ALGEBRAIC STRUCTURE OF MULTI-PARAMETER QUANTUM GROUPS

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Introduction

Let G be a connected semi-simple complex Lie group. We define and study the multi-parameter quantum group $\mathbb{C}_{q,p}[G]$ in the case where q is a complex parameter that is not a root of unity. Using a method of twisting bigraded Hopf algebras by a cocycle, [2], we develop a unified approach to the construction of $\mathbb{C}_{q,p}[G]$ and of the multi-parameter Drinfeld double $D_{q,p}$. Using a general method of deforming bigraded pairs of Hopf algebras, we construct a Hopf pairing between these algebras from which we deduce a Peter-Weyl-type theorem for $\mathbb{C}_{q,p}[G]$. We then describe the prime and primitive spectra of $\mathbb{C}_{q,p}[G]$, generalizing a result of Joseph. In the one-parameter case this description was conjectured, and established in the $SL(n)$-case, by the first and second authors in [15, 16]. It was proved in the general case by Joseph in [18, 19]. In particular the orbits in $\operatorname{Prim}\mathbb{C}_{q,p}[G]$ under the natural action of the maximal torus H are indexed, as in the one-parameter case by the elements of the double Weyl group $W \times W$. Unlike the one-parameter case there is not in general a bijection between $\operatorname{Symp} G$ and $\operatorname{Prim}\mathbb{C}_{q,p}[G]$. However in the case when the symplectic leaves are *algebraic* such a bijection does exist since the orbits corresponding to a given $w \in W \times W$ have the same dimension.

In the first section we discuss the Poisson structures on G defined by classical r-matrices of the form $r = a - u$ where $a = \sum_{\alpha\in\mathbf{R}_+} e_\alpha \wedge e_{-\alpha} \in \wedge^2\mathfrak{g}$ and $u \in \wedge^2\mathfrak{h}$. Given such an r one constructs a Manin triple of Lie groups $(G \times G, G, G_r)$. Unlike the one-parameter case (where $u = 0$), the dual group G_r will generally not be an algebraic subgroup of $G \times G$. In fact this happens if and only if $u \in \wedge^2\mathfrak{h}_{\mathbb{Q}}$. Since the quantized universal enveloping algebra $U_q(\mathfrak{g})$ is a deformation of the algebra of functions on the algebraic group G_r [11], this explains the difficulty in constructing multi-parameter versions of $U_q(\mathfrak{g})$. From [22, 30], one has that the symplectic leaves are the connected components of $G \cap G_r x G_r$ where $x \in G$. Since r is H-invariant, the symplectic leaves are permuted by H with the orbits being contained in Bruhat cells in $G \times G$ indexed by $W \times W$. In the case where G_r is algebraic, the symplectic leaves are also algebraic and an explicit formula is given for their dimension.

The philosophy of [15, 16] was that, as in the case of enveloping algebras of algebraic solvable Lie algebras, the primitive ideals of $\mathbb{C}_q[G]$ should be in bijection with the symplectic leaves of G (in the case $u = 0$). Indeed, since the Lie bracket on $\mathfrak{g}_r = \operatorname{Lie}(G_r)$ is the linearization of the Poisson structure on G, $\operatorname{Prim}\mathbb{C}_{q,p}[G]$ should resemble $\operatorname{Prim} U(\mathfrak{g}_r)$. The study of the multi-parameter versions $\mathbb{C}_{q,p}[G]$ is similar to the case of enveloping algebras of general solvable Lie algebras. In the general case $\operatorname{Prim} U(\mathfrak{g}_r)$ is in bijection with the co-adjoint orbits in $\mathfrak{g}_r^*$ under the action of the 'adjoint algebraic group' of $\mathfrak{g}_r$, [12]. It is therefore natural that, only in the case where the symplectic leaves are algebraic, does one expect and obtain a bijection between the symplectic leaves and the primitive ideals.

In section 2 we define the notion of an $\mathbf{L}$-bigraded Hopf $\mathbb{K}$-algebra, where $\mathbf{L}$ is an abelian group. When A is finitely generated such bigradings correspond bijectively to morphisms from the algebraic group $\mathbf{L}^\vee$ to the (algebraic) group $R(A)$ of one- dimensional representations of A. For any antisymmetric bicharacter p on $\mathbf{L}$, the multiplication in A may be twisted to give a new Hopf algebra A_p. Moreover, given a pair of $\mathbf{L}$-bigraded Hopf algebras A and U equipped with an $\mathbf{L}$-compatible Hopf pairing $A \times U \to \mathbb{K}$, one can deform the pairing to get a new Hopf pairing between $A_{p^{-1}}$ and U_p. This deformation commutes

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with the formation of the Drinfeld double in the following sense. Suppose that A and U are bigraded Hopf algebras equipped with a compatible Hopf pairing $A^{\mathrm{op}} \times U \to \mathbb{K}$. Then the Drinfeld double $A \bowtie U$ inherits a bigrading such that $(A \bowtie U)_p \cong A_p \bowtie U_p$.

Let $\mathbb{C}_q[G]$ denote the usual one-parameter quantum group (or quantum function algebra) and let $U_q(\mathfrak{g})$ be the quantized enveloping algebra associated to the lattice $\mathbf{L}$ of weights of G. Let $U_q(\mathfrak{b}^+)$ and $U_q(\mathfrak{b}^-)$ be the usual sub-Hopf algebras of $U_q(\mathfrak{g})$ corresponding to the Borel subalgebras $\mathfrak{b}^+$ and $\mathfrak{b}^-$ respectively. Let $D_q(\mathfrak{g}) = U_q(\mathfrak{b}^+) \bowtie U_q(\mathfrak{b}^-)$ be the Drinfeld double. Since the groups of one-dimensional representations of $U_q(\mathfrak{b}^+)$, $U_q(\mathfrak{b}^-)$, $D_q(\mathfrak{g})$ and $\mathbb{C}_q[G]$ are all isomorphic to $H = \mathbf{L}^\vee$, these algebras are all equipped with $\mathbf{L}$-bigradings. Moreover the Rosso-Tanisaki pairing is compatible with the bigradings on $U_q(\mathfrak{b}^+)$ and $U_q(\mathfrak{b}^-)$. For any anti-symmetric bicharacter p on $\mathbf{L}$ one may therefore twist simultaneously the Hopf algebras $U_q(\mathfrak{b}^+)$, $U_q(\mathfrak{b}^-)$ and $D_q(\mathfrak{g})$ in such a way that $D_{q,p}(\mathfrak{g}) \cong U_{q,p}(\mathfrak{b}^+) \bowtie U_{q,p}(\mathfrak{b}^-)$. The algebra $D_{q,p}(\mathfrak{g})$ is the 'multi-parameter quantized universal enveloping algebra' constructed by Okado and Yamane [25] and previously in special cases in [9, 32]. The canonical pairing between $\mathbb{C}_q[G]$ and $U_q(\mathfrak{g})$ induces a $\mathbf{L}$-compatible pairing between $\mathbb{C}_q[G]$ and $D_q(\mathfrak{g})$. Thus there is an induced pairing between the multi-parameter quantum group $\mathbb{C}_{q,p}[G]$ and the multi-parameter double $D_{q,p^{-1}}(\mathfrak{g})$. Recall that the Hopf algebra $\mathbb{C}_q[G]$ is defined as the restricted dual of $U_q(\mathfrak{g})$ with respect to a certain category $\mathcal{C}$ of modules over $U_q(\mathfrak{g})$. There is a natural deformation functor from this category to a category $\mathcal{C}_p$ of modules over $D_{q,p^{-1}}(\mathfrak{g})$ and $\mathbb{C}_{q,p}[G]$ turns out to be the restricted dual of $D_{q,p^{-1}}(\mathfrak{g})$ with respect to this category. This Peter-Weyl theorem for $\mathbb{C}_{q,p}[G]$ was also found by Andruskiewitsch and Enriquez in [1] using a different construction of the quantized universal enveloping algebra and in special cases in [5, 14].

The main theorem describing the primitive spectrum of $\mathbb{C}_{q,p}[G]$ is proved in the final section. Since $\mathbb{C}_{q,p}[G]$ inherits an $\mathbf{L}$-bigrading, there is a natural action of H as automorphisms of $\mathbb{C}_{q,p}[G]$. For each $w \in W \times W$, we construct an algebra $A_w = (\mathbb{C}_{q,p}[G]/I_w)_{\mathcal{E}_w}$ which is a localization of a quotient of $\mathbb{C}_{q,p}[G]$. For each prime $P \in \operatorname{Spec} \mathbb{C}_{q,p}[G]$ there is a unique $w \in W \times W$ such that $P \supset I_w$ and PA_w is proper. Thus $\operatorname{Spec} \mathbb{C}_{q,p}[G] \cong \bigsqcup_{w \in W \times W} \operatorname{Spec}_w \mathbb{C}_{q,p}[G]$ where $\operatorname{Spec}_w \mathbb{C}_{q,p}[G] \cong \operatorname{Spec} A_w$ is the set of primes of type w. The key results are then Theorems 4.14 and 4.15 which state that an ideal of A_w is generated by its intersection with the center and that H acts transitively on the maximal ideals of the center. From this it follows that the primitive ideals of $\mathbb{C}_{q,p}[G]$ of type w form an orbit under the action of H.

An earlier version of our approach to the proof of Joseph's theorem is contained in the unpublished article [17]. The approach presented here is a generalization of this proof to the multi-parameter case.

These results are algebraic analogs of results of Levendorskii [20, 21] on the irreducible representations of multi-parameter function algebras and compact quantum groups. The bijection between symplectic leaves of the compact Poisson group and irreducible $*$-representations of the compact quantum group found by Soibelman in the one-parameter case, breaks down in the multi-parameter case.

After this work was completed, the authors became aware of the work of Constantini and Varagnolo [7, 8] which has some overlap with the results in this paper.

1. Poisson Lie Groups

1.1. **Notation.** Let $\mathfrak{g}$ be a complex semi-simple Lie algebra associated to a Cartan matrix $[a_{ij}]_{1 \leqslant i,j \leqslant n}$. Let $\{d_i\}_{1 \leqslant i \leqslant n}$ be relatively prime positive integers such that $[d_i a_{ij}]_{1 \leqslant i,j \leqslant n}$ is symmetric positive definite.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\mathbf{R}$ the associated root system, $\mathbf{B} = \{\alpha_1, \dots, \alpha_n\}$ a basis of $\mathbf{R}$, $\mathbf{R}_+$ the set of positive roots and W the Weyl group. We denote by $\mathbf{P}$ and $\mathbf{Q}$ the lattices of weights and roots respectively. The fundamental weights are denoted by $\varpi_1, \dots, \varpi_n$ and the set of dominant integral weights by $\mathbf{P}^+ = \sum_{i=1}^n \mathbb{N}\varpi_i$. Let $(-,-)$ be a non-degenerate $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$; it will identify $\mathfrak{g}$, resp. $\mathfrak{h}$, with its dual $\mathfrak{g}^*$, resp. $\mathfrak{h}^*$. The form $(-,-)$ can be chosen in order to induce a perfect pairing $\mathbf{P} \times \mathbf{Q} \to \mathbb{Z}$ such that

$$(\varpi_i, \alpha_j) = \delta_{ij} d_i, \quad (\alpha_i, \alpha_j) = d_i a_{ij}.$$

Hence $d_i = (\alpha_i, \alpha_i)/2$ and $(\alpha, \alpha) \in 2\mathbb{Z}$ for all $\alpha \in \mathbf{R}$. For each α_j we denote by $h_j \in \mathfrak{h}$ the corresponding coroot: $\varpi_i(h_j) = \delta_{ij}$. We also set

$$\mathfrak{n}^{\pm} = \oplus_{\alpha \in \mathbf{R}_+} \mathfrak{g}_{\pm\alpha}, \quad \mathfrak{b}^{\pm} = \mathfrak{h} \oplus \mathfrak{n}^{\pm}, \quad \mathfrak{d} = \mathfrak{g} \times \mathfrak{g}, \quad \mathfrak{t} = \mathfrak{h} \times \mathfrak{h}, \quad \mathfrak{u}^{\pm} = \mathfrak{n}^{\pm} \times \mathfrak{n}^{\mp}.$$

Let G be a connected complex semi-simple algebraic group such that $\mathrm{Lie}(G) = \mathfrak{g}$ and set $D = G \times G$. We identify G (and its subgroups) with the diagonal copy inside D. We denote by exp the exponential map from $\mathfrak{d}$ to D. We shall in general denote a Lie subalgebra of $\mathfrak{d}$ by a gothic symbol and the corresponding connected analytic subgroup of D by a capital letter.

1.2. **Poisson Lie group structure on G.** Let $a = \sum_{\alpha \in \mathbf{R}_+} e_\alpha \wedge e_{-\alpha} \in \wedge^2 \mathfrak{g}$ where the e_α are root vectors such that $(e_\alpha, e_\beta) = \delta_{\alpha,-\beta}$. Let $u \in \wedge^2 \mathfrak{h}$ and set $r = a - u$. Then it is well known that r satisfies the modified Yang-Baxter equation [3, 20] and that therefore the tensor $\pi(g) = (l_g)_* r - (r_g)_* r$ furnishes G with the structure of a Poisson Lie group, see [13, 22, 30] ($(l_g)_*$ and $(r_g)_*$ are the differentials of the left and right translation by $g \in G$).

We may write $u = \sum_{1 \le i,j \le n} u_{ij} h_i \otimes h_j$ for a skew-symmetric $n \times n$ matrix $[u_{ij}]$. The element u can be considered either as an alternating form on $\mathfrak{h}^*$ or a linear map $u \in \mathrm{End}\, \mathfrak{h}$ by the formula

$$\forall\, x \in \mathfrak{h}, \quad u(x) = \sum_{i,j} u_{i,j}(x, h_i) h_j.$$

The Manin triple associated to the Poisson Lie structure on G given by r is described as follows. Set $u_\pm = u \pm I \in \mathrm{End}\, \mathfrak{h}$ and define

$$\vartheta : \mathfrak{h} \hookrightarrow \mathfrak{t}, \quad \vartheta(x) = -(u_-(x), u_+(x)),$$
$$\mathfrak{a} = \vartheta(\mathfrak{h}), \quad \mathfrak{g}_r = \mathfrak{a} \oplus \mathfrak{u}^+.$$

Following [30] one sees easily that the associated Manin triple is $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}_r)$ where $\mathfrak{g}$ is identified with the diagonal copy inside $\mathfrak{d}$. Then the corresponding triple of Lie groups is (D, G, G_r), where $A = \exp(\mathfrak{a})$ is an analytic torus and $G_r = AU^+$. Notice that $\mathfrak{g}_r$ is a solvable, but not in general algebraic, Lie subalgebra of $\mathfrak{d}$.

The following is an easy consequence of the definition of $\mathfrak{a}$ and the identities $u_+ + u_- = 2u, u_+ - u_- = 2I$:

$$\mathfrak{a} = \{(x, y) \in \mathfrak{t} \mid x + y = u(y - x)\} = \{(x, y) \in \mathfrak{t} \mid u_+(x) = u_-(y)\}. \tag{1.1}$$

Recall that $\exp : \mathfrak{h} \to H$ is surjective; let L_H be its kernel. We shall denote by $\mathbf{X}(K)$ the group of characters of an algebraic torus K. Any $\chi \in \mathbf{X}(H)$ is given by $\chi(\exp x) = \exp d\chi(x)$, $x \in \mathfrak{h}$, where $d\chi \in \mathfrak{h}^*$ is the differential of χ. Then

$$\mathbf{X}(H) \cong \mathbf{L} = L_H{}^\circ := \{\xi \in \mathfrak{h}^* \mid \xi(L_H) \subset 2i\pi\mathbb{Z}\}.$$

One can show that $\mathbf{L}$ has a basis consisting of dominant weights.

