2$. Then Lemma A.1 is just the well-known observation that R' is a finitely generated R-module, and thus is contained in the integral closure $\overline{R}$ of R in Q.

In this setting, $\overline{R}$ is also a finitely generated R-module and this prompts an amusing question. Suppose that S is a prime factor ring of $U(\mathfrak{g})$, for $\mathfrak{g}$ a finite dimensional Lie algebra. Then, is S contained in, and equivalent to a maximal order $\overline{S}$? (If $S \subseteq T \subseteq Q(S)$, then S is *equivalent* to T if $aTb \subseteq S$ for some regular elements $a, b \in S$. Further, T is a *maximal order* if it is equivalent to no ring T' satisfying $T \subseteq T' \subseteq Q(S)$.) If $\mathfrak{g}$ is a semi-simple, complex Lie algebra and S is a primitive ring, then a positive answer is given by [**JS,** Corollary 2.10].

A.3. The setting in which we will prove Gabber's Lemma is the following. Let $A = \sum_{n\geq 0} A_n$ be a filtered ring such that the associated graded ring, $gr\, A = \bigoplus A_n/A_{n-1}$ (where $A_{-1} = 0$) is a commutative Noetherian ring of finite homological dimension, $gldim(gr\, A) = \omega$. We assume further that this dimension is *pure;* that is $gldim(gr\, A_{\mathbf{m}}) = \omega$ for each maximal ideal $\mathbf{m}$ of $gr\, A$. Note that, by [**Bj,** Theorem 3.7, p.44], this implies that $\mu = gldim\ A \leq \omega < \infty$. The dimension of a $gr\, A$-module N, in the sense [**Bj,** §5.7, p.64], will be denoted $dim_{grA} N$ but, as usual, the subscript will be omitted whenever this can cause no confusion.

Let M be a finitely generated, non-zero left A-module, equipped with an exhaustive filtration $\{\Sigma_n : n \geq 0\}$. We will always assume that $\{\Sigma_n\}$ is a *good filtration;* that is, that $gr\, M = \bigoplus \Sigma_n/\Sigma_{n-1}$ is a finitely generated $gr\, A$-module. Define $d(M) = dim_{grA}(gr\, M)$. This is independent of the good filtration $\{\Sigma_n\}$, see for example [**Bj,** Lemma 6.2, p.69].

The hypotheses on A and any finitely generated A-module M ,as described above, will remain in force throughout this appendix.

A.4. The *grade of* the A-module M is defined by

$$j(M) \;=\; inf\{j : Ext^j_A(M,\ A) \neq 0\}.$$

The existence of $j(M)$ is, in particular, assured by the following result.

LEMMA. [**Bj,** Theorem 7.1, p.73] $d(M) + j(M) = \omega$.

We remark that this lemma also implies that $d(M) \geq \omega - \mu$ holds for every A-module M.

A.5. The significance of Lemma A.4 is that it allows one to use the homological techniques from [**Bj,** Chapter 2] in order to prove Gabber's Lemma. We need to set up the appropriate machinery. Given a finitely generated A-module M, fix a projective resolution

$$0 \longrightarrow P_\mu \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

with each P_j finitely generated. This complex will be denoted by $P_\bullet = P_\bullet(M)$ with dual complex

$$C^\bullet \;=\; C^\bullet(M) \;=\; P^*_\bullet \;=\; Hom_A(P_\bullet,\ A).$$

Observe that the homology groups of this complex are the groups $Ext^j_A(M,\ A)$ and so these Ext groups are, in particular, finitely generated A-modules. From now onwards the subscript $_A$ will be omitted from Hom and Ext.

Next, form a projective resolution $(Q^{\bullet,\bullet})$ for the complex $C^\bullet$, as defined, for example, in [**Bj,** §4.15, p.58] or [**CE,** Chapter XV]. We will not give the construction here, but merely note some of its properties. Following [**Bj**], the differentials are of degrees -1 and $+1$:

$$\partial_I : Q^{v,j} \longrightarrow Q^{v-1,j} \qquad \text{but} \qquad \partial_{II} : Q^{v,j} \longrightarrow Q^{v,j+1}.$$

Furthermore:

A.5.1. For all j, the complex $(Q^{\bullet,j})$ is a projective resolution of $C^j = C^j(M)$.

A.5.2. For all j, the cohomology groups $(H^j(Q^{v,\bullet})_v)$ (which are, again, finitely generated A-modules) form a projective resolution of $H^j(C^\bullet) = Ext^j(M, A)$.

A.5.3. There is a double complex $\mathcal{K}(M)$ of A-modules, defined by $\mathcal{K}^{p,q} = Hom(Q^{p,q}, A)$, with differentials induced from the differentials on $(Q^{\bullet,\bullet})$.

