SEMI-SIMPLICITY OF INVARIANT HOLONOMIC SYSTEMS ON A REDUCTIVE LIE ALGEBRA

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Abstract. Let $\mathfrak{g}$ be a reductive, complex Lie algebra, with adjoint group G, let G act on the ring of differential operators $\mathcal{D}(\mathfrak{g})$ via the adjoint action and write $\tau : \mathfrak{g} \to \mathcal{D}(\mathfrak{g})$ for the differential of this action. Fix $\lambda \in \mathfrak{h}^*$. Generalizing work of Hotta and Kashiwara, we prove that the invariant holonomic system

$$\mathcal{N}_\lambda = \mathcal{D}(\mathfrak{g})/\big(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \textstyle\sum_{p\in S(\mathfrak{g})^G}\mathcal{D}(\mathfrak{g})(p - p(\lambda))\big)$$

is semisimple. The simple summands of $\mathcal{N}_\lambda$ are parametrized by the irreducible representations of W_λ, the stabilizer of λ in the Weyl group. Consequently, the subcategory generated by $\mathcal{N}_\lambda$ is equivalent to the category of finite dimensional representations of W_λ.

1. Introduction

Let G be a complex connected reductive algebraic group with $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\mathcal{O}(\mathfrak{g}) = S(\mathfrak{g}^*)$ be the algebra of polynomial functions on $\mathfrak{g}$, and denote by $\mathcal{D}(\mathfrak{g})$ the algebra of differential operators on $\mathfrak{g}$, with coefficients in $\mathcal{O}(\mathfrak{g})$. We identify $S(\mathfrak{g}) \subset \mathcal{D}(\mathfrak{g})$ with the algebra of differential operators with constant coefficients. The group G acts on $\mathfrak{g}$, via the adjoint action, and hence has an induced action on $S(\mathfrak{g}^*)$, $S(\mathfrak{g})$ and $\mathcal{D}(\mathfrak{g})$. Denote the differential of this action by $\tau : \mathfrak{g} \to \mathcal{D}(\mathfrak{g})$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with associated Weyl group W.

Suppose that $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$, with adjoint group G_0, and recall that a distribution $\Theta \in \mathrm{Dist}(\mathfrak{g}_0)$ is a *G_0-invariant eigendistribution* if there exists $\lambda \in \mathfrak{h}^*$ such that $\tau(\mathfrak{g})\cdot\Theta = 0$, and $p\cdot\Theta = p(\lambda)\Theta$ for all $p \in S(\mathfrak{g})^G$. As in [10], one defines a system of linear differential equations for invariant eigendistributions or *invariant holonomic system on* $\mathfrak{g}$ by

$$\mathcal{N}_\lambda = \mathcal{D}(\mathfrak{g})/\big(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \textstyle\sum_{p\in S(\mathfrak{g})^G}\mathcal{D}(\mathfrak{g})(p - p(\lambda))\big) \quad \text{for } \lambda \in \mathfrak{h}^*.$$

This module is important, in part, because the invariant eigendistributions can be identified with $\mathrm{Hom}_{\mathcal{D}(\mathfrak{g})}(\mathcal{N}_\lambda, \mathrm{Dist}(\mathfrak{g}_0))$. Its study is also of central importance in the representation theory of $\mathfrak{g}$ and Hotta and Kashiwara have obtained deep results on the system $\mathcal{N}_\lambda$, and its Fourier transform, by means of the Riemann-Hilbert correspondence [10]. They showed, among other things, that the module $\mathcal{N}_0$ decomposes as

$$\mathcal{N}_0 = \bigoplus_{\chi\in W^\wedge} \mathcal{N}_{0,\chi} \otimes_{\mathbb{C}} V_\chi^* \tag{1.1}$$

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(see [10, Theorem 5.3]). Here, $W^\frown$ denotes the set of isomorphism classes of irreducible W-modules and V_χ is a representation in the class of χ. The modules $\mathcal{N}_{0,\chi}$ are pairwise non-isomorphic, simple $\mathcal{D}(\mathfrak{g})$-modules.

On the other hand, Wallach [19] has shown that the decomposition of the $\mathcal{D}(\mathfrak{h})^W$-module $S(\mathfrak{h}^*)$ is closely related to the study of invariant eigendistributions on $\mathfrak{g}_0$, and to the Springer correspondence. This connection is made possible by the existence of a homomorphism $\delta : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h})^W$ defined by Harish-Chandra [7]. The homomorphism δ is surjective with kernel $(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}))^G$, see [13, 14].

The aim of this paper is to prove the following theorems, which generalize (and provide different proofs of) the aforementioned results from [10] and [19]. The first theorem answers a question of M. Duflo, who asked for the decomposition of the $\mathcal{D}(\mathfrak{h})^W$-module $S(\mathfrak{h}^*)e^\lambda$.

Theorem A. *Fix $\lambda \in \mathfrak{h}^*$ and let W_λ denote the stabilizer of λ in W.*

(i) $S(\mathfrak{h}^*)e^\lambda$ is a semisimple $(\mathcal{D}(\mathfrak{h})^W, W_\lambda)$-module. Indeed:

$$S(\mathfrak{h}^*)e^\lambda = \bigoplus_{\chi \in W_\lambda^\frown} V^\chi \otimes V_\chi^*,$$

where the V^χ are simple, pairwise non-isomorphic $\mathcal{D}(\mathfrak{h})^W$-modules and V_χ^* is the dual of the simple W-module V_χ in the class of χ.

(ii) In particular, if $\chi = \mathrm{triv}$ is the trivial character, then $V^{\mathrm{triv}} = \mathcal{D}(\mathfrak{h})^W e^\lambda = S(\mathfrak{h}^*)^{W_\lambda} e^\lambda$ is a simple $\mathcal{D}(\mathfrak{h})^W$-module.

(iii) Set $\mathbf{p}_\lambda = \sum_{p \in S(\mathfrak{h})^W} S(\mathfrak{h})^W (p - p(\lambda))$. Then, $S(\mathfrak{h}^*)e^\lambda \cong \mathcal{D}(\mathfrak{h})^W / \mathcal{D}(\mathfrak{h})^W \mathbf{p}_\lambda$ as $\mathcal{D}(\mathfrak{h})^W$-modules.

This theorem follows from Theorem 3.4 and Theorem 4.4, which give analogous results in more general circumstances. The main result of the paper, see Theorem 6.9, is the following:

Theorem B. *Fix $\lambda \in \mathfrak{h}^*$. As either a $(\mathcal{D}(\mathfrak{g}), W_\lambda)$-module or a $\mathcal{D}(\mathfrak{g})$-module, $\mathcal{N}_\lambda$ is semisimple. Indeed:*

$$\mathcal{N}_\lambda \cong \bigoplus_{\chi \in W_\lambda^\frown} \mathcal{N}_{\lambda,\chi} \otimes_{\mathbb{C}} V_\chi^*, \qquad \text{where } \mathcal{N}_{\lambda,\chi} = (\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})) \otimes_{\mathcal{D}(\mathfrak{g})^G} V^\chi. \tag{1.2}$$

Moreover, the $\mathcal{N}_{\lambda,\chi}$, for $\chi \in W_\lambda^\frown$, are simple, pairwise nonisomorphic $\mathcal{D}(\mathfrak{g})$-modules.

When $\lambda = 0$ this reduces to (1.1). Hotta and Kashiwara also prove the result in the case when λ is regular [10, Lemma 4.6.1]. The analog of Theorem B for differential operators over G has been raised as a conjecture in [9, Hypothesis 4.1].

The results and proofs in [10] rely on the fact that the modules $\mathcal{N}_\lambda$, and their Fourier transforms, can be obtained via D-module constructions from $\mathcal{O}_{\tilde{\mathfrak{g}}}$, where $\tilde{\mathfrak{g}} \to \mathfrak{g}$ is the Grothendieck-Springer resolution. Our approach to Theorem B is less geometric, but in a sense more elementary. The idea is as follows: Since $\mathcal{N}_\lambda \cong (\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})) \otimes_{\mathcal{D}(\mathfrak{h})^W} S(\mathfrak{h}^*)e^\lambda$, the decomposition (1.2) follows from Theorem A. Thus, the hard part of the proof is to show that the $\mathcal{N}_{\lambda,\chi}$ are simple. To prove this, one first shows that $\mathcal{N}_{\lambda,\chi}$ is torsion-free as an $\mathcal{O}(\mathfrak{g})$-module (see Corollary 6.6). It follows that $\mathcal{N}_{\lambda,\chi}$ is simple if and only if its restriction to the set of generic elements is also simple. As this second module is an integrable connection, it is easy to analyse and this enables one to prove the theorem.

One consequence of this argument is that:

every subfactor of $\mathcal{N}_\lambda$ is torsion-free as an $\mathcal{O}(\mathfrak{g})$-module.

This generalizes work of Harish-Chandra in the sense that the essence of his result on the regularity of invariant eigendistributions [7, 8] is the assertion that $\mathcal{N}_\lambda$ has no factor that is $\{d^n\}$-torsion, where $d \in \mathcal{O}(\mathfrak{g})$ denotes the discriminant (see the discussion in [10, p.352]).

Let $\mathcal{R}_\lambda$ denote the full subcategory of finitely generated $\mathcal{D}(\mathfrak{g})$-modules generated by $\mathcal{N}_\lambda$. Then Theorem B has the following easy consequence:

Corollary C. *The category $\mathcal{R}_\lambda$ is equivalent to* W_λ-mod, through the functor

$$M \to \operatorname{Hom}_{\mathcal{D}(\mathfrak{h})^W}(M^G, S(\mathfrak{h}^*)e^\lambda).$$

As a consequence of these results, one obtains a "Springer correspondence" between representations of W_λ, local systems on G-orbits in $\mathbf{N}(\lambda) = \{x \in \mathfrak{g} : f(x) = f(\lambda) \text{ for all } f \in S(\mathfrak{g}^*)^G\}$ and W_λ-harmonics in $S(\mathfrak{h}^*)$. The details can be found in Sections 6 and 7.

2. The structure of the $\mathcal{D}(V) * W$-module generated by e^λ.

For the next four sections, we fix an algebraically closed field $\Bbbk$ of characteristic zero and a $\Bbbk$-vector space V of dimension ℓ. Set: $R = \mathcal{O}(V) = S(V^*)$, the algebra of functions on V; $\widehat{R}$ the algebra of formal power series on V; $S = S(V)$, the symmetric algebra on V, which will be identified with the algebra of constant coefficients differential operators on V; $A = \mathcal{D}(V)$, the algebra of differential operators on V. The action of $d \in A$ on $f \in \widehat{R}$ will be denoted by $d \cdot f \in \widehat{R}$.

Fix a finite subgroup W of $\mathrm{GL}(V)$. Then W and its subgroups have natural induced actions on the algebras R, $\widehat{R}$, S and A. Given $\lambda \in V^*$, we regard λ as a character $\lambda : S(V) \to \Bbbk$ and set $\mathbf{m}_\lambda = \operatorname{Ker}\lambda$ and $\mathbf{n}_\lambda = \mathbf{m}_\lambda \cap S(V)^{W_\lambda}$, where W_λ is the stabilizer of λ. An element $g \in W$ acts on λ by $(g.\lambda)(v) = \lambda(g^{-1}.v)$, $v \in V$. The element e^λ can be viewed as an element of $\widehat{R}$ and the A-submodule it generates is the simple A-module $A \cdot e^\lambda = Re^\lambda \cong A/A\mathbf{m}_\lambda$.