Recall that if $\tilde{G}$ is a connected simply connected algebraic group with Lie algebra $\mathfrak{g}$ and maximal torus $\tilde{H}$, we have

$$L_{\tilde{H}} = \mathbf{P}^\circ = \oplus_{j=1}^n 2i\pi\mathbb{Z} h_j, \quad \mathbf{X}(\tilde{H}) \cong \mathbf{P},$$
$$\mathbf{Q} \subseteq \mathbf{L} \subseteq \mathbf{P}, \quad \pi_1(G) = L_H/\mathbf{P}^\circ \cong \mathbf{P}/\mathbf{L}.$$

Notice that $L_H/\mathbf{P}^\circ$ is a finite group and $\exp u(L_H)$ is a subgroup of H. We set

$$\Gamma_0 = \{(a, a) \in T \mid a^2 = 1\}, \quad \Delta = \{(a, a) \in T \mid a^2 \in \exp u(L_H)\},$$
$$\Gamma = A \cap H = \{(a, a) \in T \mid a = \exp x = \exp y,\ x + y = u(y - x)\}.$$

It is easily seen that $\Gamma = G \cap G_r$.

Proposition 1.1. *We have $\Delta = \Gamma.\Gamma_0$.*

Proof. We obviously have $\Gamma_0 \subset \Delta$. Let $(\exp h, \exp h) \in \Gamma,\ h \in \mathfrak{h}$. By definition there exist $(x,y) \in \mathfrak{a}$, $\ell_1, \ell_2 \in L_H$ such that

$$x = h + \ell_1,\ y = h + \ell_2, \quad y + x = u(y - x).$$

Hence $y+x = 2h+\ell_1+\ell_2 = u(\ell_2-\ell_1)$ and $(\exp h)^2 = \exp 2h = \exp u(\ell_2-\ell_1)$. This shows $(\exp h, \exp h) \in \Delta$. Thus $\Gamma.\Gamma_0 \subseteq \Delta$.

Let $(a,a) \in \Delta$, $a = \exp h,\ h \in \mathfrak{h}$. From $a^2 \in \exp u(L_H)$ we get $\ell, \ell' \in L_H$ such that $2h = u(\ell') + \ell$. Set $x = h - \ell/2 - \ell'/2,\ y = h + \ell'/2 - \ell/2$. Then $y + x = u(y - x)$ and we have $\exp(-\ell/2 - \ell'/2) = \exp(\ell'/2 - \ell/2)$, since $\ell' \in L_H$. If $b = \exp(-\ell'/2 + \ell/2)$ we obtain $\exp x = \exp y = ab^{-1}$, hence $(a,a) = (\exp x, \exp y).(b,b) \in \Gamma.\Gamma_0$. Therefore $\Gamma.\Gamma_0 = \Delta$. □

Remark . When u is "generic" Γ_0 is not contained in Γ. For example, take G to be $SL(3,\mathbb{C})$ and $u = \alpha(h_1 \otimes h_2 - h_2 \otimes h_1)$ with $\alpha \notin \mathbb{Q}$.

Considered as a Poisson variety, G decomposes as a disjoint union of symplectic leaves. Denote by $\operatorname{Symp} G$ the set of these symplectic leaves. Since r is H-invariant, translation by an element of H is a Poisson morphism and hence there is an induced action of H on $\operatorname{Symp} G$. The key to classifying the symplectic leaves is the following result, cf. [22, 30].

Theorem 1.2. *The symplectic leaves of G are exactly the connected components of $G \cap G_r x G_r$ for $x \in G$.*

Remark that A, Γ and G_r are in general not closed subgroups of D. This has for consequence that the analysis of $\operatorname{Symp} G$ made in [15, Appendix A] in the case $u = 0$ does not apply in the general case.

Set $Q = HG_r = TU^+$. Then Q is a Borel subgroup of D and, recalling that the Weyl group associated to the pair (G,T) is $W \times W$, the corresponding Bruhat decomposition yields $D = \sqcup_{w \in W\times W} QwQ = \sqcup_{w\in W\times W} QwG_r$. Therefore any symplectic leaf is contained in a Bruhat cell QwQ for some $w \in W \times W$.

Definition . A leaf $\mathcal{A}$ is said to be of type w if $\mathcal{A} \subset QwQ$. The set of leaves of type w is denoted by $\operatorname{Symp}_w G$.

For each $w \in W \times W$ set $w = (w_+, w_-), w_\pm \in W$, and fix a representative $\dot{w}$ in the normaliser of T. One shows as in [15, Appendix A] that $G \cap Q\dot{w}G_r \neq \emptyset$, for all $w \in W \times W$; hence $\operatorname{Symp}_w G \neq \emptyset$ and $G \cap G_r \dot{w} G_r \neq \emptyset$, since $QwQ = \cup_{h\in H} hG_r\dot{w}G_r$.

The adjoint action of D on itself is denoted by Ad. Set

$$U_w^- = \operatorname{Ad} w(U) \cap U^+, \quad A'_w = \{a \in A \mid a\dot{w}G_r = \dot{w}G_r\},$$
$$T'_w = \{t \in T \mid tG_r\dot{w}G_r = G_r\dot{w}G_r\}, \quad H'_w = H \cap T'_w.$$

Recall that U_w^- is isomorphic to $\mathbb{C}^{l(w)}$ where $l(w) = l(w_+) + l(w_-)$ is the length of w. We set $s(w) = \dim H'_w$.

Lemma 1.3. (i) $A'_w = \operatorname{Ad} w(A) \cap A$ *and* $T'_w = A. \operatorname{Ad} w(A) = AH'_w$.

(ii) *We have* $\operatorname{Lie}(A'_w) = \mathfrak{a}'_w = \{\vartheta(x) \mid x \in \operatorname{Ker}(u_- w_-^{-1} u_+ - u_+ w_+^{-1} u_-)\}$ *and* $\dim \mathfrak{a}'_w = n - s(w)$.

Proof. (i) The first equality is obvious and the second is an easy consequence of the bijection, induced by multiplication, between $U_w^- \times T \times U^+$ and $QwQ = QwG_r$.

(ii) By definition we have $\mathfrak{a}'_w = \{\vartheta(x) \mid x \in \mathfrak{h},\ w^{-1}(\vartheta(x)) \in \mathfrak{a}\}$. From (1.1) we deduce that $\vartheta(x) \in \mathfrak{a}'_w$ if and only if $u_+ w_+^{-1}(-u_-(x)) = u_- w_-^{-1}(-u_+(x))$.

It follows from (i) that $\dim T'_w = n + \dim H'_w = 2n - \dim A'_w$, hence $\dim \mathfrak{a}'_w = n - s(w)$. □

Recall that $u \in \operatorname{End} \mathfrak{h}$ is an alternating bilinear form on $\mathfrak{h}^*$. It is easily seen that $\forall\, \lambda, \mu \in \mathfrak{h}^*$, $u(\lambda,\mu) = -({}^t u(\lambda), \mu)$, where ${}^t u \in \operatorname{End} \mathfrak{h}^*$ is the transpose of u.

Notation . Set ${}^t u = -\Phi$, $\Phi_\pm = \Phi \pm I$, $\sigma(w) = \Phi_- w_- \Phi_+ - \Phi_+ w_+ \Phi_-$, where $w_\pm \in W$ is considered as an element of $\operatorname{End} \mathfrak{h}^*$.

Observe that ${}^t u_\pm = -\Phi_\mp$ and that

$$u(\lambda,\mu) = (\Phi\lambda,\mu), \quad \text{for all } \lambda,\mu \in \mathfrak{h}^*. \tag{1.2}$$

Furthermore, since the transpose of $w_\pm \in \operatorname{End}\mathfrak{h}^*$ is $w_\pm^{-1} \in \operatorname{End}\mathfrak{h}$, we have ${}^t\sigma(w) = u_- w_-^{-1} u_+ - u_+ w_+^{-1} u_-$. Hence by Lemma 1.3

$$s(w) = \operatorname{codim}\operatorname{Ker}_{\mathfrak{h}^*}\sigma(w), \quad \dim A'_w = \dim \operatorname{Ker}_{\mathfrak{h}^*}\sigma(w). \tag{1.3}$$

1.3. **The algebraic case.** As explained in 1.1 the Lie algebra $\mathfrak{g}_r$ is in general not algebraic. We now describe its algebraic closure. Recall that a Lie subalgebra $\mathfrak{m}$ of $\mathfrak{d}$ is said to be algebraic if $\mathfrak{m}$ is the Lie algebra of a closed (connected) algebraic subgroup of D.

Definition . Let $\mathfrak{m}$ be a Lie subalgebra of $\mathfrak{d}$. The smallest algebraic Lie subalgebra of $\mathfrak{d}$ containing $\mathfrak{m}$ is called the algebraic closure of $\mathfrak{m}$ and will be denoted by $\tilde{\mathfrak{m}}$.

Recall that $\mathfrak{g}_r = \mathfrak{a} \oplus \mathfrak{u}^+$. Notice that $\mathfrak{u}^+$ is an algebraic Lie subalgebra of $\mathfrak{d}$; hence it follows from [4, Corollary II.7.7] that $\tilde{\mathfrak{g}}_r = \tilde{\mathfrak{a}} \oplus \mathfrak{u}^+$. Thus we only need to describe $\tilde{\mathfrak{a}}$. Since $\mathfrak{t}$ is algebraic we have $\tilde{\mathfrak{a}} \subseteq \mathfrak{t}$ and we are reduced to characterize the algebraic closure of a Lie subalgebra of $\mathfrak{t} = \operatorname{Lie}(T)$.

The group $T = H \times H$ is an algebraic torus (of rank $2n$). The map $\chi \mapsto d\chi$ identifies $\mathbf{X}(T)$ with $\mathbf{L}\times\mathbf{L}$. Let $\mathfrak{k} \subset \mathfrak{t}$ be a subalgebra. We set

$$\mathfrak{k}^\perp = \{\theta \in \mathbf{X}(T) \mid \mathfrak{k} \subset \operatorname{Ker}_{\mathfrak{t}}\theta\}.$$

The following proposition is well known. It can for instance be deduced from the results in [4, II. 8].

Proposition 1.4. *Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{t}$. Then $\tilde{\mathfrak{k}} = \cap_{\theta\in\mathfrak{k}^\perp}\operatorname{Ker}_{\mathfrak{t}}\theta$ and $\tilde{\mathfrak{k}}$ is the Lie algebra of the closed connected algebraic subgroup $\tilde{K} = \cap_{\theta\in\mathfrak{k}^\perp}\operatorname{Ker}_T\theta$.*

Corollary 1.5. *We have*

$$\mathfrak{a}^\perp = \{(\lambda,\mu) \in \mathbf{X}(T) \mid \Phi_+\lambda + \Phi_-\mu = 0\},$$
$$\tilde{\mathfrak{a}} = \cap_{(\lambda,\mu)\in\mathfrak{a}^\perp}\operatorname{Ker}_{\mathfrak{t}}(\lambda,\mu), \quad \tilde{A} = \cap_{(\lambda,\mu)\in\mathfrak{a}^\perp}\operatorname{Ker}_T(\lambda,\mu).$$

Proof. From the definition of $\mathfrak{a} = \vartheta(\mathfrak{h})$ we obtain

$$(\lambda,\mu) \in \mathfrak{a}^\perp \iff \forall\, x \in \mathfrak{h},\ \lambda(-u_-(x)) + \mu(-u_+(x)) = 0.$$

The first equality then follows from ${}^t u_\pm = -\Phi_\mp$. The remaining assertions are consequences of Proposition 1.4. □

Set

$$\mathfrak{h}_{\mathbb{Q}} = \mathbb{Q}\otimes_{\mathbb{Z}}\mathbf{P}^\circ = \oplus_{i=1}^n \mathbb{Q}h_i, \quad \mathfrak{h}^*_{\mathbb{Q}} = \mathbb{Q}\otimes_{\mathbb{Z}}\mathbf{P} = \oplus_{i=1}^n\mathbb{Q}\varpi_i$$
$$\mathfrak{a}^\perp_{\mathbb{Q}} = \mathbb{Q}\otimes_{\mathbb{Z}}\mathfrak{a}^\perp = \{(\lambda,\mu)\in\mathfrak{h}^*_{\mathbb{Q}}\times\mathfrak{h}^*_{\mathbb{Q}} \mid \Phi_+\lambda + \Phi_-\mu = 0\}.$$

Observe that $\dim_{\mathbb{Q}}\mathfrak{a}^\perp_{\mathbb{Q}} = \operatorname{rk}_{\mathbb{Z}}\mathfrak{a}^\perp$ and that, by Corollary 1.5,

$$\dim\tilde{\mathfrak{a}} = 2n - \dim_{\mathbb{Q}}\mathfrak{a}^\perp_{\mathbb{Q}}. \tag{1.4}$$

Lemma 1.6. $\mathfrak{a}^\perp_{\mathbb{Q}} \cong \{\nu\in\mathfrak{h}^*_{\mathbb{Q}} \mid \Phi\nu\in\mathfrak{h}^*_{\mathbb{Q}}\}$.

Proof. Define a $\mathbb{Q}$-linear map

$$\{\nu\in\mathfrak{h}^*_{\mathbb{Q}} \mid \Phi\nu\in\mathfrak{h}^*_{\mathbb{Q}}\} \longrightarrow \mathfrak{a}^\perp_{\mathbb{Q}}, \quad \nu\mapsto(-\Phi_-\nu,\Phi_+\nu).$$

It is easily seen that this provides the desired isomorphism. □

Theorem 1.7. *The following assertions are equivalent:*
(i) $\mathfrak{g}_r$ *is an algebraic Lie subalgebra of* $\mathfrak{d}$*;*
(ii) $u(\mathbf{P} \times \mathbf{P}) \subset \mathbb{Q}$;
(iii) $\exists\, m \in \mathbb{N}^*, \ \ \Phi(m\mathbf{P}) \subset \mathbf{P}$;
(iv) Γ *is a finite subgroup of* T.

Proof. Recall that $\mathfrak{g}_r$ is algebraic if and only if $\mathfrak{a} = \tilde{\mathfrak{a}}$, i.e. $n = \dim \mathfrak{a} = \dim \tilde{\mathfrak{a}}$. By (1.4) and Lemma 1.6 this is equivalent to $\Phi(\mathbf{P}) \subset \mathfrak{h}^*_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{P}$. The equivalence of (i) to (iii) then follows from the definitions, (1.2) and the fact that ${}^t u = -\Phi$.

To prove the equivalence with (iv) we first observe that, by Proposition 1.1, Γ is finite if and only if $\exp u(L_H)$ is finite. Since $L_H/\mathbf{P}^\circ$ is finite this is also equivalent to $\exp u(\mathbf{P}^\circ)$ being finite. This holds if and only if $u(m\mathbf{P}^\circ) \subset \mathbf{P}^\circ$ for some $m \in \mathbb{N}^*$. Hence the result. $\square$

When the equivalent assertions of Theorem 1.7 hold, we shall say that we are in the *algebraic case* or that u *is algebraic*. In this case all the subgroups previously introduced are closed algebraic subgroups of D and we may define the algebraic quotient varieties D/G_r and $\bar{G} = G/\Gamma$. Let p be the projection $G \to \bar{G}$. Observe that $\bar{G}$ is open in in D/G_r and that the Poisson bracket of G passes to $\bar{G}$. We set

$$\mathcal{C}_{\dot{w}} = G_r \dot{w} G_r/G_r, \quad \mathcal{C}_w = QwG_r/G_r = \cup_{h \in H} h\mathcal{C}_{\dot{w}}$$
$$\mathcal{B}_{\dot{w}} = \mathcal{C}_{\dot{w}} \cap \bar{G}, \quad \mathcal{B}_w = \mathcal{C}_w \cap \bar{G}, \quad \mathcal{A}_w = p^{-1}(\mathcal{B}_w).$$

The next theorem summarizes the description of the symplectic leaves in the algebraic case.