A.5.4. By [**CE,** Proposition 1.2, p.363], this construction is functorial in M. Thus, let $f : M \to M'$ be a homomorphism of A-modules and write $P'_\bullet = P_\bullet(M')$ for a projective resolution of M', with associated double complex $\mathcal{K}' = \mathcal{K}(M')$. Then f induces morphisms $P_\bullet \to P'_\bullet$ and $K^{\bullet,\bullet} \to (K')^{\bullet,\bullet}$. In both cases, the induced morphism will again be denoted by f.

A.6. As with any double sequence, one may associate to $\mathcal{K} = \mathcal{K}(M)$ two spectral sequences, $E_2^{p,q}(M) = H_I^p H_{II}^q(\mathcal{K})$ and ${}'E_2^{p,q} = H_{II}^p H_I^q(\mathcal{K})$. It follows easily from (A.5.2) that

$$E_2^{p,q}(M) \;=\; Ext^p(Ext^q(M, A), A).$$

Similarly, (A.5.1) implies that ${}'E_2^{p,q}$ degenerates, with the only non-zero term being ${}'E_2^{0,\mu}(M) = M$ (where we recall that $\mu = gldim\ A$). Consequently, the total cohomology $H^*(\mathcal{K})$ satisfies $H^n(\mathcal{K}) = 0$ for $n \neq \mu$ but $H^\mu(\mathcal{K}) = M$. (The detailed proofs for these assertions are given in [**Bj,** pp.60-61].)

The first filtration $\Gamma\mathcal{K}$ on the double complex $\mathcal{K}$ (see [**Bj,** §4.11, p.52]) therefore induces an exhaustive filtration $(\Gamma_p M)_{p\in\mathbb{Z}}$ on $M = H^\mu(\mathcal{K})$. Set

$$M(p) \;=\; E_\infty^{p,p} \;=\; \Gamma_p(M)/\Gamma_{p-1}(M).$$

The relevant results from [**Bj**] about these objects are contained in the next result.

THEOREM. *Assume that A and M are as in (A.3). Then:*

(i) *The filtration $(\Gamma_p M)$ is finite. Indeed,*

$$\Gamma_{-1}M = 0 \;\subseteq\; \Gamma_0 M \;\subseteq\; \cdots \;\subseteq\; \Gamma_\mu M \;=\; M.$$

Moreover, for $0 \le p \le \mu$, $\Gamma_p M$ *is the (unique) submodule* N *of* M *maximal with respect to* $d(N) \le p + (\omega - \mu)$.

(ii) $E_2^{p,q} = Ext^p(Ext^q(M, A), A) = 0$ *if* $p < q$.

(iii) *The convergence of the spectral sequence induces, for each* $0 \le p \le \mu$, *an exact sequence of* A*-modules*

$$0 \longrightarrow M(p) \xrightarrow{\sigma_p} Ext^{\mu-p}(Ext^{\mu-p}(M, A), A) \longrightarrow Q(p) \longrightarrow 0.$$

Here $Q(p)$ *is a subquotient of* $\bigoplus\{E_2^{\mu-i,\mu-i-1} : 0 \le i \le p-2\}$ *and satisfies* $d(Q(p)) \le p + (\omega - \mu) - 2$.

PROOF: This can all be found in [**Bj,** Chapter 2]. In particular, part (ii) is [**Bj,** Corollary 7.5, p.73]. Parts (i) and (iii) then follow from [**Bj,** Theorem 4.15, p.61, Theorem 7.10, p.75 and Lemma 7.3, p.74].

A.7. COROLLARY. *(Notation A.3, A.6) Let* $u : M_1 \to M_2$ *be a homomorphism between finitely generated* A*-modules. Then:*

(i) u *induces homomorphisms*

$$u_2^{p,q} : E_2^{p,q}(M_1) \longrightarrow E_2^{p,q}(M_2) \qquad \text{and} \qquad u^i : H^i(\mathcal{K}(M_1)) \longrightarrow H^i(\mathcal{K}(M_2)).$$

(ii) *Furthermore,* $u(\Gamma_p M_1) \subseteq u(\Gamma_p M_2)$ *for each* p *and this induces a commutative diagram:*

$$\begin{array}{ccccccc}
\Gamma_p M_1/\Gamma_{p-1} M_1 & = & M_1(p) & \xrightarrow{u} & M_2(p) & = & \Gamma_p M_2/\Gamma_{p-1} M_2 \\
 & & \Big\downarrow{\scriptstyle \sigma_p} & & \Big\downarrow{\scriptstyle \sigma_p} & & \\
 & & E_2^{p,p}(M_1) & \xrightarrow{u_2^{p,p}} & E_2^{p,p}(M_2) & &
\end{array}$$