The aim of the first three sections is to describe the structure of Re^λ as a module over A^W, the subring of W-invariant elements in A. The description is given in Theorem 3.4. When W is generated by pseudo-reflections a further description is given in Theorem 4.4.

The way we prove these results is to use the Morita equivalence (see §3) between A^W and the skew group ring $A * W$. This ring is defined as follows: If $d \in A$ and $w \in W$ the (left) action of w on d is denoted by $w.d$ and we set $d^w = w^{-1}.d$. The *skew group ring* $A * W$ is the free (left and right) A-module with basis $\{w\}_{w\in W}$ and multiplication defined by $dw = wd^w$ for $d \in A$ and $w \in W$. We always identify A with the subring $A\epsilon$ of $A * W$, where ϵ is the identity element of W.

Notice that $\widehat{R}$ is in a natural way a left $A * W$-module. Moreover, if $g \in W$, then $g.e^\lambda = e^{g.\lambda}$ and so Re^λ is a $A * W_\lambda$-submodule of $\widehat{R}$. To analyze this module we need to introduce the categories of modules on which $\mathbf{m}_\lambda$ or $\mathbf{n}_\lambda$ acts locally nilpotently. Let B-mod denote the category of finitely generated left B-modules over a ring B and set

$$\mathcal{S}_\lambda = \{M \in A * W_\lambda\text{-mod} \mid \forall\, x \in M,\ \exists\, k \in \mathbb{N},\ \text{such that } \mathbf{m}_\lambda^k x = 0\}$$
$$\mathcal{C}_\lambda = \{M \in A^{W_\lambda}\text{-mod} \mid \forall\, x \in M,\ \exists\, k \in \mathbb{N},\ \text{such that } \mathbf{n}_\lambda^k x = 0\}.$$

Observe that $Re^\lambda \in \mathcal{S}_\lambda$ as an $A * W_\lambda$-module, while, as an A^{W_λ}-module, $Re^\lambda \in \mathcal{C}_\lambda$. The next two lemmas give some elementary properties of Re^λ and $(A * W_\lambda) \otimes_A Re^\lambda$. The first is standard.

Lemma 2.1. (i) Let N be a finitely generated left A-module that is locally finite as a left $S(V)$-module. Then N is a finite direct sum of simple A-modules of the form Re^μ, for $\mu \in V^*$.

(ii) If Γ is a finite group of automorphisms of A and N is a semi-simple left A-module, then $(A * \Gamma) \otimes_A N$ is a semi-simple left $A * \Gamma$-module.

Proof. (i) The hypothesis ensures that N is a $\mathcal{D}(V^*)$-module supported on a finite number of points in V^*. The claim therefore follows from Kashiwara's equivalence [2, Theorem 7.11, p.264].

(ii) This is [16, Theorem 7.6(iv)]. $\square$

Now we introduce a functor from the category of left A-modules, to the category of $\Bbbk$-vector spaces by setting

$$\Phi(M) = \{x \in M \mid \mathbf{m}_\lambda x = 0\}.$$

Note that $\Phi(Re^\lambda) = \Bbbk e^\lambda$. Furthermore, $\Phi(M)$ identifies with $\mathrm{Hom}_A(Re^\lambda, M)$ under the map $f \mapsto f(e^\lambda)$, for $f \in \mathrm{Hom}_A(Re^\lambda, M)$. Moreover, if M is an $A * W_\lambda$-module, then $\Phi(M)$ is a W_λ-module. Denote by W_λ-mod the category of finite dimensional representations of W_λ.

Proposition 2.2. *Define* $\Bbbk_\lambda = S/\mathbf{m}_\lambda$. *Then* Φ *induces an equivalence of categories*

$$\mathcal{S}_\lambda \approx W_\lambda\text{-mod}$$

whose inverse functor is $\Psi : E \to Re^\lambda \otimes_{\Bbbk_\lambda} E$. *Here,* E *is considered as a trivial left* $\Bbbk_\lambda$*-module and* W_λ *acts on* $Re^\lambda \otimes E$ *via its diagonal action:* $w.(d \otimes x) = w.d \otimes w.x$.

Proof. Clearly, $E \cong \Phi\Psi(E)$, for any W_λ-module E. Let M be in $\mathcal{S}_\lambda$. By Lemma 2.1(i), $M \cong (Re^\lambda)^{(m)}$ as a left A-module and so, if $E = \Phi(M)$, then $M = AE = RE$. Thus, there is a surjection of A-modules $\psi : \Psi(E) \twoheadrightarrow M$ given by $re^\lambda \otimes \varepsilon \mapsto r\varepsilon$ for $r \in R$ and $\varepsilon \in E$. Using the fact that $w.e^\lambda = e^\lambda$, for $w \in W_\lambda$, it is easily checked that ψ is actually an $A * W_\lambda$-module map. Since $\Psi(E)$ is a semisimple right A-module, $\Phi(\mathrm{Ker}\,\psi) = 0$ and so ψ is an isomorphism. The rest of the proof is an easy verification. $\square$

Let $W_\lambda^\frown$ denote the set of isomorphism classes of irreducible representations of W_λ. For $\chi \in W_\lambda^\frown$, fix a module V_χ in the class of χ and set $m_\chi = \dim V_\chi$.

Lemma 2.3. *Let* $\chi \in W_\lambda^\frown$.

(i) As an A-module, $(A * W_\lambda) \otimes_A Re^\lambda$ is a direct sum of $|W_\lambda|$ copies of the simple module Re^λ. Moreover, $Re^\lambda \cong R$ as W_λ-modules.

(ii) Fix a left transversal $\{h_1 = \epsilon, h_2, \dots, h_s\}$ of W_λ in W. Then $Re^{h_i.\lambda} \cong Re^{h_j.\lambda}$ as A-modules if and only if $h_i = h_j$.

Proof. (i) As left A-modules, $(A * W_\lambda) \otimes_A Re^\lambda = \bigoplus_{g \in W_\lambda} A(g \otimes e^\lambda)$. If $v \in V \subset S = S(V)$ and $g \in W_\lambda$, then $\lambda(v^g) = (g.\lambda)(v) = \lambda(v)$. Thus,

$$v(g \otimes e^\lambda) = g \otimes v^g \cdot e^\lambda = g \otimes \lambda(v^g) e^\lambda = g \otimes v \cdot e^\lambda.$$

Therefore, $A(g \otimes e^\lambda) \cong Ae^\lambda = Re^\lambda$. The final assertion follows immediately from the fact that W_λ acts trivially on e^λ.

(ii) It suffices to recall that every element of $Re^{h_i\cdot\lambda}$ is killed by a power of the maximal ideal $\mathbf{m}_{h_i\cdot\lambda}$. □

By combining the previous results we obtain the following minor generalization of [19, Theorem 2.7].

Proposition 2.4. *As a left $A * W_\lambda$-module,*

$$(A * W_\lambda) \otimes_A Re^\lambda \;\cong\; \bigoplus_{\chi \in W_\lambda^\wedge} \left(Re^\lambda \otimes_{\Bbbk_\lambda} V_\chi\right)^{(m_\chi)}.$$

Moreover, the modules $Re^\lambda \otimes_{\Bbbk_\lambda} V_\chi$ are simple and pairwise non-isomorphic.

Proof. Let $\Bbbk[W_\lambda]$ denote the group ring of W_λ. Under the natural embedding, we have $\Bbbk[W_\lambda] \otimes_{\Bbbk} Re^\lambda = (A * W_\lambda) \otimes_A Re^\lambda$. Hence,

$$\Phi\big((A * W_\lambda) \otimes_A Re^\lambda\big) = \Bbbk[W_\lambda] \otimes e^\lambda \cong \Bbbk[W_\lambda],$$

thought of as a W_λ-module under the left regular representation. Since $\Bbbk[W_\lambda] \cong \bigoplus V_\chi^{(m_\chi)}$, the result follows from the equivalence of categories given by Proposition 2.2. □

We can now state the main result of this section.

Theorem 2.5. *Let $\lambda \in V^*$ and set $Z_\chi = (A * W) \otimes_{A*W_\lambda} (Re^\lambda \otimes_{\Bbbk_\lambda} V_\chi)$. Then, as an $A * W$-module,*

$$(A * W) \otimes_A Re^\lambda \cong \bigoplus_{\chi \in W_\lambda^\wedge} Z_\chi^{(m_\chi)}.$$

The Z_χ, for $\chi \in W_\lambda^\wedge$, are simple, non-isomorphic $A * W$-modules.

Proof. Note that $(A * W) \otimes_A Re^\lambda \cong (A * W) \otimes_{A*W_\lambda} (A * W_\lambda) \otimes_A Re^\lambda$ as an $A * W$-module. Thus, the displayed equation is an immediate consequence of Proposition 2.4.

Let $\{h_1 = \epsilon, h_2, \ldots, h_s\}$ be a left transversal to W_λ in W and set $Y_\chi = Re^\lambda \otimes_{\Bbbk_\lambda} V_\chi$. As A-modules, $Z_\chi \cong \bigoplus_{j=1}^s h_j Y_\chi$, and $h_i Y_\chi \cong (Re^{h_i\cdot\lambda})^{(m_\chi)}$ (use Lemma 2.3). Moreover, the A-modules $Re^{h_i\cdot\lambda}$ and $Re^{h_j\cdot\lambda}$ are simple and non-isomorphic for $i \neq j$. Hence any simple A-submodule of Z_χ is contained in some $h_j Y_\chi$.

We next show that Z_χ is a simple $A * W$-module. Since Z_χ is a direct summand of $(A * W) \otimes_A Re^\lambda$, it is semi-simple, by Lemma 2.1(iii). Thus, we may pick a simple $A * W$-summand M such that the projection from M to $h_1 Y_\chi$ is non-zero. It follows from the last paragraph that, as an A-module, $M = \bigoplus_i (M \cap h_i Y_\chi)$. Thus $M \cap h_1 Y_\chi \neq (0)$. But, Proposition 2.4 implies that $Y_\chi = h_1 Y_\chi$ is a simple $A * W_\lambda$-submodule of Z_χ, whence $h_1 Y_\chi \subseteq M$. Therefore

$$M = (A * W)M \supseteq (A * W)Y_\chi = \bigoplus_i h_i Y_\chi = Z_\chi.$$

Thus, Z_χ is indeed simple.

Finally, suppose that there exists an isomorphism $\theta : Z_\chi \to Z_\psi$ for some $\chi \neq \psi \in W_\lambda^\wedge$. If we regard θ as a homomorphism of A-modules then, as in the second paragraph, Lemma 2.3(ii) implies that θ maps $h_i Y_\chi$ to $h_i Y_\psi$. In particular, for $i = 1$, we find that θ restricts to a non-zero $A * W_\lambda$-module homomorphism $\theta' : Y_\chi \to Y_\psi$. Since Y_χ is a simple $A * W_\lambda$-module, θ' is an isomorphism and, by Proposition 2.4, $\chi = \psi$. □

3. The structure of $\mathcal{O}(V)e^{\lambda}$ as a $\mathcal{D}(V)^W$-module

As in the last section, we fix an ℓ-dimensional $\Bbbk$-vector space V and set $R = \mathcal{O}(V)$ and $A = \mathcal{D}(V)$, etc. In this section, we use the results of the last section, together with the Morita equivalence between $A * W$ and A^W to give a precise description of the structure of Re^{λ} as an A^W-module. In particular, in Theorem 3.4 we prove parts (i) and (ii) of Theorem A of the introduction.