Theorem 1.8. 1. $\operatorname{Symp}_w G \neq \emptyset$ *for all* $w \in W \times W$, $\operatorname{Symp} G = \sqcup_{w \in W \times W} \operatorname{Symp}_w G$.
2. *Each symplectic leaf of* $\bar{G}$*, resp.* G*, is of the form* $h\mathcal{B}_{\dot{w}}$*, resp.* $h\mathcal{A}_{\dot{w}}$*, for some* $h \in H$ *and* $w \in W \times W$*, where* $\mathcal{A}_{\dot{w}}$ *denotes a fixed connected component of* $p^{-1}(\mathcal{B}_{\dot{w}})$.
3. $\mathcal{C}_{\dot{w}} \cong A_w \times U_w^-$ *where* $A_w = A/A'_w$ *is a torus of rank* $s(w)$*. Hence* $\dim \mathcal{C}_{\dot{w}} = \dim \mathcal{B}_{\dot{w}} = \dim \mathcal{A}_{\dot{w}} = l(w) + s(w)$ *and* $H/\operatorname{Stab}_H \mathcal{A}_{\dot{w}}$ *is a torus of rank* $n - s(w)$.

Proof. The proofs are similar to those given in [15, Appendix A] for the case $u = 0$. $\square$

2. Deformations of Bigraded Hopf Algebras

2.1. **Bigraded Hopf Algebras and their deformations.** Let $\mathbf{L}$ be an (additive) abelian group. We will say that a Hopf algebra $(A, i, m, \epsilon, \Delta, S)$ over a field $\mathbb{K}$ is an $\mathbf{L}$*-bigraded Hopf algebra* if it is equipped with an $\mathbf{L} \times \mathbf{L}$ grading

$$A = \bigoplus_{(\lambda,\mu) \in \mathbf{L} \times \mathbf{L}} A_{\lambda,\mu}$$

such that

1. $\mathbb{K} \subset A_{0,0}$, $A_{\lambda,\mu} A_{\lambda',\mu'} \subset A_{\lambda+\lambda',\mu+\mu'}$ (i.e. A is a graded algebra)
2. $\Delta(A_{\lambda,\mu}) \subset \sum_{\nu \in \mathbf{L}} A_{\lambda,\nu} \otimes A_{-\nu,\mu}$
3. $\lambda \neq -\mu$ implies $\epsilon(A_{\lambda,\mu}) = 0$
4. $S(A_{\lambda,\mu}) \subset A_{\mu,\lambda}$.

For sake of simplicity we shall often make the following abuse of notation: If $a \in A_{\lambda,\mu}$ we will write $\Delta(a) = \sum_\nu a_{\lambda,\nu} \otimes a_{-\nu,\mu}$, $a_{\lambda,\nu} \in A_{\lambda,\nu}$, $a_{-\nu,\mu} \in A_{-\nu,\mu}$.

Let $p : \mathbf{L} \times \mathbf{L} \to \mathbb{K}^*$ be an antisymmetric bicharacter on $\mathbf{L}$ in the sense that p is multiplicative in both entries and that, for all $\lambda, \mu \in \mathbf{L}$,

$$(1)\ p(\mu,\mu) = 1 \ ; \ \ (2)\ p(\lambda,\mu) = p(\mu,-\lambda).$$

Then the map $\tilde{p} : (\mathbf{L} \times \mathbf{L}) \times (\mathbf{L} \times \mathbf{L}) \to \mathbb{K}^*$ given by

$$\tilde{p}((\lambda,\mu),(\lambda',\mu')) = p(\lambda,\lambda') p(\mu,\mu')^{-1}$$

is a 2-cocycle on $\mathbf{L} \times \mathbf{L}$ such that $\tilde{p}(0,0) = 1$.

One may then define a new multiplication, m_p, on A by

$$\forall\, a \in A_{\lambda,\mu},\ b \in A_{\lambda',\mu'}, \quad a \cdot b = p(\lambda,\lambda')p(\mu,\mu')^{-1}ab. \tag{2.1}$$

Theorem 2.1. *$A_p := (A, i, m_p, \epsilon, \Delta, S)$ is an $\mathbf{L}$-bigraded Hopf algebra.*

Proof. The proof is a slight generalization of that given in [2]. It is well known that $A_p = (A, i, m_p)$ is an associative algebra. Since Δ and ϵ are unchanged, (A, Δ, ϵ) is still a coalgebra. Thus it remains to check that ϵ, Δ are algebra morphisms and that S is an antipode.

Let $x \in A_{\lambda,\mu}$ and $y \in A_{\lambda',\mu'}$. Then

$$\begin{aligned}
\epsilon(x \cdot y) &= p(\lambda,\lambda')p(\mu,\mu')^{-1}\epsilon(xy)\\
&= p(\lambda,\lambda')p(\mu,\mu')^{-1}\delta_{\lambda,-\mu}\delta_{\lambda',-\mu'}\epsilon(x)\epsilon(y)\\
&= p(\lambda,\lambda')p(-\lambda,-\lambda')^{-1}\epsilon(x)\epsilon(y)\\
&= \epsilon(x)\epsilon(y)
\end{aligned}$$

So ϵ is a homomorphism. Now suppose that $\Delta(x) = \sum x_{\lambda,\nu} \otimes x_{-\nu,\mu}$ and $\Delta(y) = \sum y_{\lambda',\nu'} \otimes y_{-\nu',\mu'}$. Then

$$\begin{aligned}
\Delta(x) \cdot \Delta(y) &= (\sum x_{\lambda,\nu} \otimes x_{-\nu,\mu}) \cdot (\sum y_{\lambda',\nu'} \otimes y_{-\nu',\mu'})\\
&= \sum x_{\lambda,\nu} \cdot y_{\lambda',\nu'} \otimes x_{-\nu,\mu} \cdot y_{-\nu',\mu'}\\
&= p(\lambda,\lambda')p(\mu,\mu')^{-1} \sum p(\nu,\nu')^{-1}p(-\nu,-\nu')x_{\lambda,\nu}y_{\lambda',\nu'} \otimes x_{-\nu,\mu}y_{-\nu',\mu'}\\
&= p(\lambda,\lambda')p(\mu,\mu')^{-1}\Delta(xy)\\
&= \Delta(x \cdot y)
\end{aligned}$$

So Δ is also a homomorphism. Finally notice that

$$\begin{aligned}
\sum S(x_{(1)}) \cdot x_{(2)} &= \sum S(x_{\lambda,\nu}) \cdot x_{-\nu,\mu}\\
&= \sum p(\nu,-\nu)p(\lambda,\mu)^{-1}S(x_{\lambda,\nu})x_{-\nu,\mu}\\
&= p(\lambda,\mu)^{-1} \sum S(x_{\lambda,\nu}) \cdot x_{-\nu,\mu}\\
&= p(\lambda,\mu)^{-1}\epsilon(x)\\
&= \epsilon(x)
\end{aligned}$$

A similar calculation shows that $\sum x_{(1)} \cdot S(x_{(2)}) = \epsilon(x)$. Hence S is indeed an antipode. □

Remark . The isomorphism class of the algebra A_p depends only on the cohomology class $[\tilde{p}] \in H^2(\mathbf{L} \times \mathbf{L}, \mathbb{K}^*)$, [2, §3].

Remark . Theorem 2.1 is a particular case of the following more general construction. Let (A, i, m) be a $\mathbb{K}$-algebra. Assume that $F \in GL_{\mathbb{K}}(A \otimes A)$ is given such that (with the usual notation)

(1) $F(m \otimes 1) = (m \otimes 1)F_{23}F_{13}$; $F(1 \otimes m) = (1 \otimes m)F_{12}F_{13}$
(2) $F(i \otimes 1) = i \otimes 1$; $F(1 \otimes i) = 1 \otimes i$
(3) $F_{12}F_{13}F_{23} = F_{23}F_{13}F_{12}$, i.e. F satisfies the Quantum Yang-Baxter Equation.

Set $m_F = m \circ F$. Then (A, i, m_F) is a $\mathbb{K}$-algebra.
Assume furthermore that $(A, i, m, \epsilon, \Delta, S)$ is a Hopf algebra and that

(4) $F : A \otimes A \to A \otimes A$ is morphism of coalgebras
(5) $mF(S \otimes 1)\Delta = m(S \otimes 1)\Delta$; $mF(1 \otimes S)\Delta = m(1 \otimes S)\Delta$.

Then $A_F := (A, i, m_F, \epsilon, \Delta, S)$ is a Hopf algebra. The proofs are straightforward verifications and are left to the interested reader.

When A is an $\mathbf{L}$-bigraded Hopf algebra and p is an antisymmetric bicharacter as above, we may define $F \in GL_{\mathbb{K}}(A \otimes A)$ by

$$\forall a \in A_{\lambda,\mu},\ \forall b \in A_{\lambda',\mu'},\ F(a \otimes b) = p(\lambda, \lambda')p(\mu, \mu')^{-1} a \otimes b.$$

It is easily checked that F satisfies the conditions (1) to (5) and that the Hopf algebras A_F and A_p coincide.

A related construction of the quantization of a monoidal category is given in [24].

2.2. **Diagonalizable subgroups of $R(A)$.** In the case where $\mathbf{L}$ is a finitely generated group and A is a finitely generated algebra (which is the case for the multi-parameter quantum groups considered here), there is a simple geometric interpretation of $\mathbf{L}$-bigradings. They correspond to algebraic group maps from the diagonalizable group $\mathbf{L}^\vee$ to the group of one dimensional representations of A.

Assume that $\mathbb{K}$ is algebraically closed. Let $(A, i, m, \epsilon, \Delta, S)$ be a Hopf $\mathbb{K}$-algebra. Denote by $R(A)$ the multiplicative group of one dimensional representations of A, i.e. the character group of the algebra A. Notice that when A is a finitely generated $\mathbb{K}$-algebra, $R(A)$ has the structure of an affine algebraic group over $\mathbb{K}$, with algebra of regular functions given by $\mathbb{K}[R(A)] = A/J$ where J is the semi-prime ideal $\cap_{h \in R(A)} \operatorname{Ker} h$. Recall that there are two natural group homomorphisms $l, r : R(A) \to \operatorname{Aut}_{\mathbb{K}}(A)$ given by

$$l_h(x) = \sum h(S(x_{(1)}))x_{(2)} = \sum h^{-1}(x_{(1)})x_{(2)}$$
$$r_h(x) = \sum x_{(1)}h(x_{(2)}).$$

Theorem 2.2. *Let A be a finitely generated Hopf algebra and let $\mathbf{L}$ be a finitely generated abelian group. Then there is a natural bijection between:*

1. *$\mathbf{L}$-bigradings on A;*
2. *Hopf algebra maps $A \to \mathbb{K}\mathbf{L}$ (where $\mathbb{K}\mathbf{L}$ denotes the group algebra);*
3. *morphisms of algebraic groups $\mathbf{L}^\vee \to R(A)$.*

Proof. The bijection of the last two sets of maps is well-known. Given an $\mathbf{L}$-bigrading on A, we may define a map $\phi : A \to \mathbb{K}\mathbf{L}$ by $\phi(a_{\lambda,\mu}) = \epsilon(a)u_\lambda$. It is easily verified that this is a Hopf algebra map. Conversely, given a map $\mathbf{L}^\vee \to R(A)$ we may construct an $\mathbf{L}$ bigrading using the following result.

Theorem 2.3. *Let $(A, i, m, \epsilon, \Delta, S)$ be a finitely generated Hopf algebra over $\mathbb{K}$. Let H be a closed diagonalizable algebraic subgroup of $R(A)$. Denote by $\mathbf{L}$ the (additive) group of characters of H and by $\langle -, - \rangle : \mathbf{L} \times H \to \mathbb{K}^*$ the natural pairing. For $(\lambda, \mu) \in \mathbf{L} \times \mathbf{L}$ set*

$$A_{\lambda,\mu} = \{x \in A \mid \forall h \in H,\ l_h(x) = \langle \lambda, h \rangle x,\ r_h(x) = \langle \mu, h \rangle x\}.$$

Then $(A, i, m, \epsilon, \Delta, S)$ is an $\mathbf{L}$-bigraded Hopf algebra.

Proof. Recall that any element of A is contained in a finite dimensional subcoalgebra of A. Therefore the actions of H via r and l are locally finite. Since they commute and H is diagonalizable, A is $\mathbf{L} \times \mathbf{L}$ diagonalizable. Thus the decomposition $A = \bigoplus_{(\lambda,\mu) \in \mathbf{L} \times \mathbf{L}} A_{\lambda,\mu}$ is a grading.

Now let C be a finite dimensional subcoalgebra of A and let $\{c_1, \dots, c_n\}$ be a basis of $H \times H$ weight vectors. Suppose that $\Delta(c_i) = \sum t_{ij} \otimes c_j$. Then since $c_i = \sum t_{ij}\epsilon(c_j)$, the t_{ij} span C and it is easily checked that $\Delta(t_{ij}) = \sum t_{ik} \otimes t_{kj}$. Since $l_h(c_i) = \sum h^{-1}(t_{ij})c_j$ for all $h \in H$ and the c_i are weight vectors, we must have that $h(t_{ij}) = 0$ for $i \neq j$. This implies that

$$l_h(t_{ij}) = h^{-1}(t_{ii})t_{ij}, \quad r_h(t_{ij}) = h(t_{jj})t_{ij}$$

and that the map $\lambda_i(h) = h(t_{ii})$ is a character of H. Thus $t_{ij} \in A_{-\lambda_i,\lambda_j}$ and hence

$$\Delta(t_{ij}) = \sum t_{ik} \otimes t_{kj} \in \sum A_{-\lambda_i,\lambda_k} \otimes A_{-\lambda_k,\lambda_j}.$$

This gives the required condition on Δ. If $\lambda+\mu \neq 0$ then there exists an $h \in H$ such that $\langle -\lambda, h\rangle \neq \langle \mu, h\rangle$. Let $x \in A_{\lambda,\mu}$. Then

$$\langle \mu, h\rangle \epsilon(x) = \epsilon(r_h(x)) = h(x) = \epsilon(l_{h^{-1}}(x)) = \langle -\lambda, h\rangle \epsilon(x).$$

Hence $\epsilon(x) = 0$. The assertion on S follows similarly. □

Remark . In particular, if G is any algebraic group and H is a diagonalizable subgroup with character group $\mathbf{L}$, then we may deform the Hopf algebra $\mathbb{K}[G]$ using an antisymmetric bicharacter on $\mathbf{L}$. Such deformations are algebraic analogs of the deformations studied by Rieffel in [27].

2.3. **Deformations of dual pairs.** Let A and U be a dual pair of Hopf algebras. That is, there exists a bilinear pairing $\langle \mid \rangle : A \times U \to \mathbb{K}$ such that:

1. $\langle a \mid 1\rangle = \epsilon(a)$; $\langle 1 \mid u\rangle = \epsilon(u)$
2. $\langle a \mid u_1u_2\rangle = \sum \langle a_{(1)} \mid u_1\rangle\langle a_{(2)} \mid u_2\rangle$
3. $\langle a_1a_2 \mid u\rangle = \sum \langle a_1 \mid u_{(1)}\rangle\langle a_2 \mid u_{(2)}\rangle$
4. $\langle S(a) \mid u\rangle = \langle a \mid S(u)\rangle$.