(where σ_p *is defined by Theorem A.6(iii)).*

PROOF: Recall from (A.5.4) that the construction of $\mathcal{K}(M)$ is functorial in M. Thus the map u induces a map between the complexes $\mathcal{K}(M_1)$ and $\mathcal{K}(M_2)$ and hence between the various homological invariants. Moreover, the induced map on the invariants depends only upon u (see for example [**CE,** Proposition 6.1, p.332]). Combined with Theorem A.6, this proves all the assertions of the Corollary, with the possible exception of showing that the induced map from $\Gamma_p(M_1)$ to $\Gamma_p(M_2)$ (as homological invariants) is just the restriction of u. However, recall from (A.6) that $H^\mu(\mathcal{K}(M_i)) = M_i$ for $i = 1, 2$. Thus the map u^μ is none other than u, and the result follows.

A.8. REMARK. It will be useful to have a concrete description of the map $u_2^{p,q}$ of Corollary A.7. Given a map $u : M_1 \to M_2$ then, by functoriality, again, there exists an induced map in cohomology

$$Ext^j(u, A) : \ Ext^j(M_2, A) \longrightarrow Ext^j(M_1, A).$$

Of course, by (A.5.2), $Ext^j(u, A)$ is the map induced from u between the invariants $H^j(C^\bullet(M_2))$ and $H^j(C^\bullet(M_1))$. Combining this observation with the comments at the beginning of (A.6) one deduces that $u_2^{p,q}$ is the map

$$Ext^p(Ext^q(u, A), A) : \ Ext^p(Ext^q(M_1, A), A) \longrightarrow Ext^p(Ext^q(M_2, A), A).$$

A.9. A finitely generated left A-module M is called *s-homogeneous* if $d(M) = d(N) = s$ for all non-zero submodules N of M. We are now ready to prove our version of Gabber's Lemma.

PROPOSITION. *Assume that A and M satisfy the hypotheses of (A.3). Suppose further that M is s-homogeneous, for some integer s, and that E is a (not necessarily finitely generated) essential extension of M. Set*

$$\mathcal{S} = \{\text{all finitely generated left } A\text{-modules } M' \text{ such that } M \subseteq M' \subseteq E \text{ and } d(M/M') \leq s-2\}.$$

Then $\mathcal{S}$ contains a unique maximal element $\widetilde{M}$.

PROOF: Since $\mathcal{S}$ is clearly closed under finite sums, it suffices to show that $\mathcal{S}$ contains one maximal element.

Note that, if $N \in \mathcal{S}$, then $d(N) = s$ and, as M is an essential submodule of N, that N is s-homogeneous. Set $q = s - (\omega - \mu)$. Then Theorem A.6(i) implies that $\Gamma_p(N) = 0$ if $p < q$ while $\Gamma_p(N) = N$ if $q \leq p \leq \mu$. In particular, $N(q) = N$. Now, set

$$N^{max} = Ext^{\omega-s}(Ext^{\omega-s}(N, A), A).$$

Note that $N^{max} = E_2^{\omega-s,\omega-s}(N)$, in the notation of (A.6), and that Theorem A.6(iii) provides an embedding $\sigma_N : N \hookrightarrow N^{max}$.

The proposition now follows easily from the next sublemma.

A.10 SUBLEMMA. *Let $P,\ Q \in \mathcal{S}$ be such that $P \subseteq Q$ and write $u_{P,Q}$ for this inclusion map. Then*

(i) *$u_{P,Q}$ induces an isomorphism $\alpha_{P,Q} : P^{max} \to Q^{max}$. Moreover*

$$\alpha_{P,Q} \cdot \alpha_{M,P} = \alpha_{M,Q}.$$

(ii) *There exist natural embeddings $\psi_P : P \hookrightarrow M^{max}$ and $\psi_Q : Q \hookrightarrow M^{max}$. Moreover, the restriction of ψ_Q to P is just ψ_P.*

Proof of Proposition A.9: Consider the family $\{\psi_N(N) : \ N \in \mathcal{S}\}$ of submodules of M^{max} obtained by means of part (ii) of the sublemma. Since M^{max} is a finitely generated A-module (see, for example, (A.5)) this family contains a (unique) maximal element; say $\psi_{\tilde{M}}(\widetilde{M})$. By part (ii) of the sublemma, again, $\widetilde{M}$ is then the maximal element in $\mathcal{S}$.