We begin with some simple observations. Regard A as an $(A*W, A^W)$-bimodule in the natural way: $(\sum_{w_i\in W} a_i w_i) \circ r \circ b = \sum a_i (w_i.r) b$, for $\sum a_i w_i \in A*W$, $r \in A$ and $b \in A^W$. Call this module A^l. We may also regard A as an $(A^W, A*W)$-bimodule by $a \circ r \circ \sum_{w_i \in W} w_i b_i = \sum a(r^{w_i}) b_i$, for $a \in A^W$, $r \in A$ and $\sum w_i b_i \in A*W$, in which case we write the module as A^r. Of course, it will usually be clear which bimodule structure is intended, in which case the superscript will be ignored.

The basic facts that we need about the Morita equivalence are collected in the next lemma.

Lemma 3.1. (i) A^W and $A*W$ are simple, Morita equivalent rings. The equivalence is obtained by the functors:

$$I(N) = A^l \otimes_{A^W} N \qquad \text{and} \qquad F(M) = A^r \otimes_{A*W} (M),$$

for $N \in A^W$*-mod*, respectively $M \in A*W$*-mod*.

(ii) If $M \in A*W$*-mod*, then $F(M) \cong \operatorname{Hom}_{A*W}(A^l, M) \cong M^W$.

(iii) As $A*W$-bimodules, $A \otimes_{A^W} A \cong A*W$.

Proof. (i) Note that $\operatorname{Hom}_{A*W}(A^r, A*W) \cong A^l$, as $(A*W, A^W)$-bimodules, and similarly for A^l (see [15, Proposition 7.8.5]). Thus, the result follows from [16, Theorem 2.5 and Corollary 2.6].

(ii) The first isomorphism is standard, while the second isomorphism, follows from the fact that each $f \in \operatorname{Hom}_{A*W}(A, M)$ is determined by $f(1) \in M^W$.

(iii) This is just the assertion that $IF(A*W) \cong A*W$. $\square$

Remark 3.2. *The above results still apply if one replaces W by W_λ. In this situation we denote the equivalences by*

$$F_\lambda = A \otimes_{A*W_\lambda} \_\_ \qquad \textit{and} \qquad I_\lambda = A \otimes_{A^{W_\lambda}} \_\_.$$

Note that for all $a \in A$ and $j \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $\mathbf{m}_\lambda^k a \subset A\mathbf{m}_\lambda^j$. Similarly, there exists $i \in \mathbb{N}$ such that $\mathbf{m}_\lambda^i \subseteq \mathbf{n}_\lambda S$. It follows easily that, if $M \in \mathcal{C}_\lambda$ then $I_\lambda(M) \in \mathcal{S}_\lambda$, using the notation of Remark 3.2. Conversely if $N \in \mathcal{S}_\lambda$ then $F_\lambda(N)$ is obviously in $\mathcal{C}_\lambda$. Combined with Proposition 2.2 this implies:

Lemma 3.3. *The categories $\mathcal{S}_\lambda$ and $\mathcal{C}_\lambda$ are equivalent via the Morita equivalence of* Lemma 3.1. Moreover $\mathcal{C}_\lambda \approx W_\lambda$-mod via the functor ΦI_λ.

Recall that V_χ is defined to be a simple W_λ-module of type χ. Define

$$V^\chi = \operatorname{Hom}_{W_\lambda}(V_\chi^*, Re^\lambda). \tag{3.1}$$

Thus, $V^\chi \cong \left(Re^\lambda \otimes_{\Bbbk_\lambda} V_\chi\right)^{W_\lambda}$. Since the A^{W_λ}- and W_λ-module structures on Re^λ commute, the action of A^{W_λ} on Re^λ induces a left A^{W_λ}-module structure on V^χ. Note that $V^\chi \otimes V_\chi^*$ is isomorphic with the isotypic component of type χ^* in the W_λ-module Re^λ.

We can now complete the description of Re^λ as an A^W-module.

Theorem 3.4. (i) The (A^W, W_λ)-module Re^λ decomposes as:

$$Re^\lambda \cong \bigoplus_{\chi \in \widehat{W_\lambda}} V^\chi \otimes V_\chi^*. \tag{3.2}$$

The A^W-modules V^χ, for $\chi \in \widehat{W_\lambda}$, are simple and pairwise non-isomorphic.

(ii) If $\chi = \mathrm{triv}$ is the trivial character, then $V^{\mathrm{triv}} = A^W \cdot e^\lambda = R^{W_\lambda} e^\lambda$ is a simple A^W-module.

Remark 3.5. (1) By their construction, the V^χ are A^{W_λ}-modules. Therefore, the decomposition of part *(i)* of the theorem is also a decomposition of Re^λ into a direct sum of simple A^{W_λ}-modules.

(2) When $\lambda = 0$, one has $W_\lambda = W$ and $\mathcal{C}_0 \approx W$-mod. When λ is *regular*, that is, $W_\lambda = \{1\}$, Re^λ is a simple A^W-module (compare with [4, Lemmes 9 & 11]).

Proof. Note that one has a natural isomorphism of $(A^{W_\lambda}, W_\lambda)$-modules

$$\bigoplus_{\chi \in \widehat{W_\lambda}} V^\chi \otimes V_\chi^* \xrightarrow{\sim} Re^\lambda$$

given by $f \otimes v \mapsto f(v)$, for $f \in V^\chi$ and $v \in V_\chi^*$. Thus (3.2) holds. In order to prove the rest of the theorem, we wish to reduce the result to that given by Theorem 2.5. As in §2, set $Y_\chi = Re^\lambda \otimes_{\Bbbk_\lambda} V_\chi$ and $Z_\chi = (A * W) \otimes_{A * W_\lambda} Y_\chi$. Write $U_\chi = A \otimes_{A * W_\lambda} Y_\chi$, regarded as an A^W-module and note that $U_\chi = F(Z_\chi) = F_\lambda(Y_\chi)$. Clearly, $A \cong A \otimes_{A*W} (A * W)$, as an (A^W, A)-bimodule. Thus, by Theorem 2.5, one obtains the following isomorphism of A^W-modules:

$$Re^\lambda \cong A \otimes_{A*W} (A * W) \otimes_A Re^\lambda \cong \bigoplus_\chi (A \otimes_{A*W} Z_\chi)^{(m_\chi)} = \bigoplus_\chi U_\chi^{(m_\chi)}.$$

By Morita equivalence and Theorem 2.5, again, the A^W-modules U_χ are simple and pairwise non-isomorphic. Thus, in order to prove the theorem, it suffices to show that $U_\chi \cong V^\chi$ for each χ and that $V^{\mathrm{triv}} = R^{W_\lambda} e^\lambda$.

Since $U_\chi = F_\lambda(Y_\chi)$, it follows from Lemma 3.1(ii) that $U_\chi \cong (Y_\chi)^{W_\lambda}$ as A^{W_λ}-modules and therefore as A^W-modules. Thus,

$$U_\chi = (Re^\lambda \otimes_{\Bbbk_\lambda} V_\chi)^{W_\lambda} = \mathrm{Hom}_{W_\lambda}(V_{\mathrm{triv}}, Re^\lambda \otimes_{\Bbbk_\lambda} V_\chi) = \mathrm{Hom}_{W_\lambda}(V_\chi^*, Re^\lambda) = V^\chi.$$

Finally, when $\chi = \mathrm{triv}$, we have $V^{\mathrm{triv}} = \mathrm{Hom}_{W_\lambda}(V_{\mathrm{triv}}, Re^\lambda) = R^{W_\lambda} e^\lambda$. Since this A^W-module is simple and contains $A^W \cdot e^\lambda$, part (ii) follows. ☐

Corollary 3.6. *Let $\lambda, \mu \in V^*$. Then the following are equivalent:*

(i) $Re^\lambda \cong Re^\mu$ as A^W-modules

(ii) Re^λ and Re^μ have a simple A^W-submodule in common

(iii) $\mu \in W.\lambda$

Proof. (iii) ⇒ (i) Assume that $\mu = w.\lambda$ for some $w \in W$. We obviously have an isomorphism of vector spaces $\tilde{w} : Re^\lambda \to Re^\mu$, given by $fe^\lambda \mapsto w.(fe^\lambda) = (w.f)e^{w.\lambda}$. By definition $w.(d \cdot fe^\lambda) = (w.d) \cdot (w.(fe^\lambda))$ for all $d \in A$. Hence the map $\tilde{w}$ is an isomorphism of A^W-modules.

(i) ⇒ (ii) is trivial.

(ii) ⇒ (iii) Generalizing the previous notation we set $W = \bigsqcup_i h_i(\lambda) W_\lambda$ with $h_1(\lambda) = \epsilon$ and write

$$(A * W) \otimes_{A*W_\lambda} Re^\lambda = \bigoplus_{\chi \in \widehat{W_\lambda}} (A * W) \otimes_{A*W_\lambda} Z_\chi(\lambda),$$

etc. Assume that the A^W-modules Re^λ and Re^μ have a simple A^W-module in common. Then there there is an $A*W$-isomorphism

$$\sigma : (A*W)\otimes_{A*W_\lambda} Z_\chi(\lambda) \to (A*W)\otimes_{A*W_\mu} Z_\psi(\mu)$$

for some $\chi \in W_\lambda^\frown$ and $\psi \in W_\mu^\frown$. As in the proof of Theorem 2.5, any simple A-submodule of $(A*W)\otimes_{A*W_\lambda} Z_\chi(\lambda)$ is of the form $Re^{h_i(\lambda).\lambda}$, for some i. Thus, there exist $h_i(\lambda)$, $h_j(\mu)$ such that $Re^{h_i(\lambda).\lambda} \cong Re^{h_j(\mu).\mu}$ as A-modules; this implies that $h_i(\lambda).\lambda = h_j(\mu).\mu$. Thus, $\mu \in W.\lambda$. □

4. The case of a group generated by pseudo-reflections

We continue with the notation of the last two sections. The A^W-module Re^λ clearly has finite length and so, by [1, Theorem 1.8.18] it is cyclic. In this section we will give a presentation $Re^\lambda \cong A^W/L$ in the case when W is generated by pseudo-reflections. It is this presentation that will allow us to relate Re^λ to invariant holonomic systems, as described in the introduction to the paper.

We begin with some general results.

Lemma 4.1. *Let C be a subalgebra of the $\Bbbk$-algebra B. Let $B/J = \bigoplus_{i=1}^t S_i^{n_i}$ be a cyclic, semi-simple B-module. Assume that the C-modules S_i are simple and non-isomorphic, and that $\operatorname{End}_C(S_i) = \operatorname{End}_B(S_i) = \Bbbk$. Then $B/J = (C+J)/J$.*

Proof. The hypotheses imply that $\operatorname{End}_C(B/J) = \operatorname{End}_B(B/J)$ and so any decomposition of B/J as a C-module is also a B-module decomposition. But, as a C-module, $B/J = N\bigoplus(C+J)/J$, for some module N. Therefore, this is a B-module decomposition with its generator $[1+J]$ in the second factor. Thus $B/J = (C+J)/J$. □

Lemma 4.2. *Let E be a finite dimensional $S*W_\lambda$-module such that $\mathbf{m}_\lambda^j E = (0)$ for some $j \in \mathbb{N}$. Then $\operatorname{Hom}_A(Re^\lambda, A\otimes_S E) \cong E$, as W_λ-modules.*

Proof. Set $E_i = \mathbf{m}_\lambda^i E/\mathbf{m}_\lambda^{i+1}E$, and note that each E_i is a subfactor of E as an $S*W_\lambda$-module and a summand of E as a W_λ-module. Since the elements of S act ad-nilpotently on A, clearly $A\otimes_S E \in \mathcal{S}_\lambda$. Therefore, by Lemma 2.1, $A\otimes_S E$ is semi-simple and hence $A\otimes_S E = \bigoplus_i A\otimes_S E_i$. Thus, it suffices to prove the result with E replaced by some E_i. In this case, $A\otimes_S E_i \cong A\otimes_{\Bbbk_\lambda} E_i$ and the lemma follows from Proposition 2.2 and the definition of Φ. □

For the rest of this section we will be concerned with the case when the group $W \subset \mathrm{GL}(V)$ is generated by pseudo-reflections. Recall that this has the following consequences, see [3, Chapitre V §5]: S is a free S^W-module of rank $|W|$ and the subgroup W_λ is also generated by pseudo-reflections. Also, there exists a W_λ-stable subspace $\mathcal{K} \subset S$ such that $S = \mathcal{K}\otimes_\Bbbk S^{W_\lambda}$. As a W_λ-module, $\mathcal{K} \cong \Bbbk[W_\lambda]$, the regular representation.