Assume that A is bigraded by $\mathbf{L}$, U is bigraded by an abelian group $\mathbf{Q}$ and that there is a homomorphism $\breve{\ } : \mathbf{Q} \to \mathbf{L}$ such that

$$\langle A_{\lambda,\mu} \mid U_{\gamma,\delta}\rangle \neq 0 \;\text{ only if }\; \lambda + \mu = \breve{\gamma} + \breve{\delta}. \tag{2.2}$$

In this case we will call the pair $\{A, U\}$ an $\mathbf{L}$-*bigraded dual pair.* We shall be interested in §3 and §4 in the case where $\mathbf{Q} = \mathbf{L}$ and $\breve{\ } = Id$.

Remark . Suppose that the bigradings above are induced from subgroups H and $\breve{H}$ of $R(A)$ and $R(U)$ respectively and that the map $\mathbf{Q} \to \mathbf{L}$ is induced from a map $h \mapsto \breve{h}$ from H to $\breve{H}$. Then the condition on the pairing may be restated as the fact that the form is ad-invariant in the sense that for all $a \in A$, $u \in U$ and $h \in H$,

$$\langle \mathrm{ad}_h\, a \mid u\rangle = \langle a \mid \mathrm{ad}_{\breve{h}}\, u\rangle$$

where $\mathrm{ad}_h\, a = r_h l_h(a)$.

Theorem 2.4. *Let $\{A, U\}$ be the bigraded dual pair as described above. Let p be an antisymmetric bicharacter on $\mathbf{L}$ and let $\breve{p}$ be the induced bicharacter on $\mathbf{Q}$. Define a bilinear form $\langle \mid \rangle_p : A_{p^{-1}} \times U_{\breve{p}} \to \mathbb{K}$ by:*

$$\langle a_{\lambda,\mu} \mid u_{\gamma,\delta}\rangle_p = p(\lambda, \breve{\gamma})^{-1} p(\mu, \breve{\delta})^{-1} \langle a_{\lambda,\mu} \mid u_{\gamma,\delta}\rangle.$$

Then $\langle \mid \rangle_p$ is a Hopf pairing and $\{A_{p^{-1}}, U_{\breve{p}}\}$ is an $\mathbf{L}$-bigraded dual pair.

Proof. Let $a \in A_{\lambda,\mu}$ and let $u_i \in U_{\gamma_i,\delta_i}$, $i = 1, 2$. Then

$$\langle a \mid u_1u_2\rangle_p = p(\breve{\gamma}_1, \breve{\gamma}_2) p(\breve{\delta}_1, \breve{\delta}_2)^{-1} p(\lambda, \breve{\gamma}_1 + \breve{\gamma}_2)^{-1} p(\mu, \breve{\delta}_1 + \breve{\delta}_2)^{-1} \langle a \mid u_1u_2\rangle.$$

Suppose that $\Delta(a) = \sum_\nu a_{\lambda,\nu} \otimes a_{-\nu,\mu}$. Then by the assumption on the pairing, the only possible value of ν for which $\langle a_{\lambda,\nu} \mid u_1\rangle\langle a_{-\nu,\mu} \mid u_2\rangle$ is non-zero is $\nu = \breve{\gamma}_1 + \breve{\delta}_1 - \lambda = \mu - \breve{\gamma}_2 - \breve{\delta}_2$. Therefore

$$\begin{aligned}
\langle a_{(1)} \mid u_1\rangle_p \langle a_{(2)} \mid u_2\rangle_p &= p(\lambda, \breve{\gamma}_1)^{-1} p(\nu, \breve{\delta}_1)^{-1} p(-\nu, \breve{\gamma}_2)^{-1} p(\mu, \breve{\delta}_2)^{-1} \langle a_{(1)} \mid u_1\rangle\langle a_{(2)} \mid u_2\rangle \\
&= p(\lambda, \breve{\gamma}_1)^{-1} p(\mu - \breve{\gamma}_2 - \breve{\delta}_2, \breve{\delta}_1)^{-1} p(\lambda - \breve{\gamma}_1 - \breve{\delta}_1, \breve{\gamma}_2)^{-1} p(\mu, \breve{\delta}_2)^{-1} \langle a_{(1)} \mid u_1\rangle\langle a_{(2)} \mid u_2\rangle \\
&= p(\breve{\gamma}_1, \breve{\gamma}_2) p(\breve{\delta}_1, \breve{\delta}_2)^{-1} p(\lambda, \breve{\gamma}_1 + \breve{\gamma}_2)^{-1} p(\mu, \breve{\delta}_1 + \breve{\delta}_2)^{-1} \langle a \mid u_1u_2\rangle = \langle a \mid u_1u_2\rangle_p.
\end{aligned}$$

This proves the first axiom. The others are verified similarly. □

Corollary 2.5. *Let $\{A, U, p\}$ be as in Theorem 2.4. Let M be a right A-comodule with structure map $\rho : M \to M \otimes A$. Then M is naturally endowed with U and $U_{\breve{p}}$ left module structures, denoted by $(u, x) \mapsto ux$ and $(u, x) \mapsto u \cdot x$ respectively. Assume that $M = \oplus_{\lambda \in \mathbf{L}} M_\lambda$ for some $\mathbb{K}$-subspaces such that $\rho(M_\lambda) \subset \sum_\nu M_{-\nu} \otimes A_{\nu,\lambda}$. Then we have $U_{\gamma,\delta} M_\lambda \subset M_{\lambda - \breve{\gamma} - \breve{\delta}}$ and the two structures are related by*

$$\forall u \in U_{\gamma,\delta},\ \forall x \in M_\lambda, \quad u \cdot x = p(\lambda, \breve{\gamma} - \breve{\delta}) p(\breve{\gamma}, \breve{\delta}) ux.$$

Proof. Notice that the coalgebras A and $A_{p^{-1}}$ are the same. Set $\rho(x) = \sum x_{(0)} \otimes x_{(1)}$ for all $x \in M$. Then it is easily checked that the following formulas define the desired U and $U_{\breve{p}}$ module structures:

$$\forall u \in U, \quad ux = \sum x_{(0)} \langle x_{(1)} \mid u \rangle, \quad u \cdot x = \sum x_{(0)} \langle x_{(1)} \mid u \rangle_p.$$

When $x \in M_\lambda$ and $u \in U_{\gamma,\delta}$ the additional condition yields

$$u \cdot x = \sum x_{(0)} p(\nu, -\breve{\gamma}) p(\lambda, -\breve{\delta}) \langle x_{(1)} \mid u \rangle.$$

But $\langle x_{(1)} \mid u \rangle \neq 0$ forces $-\nu = \lambda - \breve{\gamma} - \breve{\delta}$, hence $u \cdot x = p(\lambda, \breve{\gamma} - \breve{\delta}) p(\breve{\gamma}, \breve{\delta}) \sum x_{(0)} \langle x_{(1)} \mid u \rangle = p(\lambda, \breve{\gamma} - \breve{\delta}) p(\breve{\gamma}, \breve{\delta}) ux$. ☐

Denote by A^{op} the opposite algebra of the $\mathbb{K}$-algebra A. Let $\{A^{\mathrm{op}}, U, \langle\ \mid\ \rangle\}$ be a dual pair of Hopf algebras. The double $A \bowtie U$ is defined as follows, [10, 3.3]. Let I be the ideal of the tensor algebra $T(A \otimes U)$ generated by elements of type

(a) $$1 - 1_A, \quad 1 - 1_U$$

(b) $$xx' - x \otimes x',\ x, x' \in A, \quad yy' - y \otimes y',\ y, y' \in U$$

(c) $$x_{(1)} \otimes y_{(1)} \langle x_{(2)} \mid y_{(2)} \rangle - \langle x_{(1)} \mid y_{(1)} \rangle y_{(2)} \otimes x_{(2)},\ x \in A,\ y \in U$$

Then the algebra $A \bowtie U := T(A \otimes U)/I$ is called *the Drinfeld double of* $\{A, U\}$. It is a Hopf algebra in a natural way:

$$\Delta(a \otimes u) = (a_{(1)} \otimes u_{(1)}) \otimes (a_{(2)} \otimes u_{(2)}),$$
$$\epsilon(a \otimes u) = \epsilon(a)\epsilon(u), \quad S(a \otimes u) = (S(a) \otimes 1)(1 \otimes S(u)).$$

Notice for further use that $A \bowtie U$ can equally be defined by relations of type (a), (b), ($c_{x,y}$) or (a), (b), ($c_{y,x}$), where we set

($c_{x,y}$) $$x \otimes y = \langle x_{(1)} \mid y_{(1)} \rangle \langle x_{(3)} \mid S(y_{(3)}) \rangle y_{(2)} \otimes x_{(2)},\ x \in A,\ y \in U$$

($c_{y,x}$) $$y \otimes x = \langle x_{(1)} \mid S(y_{(1)}) \rangle \langle x_{(3)} \mid y_{(3)} \rangle x_{(2)} \otimes y_{(2)},\ x \in A,\ y \in U$$

Theorem 2.6. *Let $\{A^{op}, U\}$ be an $\mathbf{L}$-bigraded dual pair, p be an antisymmetric bicharacter on $\mathbf{L}$ and $\breve{p}$ be the induced bicharacter on $\mathbf{Q}$. Then $A \bowtie U$ inherits an $\mathbf{L}$-bigrading and there is a natural isomorphism of $\mathbf{L}$-bigraded Hopf algebras:*

$$(A \bowtie U)_p \cong A_p \bowtie U_{\breve{p}}.$$

Proof. Recall that as a $\mathbb{K}$-vector space $A \bowtie U$ identifies with $A \otimes U$. Define an $\mathbf{L}$-bigrading on $A \bowtie U$ by

$$\forall \alpha, \beta \in \mathbf{L}, \quad (A \bowtie U)_{\alpha,\beta} = \sum_{\lambda - \breve{\gamma} = \alpha, \mu - \breve{\delta} = \beta} A_{\lambda,\mu} \otimes U_{\gamma,\delta}.$$

To verify that this yields a structure of graded algebra on $A \bowtie U$ it suffices to check that the defining relations of $A \bowtie U$ are homogeneous. This is clear for relations of type (a) or (b). Let $x_{\lambda,\mu} \in A_{\lambda,\mu}$ and $y_{\gamma,\delta} \in U_{\gamma,\delta}$. Then the corresponding relation of type (c) becomes

($\star$) $$\sum_{\nu,\xi} x_{\lambda,\nu} y_{\gamma,\xi} \langle x_{-\nu,\mu} \mid y_{-\xi,\delta} \rangle - \langle x_{\lambda,\mu} \mid y_{\gamma,\xi} \rangle y_{-\xi,\delta} x_{-\nu,\mu}.$$

When a term of this sum is non-zero we obtain $-\nu+\mu = -\breve{\xi}+\breve{\delta}$, $\lambda+\nu = \breve{\gamma}+\breve{\xi}$. Hence $\lambda-\breve{\gamma} = -\nu+\breve{\xi} = -\mu+\breve{\delta}$, which shows that the relation $(\star)$ is homogeneous. It is easy to see that the conditions (2), (3), (4) of 2.1 hold. Hence $A \bowtie U$ is an $\mathbf{L}$-bigraded Hopf algebra.

Notice that $(A_p)^{\mathrm{op}} \cong (A^{\mathrm{op}})_{p^{-1}}$, so that Theorem 2.4 defines a suitable pairing between $(A_p)^{\mathrm{op}}$ and $U_{\breve{p}}$. Thus $A_p \bowtie U_{\breve{p}}$ is defined. Let ϕ be the natural surjective homomorphism from $T(A \otimes U)$ onto $A_p \bowtie U_{\breve{p}}$. To check that ϕ induces an isomorphism it again suffices to check that ϕ vanishes on the defining relations of $(A \bowtie U)_p$. Again, this is easy for relations of type (a) and (b). The relation $(\star)$ says that

$$p(\lambda, \breve{\gamma})p(-\nu, \breve{\xi})\langle x_{-\nu,\mu} \mid y_{-\xi,\delta}\rangle x_{\lambda,\nu} \cdot y_{\gamma,\xi} - p(\breve{\xi}, \nu)p(\breve{\delta}, -\mu)\langle x_{\lambda,\mu} \mid y_{\gamma,\xi}\rangle y_{-\xi,\delta} \cdot x_{-\nu,\mu} = 0$$

in $(A \bowtie U)_p$. Multiply the left hand side of this equation by $p(\lambda, -\breve{\gamma})p(\mu, -\breve{\delta})$ and apply ϕ. We obtain the following expression in $A_p \bowtie U_{\breve{p}}$:

$$p(-\nu, \breve{\xi})p(\mu, -\breve{\delta})\langle x_{-\nu,\mu} \mid y_{-\xi,\delta}\rangle x_{\lambda,\nu} y_{\gamma,\xi} - p(\lambda, -\breve{\gamma})p(\nu, -\breve{\xi})\langle x_{\lambda,\mu} \mid y_{\gamma,\xi}\rangle y_{-\xi,\delta} x_{-\nu,\mu}$$

which is equal to

$$\langle x_{-\nu,\mu} \mid y_{-\xi,\delta}\rangle_p x_{\lambda,\nu} y_{\gamma,\xi} - \langle x_{\lambda,\mu} \mid y_{\gamma,\xi}\rangle_p y_{-\xi,\delta} x_{-\nu,\mu}.$$

But this is a defining relation of type (c) in $A_p \bowtie U_{\breve{p}}$, hence zero.

It remains to see that ϕ induces an isomorphism of Hopf algebras, which is a straightforward consequence of the definitions. □

2.4. **Cocycles.** Let $\mathbf{L}$ be, in this section, an arbitrary free abelian group with basis $\{\omega_1, \ldots, \omega_n\}$ and set $\mathfrak{h}^* = \mathbb{C} \otimes_{\mathbb{Z}} \mathbf{L}$. We freely use the terminology of [2]. Recall that $H^2(\mathbf{L}, \mathbb{C}^*)$ is in bijection with the set $\mathcal{H}$ of multiplicatively antisymmetric $n \times n$-matrices $\gamma = [\gamma_{ij}]$. This bijection maps the class $[c]$ onto the matrix defined by $\gamma_{ij} = c(\omega_i, \omega_j)/c(\omega_j, \omega_i)$. Furthermore it is an isomorphism of groups with respect to component-wise multiplication of matrices.

Remark . The notation is as in 2.1. We recalled that the isomorphism class of the algebra A_p depends only on the cohomology class $[\tilde{p}] \in H^2(\mathbf{L} \times \mathbf{L}, \mathbb{K}^*)$. Let $\gamma \in \mathcal{H}$ be the matrix associated to p and γ^{-1} its inverse in $\mathcal{H}$. Notice that the multiplicative matrix associated to $[\tilde{p}]$ is then $\tilde{\gamma} = \left[\begin{smallmatrix} \gamma & 1 \\ 1 & \gamma^{-1} \end{smallmatrix}\right]$ in the basis given by the $(\omega_i, 0), (0, \omega_i) \in \mathbf{L} \times \mathbf{L}$. Therefore the isomorphism class of the algebra A_p depends only on the cohomology class $[p] \in H^2(\mathbf{L}, \mathbb{K}^*)$.

Let $\hbar \in \mathbb{C}^*$. If $x \in \mathbb{C}$ we set $q^x = \exp(-x\hbar/2)$. In particular $q = \exp(-\hbar/2)$. Let $u : \mathbf{L} \times \mathbf{L} \to \mathbb{C}$ be a complex alternating $\mathbb{Z}$-bilinear form. Define

$$p : \mathbf{L} \times \mathbf{L} \to \mathbb{C}^*, \quad p(\lambda, \mu) = \exp\big(-\frac{\hbar}{4}u(\lambda, \mu)\big) = q^{\frac{1}{2}u(\lambda,\mu)}. \tag{2.3}$$

Then it is clear that p is an antisymmetric bicharacter on $\mathbf{L}$.