Proof of the sublemma: **(i)** Write $u = u_{P,Q}$ and consider the short exact sequence

$$0 \longrightarrow P \xrightarrow{u} Q \longrightarrow Q/P \longrightarrow 0.$$

Applying $Hom(\ \ ,A)$ gives the exact sequence

$$\cdots \longrightarrow Ext^{\omega-s}(Q/P,\ A) \longrightarrow Ext^{\omega-s}(Q,\ A) \longrightarrow Ext^{\omega-s}(P,\ A) \longrightarrow$$
$$\longrightarrow Ext^{\omega-s+1}(Q/P,\ A) \longrightarrow \cdots.$$

But, by hypothesis, $d(Q/P) \leq s-2$ and so Lemma A.4 implies that $j(Q/P) \geq \omega - s + 2$. Thus this long exact sequence reduces to an isomorphism

$$Ext^{\omega-s}(Q,\ A) \xrightarrow{\sim} Ext^{\omega-s}(P,\ A).$$

Applying $Ext^{\omega-s}(\ \ ,A)$ gives the isomorphism $\alpha_{P,Q}$. By Remark A.8, this is precisely the construction of the map $u_2^{\omega-s,\omega-s}$ of Corollary A.7(i). Also, by the functoriality of Ext, one obtains the equation

$$\alpha_{P,Q}\alpha_{M,P} = \alpha_{M,Q}.$$

(ii) By the remarks made before the statement of the sublemma, Theorem A.6(iii) provides an embedding $\sigma_P : P \hookrightarrow P^{max}$. Thus ψ_P and ψ_Q are defined by $\psi_P = (\alpha_{M,P})^{-1}\sigma_P$ and $\psi_Q = (\alpha_{M,Q})^{-1}\sigma_Q$. Finally, by Corollary A.7(ii) and the final assertion of part (i) of the sublemma, the restriction of ψ_Q to P satisfies

$$\psi_Q \mid P = (\alpha_{M,Q})^{-1}\sigma_Q \mid P = (\alpha_{M,Q})^{-1}(\alpha_{P,Q}\sigma_P) = \psi_P.$$

This completes the proof of the sublemma, and hence of the proposition.

A.11. REMARKS. Suppose, in the statement of Proposition A.9, that E is the injective hull of M. Then it is easy to see that $\widetilde{M}$ is naturally isomorphic to

$$M^{max} \;=\; Ext^{\omega-s}(Ext^{\omega-s}(M, A), A).$$

For, [**Bj,** Theorem 7.10, p.74] shows that M^{max} is s-homogeneous and Theorem A.6(iii) implies that $d(M^{max}/M) \leq s-2$. It follows easily that M^{max} is an essential extension of M, and our claim now follows from the sublemma.

A.12 If $A = U(\mathfrak{g})$ is the enveloping algebra of a finite dimensional Lie algebra $\mathfrak{g}$ equipped with its standard filtration, then $gr\,A \cong S(\mathfrak{g})$ is a polynomial ring, and so certainly satisfies the hypotheses of (A.3). Moreover, in this case, the dimension $d(\ )$ is just Gelfand-Kirillov dimension. Thus Gabber's Lemma (A.1) is indeed a special case of Proposition A.9.

However, Proposition A.9 also applies in other situations – the most obvious case is when $A = \mathcal{D}(\mathcal{X})$ is the ring of differential operators over a non-singular, irreducible, affine algebraic variety $\mathcal{X}$. In this case A is filtered by the order of differential operators, and so this is certainly not a finite dimensional filtering.

There is a second class of rings of differential operators to which Proposition A.9 cannot be applied in its present form. For, suppose that $R = \mathbb{C}[\![x_1, \ldots, x_n]\!]$ is a ring of formal power series and that

$$\widehat{\mathcal{D}}_n \;=\; \mathcal{D}(R) \;=\; R[\partial/\partial x_1, \ldots, \partial/\partial x_n],$$

which we again filter by the order of differential operators. Then $gr(\widehat{\mathcal{D}}_n) \cong R[y_1, \ldots, y_n]$ and it is readily checked that this ring does not have pure global dimension in the sense of (A.3). Thus Proposition A.9 cannot be applied.

One can get around this problem in the following way. Suppose that $B = \bigoplus\{B_n : n \geq 0\}$ is a commutative, graded, Noetherian ring with $gldim\ B = \omega < \infty$. Define B to have *pure graded dimension* ω, if $gldim\ B_{\mathbf{m}} = \omega$ for every *graded* maximal ideal $\mathbf{m}$ of B. It is fairly easy to check that the results of [**Bj,** Chapter 2, §7] still hold with this assumption in place of B having pure dimension ω. Since it is only in applying those results from [**Bj**] that the concept of pure dimension was used, we obtain:

Proposition A.9 still holds if the assumption that $gr\,A$ has pure dimension ω is replaced by the assumption that $gr\,A$ has pure graded dimension ω.

Of course, the point behind these comments is that the ring $gr(\widehat{\mathcal{D}}_n)$ *does* have pure graded dimension ω, and so Proposition A.9 can now be applied to this ring.

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