Corollary 4.3. *Assume that W is generated by pseudo-reflections. Then, $Re^\lambda \cong A^{W_\lambda}/A^{W_\lambda}\mathbf{n}_\lambda$, as A^{W_λ}-modules.*

Proof. We first note that $\Phi I_\lambda(Re^\lambda) = \Bbbk[W_\lambda]$ under the left regular representation of W_λ. Indeed, by definition and Lemma 3.1(iii), $I_\lambda(Re^\lambda) = (A*W_\lambda)\otimes_A Re^\lambda$, while, by the proof of Proposition 2.4, $\Phi(A*W_\lambda \otimes_A Re^\lambda) = \Bbbk[W_\lambda]$, as required.

Note that $M = A^{W_\lambda}/A^{W_\lambda}\mathbf{n}_\lambda \in \mathcal{C}_\lambda$. Therefore by the last paragraph and Lemma 3.3, it is enough to show that $\Phi I_\lambda(M) = \Bbbk[W_\lambda]$. However, as W_λ-modules,

$$\begin{aligned}\Phi I_\lambda(M) &= \operatorname{Hom}_A(Re^\lambda, A \otimes_{A^{W_\lambda}} M) = \operatorname{Hom}_A(Re^\lambda, A/A\mathbf{n}_\lambda)\\ &= \operatorname{Hom}_A(Re^\lambda, A \otimes_S S/S\mathbf{n}_\lambda) = S/S\mathbf{n}_\lambda,\end{aligned}$$

where the final equality follows from Lemma 4.2. Since W_λ is generated by pseudo-reflections,

$$S/S\mathbf{n}_\lambda = \mathcal{K} \otimes_\Bbbk (S^{W_\lambda}/\mathbf{n}_\lambda) \cong \Bbbk[W_\lambda]$$

as W_λ-modules. Hence the result. □

Theorem 4.4. *Assume that W is generated by pseudo-reflections and set $\mathbf{p}_\lambda = S^W \cap \mathbf{m}_\lambda$. Then $Re^\lambda \cong A^W/A^W\mathbf{p}_\lambda$ as A^W-modules.*

Proof. By Theorem 3.4, the hypotheses of Lemma 4.1 are satisfied by $C = A^W \subseteq B = A^{W_\lambda}$ and $B/J = Re^\lambda$. Thus, Corollary 4.3 and Lemma 4.1 imply that $Re^\lambda \cong A^W/(A^W \cap A^{W_\lambda}\mathbf{n}_\lambda)$. Consequently, there exists a surjective morphism of A^W-modules $\phi : A^W/A^W\mathbf{p}_\lambda \twoheadrightarrow Re^\lambda$.

Using Lemma 3.1(iii), this induces a surjection of $A * W$-modules

$$\phi' : A \otimes_{A^W} (A^W/A^W\mathbf{p}_\lambda) \twoheadrightarrow A \otimes_{A^W} Re^\lambda \cong (A * W) \otimes_A Re^\lambda.$$

Thus, by Morita equivalence, it suffices to prove that ϕ' is an isomorphism and, to do so, we need only to consider the A-module structure. By Lemma 2.3, $(A * W) \otimes_A Re^\lambda$ has length $|W|$. On the other hand $A \otimes_{A^W} (A^W/A^W\mathbf{p}_\lambda) \cong A \otimes_S (S/S\mathbf{p}_\lambda)$. Since W is generated by pseudo-reflections, $S/S\mathbf{p}_\lambda$ is a $\Bbbk$-vector space of dimension $|W|$ and hence is an S-module of length $|W|$, with each simple factor isomorphic to $\Bbbk_\lambda$. Thus, the A-module $A \otimes_{A^W} (A^W/A^W\mathbf{p}_\lambda)$ has length $|W|$, with each factor isomorphic to Re^λ. Therefore both ϕ' and ϕ are isomorphisms. □

5. On the equivalence $\mathcal{C}_\lambda \approx W_\lambda$-mod

By Lemma 3.3 the category $\mathcal{C}_\lambda$ is equivalent to W_λ-mod via the functor $\Phi I_\lambda = \operatorname{Hom}_A(Re^\lambda, A \otimes_{A^{W_\lambda}} \underline{\quad})$, with inverse functor $F_\lambda \Psi$. We study these functors in more depth in this section. Note that, by Remark 3.5, $\mathcal{C}_\lambda$ is also the full subcategory of A^W-modules generated by Re^λ and it makes little difference in this section whether we work with A^W-modules or A^{W_λ}-modules.

Note that ΦI_λ satisfies (and hence is defined by) $\Phi I_\lambda(V^\chi) = V_\chi$ for all χ. This permits one to give an alternative interpretation of this functor. Define functors from $\mathcal{C}_\lambda$ to W_λ-mod by

$$\operatorname{Sol}(M) = \operatorname{Hom}_{A^{W_\lambda}}(M, Re^\lambda) = \operatorname{Hom}_{A^W}(M, Re^\lambda),$$

and

$$\operatorname{DR}(M) = \operatorname{Hom}_{A^{W_\lambda}}(Re^\lambda, M) = \operatorname{Hom}_{A^W}(Re^\lambda, M),$$

By analogy with [2, VIII.13] we call these functors the *solution functor,* respectively the *de Rham functor*. Since Re^λ is an $(A^{W_\lambda}, W_\lambda)$-module, their images do indeed lie in W_λ-mod.

Proposition 5.1. (i) $\mathrm{DR} = \Phi I_\lambda$ provides an equivalence of categories from $\mathcal{C}_\lambda$ to W_λ-mod. Similarly, Sol gives an equivalence of categories between $\mathcal{C}_\lambda$ and $(W_\lambda\text{-mod})^{\mathrm{op}}$.

(ii) $\operatorname{Sol}(Re^\lambda) = \operatorname{DR}(Re^\lambda) = \Bbbk[W_\lambda]$ as $(W_\lambda \times W_\lambda)$-module.

Proof. (i) It follows from Theorem 3.4 that $\mathrm{DR}(V^{\chi}) = V_{\chi}$ and that $\mathrm{Sol}(V^{\chi}) = (V_{\chi})^*$. Thus, the result follows from the earlier observations.

(ii) By Theorem 3.4, there is a natural isomorphism:

$$\mathrm{End}_{A^{W_\lambda}}(Re^{\lambda}) \cong \bigoplus_{\psi,\chi \in W_\lambda^{\wedge}} \mathrm{Hom}_{A^{W_\lambda}}(V^{\psi}, V^{\chi}) \otimes (V_{\psi} \otimes V_{\chi}^*) = \bigoplus_{\psi \in W_\lambda^{\wedge}} V_{\psi} \otimes V_{\psi}^* \cong \Bbbk[W_\lambda].$$

Since the A^{W_λ} and W_λ actions on Re^{λ} commute, this is an isomorphism of $(W_\lambda \times W_\lambda)$-modules. □

Now, assume that W is generated by pseudo-reflections and adopt the notation of §4. By Theorem 4.4, $\mathrm{Sol}(Re^{\lambda}) = \mathrm{Hom}_{A^W}(A^W/A^W\mathbf{p}_\lambda, Re^{\lambda})$ and so it can be identified with:

$$\mathcal{H}_\lambda = \{f \in Re^{\lambda} \;:\; \mathbf{p}_\lambda \cdot f = 0\} = \left\{f \in Re^{\lambda} \;:\; p\cdot f = p(\lambda)f, \text{ for all } p \in S^W\right\}.$$

By Proposition 5.1, $\mathcal{H}_\lambda \cong \Bbbk[W_\lambda]$ as a W_λ-module. If $M \in \mathcal{C}_\lambda$ is a quotient of Re^{λ}, then $\mathrm{Sol}(M)$ is a submodule of $\mathrm{Sol}(Re^{\lambda})$ and can therefore be identified with a W_λ-submodule of $\mathcal{H}_\lambda$.

We want to relate the subspace $\mathcal{H}_\lambda$ with a subspace of W_λ-harmonic elements in R. Let $V(\lambda) \subset \operatorname{Ker}\lambda$ be the unique W_λ-stable complement of the subspace of invariants $V^{W_\lambda} \subseteq V$. We will identify $V(\lambda)^*$ with the W_λ-submodule $\left(V/V^{W_\lambda}\right)^*$ of V^*. Note that $S(V(\lambda)^*)e^{\lambda} \cong S(V(\lambda)^*)$ as W_λ-modules. Define

$$\mathcal{H}(\lambda) = \left\{f \in S(V(\lambda)^*) \;:\; p\cdot f = p(0)f, \text{ for all } p \in S(V(\lambda))^{W_\lambda}\right\} \subseteq S(V(\lambda)^*).$$

Replacing V by $V(\lambda)$ in the earlier argument, it follows that $\mathcal{H}(\lambda) \cong \Bbbk[W_\lambda]$, as a W_λ-module.

Lemma 5.2. *If W is generated by pseudo-reflections, then $\mathcal{H}_\lambda = \mathcal{H}(\lambda)e^{\lambda}$.*

Proof. Since $\mathcal{H}(\lambda)e^{\lambda}$ and $\mathcal{H}_\lambda$ are subspaces of Re^{λ} of the same dimension, $|W_\lambda|$, it suffices to show that $\mathcal{H}_\lambda \subseteq \mathcal{H}(\lambda)e^{\lambda}$. Note that that $q\cdot(\varphi e^{\lambda}) = (q\cdot\varphi)e^{\lambda}$, for all $q \in S(V(\lambda))$ and $\varphi \in R$. Similarly, $r\cdot(fe^{\lambda}) = r(\lambda)fe^{\lambda}$, for all $r \in S(V^{W_\lambda})$ and $f \in S(V(\lambda)^*)$. Let $f \in \mathcal{H}(\lambda)$ and $p \in S^W$. Write $p = \sum_j r_j q_j$, for $r_j \in S(V^{W_\lambda})$, and $q_j \in S(V(\lambda))^{W_\lambda}$. Observe that $p(\lambda) = \sum_j r_j(\lambda)q_j(0)$.