Observe that, since $\mathfrak{h}^* = \mathbb{C} \otimes_{\mathbb{Z}} \mathbf{L}$, there is a natural isomorphism of additive groups between $\wedge^2\mathfrak{h}$ and the group of complex alternating $\mathbb{Z}$-bilinear forms on $\mathbf{L}$, where $\mathfrak{h}$ is the $\mathbb{C}$-dual of $\mathfrak{h}^*$. Set $\mathcal{Z}_\hbar = \{u \in \wedge^2\mathfrak{h} \mid u(\mathbf{L} \times \mathbf{L}) \subset \frac{4i\pi}{\hbar}\mathbb{Z}\}$.

Theorem 2.7. *There are isomorphisms of abelian groups:*

$$H^2(\mathbf{L}, \mathbb{C}^*) \cong \mathcal{H} \cong \wedge^2\mathfrak{h}/\mathcal{Z}_\hbar.$$

Proof. The first isomorphism has been described above. Let $\gamma = [\gamma_{ij}] \in \mathcal{H}$ and choose u_{ij}, $1 \le i < j \le n$ such that $\gamma_{ij} = \exp(-\frac{\hbar}{2}u_{ij})$. We can define $u \in \wedge^2\mathfrak{h}$ by setting $u(\omega_i, \omega_j) = u_{ij}$, $1 \le i < j \le n$. It is then easily seen that one can define an injective morphism of abelian groups

$$\varphi : H^2(\mathbf{L}, \mathbb{C}^*) \cong \mathcal{H} \longrightarrow \wedge^2\mathfrak{h}/\mathcal{Z}_\hbar, \quad \varphi(\gamma) = [u]$$

where $[u]$ is the class of u. If $u \in \wedge^2\mathfrak{h}$, define a 2-cocycle p by the formula (2.3). Then the multiplicative matrix associated to $[p] \in H^2(\mathbf{L}, \mathbb{C}^*)$ is given by

$$\gamma_{ij} = p(\omega_i, \omega_j)/p(\omega_j, \omega_i) = p(\omega_i, \omega_j)^2 = \exp(-\frac{\hbar}{2}u(\omega_i, \omega_j)).$$

This shows that $[u]=\varphi([\gamma_{ij}])$; thus φ is an isomorphism. $\square$

We list some consequences of Theorem 2.7. We denote by $[u]$ an element of $\wedge^2\mathfrak{h}/\mathcal{Z}_\hbar$ and we set $[p]=\varphi^{-1}([u])$. We have seen that we can define a representative p by the formula (2.3).

1. $[p]$ of finite order in $H^2(\mathbf{L},\mathbb{C}^*)\Leftrightarrow u(\mathbf{L}\times\mathbf{L})\subset\frac{i\pi}{\hbar}\mathbb{Q}$, and q root of unity $\Leftrightarrow \hbar\in i\pi\mathbb{Q}$.

2. Notice that $u=0$ is algebraic, whether q is a root of unity or not. Assume that q is a root of unity; then we get from 1 that

$$[p]\ \text{ of finite order} \Leftrightarrow u \text{ is algebraic}.$$

3. Assume that q is not a root of unity and that $u\neq 0$. Then $[p]$ of finite order implies $(0)\neq u(\mathbf{L}\times\mathbf{L})\subset\frac{i\pi}{\hbar}\mathbb{Q}$. This shows that

$$0\neq u \text{ algebraic} \Rightarrow [p]\ \text{ is not of finite order}.$$

Definition . The bicharacter $p:(\lambda,\mu)\mapsto q^{\frac{1}{2}u(\lambda,\mu)}$ is called q-rational if $u\in\wedge^2\mathfrak{h}$ is algebraic.

The following technical result will be used in the next section. Recall the definition of $\Phi_-=\Phi-I$ given in the Section 1.

Proposition 2.8. *Let* $\mathbf{K}=\left\{\lambda\in\mathbf{L}\,:\,(\Phi_-\lambda,\mathbf{L})\subset\frac{4i\pi}{\hbar}\mathbb{Z}\right\}$. *If q is not a root of unity, then* $\mathbf{K}=0$.

Proof. Let $\lambda\in\mathbf{K}$. We can define $z:\mathfrak{h}^*_{\mathbb{Q}}\to\mathbb{Q}$, by

$$\forall\,\mu\in\mathfrak{h}^*_{\mathbb{Q}},\quad(\Phi_-\lambda,\mu)=\frac{4i\pi}{\hbar}z(\mu).$$

The map z is clearly $\mathbb{Q}$-linear. It follows, since $(\ ,\ )$ is non-degenerate on $\mathfrak{h}^*_{\mathbb{Q}}$, that there exists $\nu\in\mathfrak{h}^*_{\mathbb{Q}}$ such that $z(\mu)=(\nu,\mu)$ for all $\mu\in\mathfrak{h}^*_{\mathbb{Q}}$. Therefore $\Phi_-\lambda=\frac{4i\pi}{\hbar}\nu$, and so $\Phi\lambda=\lambda+\frac{4i\pi}{\hbar}\nu$.

Now, $(\Phi\lambda,\lambda)=u(\lambda,\lambda)=0$ implies that

$$\frac{4i\pi}{\hbar}(\nu,\lambda)=-(\lambda,\lambda)$$

If $(\lambda,\lambda)\neq 0$ then $\hbar\in i\pi\mathbb{Q}$, contradicting the assumption that q is not a root of unity. Hence $(\lambda,\lambda)=0$, which forces $\lambda=0$ since $\lambda\in\mathbf{L}\subset\mathfrak{h}^*_{\mathbb{Q}}$. $\square$

3. Multiparameter Quantum Groups

3.1. **One-parameter quantized enveloping algebras.** The notation is as in sections 1 and 2. In particular we fix a lattice $\mathbf{L}$ such that $\mathbf{Q}\subset\mathbf{L}\subset\mathbf{P}$ and we denote by G the connected semi-simple algebraic group with maximal torus H such that $\mathrm{Lie}(G)=\mathfrak{g}$ and $\mathbf{X}(H)\cong\mathbf{L}$.

Let $q\in\mathbb{C}^*$ and assume that q *is not a root of unity.* Let $\hbar\in\mathbb{C}\setminus i\pi\mathbb{Q}$ such that $q=\exp(-\hbar/2)$ as in 2.4. We set

$$q_i=q^{d_i},\quad \hat{q}_i=(q_i-q_i^{-1})^{-1},\quad 1\leq i\leq n.$$

Denote by U^0 the group algebra of $\mathbf{X}(H)$, hence

$$U^0=\mathbb{C}[k_\lambda\,;\,\lambda\in\mathbf{L}],\quad k_0=1,\quad k_\lambda k_\mu=k_{\lambda+\mu}.$$

Set $k_i=k_{\alpha_i}$, $1\leq i\leq n$. The one parameter quantized enveloping algebra associated to this data, cf. [33], is the Hopf algebra

$$U_q(\mathfrak{g})=U^0[e_i,f_i\,;\,1\leq i\leq n]$$

with defining relations:

$$k_\lambda e_j k_\lambda^{-1} = q^{(\lambda,\alpha_j)} e_j, \quad k_\lambda f_j k_\lambda^{-1} = q^{-(\lambda,\alpha_j)} f_j$$
$$e_i f_j - f_j e_i = \delta_{ij} \hat{q}_i (k_i - k_i^{-1})$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[{1-a_{ij} \atop k} \right]_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0, \text{ if } i \neq j$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[{1-a_{ij} \atop k} \right]_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0, \text{ if } i \neq j$$

where $[m]_t = (t - t^{-1}) \dots (t^m - t^{-m})$ and $\left[{m \atop k} \right]_t = \frac{[m]_t}{[k]_t [m-k]_t}$. The Hopf algebra structure is given by

$$\Delta(k_\lambda) = k_\lambda \otimes k_\lambda, \quad \epsilon(k_\lambda) = 1, \quad S(k_\lambda) = k_\lambda^{-1}$$
$$\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i$$
$$\epsilon(e_i) = \epsilon(f_i) = 0, \quad S(e_i) = -k_i^{-1} e_i, \quad S(f_i) = -f_i k_i.$$

We define subalgebras of $U_q(\mathfrak{g})$ as follows

$$U_q(\mathfrak{n}^+) = \mathbb{C}[e_i\,,\,;\,1 \leq i \leq n], \quad U_q(\mathfrak{n}^-) = \mathbb{C}[f_i\,,\,;\,1 \leq i \leq n]$$
$$U_q(\mathfrak{b}^+) = U^0[e_i\,,\,;\,1 \leq i \leq n], \quad U_q(\mathfrak{b}^-) = U^0[f_i\,,\,;\,1 \leq i \leq n].$$

For simplicity we shall set $U^\pm = U_q(\mathfrak{n}^\pm)$. Notice that U^0 and $U_q(\mathfrak{b}^\pm)$ are Hopf subalgebras of $U_q(\mathfrak{g})$. Recall [23] that the multiplication in $U_q(\mathfrak{g})$ induces isomorphisms of vector spaces

$$U_q(\mathfrak{g}) \cong U^- \otimes U^0 \otimes U^+ \cong U^+ \otimes U^0 \otimes U^-.$$

Set $\mathbf{Q}_+ = \oplus_{i=1}^n \mathbb{N}\alpha_i$ and

$$\forall \beta \in \mathbf{Q}_+, \quad U^\pm_\beta = \{u \in U^\pm \mid \forall \lambda \in \mathbf{L},\, k_\lambda u k_\lambda^{-1} = q^{(\lambda, \pm\beta)} u\}.$$

Then one gets: $U^\pm = \oplus_{\beta \in \mathbf{Q}_+} U^\pm_{\pm\beta}$.

3.2. **The Rosso-Tanisaki-Killing form.** Recall the following result, [28, 33].

Theorem 3.1. 1. *There exists a unique non degenerate Hopf pairing*

$$\langle \mid \rangle : U_q(\mathfrak{b}^+)^{op} \otimes U_q(\mathfrak{b}^-) \longrightarrow \mathbb{C}$$

satisfying the following conditions:
(i) $\langle k_\lambda \mid k_\mu \rangle = q^{-(\lambda,\mu)}$;
(ii) $\forall \lambda \in \mathbf{L}, 1 \leq i \leq n$, $\langle k_\lambda \mid f_i \rangle = \langle e_i \mid k_\lambda \rangle = 0$;
(iii) $\forall 1 \leq i, j \leq n$, $\langle e_i \mid f_j \rangle = -\delta_{ij} \hat{q}_i$.
2. *If $\gamma, \eta \in \mathbf{Q}_+$, $\langle U^+_\gamma \mid U^-_{-\eta} \rangle \neq 0$ implies $\gamma = \eta$.*

The results of §2.3 then apply and we may define the associated double:

$$D_q(\mathfrak{g}) = U_q(\mathfrak{b}^+) \bowtie U_q(\mathfrak{b}^-).$$

It is well known, e.g. [10], that

$$D_q(\mathfrak{g}) = \mathbb{C}[s_\lambda, t_\lambda, e_i, f_i\,;\, \lambda \in \mathbf{L}, 1 \leq i \leq n]$$

where $s_\lambda = k_\lambda \otimes 1$, $t_\lambda = 1 \otimes k_\lambda$, $e_i = e_i \otimes 1$, $f_i = 1 \otimes f_i$. The defining relations of the double given in §2.3 imply that

$$s_\lambda t_\mu = t_\mu s_\lambda, \quad e_i f_j - f_j e_i = \delta_{ij} \hat{q}_i (s_{\alpha_i} - t_{\alpha_i}^{-1})$$
$$s_\lambda e_j s_\lambda^{-1} = q^{(\lambda,\alpha_j)} e_j,\ t_\lambda e_j t_\lambda^{-1} = q^{(\lambda,\alpha_j)} e_j,\ s_\lambda f_j s_\lambda^{-1} = q^{-(\lambda,\alpha_j)} f_j,\ t_\lambda f_j t_\lambda^{-1} = q^{-(\lambda,\alpha_j)} f_j.$$

It follows that

$$D_q(\mathfrak{g})/(s_\lambda - t_\lambda\,;\, \lambda \in \mathbf{L}) \xrightarrow{\sim} U_q(\mathfrak{g}),\ e_i \mapsto e_i,\ f_i \mapsto f_i,\ s_\lambda \mapsto k_\lambda,\ t_\lambda \mapsto k_\lambda.$$

Observe that this yields an isomorphism of Hopf algebras. The next proposition collects some well known elementary facts.

Proposition 3.2. 1. *Any finite dimensional simple $U_q(\mathfrak{b}^\pm)$-module is one dimensional and $R(U_q(\mathfrak{b}^\pm))$ identifies with H via*

$$\forall h \in H, \quad h(k_\lambda) = \langle \lambda, h\rangle, \quad h(e_i) = 0, \quad h(f_i) = 0.$$

2. *$R(D_q(\mathfrak{g}))$ identifies with H via*

$$\forall h \in H, \quad h(s_\lambda) = \langle \lambda, h\rangle, \quad h(t_\lambda) = \langle \lambda, h\rangle^{-1}, \quad h(e_i) = h(f_i) = 0.$$

Corollary 3.3. 1. *$\{U_q(\mathfrak{b}^+)^{op}, U_q(\mathfrak{b}^-)\}$ is an $\mathbf{L}$-bigraded dual pair. We have*

$$k_\lambda \in U_q(\mathfrak{b}^\pm)_{-\lambda,\lambda}, \quad e_i \in U_q(\mathfrak{b}^+)_{-\alpha_i,0}, \quad f_i \in U_q(\mathfrak{b}^-)_{0,-\alpha_i}.$$

2. *$D_q(\mathfrak{g})$ is an $\mathbf{L}$-bigraded Hopf algebra where*

$$s_\lambda \in D_q(\mathfrak{g})_{-\lambda,\lambda}, \quad t_\lambda \in D_q(\mathfrak{g})_{\lambda,-\lambda}, \quad e_i \in D_q(\mathfrak{g})_{-\alpha_i,0}, \quad f_i \in D_q(\mathfrak{g})_{0,\alpha_i}.$$

Proof. 1. Observe that for all $h \in H$,

$$l_h(k_\lambda) = h^{-1}(k_\lambda) = \langle -\lambda, h\rangle k_\lambda, \quad r_h(k_\lambda) = h(k_\lambda) = \langle \lambda, h\rangle k_\lambda,$$
$$l_h(e_i) = h^{-1}(k_i)e_i = \langle -\alpha_i, h\rangle e_i, \quad r_h(e_i) = e_i,$$
$$l_h(f_i) = f_i, \quad r_h(f_i) = h(k_i^{-1})f_i = \langle -\alpha_i, h\rangle f_i.$$

It is then clear that $U^+_{-\gamma,0} = U^+_\gamma$ and $U^-_{0,-\gamma} = U^-_{-\gamma}$ for all $\gamma \in \mathbf{Q}_+$. The claims then follow from these formulas, Theorem 2.3, Theorem 3.1, and the definitions.

2. The fact that $D_q(\mathfrak{g})$ is an $\mathbf{L}$-bigraded Hopf algebra follows from Theorem 2.3. The assertions about the $\mathbf{L} \times \mathbf{L}$ degree of the generators is proved by direct computation using Proposition 3.2. □

Remark . We have shown in Theorem 2.6 that, as a double, $D_q(\mathfrak{g})$ inherits an $\mathbf{L}$-bigrading given by:

$$D_q(\mathfrak{g})_{\alpha,\beta} = \sum_{\lambda-\gamma=\alpha, \mu-\delta=\beta} U_q(\mathfrak{b}^+)_{\lambda,\mu} \otimes U_q(\mathfrak{b}^-)_{\gamma,\delta}.$$

It is easily checked that this bigrading coincides with the bigrading obtained in the above corollary by means of Theorem 2.3.