Now, if $p \in \mathbf{p}_\lambda$, equivalently $p(\lambda) = 0$, then these observations show that

$$\begin{aligned} p\cdot(fe^{\lambda}) &= \sum_j r_j\cdot(q_j\cdot(fe^{\lambda})) = \sum_j r_j\cdot((q_j\cdot f)e^{\lambda}) \\ &= \sum_j r_j\cdot(q_j(0)fe^{\lambda}) \quad (\text{since } f\in\mathcal{H}(\lambda)) \\ &= \sum_j q_j(0)r_j(\lambda)fe^{\lambda} = p(\lambda)fe^{\lambda} = 0. \end{aligned}$$

Thus $fe^{\lambda} \in \mathcal{H}_\lambda$. □

This lemma gives us another way of defining the equivalence $\mathcal{C}_\lambda \approx W_\lambda$-mod:

Corollary 5.3. *Assume that W is generated by pseudo-reflections If M is a quotient of Re^{λ}, define*

$$\mathrm{Sol}(M)e^{-\lambda} = \{f \in \mathcal{H}(\lambda) \;:\; fe^{\lambda} \in \mathrm{Sol}(M)\}.$$

Then the functor $M \to \mathrm{Sol}(M)e^{-\lambda}$ induces an equivalence of categories between $\mathcal{C}_\lambda$ and $(W_\lambda\text{-mod})^{\mathrm{op}}$.

6. Invariant holonomic systems

Let G be a connected reductive complex algebraic group with Lie algebra $\mathfrak{g}$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and denote by W the associated Weyl group. Let $\tau : \mathfrak{g} \to \mathcal{D}(\mathfrak{g})$ be the differential of the adjoint action. Set $\ell = \operatorname{rk} \mathfrak{g}$ and $n = \dim \mathfrak{g}$. Throughout the section, we fix $\lambda \in \mathfrak{h}^*$. Following [10] we define the system of linear differential equations for invariant eigendistributions, also called the "invariant holonomic system on $\mathfrak{g}$", by

$$\mathcal{N}_\lambda = \mathcal{D}(\mathfrak{g}) / \big(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \textstyle\sum_{p \in S(\mathfrak{g})^G} \mathcal{D}(\mathfrak{g})(p - p(\lambda))\big). \tag{6.1}$$

It is not difficult to see that $\mathcal{N}_\lambda$ is an holonomic $\mathcal{D}(\mathfrak{g})$-module, and therefore has finite length. The aim of this section is to give a complete description of the structure of $\mathcal{N}_\lambda$ (see Theorem 6.9), thereby proving Theorem B from the introduction. This generalizes the results of [10] which did this in the cases when $\lambda = 0$ and when λ is regular. The results of the previous sections apply here with $V = \mathfrak{h}$. We first need to relate $\mathcal{N}_\lambda$ to the $\mathcal{D}(\mathfrak{h})^W$-module $\mathcal{O}(\mathfrak{h})e^\lambda$.

In [6], Harish-Chandra defines a homomorphism $\delta : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h})^W$ such that the restriction of δ to $\mathcal{O}(\mathfrak{g})^G$ and to $S(\mathfrak{g})^G$ are just the Chevalley isomorphisms $\delta : \mathcal{O}(\mathfrak{g})^G \to \mathcal{O}(\mathfrak{h})^W$ and $\delta : S(\mathfrak{g})^G \to S(\mathfrak{h})^W$. Moreover, by [13] and [14], δ induces an isomorphism $\mathcal{D}(\mathfrak{h})^W \cong \mathcal{D}(\mathfrak{g})^G/\mathcal{I}$, where $\mathcal{I} = \mathcal{D}(\mathfrak{g})^G \cap \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$. In particular, since $\tau(\mathfrak{g})$ commutes element-wise with $\mathcal{D}(\mathfrak{g})^G$, we obtain a natural right $\mathcal{D}(\mathfrak{h})^W$-module structure on $\mathcal{N} = \mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$.

Recall the following definition from [20, §5]. Denote by Ad the adjoint action of G on $\mathfrak{g}$ or $\mathcal{D}(\mathfrak{g})$. A $\mathcal{D}(\mathfrak{g})$-module M is said to be a *compatible* $(\mathcal{D}(\mathfrak{g}), G)$-module (or G-equivariant $\mathcal{D}_{\mathfrak{g}}$-module as in [2, 12.10]) if there is an action of G on M such that

(i) If $p \in \mathcal{D}(\mathfrak{g})$, $g \in G$, then $g.(px) = (\operatorname{Ad} g(p))\, g.x$ for all $x \in M$;
(ii) As a G-module, M is locally finite;
(iii) The differential of the G-action on M is given via $\tau : \mathfrak{g} \to \mathcal{D}(\mathfrak{g})$.

Lemma 6.1. *Any subquotient of* $\mathcal{N}$ *is a compatible* $(\mathcal{D}(\mathfrak{g}), G)$*-module.*

Proof. Let $P \subseteq L$ be left ideals of $\mathcal{D}(\mathfrak{g})$ such that $\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) \subseteq P$. Observe that, for all $x \in L$ and $\xi \in \mathfrak{g}$, $\tau(\xi)x \equiv \operatorname{ad} \circ \tau(\xi)(x) = [\tau(\xi), x]$ modulo $\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$. Recall that the differential of the G-action on $\mathcal{D}(\mathfrak{g})$ is given by $\operatorname{ad} \circ \tau$; it follows that L/P is a locally finite $\mathfrak{g}$-module via $\operatorname{ad} \circ \tau$. Therefore, since G is connected, L/P is a locally finite G-module. It is then clear that L/P is a compatible $(\mathcal{D}(\mathfrak{g}), G)$-module. □

Set $\mathbf{m}_\lambda = \operatorname{Ker} \lambda \subset S(\mathfrak{h})$ and $\mathbf{p}_\lambda = \mathbf{m}_\lambda \cap S(\mathfrak{h})^W$. By Theorem 4.4, $\mathcal{O}(\mathfrak{h})e^\lambda$ is a $(\mathcal{D}(\mathfrak{h})^W, W_\lambda)$-module isomorphic, as a $\mathcal{D}(\mathfrak{h})^W$-module, to $\mathcal{D}(\mathfrak{h})^W/\mathcal{D}(\mathfrak{h})^W \mathbf{p}_\lambda$. Since $\delta : S(\mathfrak{g})^G \to S(\mathfrak{h})^W$ is the Chevalley isomorphism,

$$\delta\big(\textstyle\sum_{p \in S(\mathfrak{g})^G} S(\mathfrak{g})^G (p - p(\lambda))\big) = \mathbf{p}_\lambda.$$

Thus, we have proved the first assertion of:

Lemma 6.2. (i) As left $\mathcal{D}(\mathfrak{g})$-modules, $\mathcal{N}_\lambda \cong \mathcal{N} \otimes_{\mathcal{D}(\mathfrak{h})^W} \mathcal{O}(\mathfrak{h})e^\lambda$.

(ii) Let $\{V^\chi = \operatorname{Hom}_{W_\lambda}(V_\chi^*, \mathcal{O}(\mathfrak{h})e^\lambda)\}$ be the $\mathcal{D}(\mathfrak{h})^W$-modules defined in (3.1). Then, there is a decomposition of $(\mathcal{D}(\mathfrak{g}), W_\lambda)$-modules:

$$\mathcal{N}_\lambda = \bigoplus_{\chi \in W_\lambda^\wedge} \mathcal{N}_{\lambda,\chi} \otimes_{\mathbb{C}} V_\chi^*, \qquad \text{where } \mathcal{N}_{\lambda,\chi} = \mathcal{N} \otimes_{\mathcal{D}(\mathfrak{h})^W} V^\chi.$$

Here, $\mathcal{D}(\mathfrak{g})$ acts on $\mathcal{N}_{\lambda,\chi}$ while W_λ acts on V_χ^*. Finally, $(\mathcal{N}_{\lambda,\chi})^G = V^\chi$, for each χ.

Proof. (ii) The decomposition of $\mathcal{N}_\lambda$ follows from Theorem 3.4. The final assertion of the lemma is an immediate consequence of the fact that, as a left or right $\mathcal{D}(\mathfrak{g})^G$-module, $\mathcal{N} \cong \mathcal{D}(\mathfrak{g})^G/\mathcal{I} \oplus \big(\oplus_{0\neq\nu\in G^\wedge}\mathcal{N}[\nu]\big)$, where $\mathcal{N}[\nu]$ denotes the sum of finite dimensional irreducible G-modules of type ν in $\mathcal{N}$. □

This result raises the question of whether the $\mathcal{N}_{\lambda,\chi}$ are simple, pairwise non-isomorphic $\mathcal{D}(\mathfrak{g})$-modules. That this is true will form the main result, Theorem 6.9, of this section. This will follow easily provided we can prove that each $\mathcal{D}(\mathfrak{g})$-submodule of $\mathcal{N}_\lambda$ contains non-zero G-invariant elements.

In order to simplify the notation, we will write $\mathcal{D} = \mathcal{D}(\mathfrak{g})$, $A = \mathcal{D}(\mathfrak{h})$ and $B = \mathcal{N} \otimes_{A^W} A$, regarded as a $(\mathcal{D}, A)$-bimodule. Let d denote the discriminant of $\mathfrak{g}$. Thus, $d \in \mathcal{O}(\mathfrak{g})^G$ and under the Chevalley isomorphism $\mathcal{O}(\mathfrak{g})^G \to \mathcal{O}(\mathfrak{h})^W$, d maps to $\widetilde{d} = \pi^2$, where π is the product of the positive roots of $\mathfrak{h}$ in $\mathfrak{g}$. Since $d \in \mathcal{O}(\mathfrak{g})^G$, it acts locally ad-nilpotently on $\mathcal{D}$ and so $\mathcal{C} = \{d^m : m \in \mathbb{N}\}$ is an Ore set in both $\mathcal{D}^G$ and $\mathcal{D}$. Similarly, $\widetilde{\mathcal{C}} = \{\widetilde{d^m}\}$ is an Ore set in both A^W and in A. Recall that $\mathfrak{g}' = \{x \in \mathfrak{g} : d(x) \neq 0\}$ is the set of generic elements. Thus, $\mathcal{O}(\mathfrak{g}') = \mathcal{O}(\mathfrak{g})_{\mathcal{C}}$ and $\mathcal{D}(\mathfrak{g}') = \mathcal{D}_{\mathcal{C}}$.

The following identifications will prove useful:

Lemma 6.3. (i) As left $\mathcal{D}$-modules, $\mathcal{N}_\lambda \cong B \otimes_A A/A\mathbf{m}_\lambda$.

(ii) There is a natural isomorphism of localized modules:

$$\mathcal{D}_{\mathcal{C}} \otimes_{\mathcal{D}} \mathcal{N}_\lambda \cong B \otimes_A \big(A/A\mathbf{m}_\lambda\big)_{\widetilde{\mathcal{C}}}$$

Proof. (i) As remarked in §2, $\mathcal{O}(\mathfrak{h})e^\lambda \cong A/A\mathbf{m}_\lambda$ as A^W-modules, via the natural A^W-module structure on A. Thus, part (i) follows from Lemma 6.2(i).

(ii) Since d acts locally ad-nilpotently on $\mathcal{D}$, given any $f \in \mathcal{D}$, there exists $m \in \mathbb{N}$ and $f_1 \in \mathcal{D}$ such that $fd^m = df_1$. Hence, $d^{-1}f = f_1 d^{-m}$, whenever the terms make sense. By the usual universality arguments, this implies that there is a natural isomorphism:

$$\mathcal{D}_{\mathcal{C}} \otimes_{\mathcal{D}} \mathcal{N} \cong \mathcal{N} \otimes_{\mathcal{D}^G} \mathcal{D}^G_{\mathcal{C}} \tag{6.2}$$

Under the identification of $\mathcal{D}^G/\mathcal{I}$ with A^W, d identifies with $\widetilde{d}$ and so $\mathcal{C}$ can be identified with $\widetilde{\mathcal{C}}$. Thus, combined with (6.2), part (i) implies that

$$\mathcal{D}_{\mathcal{C}} \otimes_{\mathcal{D}} \mathcal{N}_\lambda \cong \mathcal{N} \otimes_{A^W} A^W_{\widetilde{\mathcal{C}}} \otimes_{A^W} A/A\mathbf{m}_\lambda \cong B \otimes_A \big(A/A\mathbf{m}_\lambda\big)_{\widetilde{\mathcal{C}}}$$

as desired. □

One consequence of Lemma 6.3 is that localizing with respect to $\mathcal{C}$ or $\widetilde{\mathcal{C}}$ is always going to have the same result and so, *from now on we will identify $\widetilde{d}$ with d and hence identify $\widetilde{\mathcal{C}}$ with $\mathcal{C}$.*

We will need the following general result:

Lemma 6.4. *Let $C = \Bbbk[y_1, \dots, y_\ell]$ be a commutative polynomial ring over an algebraically closed field $\Bbbk$ of characteristic zero and set $A_\ell = \mathcal{D}(C)$ (the ℓ-th Weyl algebra). Let D be a left Noetherian ring and L be a (D, A_ℓ)-bimodule, finitely generated as a left D-module. Then L is a flat right C-module.*

Proof. Use the first two paragraphs of the proof of [14, Theorem 2.4]. □

Proposition 6.5. *Let $\mathbf{m}$ be a maximal ideal of $S(\mathfrak{h})$ and set $P = A/A\mathbf{m}$. Then,* $\mathrm{Tor}_1^A(B, P_{\mathcal{C}}/P) = 0$.

Proof. To begin with, let $\alpha \in \mathfrak{h}^*$ be one positive root. Thus, we may choose a basis $x_1 = \alpha, x_2, \ldots, x_\ell$ of $\mathfrak{h}^*$ and regard $\mathcal{O}(\mathfrak{h})$ as the polynomial ring in the $\{x_i\}$. Write $\partial_i = \frac{\partial}{\partial x_i}$; thus the $\{\partial_i\}$ are generators of $S(\mathfrak{h})$ and $\mathbf{m} = \sum_i S(\mathfrak{h})(\partial_i - \lambda_i)$, for some scalars λ_i. Set $\mathcal{S} = \mathcal{S}_\alpha = \{x_1^m : m \in \mathbb{N}\}$.

We first consider the module $P_\mathcal{S}/P = A_\mathcal{S}/\big(A + \sum_i A_\mathcal{S}(\partial_i - \lambda_i)\big)$. Certainly, this is generated by the images $[x_1^{-m}] \in P_\mathcal{S}/P$ of the x_1^{-m}, for $m \in \mathbb{N}$. However,

$$(\partial_1 - \lambda_1)[x_1^{-m}] = [x_1^{-m}(\partial_1 - \lambda_1) + (-m)x_1^{-m-1}] = [(-m)x_1^{-m-1}],$$

and so, in fact, $P_\mathcal{S}$ is generated by $[x_1^{-1}]$. This element is clearly killed by x_1 and by $(\partial_i - \lambda_i)$ for $2 \le i \le \ell$. Hence, either $P_\mathcal{S}/P = 0$ or $P_\mathcal{S}/P$ is isomorphic to the simple A-module

$$P_\mathcal{S}/P \;\cong\; A/\big(Ax_1 + \textstyle\sum_{i=2}^{\ell} A(\partial_i - \lambda_i)\big). \tag{6.3}$$

Now, consider $P_\mathcal{C}/P$. By [1, Theorem 1.5.5] this is a finitely generated A-module, generated by the elements $[d^{-q}]$, for $q \in \mathbb{N}$. Hence, there exists $m \in \mathbb{N}$, such that $P_\mathcal{C}/P = A[d^{-m}]$. Write $d^{-m} = \prod_{i \in Q} \alpha_i^{-2m}$, for distinct positive roots α_i. By the Chinese Remainder Theorem, $d^{-m} = \sum_i g_i \alpha_i^{-j(i)}$ for some $g_i \in \mathcal{O}(\mathfrak{h})$. Thus, as in the last paragraph,

$$A[d^{-m}] = A\textstyle\sum_i g_i[\alpha_i^{-j(i)}] \;\subseteq\; \sum_i A[\alpha_i^{-j(i)}] = \sum_i A[\alpha_i^{-1}].$$

Therefore, for some subset Q' of Q (in fact, one can prove that $Q' = Q$),

$$P_\mathcal{C}/P = A[d^{-m}] = \bigoplus_{i \in Q'} A[\alpha_i^{-1}] = \bigoplus_{i \in Q'} P_{\mathcal{S}_{\alpha_i}}/P.$$

Thus, in order to prove the proposition, it suffices to prove that

$$\mathrm{Tor}_1^A\big(B, P_{\mathcal{S}_{\alpha_i}}/P\big) = 0 \quad \text{for each } i.$$

We may assume that $\alpha_i = \alpha_1$ and revert to the notation of the first two paragraphs. By (6.3), we may write $P_\mathcal{S}/P = A/\sum_{i=1}^{\ell} Ay_i$, where $y_1 = x_1$ and $y_i = \partial_i - \lambda_i$ for $i > 1$. Moreover, the ring $C = \mathbb{C}[y_1, \ldots, y_\ell]$ is a polynomial ring in the $\{y_j\}$ and $A \cong \mathcal{D}(C)$. Then, by Lemma 6.4, B is a flat right C-module. Thus,

$$\mathrm{Tor}_1^A\big(B, A/\textstyle\sum_i Ay_i\big) \cong \mathrm{Tor}_1^A\big(B, A \otimes_C C/\sum_i Cy_i\big) \cong \mathrm{Tor}_1^C\big(B, C/\sum_i Cy_i\big) \;=\; 0,$$

where the second isomorphism comes from [17, Theorem 11.53]. This completes the proof of the proposition. □

The significance of this result is that it implies:

Corollary 6.6. *The module $\mathcal{N}_\lambda$ is torsion-free as a left $\mathcal{O}(\mathfrak{g})$-module.*

Proof. By Proposition 6.5, $\mathrm{Tor}_1^A(B, P_\mathcal{C}/P) = 0$, where $P = A/A\mathbf{m}_\lambda$. It follows then from Lemma 6.3 that the natural map $\mathcal{N}_\lambda \to (\mathcal{N}_\lambda)_\mathcal{C}$ is injective.

By [14, Lemma 3.1], the characteristic variety $\mathrm{Ch}\,(\mathcal{N}_\lambda)_\mathcal{C}$ of the $\mathcal{D}(\mathfrak{g}')$-module $(\mathcal{N}_\lambda)_\mathcal{C}$ is contained in $\mathcal{C}(\mathfrak{g}) \cap (\mathfrak{g}' \times \mathbf{N})$, where $\mathcal{C}(\mathfrak{g})$ denotes the commuting variety $\{(x, y) \in \mathfrak{g} \times \mathfrak{g} : [x, y] = 0\}$ and $\mathbf{N}$ is the cone of nilpotent elements of $\mathfrak{g}$. Now, for any element $x \in \mathfrak{g}'$, the commutant $\mathfrak{g}^x$ of x is a Cartan subalgebra of $\mathfrak{g}$, and so $\mathfrak{g}^x \cap \mathbf{N} = 0$. Hence, $\mathrm{Ch}\,(\mathcal{N}_\lambda)_\mathcal{C} \subseteq (\mathfrak{g}' \times \{0\})$.

This implies that $(\mathcal{N}_\lambda)_\mathcal{C}$ is an integrable connection on $\mathfrak{g}'$, see [2, VI.1.7 and VII.10.4]. In particular, $(\mathcal{N}_\lambda)_\mathcal{C}$ is a finitely generated, locally free $\mathcal{O}(\mathfrak{g}')$-module. Consequently, $\mathcal{N}_\lambda \hookrightarrow (\mathcal{N}_\lambda)_\mathcal{C}$ is a torsion-free $\mathcal{O}(\mathfrak{g})$-module. □

The fact that $\mathcal{N}_\lambda$ has no $\mathcal{C}$-torsion also follows from the proof of [10, Theorem 6.1]. The corollary provides a nice analogue of a deep result of Harish-Chandra [7, 8] that proves that $\mathcal{N}_\lambda$ has no factor that is $\mathcal{O}(\mathfrak{g})$-torsion.

Let κ be a G-invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$. If $u \in \mathcal{O}(\mathfrak{g})$, denote by $\operatorname{grad}(u) \in \operatorname{Der}\mathcal{O}(\mathfrak{g})$ the vector field defined by $\kappa(\operatorname{grad}(u)_x, y) = \frac{d}{dt}_{|t=0} u(x+ty)$, for $x, y \in \mathfrak{g}$.

Lemma 6.7. *Let $\{u_1, \dots, u_\ell\}$ be algebraically independent generators of $\mathcal{O}(\mathfrak{g})^G$ and set $\theta_i = \operatorname{grad}(u_i)$. Then, each θ_i is G-invariant and*

$$\operatorname{Der}\mathcal{O}(\mathfrak{g}') = \mathcal{O}(\mathfrak{g}')\,\tau(\mathfrak{g}) \oplus \big(\oplus_{i=1}^{\ell} \mathcal{O}(\mathfrak{g}')\,\theta_i\big).$$

Consequently, $\mathcal{N}_{\mathcal{C}} = \mathcal{D}(\mathfrak{g}')/\mathcal{D}(\mathfrak{g}')\tau(\mathfrak{g})$ is a free left $\mathcal{O}(\mathfrak{g}')$-module with basis

$$\big\{\Theta^{\boldsymbol{i}} = \theta_1^{i_1} \cdots \theta_\ell^{i_\ell} \;:\; \boldsymbol{i} = (i_1, \dots, i_\ell) \in \mathbb{N}^\ell\big\}.$$

Proof. It follows from the definition that, if $u \in \mathcal{O}(\mathfrak{g})^G$, then the vector field $\operatorname{grad}(u)$ is G-invariant. Furthermore, it is easily seen that $\operatorname{grad}(u)_x \in \mathfrak{g}^x$, for all $x \in \mathfrak{g}$. Let $(\mathcal{O}_x, \mathbf{m}_x)$ be the local ring of $\mathfrak{g}$ at x. Identify the tangent space $T_x\mathfrak{g} \equiv \mathfrak{g}$ with $\operatorname{Der}\mathcal{O}_x/\mathbf{m}_x \operatorname{Der}\mathcal{O}_x$, and $T_x(G.x)$ with $[x, \mathfrak{g}] = \{\tau(\xi)_x : \xi \in \mathfrak{g}\}$.

Define a submodule of $\operatorname{Der}\mathcal{O}(\mathfrak{g})$ by setting $\mathcal{L} = \sum_{i=1}^{\ell} \mathcal{O}(\mathfrak{g})\,\theta_i$. We first want to show that $\mathcal{L} = \oplus_i \mathcal{O}(\mathfrak{g})\,\theta_i$ is a free $\mathcal{O}(\mathfrak{g})$-module of rank ℓ, and that $\mathcal{L} \cap \mathcal{O}(\mathfrak{g})\tau(\mathfrak{g}) = 0$. Thus, suppose that $\sum_{i=1}^{\ell} a_i\theta_i \in \mathcal{O}(\mathfrak{g})\tau(\mathfrak{g})$, for some $a_i \in \mathcal{O}(\mathfrak{g})$. Let $x \in \mathfrak{g}'$. By the identifications of the last paragraph, $\sum_i a_i(x)\operatorname{grad}(u_i)_x \in \tau(\mathfrak{g})_x = [x, \mathfrak{g}]$. Therefore, $\sum_i a_i(x)\operatorname{grad}(u_i)_x \in \mathfrak{g}^x \cap [x, \mathfrak{g}] = 0$. Recall [12, Theorem 0.1] that the $\{\operatorname{grad}(u_i)_x\}$ are linearly independent. Hence, for all i and $x \in \mathfrak{g}'$, $a_i(x) = 0$. Since $\mathfrak{g}'$ is dense in $\mathfrak{g}$, it follows that $a_i = 0$, for all i. Thus, $\mathcal{L}$ is free and $\operatorname{Der}\mathcal{O}(\mathfrak{g}) \supset \mathcal{L} \oplus \mathcal{O}(\mathfrak{g})\tau(\mathfrak{g})$, as desired.