3.3. **One-parameter quantized function algebras.** Let M be a left $D_q(\mathfrak{g})$-module. The dual M^* will be considered in the usual way as a left $D_q(\mathfrak{g})$-module by the rule: $(uf)(x) = f(S(u)x)$, $x \in M, f \in M^*$, $u \in D_q(\mathfrak{g})$. Assume that M is an $U_q(\mathfrak{g})$-module. An element $x \in M$ is said to have *weight* $\mu \in \mathbf{L}$ if $k_\lambda x = q^{(\lambda,\mu)}x$ for all $\lambda \in \mathbf{L}$; we denote by M_μ the subspace of elements of weight μ.

It is known, [13], that the category of finite dimensional (left) $U_q(\mathfrak{g})$-modules is a completely reducible braided rigid monoidal category. Set $\mathbf{L}^+ = \mathbf{L} \cap \mathbf{P}^+$ and recall that for each $\Lambda \in \mathbf{L}^+$ there exists a finite dimensional simple module of highest weight Λ, denoted by $L(\Lambda)$, cf. [29] for instance. One has $L(\Lambda)^* \cong L(w_0\Lambda)$ where w_0 is the longest element of W. (Notice that the results quoted usually cover the case where $\mathbf{L} = \mathbf{Q}$. One defines the modules $L(\lambda)$ in the general case in the following way. Let us denote temporarily the algebra $U_q(\mathfrak{g})$ for a given choice of $\mathbf{L}$ by $U_{q,\mathbf{L}}(\mathfrak{g})$. Given a module $L(\lambda)$ defined on $U_{q,\mathbf{Q}}(\mathfrak{g})$ we may define an action of $U_{q,\mathbf{L}}(\mathfrak{g})$ by setting $k_\lambda.x = q^{(\lambda,\mu)}x$ for all elements x of weight μ, where $q^{(\lambda,\mu)}$ is as defined in section 2.4.)

Let $\mathcal{C}_q$ be the subcategory of finite dimensional $U_q(\mathfrak{g})$-modules consisting of finite direct sums of $L(\Lambda)$, $\Lambda \in \mathbf{L}^+$. The category $\mathcal{C}_q$ is closed under tensor products and the formation of duals. Notice that $\mathcal{C}_q$ can be considered as a braided rigid monoidal category of $D_q(\mathfrak{g})$-modules where s_λ, t_λ act as k_λ on an object of $\mathcal{C}_q$.

Let $M \in \mathrm{obj}(\mathcal{C}_q)$, then $M = \oplus_{\mu\in\mathbf{L}} M_\mu$. For $f \in M^*$, $v \in M$ we define the coordinate function $c_{f,v} \in U_q(\mathfrak{g})^*$ by

$$\forall u \in U_q(\mathfrak{g}), \quad c_{f,v}(u) = \langle f, uv\rangle$$

where $\langle\ ,\ \rangle$ is the duality pairing. Using the standard isomorphism $(M\otimes N)^* \cong N^*\otimes M^*$ one has the following formula for multiplication,

$$c_{f,v}c_{f',v'} = c_{f'\otimes f, v\otimes v'}.$$

Definition . The quantized function algebra $\mathbb{C}_q[G]$ is the restricted dual of $\mathcal{C}_q$: that is to say

$$\mathbb{C}_q[G] = \mathbb{C}[c_{f,v}\ ;\ v\in M, f\in M^*,\ M\in \mathrm{obj}(\mathcal{C}_q)].$$

The algebra $\mathbb{C}_q[G]$ is a Hopf algebra; we denote by Δ, ϵ, S the comultiplication, counit and antipode on $\mathbb{C}_q[G]$. If $\{v_1,\dots,v_s; f_1,\dots,f_s\}$ is a dual basis for $M\in\mathrm{obj}(\mathcal{C}_q)$ one has

$$\Delta(c_{f,v}) = \sum_i c_{f,v_i}\otimes c_{f_i,v}, \quad \epsilon(c_{f,v}) = \langle f, v\rangle, \quad S(c_{f,v}) = c_{v,f}. \tag{3.1}$$

Notice that we may assume that $v_j \in M_{\nu_j}$, $f_j \in M^*_{-\nu_j}$. We set

$$C(M) = \mathbb{C}\langle c_{f,v}\ ;\ f\in M^*,\ v\in M\rangle, \quad C(M)_{\lambda,\mu} = \mathbb{C}\langle c_{f,v}\ ;\ f\in M^*_\lambda,\ v\in M_\mu\rangle.$$

Then $C(M)$ is a subcoalgebra of $\mathbb{C}_q[G]$ such that $C(M) = \bigoplus_{(\lambda,\mu)\in\mathbf{L}\times\mathbf{L}} C(M)_{\lambda,\mu}$. When $M = L(\Lambda)$ we abbreviate the notation to $C(M) = C(\Lambda)$. It is then classical that

$$\mathbb{C}_q[G] = \bigoplus_{\Lambda\in\mathbf{L}^+} C(\Lambda).$$

Since $\mathbb{C}_q[G]\subset U_q(\mathfrak{g})^*$ we have a duality pairing

$$\langle\ ,\ \rangle : \mathbb{C}_q[G]\times D_q(\mathfrak{g}) \longrightarrow \mathbb{C}.$$

Observe that there is a natural injective morphism of algebraic groups

$$H \longrightarrow R(\mathbb{C}_q[G]), \quad h(c_{f,v}) = \langle \mu, h\rangle \epsilon(c_{f,v}) \text{ for all } v\in M_\mu,\ M\in\mathrm{obj}(\mathcal{C}_q).$$

The associated automorphisms $r_h, l_h \in \mathrm{Aut}(\mathbb{C}_q[G])$ are then described by

$$\forall c_{f,v}\in C(M)_{\lambda,\mu}, \quad r_h(c_{f,v}) = \langle\mu,h\rangle c_{f,v},\ l_h(c_{f,v}) = \langle\lambda,h\rangle c_{f,v}.$$

Define

$$\forall(\lambda,\mu)\in\mathbf{L}\times\mathbf{L}, \quad \mathbb{C}_q[G]_{\lambda,\mu} = \{a\in\mathbb{C}_q[G] \mid r_h(a) = \langle\mu,h\rangle a,\ l_h(a) = \langle\lambda,h\rangle a\}.$$

Theorem 3.4. *The pair of Hopf algebras $\{\mathbb{C}_q[G], D_q(\mathfrak{g})\}$ is an $\mathbf{L}$-bigraded dual pair.*

Proof. It follows from (3.1) that $\mathbb{C}_q[G]$ is an $\mathbf{L}$-bigraded Hopf algebra. The axioms (1) to (4) of 2.3 are satisfied by definition of the Hopf algebra $\mathbb{C}_q[G]$. We take $\check{}$ to be the identity map of $\mathbf{L}$. The condition (2.2) is consequence of $D_q(\mathfrak{g})_{\gamma,\delta}M_\mu \subset M_{\mu-\gamma-\delta}$ for all $M\in\mathcal{C}_q$. To verify this inclusion, notice that

$$e_j \in D_q(\mathfrak{g})_{-\alpha_j,0},\ f_j\in D_q(\mathfrak{g})_{0,\alpha_j}, \quad e_jM_\mu\subset M_{\mu+\alpha_j},\ f_jM_\mu\subset M_{\mu-\alpha_j}.$$

The result then follows easily □

Consider the algebras $D_{q^{-1}}(\mathfrak{g})$ and $\mathbb{C}_{q^{-1}}[G]$ and use $\hat{}$ to distinguish elements, subalgebras, etc. of $D_{q^{-1}}(\mathfrak{g})$ and $\mathbb{C}_{q^{-1}}[G]$. It is easily verified that the map $\sigma : D_q(\mathfrak{g})\to D_{q^{-1}}(\mathfrak{g})$ given by

$$s_\lambda\mapsto\hat{s}_\lambda,\ t_\lambda\mapsto\hat{t}_\lambda,\ e_i\mapsto q_i^{1/2}\hat{f}_i\hat{t}_{\alpha_i},\ f_i\mapsto q_i^{1/2}\hat{e}_i\hat{s}_{\alpha_i}^{-1}$$

is an isomorphism of Hopf algebras.

For each $\Lambda \in \mathbf{L}^+$, σ gives a bijection $\sigma : L(-w_0\Lambda) \to \hat{L}(\Lambda)$ which sends $v \in L(-w_0\Lambda)_\mu$ onto $\hat{v} \in \hat{L}(\Lambda)_{-\mu}$. Therefore we obtain an isomorphism $\sigma : \mathbb{C}_{q^{-1}}[G] \to \mathbb{C}_q[G]$ such that

$$\forall\, f \in L(-w_0\Lambda)^*_{-\lambda},\, v \in L(-w_0\Lambda)_\mu, \quad \sigma(\hat{c}_{\hat{f},\hat{v}}) = c_{f,v}. \tag{3.2}$$

Notice that

$$\sigma(D_q(\mathfrak{g})_{\gamma,\delta}) = D_{q^{-1}}(\mathfrak{g})_{-\gamma,-\delta} \ \text{ and } \ \sigma(\mathbb{C}_{q^{-1}}[G]_{\lambda,\mu}) = \mathbb{C}_q[G]_{-\lambda,-\mu}. \tag{3.3}$$

3.4. **Deformation of one-parameter quantum groups.** We continue with the same notation. Let $[p] \in H^2(\mathbf{L}, \mathbb{C}^*)$. As seen in §2.4 we can, and we do, choose p to be an antisymmetric bicharacter such that

$$\forall \lambda, \mu \in \mathbf{L}, \quad p(\lambda,\mu) = q^{\frac{1}{2}u(\lambda,\mu)}$$

for some $u \in \wedge^2\mathfrak{h}$. Recall that $\tilde{p} \in Z^2(\mathbf{L}\times\mathbf{L}, \mathbb{C}^*)$, cf. 2.1.

We now apply the results of §2.1 to $D_q(\mathfrak{g})$ and $\mathbb{C}_q[G]$. Using Theorem 2.1 we can twist $D_q(\mathfrak{g})$ by $\tilde{p}^{-1}$ and $\mathbb{C}_q[G]$ by $\tilde{p}$. The resulting $\mathbf{L}$-bigraded Hopf algebras will be denoted by $D_{q,p^{-1}}(\mathfrak{g})$ and $\mathbb{C}_{q,p}[G]$. The algebra $\mathbb{C}_{q,p}[G]$ will be referred to as the *multi-parameter quantized function algebra.* Versions of $D_{q,p^{-1}}(\mathfrak{g})$ are referred to by some authors as the *multi-parameter quantized enveloping algebra.* Alternatively, this name can be applied to the quotient of $D_{q,p^{-1}}(\mathfrak{g})$ by the radical of the pairing with $\mathbb{C}_{q,p}[G]$.

Theorem 3.5. *Let $U_{q,p^{-1}}(\mathfrak{b}^+)$ and $U_{q,p^{-1}}(\mathfrak{b}^-)$ be the deformations by p^{-1} of $U_q(\mathfrak{b}^+)$ and $U_q(\mathfrak{b}^-)$ respectively. Then the deformed pairing*

$$\langle\ |\ \rangle_{p^{-1}} : U_{q,p^{-1}}(\mathfrak{b}^+)^{op} \otimes U_{q,p^{-1}}(\mathfrak{b}^-) \to \mathbb{C}$$

is a non-degenerate Hopf pairing satisfying:

$$\forall\, x \in U^+,\, y \in U^-,\, \lambda, \mu \in \mathbf{L}, \quad \langle x\cdot k_\lambda \mid y \cdot k_\mu\rangle_{p^{-1}} = q^{(\Phi_-\lambda,\mu)}\langle x \mid y\rangle. \tag{3.4}$$

Moreover,

$$U_{q,p^{-1}}(\mathfrak{b}^+) \bowtie U_{q,p^{-1}}(\mathfrak{b}^-) \cong (U_q(\mathfrak{b}^+) \bowtie U_q(\mathfrak{b}^-))_{p^{-1}} = D_{q,p^{-1}}(\mathfrak{g}).$$

Proof. By Theorem 2.4 the deformed pairing is given by

$$\langle a_{\lambda,\mu} \mid u_{\gamma,\delta}\rangle_{p^{-1}} = p(\lambda,\gamma)p(\mu,\delta)\langle a_{\lambda,\mu} \mid u_{\gamma,\delta}\rangle.$$

To prove (3.4) we can assume that $x \in U^+_{-\gamma,0}$, $y \in U^-_{0,-\nu}$. Then we obtain

$$\begin{aligned}\langle x\cdot k_\lambda \mid y\cdot k_\mu\rangle_{p^{-1}} &= p(\lambda+\gamma,\mu)p(\lambda,\mu-\nu)\langle x\cdot k_\lambda \mid y\cdot k_\mu\rangle \\ &= p(\lambda,2\mu)p(\lambda-\mu,\gamma-\nu)q^{-(\lambda,\mu)}\langle x \mid y\rangle\end{aligned}$$

by the definition of the product $\cdot$ and [33, 2.1.3]. But $\langle x \mid y\rangle = 0$ unless $\gamma = \nu$, hence the result. Observe in particular that $\langle x \mid y\rangle_{p^{-1}} = \langle x \mid y\rangle$. Therefore [33, 2.1.4] shows that $\langle\ |\ \rangle_{p^{-1}}$ is non-degenerate on $U^+_\gamma \times U^-_{-\gamma}$. It then follows from (3.4) and Proposition 2.8 that $\langle\ |\ \rangle_{p^{-1}}$ is non-degenerate. The remaining isomorphism follows from Theorem 2.6. □

Many authors have defined multi-parameter quantized enveloping algebras. In [14, 25] a definition is given using explicit generators and relations, and in [1] the construction is made by twisting the comultiplication, following [26]. It can be easily verified that these algebras and the algebras $D_{q,p^{-1}}(\mathfrak{g})$ coincide. The construction of a multi-parameter quantized function algebra by twisting the multiplication was first performed in the $GL(n)$-case in [2].

The fact that $D_{q,p^{-1}}(\mathfrak{g})$ and $\mathbb{C}_{q,p}[G]$ form a Hopf dual pair has already been observed in particular cases, see e.g. [14]. We will now deduce from the previous results that this phenomenon holds for an arbitrary semi-simple group.

Theorem 3.6. *$\{\mathbb{C}_{q,p}[G], D_{q,p^{-1}}(\mathfrak{g})\}$ is an $\mathbf{L}$-bigraded dual pair. The associated pairing is given by*

$$\forall a \in \mathbb{C}_{q,p}[G]_{\lambda,\mu},\, \forall u \in D_{q,p^{-1}}(\mathfrak{g})_{\gamma,\delta}, \quad \langle a, u\rangle_p = p(\lambda,\gamma)p(\mu,\delta)\langle a, u\rangle.$$

Proof. This follows from Theorem 2.4 applied to the pair $\{A, U\} = \{\mathbb{C}_q[G], D_q(\mathfrak{g})\}$ and the bicharacter p^{-1} (recall that the map $\breve{}$ is the identity). □

Let $M \in \mathrm{obj}(\mathcal{C}_q)$. The left $D_q(\mathfrak{g})$-module structure on M yields a right $\mathbb{C}_q[G]$-comodule structure in the usual way. Let $\{v_1, \ldots, v_s; f_1, \ldots, f_s\}$ be a dual basis for M. The structure map $\rho : M \to M \otimes \mathbb{C}_q[G]$, is given by $\rho(x) = \sum_j v_j \otimes c_{f_j,x}$ for $x \in M$. Using this comodule structure on M, one can check that

$$M_\mu = \{x \in M \mid \forall h \in H,\ r_h(x) = \langle \mu, h\rangle x\}.$$

Proposition 3.7. *Let $M \in obj(\mathcal{C}_q)$. Then M has a natural structure of left $D_{q,p^{-1}}(\mathfrak{g})$ module. Denote by $M\breve{}$ this module and by $(u, x) \mapsto u \cdot x$ the action of $D_{q,p^{-1}}(\mathfrak{g})$. Then*

$$\forall u \in D_q(\mathfrak{g})_{\gamma,\delta},\ \forall x \in M_\lambda, \quad u \cdot x = p(\lambda, \delta - \gamma)p(\delta, \gamma)ux.$$

Proof. The proposition is a translation in this particular setting of Corollary 2.5. □

Denote by $\mathcal{C}_{q,p}$ the subcategory of finite dimensional left $D_{q,p^{-1}}(\mathfrak{g})$-modules whose objects are the $M\breve{}$, $M \in \mathrm{obj}(\mathcal{C}_q)$. It follows from Proposition 3.7 that if $M \in \mathrm{obj}(\mathcal{C}_q)$, then $M\breve{} = \oplus_{\mu\in\mathbf{L}} M\breve{}_\mu$, where

$$M\breve{}_\mu = \{x \in M \mid \forall\alpha \in \mathbf{L},\ s_\alpha \cdot x = p(\mu, 2\alpha)q^{(\mu,\alpha)}x,\ t_\alpha \cdot x = p(\mu, -2\alpha)q^{(\mu,\alpha)}x\}.$$

Notice that $p(\mu, \pm 2\alpha)q^{(\mu,\alpha)} = q^{\pm(\Phi_\pm\mu,\alpha)}$.