Let $x \in \mathfrak{g}'$. Since $\mathfrak{g} = \mathfrak{g}^x \oplus [x, \mathfrak{g}]$, we know that $\operatorname{Der}\mathcal{O}_x/\mathbf{m}_x \operatorname{Der}\mathcal{O}_x$ is spanned by the $\{\operatorname{grad}(u_i)_x\}$ and $\tau(\mathfrak{g})_x$. Thus, by Nakayama's Lemma, $\operatorname{Der}\mathcal{O}_x = \mathcal{L}_x \oplus \mathcal{O}_x\tau(\mathfrak{g})$. This proves the second assertion of the lemma, from which the final assertion follows easily. $\square$

Recall the following well known result:

Lemma 6.8. *Let I be a non-zero G-stable ideal in $\mathcal{O}(\mathfrak{g})$. Then, $I^G \neq 0$.*

Proof. Since I contains a product of prime ideals, we may assume that I is prime. If $d \in I$ there is nothing to prove. If $d \notin I$, there exists a generic point in the variety of zeroes of I and, by [12, Theorem 0.5], I is generated by $I \cap \mathcal{O}(\mathfrak{g})^G$. $\square$

We can now prove the main result of this section:

Theorem 6.9. *Fix $\lambda \in \mathfrak{h}^*$ and let*

$$\mathcal{N}_\lambda = \mathcal{D}(\mathfrak{g})\big/\big(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \textstyle\sum_{p \in S(\mathfrak{g})^G} \mathcal{D}(\mathfrak{g})(p - p(\lambda))\big).$$

Then, as $(\mathcal{D}(\mathfrak{g}), W_\lambda)$-modules:

$$\mathcal{N}_\lambda = \bigoplus_{\chi \in W_\lambda^\wedge} \mathcal{N}_{\lambda,\chi} \otimes_{\mathbb{C}} V_\chi^*,$$

where each $\mathcal{N}_{\lambda,\chi} = \mathcal{N} \otimes_{\mathcal{D}(\mathfrak{g})^G} V^\chi$ is a simple $\mathcal{D}(\mathfrak{g})$-module. Moreover, if $\chi \neq \psi$, then $\mathcal{N}_{\lambda,\chi} \not\cong \mathcal{N}_{\lambda,\psi}$.

Proof. The main aim of the proof is to show that

(6.4) If L is a non-zero submodule of $\mathcal{N}_\lambda$, then $L^G \neq 0$.

Indeed, suppose that (6.4) holds. By Lemma 6.2, it remains to prove that the $\mathcal{N}_{\lambda,\chi}$ are simple and non-isomorphic. Suppose that $\mathcal{N}_{\lambda,\chi}$ has a non-zero submodule L. Since $L \subseteq \mathcal{N}_{\lambda,\chi} \subseteq \mathcal{N}_\lambda$, (6.4) implies that $L^G \neq 0$. By Theorem 3.4, L^G is then a non-zero submodule of the simple $\mathcal{D}(\mathfrak{h})^W$-module $(\mathcal{N}_{\lambda,\chi})^G = V^\chi$. Thus, $L^G = V^\chi$ and hence $L \supseteq \mathcal{D}(\mathfrak{g})V^\chi = \mathcal{N}_{\lambda,\chi}$. Thus, $\mathcal{N}_{\lambda,\chi}$ is simple. Lemma 6.1 implies that $(\mathcal{N}_{\lambda,\chi})^G = \{\theta \in \mathcal{N}_{\lambda,\chi} : \tau(\mathfrak{g})\,\theta = 0\}$. Hence, if $\mathcal{N}_{\lambda,\chi} \cong \mathcal{N}_{\lambda,\psi}$, then $(\mathcal{N}_{\lambda,\chi})^G \cong (\mathcal{N}_{\lambda,\psi})^G$, as $\mathcal{D}(\mathfrak{g})^G$-modules. Thus, $\chi = \psi$, by Theorem 3.4(i) and Lemma 6.2(ii).

Thus, it remains to prove (6.4). By Corollary 6.6, $L \subseteq L_{\mathcal{C}}$. Moreover, since $d \in \mathcal{O}(\mathfrak{g})^G$, the action of G on $\mathcal{D}$ extends to an action on $\mathcal{D}_{\mathcal{C}}$ and hence to $L_{\mathcal{C}}$. If $0 \neq f \in L_{\mathcal{C}}^G$, then some $d^m f \in L^G$ and hence $L^G \neq 0$ (Corollary 6.6). Thus, it suffices to prove:

(6.5) If L is a non-zero submodule of $\mathcal{N}_\lambda$, then $L_{\mathcal{C}}^G \neq 0$.

We next need to study $\mathcal{N}_{\mathcal{C}} = \mathcal{D}_{\mathcal{C}}/\mathcal{D}_{\mathcal{C}}\tau(\mathfrak{g})$, for which we use the notation of Lemma 6.7. Fix a lexicographic ordering $\preceq$ on $\mathbb{N}^\ell$. Then, by Lemma 6.7, any element $\alpha \in \mathcal{N}_{\mathcal{C}}$ can be uniquely written $\alpha = a_{\boldsymbol{j}}\Theta^{\boldsymbol{j}} + \sum_{\boldsymbol{i}\prec\boldsymbol{j}} a_{\boldsymbol{i}}\Theta^{\boldsymbol{i}}$, for some $a_{\boldsymbol{i}} \in \mathcal{O}(\mathfrak{g}')$. For $\boldsymbol{i} \in \mathbb{N}^\ell$, set

$$F_{\boldsymbol{i}}\mathcal{N}_{\mathcal{C}} = \sum_{\boldsymbol{j}\preceq\boldsymbol{i}} \mathcal{O}(\mathfrak{g}')\Theta^{\boldsymbol{j}} \subset \mathcal{N}_{\mathcal{C}}.$$

Since the $\{\theta_i\}$ are G-invariant, each $F_{\boldsymbol{i}}\mathcal{N}_{\mathcal{C}}$ is an $(\mathcal{O}(\mathfrak{g}'), G)$-module. It follows from Lemma 6.7 that the $\{F_{\boldsymbol{i}}\mathcal{N}_{\mathcal{C}}\}$ form a filtration on $\mathcal{N}_{\mathcal{C}}$. Let P be a $\mathcal{D}_{\mathcal{C}}$-submodule of $\mathcal{N}_{\mathcal{C}}$ and set $F_{\boldsymbol{i}}P = P \cap F_{\boldsymbol{i}}\mathcal{N}_{\mathcal{C}}$. By Lemma 6.1, the $\{F_{\boldsymbol{i}}P\}$ provide a filtration of P by $(\mathcal{O}(\mathfrak{g}'), G)$-modules that are locally finite as G-modules. For $\boldsymbol{j} \in \mathbb{N}^\ell$, set $\operatorname{gr}_{\boldsymbol{j}} P = F_{\boldsymbol{j}}P/\big(\sum_{\boldsymbol{i}\prec\boldsymbol{j}} F_{\boldsymbol{i}}P\big)$. Thus, $\operatorname{gr}_{\boldsymbol{j}} P \subseteq \operatorname{gr}_{\boldsymbol{j}} \mathcal{N}_{\mathcal{C}} \cong \mathcal{O}(\mathfrak{g}')\Theta^{\boldsymbol{j}}$. Observe that, since G is reductive, $F_{\boldsymbol{q}}P \cong \operatorname{gr}_{\boldsymbol{q}} P \oplus \sum_{\boldsymbol{i}\prec\boldsymbol{q}} F_{\boldsymbol{i}}P$ as G-modules. Thus, $(F_{\boldsymbol{q}}P)^G \neq 0$ whenever $(\operatorname{gr}_{\boldsymbol{q}} P)^G \neq 0$. Suppose that $\operatorname{gr}_{\boldsymbol{q}} P \neq 0$. Then $\operatorname{gr}_{\boldsymbol{q}} P = I\,\Theta^{\boldsymbol{q}}$ for some non-zero, G-stable ideal I of $\mathcal{O}(\mathfrak{g}')$. Lemma 6.8 implies that $I^G \neq 0$, and so Lemma 6.7 implies that $(\operatorname{gr}_{\boldsymbol{q}} P)^G = I^G\,\Theta^{\boldsymbol{q}} \neq 0$. To summarize, we have proved:

(6.6) $$\operatorname{gr}_{\boldsymbol{q}} P \neq 0 \implies (F_{\boldsymbol{q}}P)^G \neq 0$$

Let $Q \subsetneq P$ be submodules of $\mathcal{N}_{\mathcal{C}}$ and assume that P/Q is a torsion-free $\mathcal{O}(\mathfrak{g}')$-module. Pick $\boldsymbol{q} \in \mathbb{N}^\ell$ minimal such that $F_{\boldsymbol{q}}Q \subsetneq F_{\boldsymbol{q}}P$. We claim that $F_{\boldsymbol{q}}Q = 0$. If not, we may pick $b = \beta\Theta^{\boldsymbol{q}} + b' \in F_{\boldsymbol{q}}Q$ and $f = \alpha\Theta^{\boldsymbol{q}} + f' \in F_{\boldsymbol{q}}P \setminus F_{\boldsymbol{q}}Q$, with $0 \neq \beta, \alpha \in \mathcal{O}(\mathfrak{g}')$ and $b', f' \in \sum_{\boldsymbol{i}\prec\boldsymbol{q}} F_{\boldsymbol{i}}\mathcal{N}'$. Then, $\beta f - \alpha b = \beta f' - \alpha b' \in F_{\boldsymbol{j}}\mathcal{N}' \cap P = F_{\boldsymbol{j}}P$ for some $\boldsymbol{j} \prec \boldsymbol{q}$. Thus, $\beta f \in \alpha b + F_{\boldsymbol{j}}Q \subseteq Q$. Since P/Q is torsion-free, this forces $\beta = 0$ and a contradiction. Thus, $F_{\boldsymbol{q}}Q = 0$.

Finally, write $\mathcal{N}'_\lambda = (\mathcal{N}_\lambda)_{\mathcal{C}} = \mathcal{D}_{\mathcal{C}}/Q$, where $Q \supseteq \mathcal{D}_{\mathcal{C}}\tau(\mathfrak{g})$ denotes the annihilator of the natural generator of $\mathcal{N}'_\lambda$. Let $L_{\mathcal{C}} = P/Q \subseteq \mathcal{N}'_\lambda$ be a non-zero submodule of $\mathcal{N}'_\lambda$. By Corollary 6.6, $L_{\mathcal{C}} \subseteq \mathcal{N}'_\lambda$ is a torsion-free $\mathcal{O}(\mathfrak{g}')$-module and so the comments of the last paragraph apply. If $\boldsymbol{q}$ is chosen as there, then $F_{\boldsymbol{q}}Q = 0$ and $\operatorname{gr}_{\boldsymbol{q}} P \neq 0$. Thus, (6.6) implies that $L_{\mathcal{C}}^G \supseteq (F_{\boldsymbol{q}}P)^G \neq 0$. By (6.5), this completes the proof of the theorem. $\square$

Let $\mathcal{R}_\lambda$ denote the full subcategory of $\mathcal{D}(\mathfrak{g})$-mod generated by $\mathcal{N}_\lambda$ or, equivalently, by the modules $\mathcal{N}_{\lambda,\chi}$, $\chi \in \widehat{W_\lambda}$. We end this section by giving two alternative descriptions of this category.