Theorem 3.8. 1. *The functor $M \to M\breve{}$ from $\mathcal{C}_q$ to $\mathcal{C}_{q,p}$ is an equivalence of rigid monoidal categories.*
2. *The Hopf pairing $\langle\ ,\ \rangle_p$ identifies the Hopf algebra $\mathbb{C}_{q,p}[G]$ with the restricted dual of $\mathcal{C}_{q,p}$, i.e. the Hopf algebra of coordinate functions on the objects of $\mathcal{C}_{q,p}$.*

Proof. 1. One needs in particular to prove that, for $M, N \in \mathrm{obj}(\mathcal{C}_q)$, there are natural isomorphisms of $D_{q,p^{-1}}(\mathfrak{g})$-modules: $\varphi_{M,N} : (M \otimes N)\breve{} \to M\breve{} \otimes N\breve{}$. These isomorphisms are given by $x \otimes y \mapsto p(\lambda, \mu)x \otimes y$ for all $x \in M_\lambda$, $y \in N_\mu$. The other verifications are elementary.
2. We have to show that if $M \in \mathrm{obj}(\mathcal{C}_q)$, $f \in M^*$, $v \in M$ and $u \in D_{q,p^{-1}}(\mathfrak{g})$, then $\langle c_{f,v}, u\rangle_p = \langle f, u \cdot v\rangle$. It suffices to prove the result in the case where $f \in M^*_\lambda$, $v \in M_\mu$ and $u \in D_{q,p^{-1}}(\mathfrak{g})_{\gamma,\delta}$. Then

$$\begin{aligned}\langle f, u \cdot v\rangle &= p(\mu, \delta - \gamma)p(\delta, \gamma)\langle f, uv\rangle \\ &= \delta_{-\lambda+\gamma+\delta,\mu}\, p(-\lambda + \gamma + \delta, \delta - \gamma)p(\delta, \gamma)\langle f, uv\rangle \\ &= p(\lambda, \gamma)p(\mu, \delta)\langle f, uv\rangle \\ &= \langle c_{f,v}, u\rangle_p\end{aligned}$$

by Theorem 3.6. □

Recall that we introduced in §3.3 isomorphisms $\sigma : D_q(\mathfrak{g}) \to D_{q^{-1}}(\mathfrak{g})$ and $\sigma : \mathbb{C}_q[G] \to \mathbb{C}_{q^{-1}}[G]$. From (3.3) it follows that, after twisting by $\tilde{p}^{-1}$ or $\tilde{p}$, σ induces isomorphisms

$$D_{q,p^{-1}}(\mathfrak{g}) \xrightarrow{\sim} D_{q^{-1},p^{-1}}(\mathfrak{g}), \quad \mathbb{C}_{q^{-1},p}[G] \xrightarrow{\sim} \mathbb{C}_{q,p}[G]$$

which satisfy (3.2).

3.5. **Braiding isomorphisms.** We remarked above that the categories $\mathcal{C}_{q,p}$ are braided. In the one parameter case this braiding is well-known. Let M and N be objects of $\mathcal{C}_q$. Let $E : M \otimes N \to M \otimes N$ be the operator given by

$$E(m \otimes n) = q^{(\lambda,\mu)} m \otimes n$$

for $m \in M_\lambda$ and $n \in N_\mu$. Let $\tau : M \otimes N \to N \otimes M$ be the usual twist operator. Finally let C be the operator given by left multiplication by

$$C = \sum_{\beta\in\mathbf{Q}_+} C_\beta$$

where C_β is the canonical element of $D_q(\mathfrak{g})$ associated to the non-degenerate pairing $U^+_\beta \otimes U^-_{-\beta} \to \mathbb{C}$ described above. Then one deduces from [33, 4.3] that the operators

$$\theta_{M,N} = \tau \circ C \circ E^{-1} : M \otimes N \to N \otimes M$$

define the braiding on $\mathcal{C}_q$.

As mentioned above, the category $\mathcal{C}_{q,p}$ inherits a braiding given by

$$\psi_{M,N} = \varphi_{N,M} \circ \theta_{M,N} \circ \varphi^{-1}_{M,N}$$

where $\varphi_{M,N}$ is the isomorphism $(M \otimes N)\check{} \xrightarrow{\sim} M\check{} \otimes N\check{}$ introduced in the proof of Theorem 3.8 (the same formula can be found in [1, §10] and in a more general situation in [24]). We now note that these general operators are of the same form as those in the one parameter case. Let M and N be objects of $\mathcal{C}_{q,p}$ and let $E : M \otimes N \to M \otimes N$ be the operator given by

$$E(m \otimes n) = q^{(\Phi_+\lambda,\mu)} m \otimes n$$

for $m \in M_\lambda$ and $n \in N_\mu$. Denote by C_β the canonical element of $D_{q,p^{-1}}(\mathfrak{g})$ associated to the nondegenerate pairing $U_{q,p^{-1}}(\mathfrak{b}^+)_{-\beta,0} \otimes U_{q,p^{-1}}(\mathfrak{b}^-)_{0,-\beta} \to \mathbb{C}$ and let $C : M \otimes N \to M \otimes N$ be the operator given by left multiplication by

$$C = \sum_{\beta \in \mathbf{Q}_+} C_\beta.$$

Theorem 3.9. *The braiding operators $\psi_{M,N}$ are given by*

$$\psi_{M,N} = \tau \circ C \circ E^{-1}.$$

Moreover $(\psi_{M,N})^* = \psi_{M^*,N^*}$.

Proof. The assertions follow easily from the analogous assertions for $\theta_{M,N}$. □

The following commutation relations are well known [31], [21, 4.2.2]. We include a proof for completeness.

Corollary 3.10. *Let $\Lambda, \Lambda' \in \mathbf{L}^+$, let $g \in L(\Lambda')^*_{-\eta}$ and $f \in L(\Lambda)^*_{-\mu}$ and let $v_\Lambda \in L(\Lambda)_\Lambda$. Then for any $v \in L(\Lambda')_\gamma$,*

$$c_{g,v} \cdot c_{f,v_\Lambda} = q^{(\Phi_+\Lambda,\gamma)-(\Phi_+\mu,\eta)} c_{f,v_\Lambda} \cdot c_{g,v} + q^{(\Phi_+\Lambda,\gamma)-(\Phi_+\mu,\eta)} \sum_{\nu \in \mathbf{Q}_+} c_{f_\nu,v_\Lambda} \cdot c_{g_\nu,v}$$

where $f_\nu \in (U_{q,p^{-1}}(\mathfrak{b}^+)f)_{-\mu+\nu}$ and $g_\nu \in (U_{q,p^{-1}}(\mathfrak{b}^-)g)_{-\eta-\nu}$ are such that $\sum f_\nu \otimes g_\nu = \sum_{\beta \in \mathbf{Q}^+ \setminus \{0\}} C_\beta(f \otimes g)$.

Proof. Let $\psi = \psi_{L(\Lambda),L(\Lambda')}$. Notice that

$$c_{f\otimes g, \psi(v_\Lambda \otimes v)} = c_{\psi^*(f\otimes g), v_\Lambda \otimes v}.$$

Using the theorem above we obtain

$$\psi^*(f \otimes g) = q^{-(\Phi_+\mu,\eta)}(g \otimes f + \sum g_\nu \otimes f_\nu)$$

and

$$\psi(v_\Lambda \otimes v) = q^{-(\Phi_+\Lambda,\gamma)}(v \otimes v_\Lambda). \tag{3.5}$$

Combining these formulae yields the required relations. □

4. Prime and Primitive Spectrum of $\mathbb{C}_{q,p}[G]$

In this section we prove our main result on the primitive spectrum of $\mathbb{C}_{q,p}[G]$; namely that the H orbits inside $\mathrm{Prim}_w\,\mathbb{C}_{q,p}[G]$ are parameterized by the double Weyl group. For completeness we have attempted to make the proof more or less self-contained. The overall structure of the proof is similar to that used in [16] except that the proof of the key 4.12 (and the lemmas leading up to it) form a modified and abbreviated version of Joseph's proof of this result in the one-parameter case [18]. One of the main differences with the approach of [18] is the use of the Rosso-Tanisaki form introduced in 3.2 which simplifies the analysis of the adjoint action of $\mathbb{C}_{q,p}[G]$. The ideas behind the first few results of this section go back to Soibelman's work in the one-parameter 'compact' case [31]. These ideas were adapted to the multi-parameter case by Levendorskii [20].

4.1. **Parameterization of the prime spectrum.** Let q, p be as in §3.4. For simplicity we set

$$A = \mathbb{C}_{q,p}[G]$$

and the product $a \cdot b$ as defined in (2.1) will be denoted by ab.

For each $\Lambda \in \mathbf{L}^+$ choose weight vectors

$$v_\Lambda \in L(\Lambda)_\Lambda,\ v_{w_0\Lambda} \in L(\Lambda)_{w_0\Lambda},\ f_{-\Lambda} \in L(\Lambda)^*_{-\Lambda},\ f_{-w_0\Lambda} \in L(\Lambda)^*_{-w_0\Lambda}$$

such that $\langle f_{-\Lambda}, v_\Lambda\rangle = \langle f_{-w_0\Lambda}, v_{w_0\Lambda}\rangle = 1$. Set

$$A^+ = \sum_{\mu\in\mathbf{L}^+}\sum_{f\in L(\mu)^*}\mathbb{C}c_{f,v_\mu}, \quad A^- = \sum_{\mu\in\mathbf{L}^+}\sum_{f\in L(\mu)^*}\mathbb{C}c_{f,v_{w_0\mu}}.$$

Recall the following result.

Theorem 4.1. *The multiplication map $A^+\otimes A^- \to A$ is surjective.*

Proof. Clearly it is enough to prove the theorem in the one-parameter case. When $\mathbf{L} = \mathbf{P}$ the result is proved in [31, 3.1] and [18, Theorem 3.7].

The general case can be deduced from the simply-connected case as follows. One first observes that $\mathbb{C}_q[G] \subset \mathbb{C}_q[\tilde{G}] = \bigoplus_{\Lambda\in\mathbf{P}^+} C(\Lambda)$. Therefore any $a \in \mathbb{C}_q[G]$ can be written in the form $a = \sum_{\Lambda',\Lambda''\in\mathbf{P}^+} c_{f,v_{\Lambda'}} c_{g,v_{-\Lambda''}}$ where $\Lambda' - \Lambda'' \in \mathbf{L}$. Let $\Lambda \in \mathbf{P}$ and $\{v_i; f_i\}_i$ be a dual basis of $L(\Lambda)$. Then we have

$$1 = \epsilon(c_{v_\Lambda, f_{-\Lambda}}) = \sum_i c_{f_i,v_\Lambda} c_{v_i,f_{-\Lambda}}.$$

Let Λ' be as above and choose Λ such that $\Lambda + \Lambda' \in \mathbf{L}^+$. Then, for all i, $c_{f,v_{\Lambda'}} c_{f_i,v_\Lambda} \in C(\Lambda+\Lambda')\cap A^+$ and $c_{v_i,f_{-\Lambda}} c_{g,v_{-\Lambda''}} \in C(-w_0(\Lambda+\Lambda''))\cap A^-$. The result then follows by inserting 1 between the terms $c_{f,v_{\Lambda'}}$ and $c_{g,v_{-\Lambda''}}$. □

Remark . The algebra A is a Noetherian domain (this result will not be used in the sequel). The fact that A is a domain follows from the same result in [18, Lemma 3.1]. The fact that A is Noetherian is consequence of [18, Proposition 4.1] and [6, Theorem 3.7].

For each $y \in W$ define the following ideals of A

$$I_y^+ = \langle c_{f,v_\Lambda} \mid f \in (U_{q,p^{-1}}(\mathfrak{b}^+)L(\Lambda)_{y\Lambda})^\perp,\ \Lambda\in\mathbf{L}^+\rangle,$$
$$I_y^- = \langle c_{f,v_{w_0\Lambda}} \mid f \in (U_{q,p^{-1}}(\mathfrak{b}^-)L(\Lambda)_{yw_0\Lambda})^\perp,\ \Lambda\in\mathbf{L}^+\rangle$$

where $(\ )^\perp$ denotes the orthogonal in $L(\Lambda)^*$. Notice that $I_y^- = \sigma(\hat{I}_y^+)$, σ as in §3.4, and that $I_y^\pm$ is an $\mathbf{L}\times\mathbf{L}$ homogeneous ideal of A.

Notation . For $w = (w_+, w_-) \in W\times W$ set $I_w = I^+_{w_+} + I^-_{w_-}$. For $\Lambda \in \mathbf{L}^+$ set $c_{w\Lambda} = c_{f_{-w_+\Lambda},v_\Lambda} \in C(\Lambda)_{-w_+\Lambda,\Lambda}$ and $\tilde{c}_{w\Lambda} = c_{v_{w_-\Lambda},f_{-\Lambda}} \in C(-w_0\Lambda)_{w_-\Lambda,-\Lambda}$.

Lemma 4.2. *Let $\Lambda \in \mathbf{L}^+$ and $a \in A_{-\eta,\gamma}$. Then*

$$c_{w\Lambda} a \equiv q^{(\Phi_+ w_+\Lambda,\eta)-(\Phi_+\Lambda,\gamma)} a c_{w\Lambda} \mod I^+_{w_+}$$
$$\tilde{c}_{w\Lambda} a \equiv q^{(\Phi_-\Lambda,\gamma)-(\Phi_- w_-\Lambda,\eta)} a \tilde{c}_{w\Lambda} \mod I^-_{w_-}.$$

Proof. The first identity follows from Corollary 3.10 and the definition of $I^+_{w_+}$. The second identity can be deduced from the first one by applying σ. $\square$

We continue to denote by $c_{w\Lambda}$ and $\tilde{c}_{w\Lambda}$ the images of these elements in A/I_w. It follows from Lemma 4.2 that the sets

$$\mathcal{E}_{w_+} = \{\alpha c_{w\Lambda} \mid \alpha \in \mathbb{C}^*, \Lambda \in \mathbf{L}^+\},\ \mathcal{E}_{w_-} = \{\alpha \tilde{c}_{w\Lambda} \mid \alpha \in \mathbb{C}^*, \Lambda \in \mathbf{L}^+\},\ \mathcal{E}_w = \mathcal{E}_{w_+}\mathcal{E}_{w_-}$$

are multiplicatively closed sets of normal elements in A/I_w. Thus $\mathcal{E}_w$ is an Ore set in A/I_w. Define

$$A_w = (A/I_w)_{\mathcal{E}_w}.$$

Notice that σ extends to an isomorphism $\sigma : \hat{A}_{\hat{w}} \to A_w$, where $\hat{w} = (w_-, w_+)$.