Corollary 6.10. *Set* $\mathbf{p}_\lambda = \sum_{p \in S(\mathfrak{g})^G} S(\mathfrak{g})^G(p - p(\lambda))$. *The category* $\mathcal{R}_\lambda$ *is the subcategory of* $\mathcal{D}(\mathfrak{g})$*-modules* M *such that:*

(1) M *is a compatible* $(\mathcal{D}(\mathfrak{g}), G)$*-module, finitely generated as a* $\mathcal{D}(\mathfrak{g})$*-module;*
(2) $M = \mathcal{D}(\mathfrak{g})M^G$*;*
(3) *For all* $x \in M$*, there exists* $k \in \mathbb{N}$ *such that* $\mathbf{p}_\lambda^k x = 0$.

Proof. It is clear that any object in $\mathcal{R}_\lambda$ satisfies these conditions. Conversely, let M satisfy (1), (2) and (3). Since M is finitely generated we may write $M = \mathcal{D}(\mathfrak{g})E$ for some finite dimensional subspace $E \subset M^G$. Then, $X = \mathcal{D}(\mathfrak{g})^G E$ can be viewed (through δ) as a finitely generated $\mathcal{D}(\mathfrak{h})^W$-module. It follows from the hypotheses that $X \in \mathcal{C}_\lambda$. Hence, $\mathcal{N}_\lambda \otimes_{\mathcal{D}(\mathfrak{h})^W} X \in \mathcal{R}_\lambda$ and there is a natural surjection $\mathcal{N}_\lambda \otimes_{\mathcal{D}(\mathfrak{h})^W} X \twoheadrightarrow M$. This proves that $M \in \mathcal{R}_\lambda$. $\square$

It is implicit in Theorem 6.9 that the categories $\mathcal{R}_\lambda$ and W_λ-mod are equivalent. We make this explicit in the next result, in which we freely use the notation of Section 5 for the case $V = \mathfrak{h}$. We state the result using the functor Sol, although one could as easily use DR.

Corollary 6.11. *The category* $\mathcal{R}_\lambda$ *is equivalent to* $(W_\lambda\text{-mod})^{\mathrm{op}}$*, through the functor* $M \to \mathrm{Sol}(M^G) = \mathrm{Hom}_{\mathcal{D}(\mathfrak{h})^W}(M^G, \mathcal{O}(\mathfrak{h})e^\lambda)$. *Moreover, if* $M \in \mathcal{R}_\lambda$,

(i) $\mathrm{Sol}(M^G) \cong \mathrm{Hom}_{\mathcal{D}(\mathfrak{g})}(M, \mathcal{N}_\lambda)$ *as* W_λ*-modules;*
(ii) *When* M *is a quotient of* $\mathcal{N}_\lambda$, $\mathrm{Sol}(M^G)$ *identifies naturally with a* W_λ*-submodule of* $\mathcal{H}(\lambda)e^\lambda$*;*
(iii) $\mathrm{End}_{\mathcal{D}(\mathfrak{g})} \mathcal{N}_\lambda \cong \mathbb{C}[W_\lambda]$ *as* $(W_\lambda \times W_\lambda)$*-modules.*

Proof. By Theorem 6.9, the functor $M \to M^G$ provides an equivalence of categories $\mathcal{R}_\lambda \approx \mathcal{C}_\lambda$. Thus, by Proposition 5.1, the map $M \mapsto \mathrm{Sol}(M^G)$ gives an equivalence $\mathcal{R}_\lambda \approx (W_\lambda\text{-mod})^{\mathrm{op}}$. The proof now follows easily from the results of Section 5 and the details are left to the interested reader. $\square$

7. Fourier transforms and the Springer correspondence

As has been remarked in the introduction, the module $\mathcal{N}_\lambda$ was defined and intensively studied in [10]. In that paper, $\mathcal{N}_\lambda$ is constructed as a "Fourier transform" of a second module

$$\mathcal{M}_\lambda = \mathcal{D}(\mathfrak{g}) \big/ \big(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \textstyle\sum_{q \in S(\mathfrak{g}^*)^G} \mathcal{D}(\mathfrak{g})(q - q(\lambda))\big). \tag{7.1}$$

(see [10, (6.1.2)]). In this section, we apply our results to give a detailed description of $\mathcal{M}_\lambda$ and use this to obtain an analogue of Springer correspondence for non-nilpotent orbits.

The Fourier transformation is defined as follows: Use the G-invariant bilinear form κ to define an isomorphism $\kappa : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$. Since one has a G-module isomorphism $\mathcal{D}(\mathfrak{g}) \cong S(\mathfrak{g}^*) \otimes S(\mathfrak{g})$, κ induces an algebra automorphism F of $\mathcal{D}(\mathfrak{g})$ by $F(f) = -\kappa^{-1}(f)$ and $F(x) = \kappa(x)$ for $f \in \mathfrak{g}^*$ and $x \in \mathfrak{g}$. It is easily checked that F is a G-automorphism of $\mathcal{D}(\mathfrak{g})$ such that $F(\tau(x)) = \tau(x)$ for all $x \in \mathfrak{g}$. Given a $\mathcal{D}(\mathfrak{g})$-module M, define the *Fourier transform* M^F *of* M to be the abelian group M

with multiplication defined by $a \circ m = F(a)m$, for $a \in \mathcal{D}(\mathfrak{g})$ and $m \in M$. Then, as in [10, Section 6], $\mathcal{N}_\lambda$ is the Fourier transform $\mathcal{N}_\lambda = \mathcal{M}_\lambda^F$.

Using the automorphism F, Theorem 6.9 immediately gives the following generalization of [10, Theorem 5.3].

Corollary 7.1. *For each $\chi \in W_\lambda^\wedge$, set $\mathcal{M}_{\lambda,\chi} = \mathcal{N}_{\lambda,\chi}^{F^{-1}}$. As $(\mathcal{D}(\mathfrak{g}), W_\lambda)$-modules:*

$$\mathcal{M}_\lambda = \bigoplus_{\chi \in W_\lambda^\wedge} \mathcal{M}_{\lambda,\chi} \otimes_{\mathbb{C}} V_\chi^*,$$

where each $\mathcal{M}_{\lambda,\chi}$ is a simple $\mathcal{D}(\mathfrak{g})$-module. If $\chi \neq \psi$, then $\mathcal{M}_{\lambda,\chi} \not\cong \mathcal{M}_{\lambda,\psi}$. □

This decomposition of $\mathcal{M}_\lambda$ is also related to the G-orbit structure of $\mathfrak{g}$, for which we need some definitions. Let $\pi : \mathfrak{g} \twoheadrightarrow \mathfrak{g}/G \rightsquigarrow \mathfrak{h}/W$ be the natural projection from $\mathfrak{g}$ to the space of W-orbits in $\mathfrak{h}$. Set $\mathbf{N}(\lambda) = \pi^{-1}(W.\kappa^{-1}(\lambda)) \subset \mathfrak{g}$. By [12, Theorem 3], $\mathbf{N}(\lambda)$ is a finite union of G-orbits:

$$\mathbf{N}(\lambda) = \overline{\mathbf{O}_\lambda^r} = \mathbf{O}_\lambda^r \cup \cdots \cup \mathbf{O}_\lambda^s,$$

where $\dim \mathbf{O}_\lambda^r = n - \ell$ and $\mathbf{O}_\lambda^s = G.\kappa^{-1}(\lambda)$ is the unique closed orbit in $\mathbf{N}(\lambda)$. In the sequel, we adopt the notation of [2] for concepts related to algebraic D-modules. In particular, for any $\mathcal{D}(\mathfrak{g})$-module M, one defines the closed subset $\operatorname{Supp} M \subseteq \mathfrak{g}$ to be the support of the $\mathcal{O}(\mathfrak{g})$-module M.

Corollary 7.2. *The support of $\mathcal{M}_{\lambda,\chi}$ is the closure of an orbit $\mathbf{O}(\chi) \subseteq \mathbf{N}(\lambda)$.*

Proof. Note first that each $\mathcal{M}_{\lambda,\chi}$ is a compatible $(\mathcal{D}(\mathfrak{g}), G)$-module (use Lemma 6.1 and the equivariance of F). It therefore follows from (7.1) that $\operatorname{Supp} \mathcal{M}_\chi$ is a closed G-stable subset of $\mathbf{N}(\lambda)$. But, since $\mathcal{M}_{\lambda,\chi}$ is simple, its support is irreducible, (see the proof of [1, Lemmas 3.3.16 and 3.3.17]). Therefore, $\operatorname{Supp} \mathcal{M}_{\lambda,\chi}$ must be the closure of a single orbit contained in $\mathbf{N}(\lambda)$. □

As in [11], one can relate the simple modules $\mathcal{M}_{\lambda,\chi}$ to connections on the orbits contained in $\mathbf{N}(\lambda)$, and interpret the equivalence $\mathcal{R}_\lambda \approx W_\lambda$-mod through a "Springer correspondence".

Let $M = \mathcal{M}_{\lambda,\chi}$ be a simple quotient of $\mathcal{M}_\lambda$ and, by Corollary 7.2, write $\operatorname{Supp} M = \overline{\mathbf{O}}$, for the appropriate orbit $\mathbf{O} = \mathbf{O}(\chi)$. If $\imath : \mathbf{O} \hookrightarrow \mathfrak{g}$ is the natural embedding, then [2, Proposition 10.4] implies that $\imath^! M$ is a connection, $\mathcal{E}$, on a dense affine Zariski open subset of $\mathbf{O}$. Note that, since $\mathbf{O}$ is a G-space, $\imath^! M$ is a connection on $\mathbf{O}$. Moreover, $\imath^! M$ is regular holonomic, and is uniquely determined by a representation $\psi \in A(\mathbf{O})^\wedge$, [18, Lemma 2.1], where $A(\mathbf{O}) = G^x/(G^x)^0$, $x \in \mathbf{O}$, is the component group of $\mathbf{O}$. Then, by [2, Theorem 10.6], M is of the form $\imath_{!+}\mathcal{E}$ (this is a regular holonomic module, minimal extension of $\mathcal{E}$ in the algebraic category). Thus, we have proved:

Corollary 7.3. *The module $\mathcal{M}_{\lambda,\chi}$ is uniquely determined by the associated datum $(\mathbf{O}, \psi)$, where $\mathbf{O}$ an orbit in $\mathbf{N}(\lambda)$ and $\psi \in A(\mathbf{O})^\wedge$. Consequently, any representation $\chi \in W_\lambda{}^\wedge$ is also uniquely determined by this datum; written $\chi = \sigma(\mathbf{O}, \psi)$.* □

When $\lambda = 0$, the correspondence $(\mathbf{O}, \psi) \to \sigma(\mathbf{O}, \psi)$ is closely related to the Springer correspondence (see [5], [10, Section 8] and [19, Theorem 6.10]).

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