Proposition 4.3. *For all $w \in W \times W$, $A_w \neq (0)$.*

Proof. Notice first that since the generators of A_w and the elements of $\mathcal{E}_w$ are $\mathbf{L} \times \mathbf{L}$ homogeneous, it suffices to work in the one-parameter case. The proof is then similar to that of [15, Theorem 2.2.3] (written in the $SL(n)$-case). For completeness we recall the steps of this proof. The technical details are straightforward generalizations to the general case of [15, loc. cit.].

For $1 \leq i \leq n$ denote by $U_q(\mathfrak{sl}_i(2))$ the Hopf subalgebra of $U_q(\mathfrak{g})$ generated by $e_i, f_i, k_i^{\pm 1}$. The associated quantized function algebra $A_i \cong \mathbb{C}_q[SL(2)]$ is naturally a quotient of A. Let σ_i be the reflection associated to the root α_i. It is easily seen that there exist M_i^+ and M_i^-, non-zero $(A_i)_{(\sigma_i,e)}$ and $(A_i)_{(e,\sigma_i)}$ modules respectively. These modules can then be viewed as non-zero A-modules.

Let $w_+ = \sigma_{i_1} \dots \sigma_{i_k}$ and $w_- = \sigma_{j_1} \dots \sigma_{j_m}$ be reduced expressions for $w_\pm$. Then

$$M^+_{i_1} \otimes \cdots \otimes M^+_{i_k} \otimes M^-_{j_1} \otimes \cdots \otimes M^-_{j_m}$$

is a non-zero A_w-module. $\square$

In the one-parameter case the proof of the following result was found independently by the authors in [16, 1.2] and Joseph in [18, 6.2].

Theorem 4.4. *Let $P \in \operatorname{Spec} \mathbb{C}_{q,p}[G]$. There exists a unique $w \in W \times W$ such that $P \supset I_w$ and $(P/I_w) \cap \mathcal{E}_w = \emptyset$.*

Proof. Fix a dominant weight Λ. Define an ordering on the weight vectors of $L(\Lambda)^*$ by $f \leq f'$ if $f' \in U_{q,p^{-1}}(\mathfrak{b}^+)f$. This is a preordering which induces a partial ordering on the set of one dimensional weight spaces. Consider the set:

$$\mathcal{F}(\Lambda) = \{f \in L(\Lambda)^*_\mu \mid c_{f,v_\Lambda} \notin P\}.$$

Let f be an element of $\mathcal{F}(\Lambda)$ which is maximal for the above ordering. Suppose that f' has the same property and that f and f' have weights μ and μ' respectively. By Corollary 3.10 the two elements c_{f,v_Λ} and c_{f',v_Λ} are normal modulo P. Therefore we have, modulo P,

$$c_{f,v_\Lambda} c_{f',v_\Lambda} = q^{(\Phi_+\Lambda,\Lambda)-(\Phi_+\mu,\mu')} c_{f',v_\Lambda} c_{f,v_\Lambda} = q^{2(\Phi_+\Lambda,\Lambda)-(\Phi_+\mu,\mu')-(\Phi_+\mu',\mu)} c_{f,v_\Lambda} c_{f',v_\Lambda}.$$

But, since u is alternating, $2(\Phi_+\Lambda,\Lambda) - (\Phi_+\mu,\mu') - (\Phi_+\mu',\mu) = 2(\Lambda,\Lambda) - 2(\mu,\mu')$. Since P is prime and q is not a root of unity we can deduce that $(\Lambda,\Lambda) = (\mu,\mu')$. This forces $\mu = \mu' \in W(-\Lambda)$. In conclusion, we have shown that for all dominant Λ there exists a unique (up to scalar multiplication) maximal element $g_\Lambda \in \mathcal{F}(\Lambda)$ with weight $-w_\Lambda\Lambda$, $w_\Lambda \in W$. Applying the argument above to a pair of such elements, c_{g_Λ,v_Λ}

and $c_{g_\Lambda, v_{\Lambda'}}$, yields that $(w_\Lambda \Lambda, w_{\Lambda'}\Lambda') = (\Lambda, \Lambda')$ for all $\Lambda, \Lambda' \in \mathbf{L}^+$. Then it is not difficult to show that this furnishes a unique $w_+ \in W$ such that $w_+\Lambda = w_\Lambda \Lambda$ for all $\Lambda \in \mathbf{L}^+$. Thus for each $\Lambda \in \mathbf{L}^+$,

$$c_{g,v_\Lambda} \in P \iff g \not\leq f_{-w_+\Lambda}.$$

Hence $P \supset I^+_{w_+}$ and $P \cap \mathcal{E}_{w_+} = \emptyset$. It is easily checked that such a w_+ must be unique. Using σ one deduces the existence and uniqueness of w_-. □

Definition . A prime ideal P such that $P \supset I_w$ and $P \cap \mathcal{E}_w = \emptyset$ will be called a prime ideal of type w. We denote by $\operatorname{Spec}_w \mathbb{C}_{q,p}[G]$, resp. $\operatorname{Prim}_w \mathbb{C}_{q,p}[G]$, the subset of $\operatorname{Spec} \mathbb{C}_{q,p}[G]$ consisting of prime, resp. primitive, ideals of type w.

Clearly $\operatorname{Spec}_w \mathbb{C}_{q,p}[G] \cong \operatorname{Spec} A_w$ and $\sigma(\operatorname{Spec}_{\hat{w}} \mathbb{C}_{q^{-1},p}[G]) = \operatorname{Spec}_w \mathbb{C}_{q,p}[G]$. The following corollary is therefore clear.

Corollary 4.5. *One has*

$$\operatorname{Spec} \mathbb{C}_{q,p}[G] = \sqcup_{w \in W\times W} \operatorname{Spec}_w \mathbb{C}_{q,p}[G], \quad \operatorname{Prim} \mathbb{C}_{q,p}[G] = \sqcup_{w\in W\times W} \operatorname{Prim}_w \mathbb{C}_{q,p}[G].$$

We end this section by a result which is the key idea in [18] for analyzing the adjoint action of A on A_w. It says that in the one parameter case the quantized function algebra $\mathbb{C}_q[B^-]$ identifies with $U_q(\mathfrak{b}^+)$ through the Rosso-Tanisaki-Killing form, [10, 17, 18]. Evidently this continues to hold in the multi-parameter case. For completeness we include a proof of that result.

Set $\mathbb{C}_{q,p}[B^-] = A/I_{(w_0,e)}$. The embedding $U_{q,p^{-1}}(\mathfrak{b}^-) \to D_{q,p^{-1}}(\mathfrak{g})$ induces a Hopf algebra map $\phi : A \to U_{q,p^{-1}}(\mathfrak{b}^-)^\circ$, where $U_{q,p^{-1}}(\mathfrak{b}^-)^\circ$ denotes the cofinite dual. On the other hand the non-degenerate Hopf algebra pairing $\langle\ |\ \rangle_{p^{-1}}$ furnishes an injective morphism $\theta : U_{q,p^{-1}}(\mathfrak{b}^+)^{\mathrm{op}} \to U_{q,p^{-1}}(\mathfrak{b}^-)^*$.

Proposition 4.6. 1. $\mathbb{C}_{q,p}[B^-]$ *is an* $\mathbf{L}$*-bigraded Hopf algebra.*
2. *The map* $\gamma = \theta^{-1}\phi : \mathbb{C}_{q,p}[B^-] \to U_{q,p^{-1}}(\mathfrak{b}^+)^{op}$ *is an isomorphism of Hopf algebras.*

Proof. 1. It is easy to check that $I_{(w_0,e)}$ is an $\mathbf{L}\times\mathbf{L}$ graded bi-ideal of the bialgebra A. Let $\mu \in \mathbf{L}^+$ and fix a dual basis $\{v_\nu; f_{-\nu}\}_\nu$ of $L(\mu)$ (with the usual abuse of notation). Then

$$\sum_\nu c_{v_\nu, f_{-\eta}} c_{f_{-\nu}, v_\gamma} = \sum_\nu S(c_{f_{-\eta}, v_\nu}) c_{f_{-\nu}, v_\gamma} = \epsilon(c_{f_{-\eta}, v_\gamma}).$$

Taking $\gamma = \eta = \mu$ yields $\tilde{c}_\mu c_\mu = 1$ modulo $I_{(w_0,e)}$. If $\gamma = w_0\mu$ and $\eta \neq w_0\mu$, the above relation shows that $S(c_{f_{-\eta}, v_{w_0\mu}})\tilde{c}_{-w_0\mu} \in I_{(w_0,e)}$. Thus $I_{(w_0,e)}$ is a Hopf ideal.

2. We first show that

$$\forall\, \Lambda \in \mathbf{L}^+, c_{f,v_\Lambda} \in C(\Lambda)_{-\lambda,\Lambda},\ \exists!\, x_\lambda \in U^+_{\Lambda-\lambda}, \quad \phi(c_{f,v_\Lambda}) = \theta(x_\lambda \cdot k_{-\Lambda}). \tag{4.1}$$

Set $c = c_{f,v_\Lambda}$. Then $c(U^-_{-\eta}) = 0$ unless $\eta = \Lambda - \lambda$; denote by $\bar{c}$ the restriction of c to U^-. By the non-degeneracy of the pairing on $U^+_{\Lambda-\lambda} \times U^-_{\lambda-\Lambda}$ we know that there exists a unique $x_\lambda \in U^+_{\Lambda-\lambda}$ such that $\bar{c} = \theta(x_\lambda)$. Then, for all $y \in U^-_{\lambda-\Lambda}$, we have

$$\begin{aligned} c(y\cdot k_\mu) &= \langle f, y\cdot k_\mu \cdot v_\Lambda\rangle = q^{-(\Phi_-\Lambda,\mu)}\bar{c}(y) = q^{-(\Phi_-\Lambda,\mu)}\langle x_\lambda \mid y\rangle \\ &= \langle x_\lambda \cdot k_{-\Lambda} \mid y \cdot k_\mu\rangle_{p^{-1}} \end{aligned}$$

by (3.4). This proves (4.1).

We now show that ϕ is injective on A^+. Suppose that $c = c_{f,v_\Lambda} \in C(\Lambda)_{-\lambda,\Lambda} \cap \operatorname{Ker}\phi$, hence $c = 0$ on $U_{q,p^{-1}}(\mathfrak{b}^-)$. Since $L(\Lambda) = U_{q,p^{-1}}(\mathfrak{b}^-)v_\Lambda = D_{q,p^{-1}}(\mathfrak{g})v_\Lambda$ it follows that $c = 0$. An easy weight argument using (4.1) shows then that ϕ is injective on A^+.

It is clear that $\operatorname{Ker}\phi \supset I_{(w_0,e)}$, and that $A^+A^- = A$ implies $\phi(A) = \phi(A^+[\tilde{c}_\mu\,;\, \mu \in \mathbf{L}^+])$. Since $\tilde{c}_\mu = c_\mu^{-1}$ modulo $I_{(w_0,e)}$ by part 1, if $a \in A$ there exists $\nu \in \mathbf{L}^+$ such that $\phi(c_\nu)\phi(a) \in \phi(A^+)$. The inclusion $\operatorname{Ker}\phi \subset I_{(w_0,e)}$ follows easily. Therefore γ is a well defined Hopf algebra morphism.

If $\alpha_j \in \mathbf{B}$, there exists $\Lambda \in \mathbf{L}^+$ such that $L(\Lambda)_{\Lambda-\alpha_j} \neq (0)$. Pick $0 \neq f \in L(\Lambda)^*_{-\Lambda+\alpha_j}$. Then (4.1) shows that, up to some scalar, $\phi(c_{f,v_\Lambda}) = \theta(e_j \cdot k_{-\Lambda})$. If $\lambda \in \mathbf{L}$, there exists $\Lambda \in W\lambda \cap \mathbf{L}^+$; in particular $L(\Lambda)_\lambda \neq (0)$. Let $v \in L(\Lambda)_\lambda$ and $f \in L(\Lambda)^*_{-\lambda}$ such that $\langle f, v\rangle = 1$. Then it is easily verified that $\phi(c_{f,v}) = \theta(k_{-\lambda})$. This proves that γ is surjective, and the proposition. □

4.2. **The adjoint action.** Recall that if M is an arbitrary A-bimodule one defines the adjoint action of A on M by

$$\forall\, a \in A,\ x \in M, \quad \mathrm{ad}(a).x = \sum a_{(1)} x S(a_{(2)}).$$

Then it is well known that the subspace of ad-invariant elements $M^{\mathrm{ad}} = \{x \in M \mid \forall a \in A,\ \mathrm{ad}(a).x = \epsilon(a)x\}$ is equal to $\{x \in M \mid \forall a \in A,\ ax = xa\}$.

Henceforth we fix $w \in W \times W$ and work inside A_w. For $\Lambda \in \mathbf{L}^+$, $f \in L(\Lambda)^*$ and $v \in L(\Lambda)$ we set

$$z_f^+ = c_{w\Lambda}^{-1} c_{f,v_\Lambda}, \quad z_v^- = \tilde{c}_{w\Lambda}^{-1} c_{v,f_{-\Lambda}}.$$

Let $\{\omega_1, \dots, \omega_n\}$ be a basis of $\mathbf{L}$ such that $\omega_i \in \mathbf{L}^+$ for all i. Observe that $c_{w\Lambda}c_{w\Lambda'}$ and $c_{w\Lambda'}c_{w\Lambda}$ differ by a non-zero scalar (similarly for $\tilde{c}_{w\Lambda}\tilde{c}_{w\Lambda'}$). For each $\lambda = \sum_i \ell_i \omega_i \in \mathbf{L}$ we define normal elements of A_w by

$$c_{w\lambda} = \prod_{i=1}^n c_{w\omega_i}^{\ell_i}, \quad \tilde{c}_{w\lambda} = \prod_{i=1}^n \tilde{c}_{w\omega_i}^{\ell_i}, \quad d_\lambda = (\tilde{c}_{w\lambda} c_{w\lambda})^{-1}.$$

Notice then that, for $\Lambda \in \mathbf{L}^+$, the "new" $c_{w\Lambda}$ belongs to $\mathbb{C}^* c_{f_{-w_+\Lambda}, v_\Lambda}$ (similarly for $\tilde{c}_{w\Lambda}$). Define subalgebras of A_w by

$$C_w = \mathbb{C}[z_f^+, z_v^-, c_{w\lambda}\,;\, f \in L(\Lambda)^*, v \in L(\Lambda),\ \Lambda \in \mathbf{L}^+,\ \lambda \in \mathbf{L}]$$
$$C_w^+ = \mathbb{C}[z_f^+\,;\, f \in L(\Lambda)^*,\ \Lambda \in \mathbf{L}^+], \quad C_w^- = \mathbb{C}[z_v^-\,;\, v \in L(\Lambda),\ \Lambda \in \mathbf{L}^+].$$

Recall that the torus H acts on $A_{\lambda,\mu}$ by $r_h(a) = \mu(h)a$, where $\mu(h) = \langle \mu, h\rangle$. Since the generators of I_w and the elements of $\mathcal{E}_w$ are eigenvectors for H, the action of H extends to an action on A_w. The algebras C_w and $C_w^\pm$ are obviously H-stable.

Theorem 4.7. 1. $C_w^H = \mathbb{C}[z_f^+, z_v^-\,;\, f \in L(\Lambda)^*, v \in L(\Lambda),\ \Lambda \in \mathbf{L}^+]$.

2. *The set $\mathcal{D} = \{d_\Lambda\,;\, \Lambda \in \mathbf{L}^+\}$ is an Ore subset of C_w^H. Furthermore $A_w = (C_w)_{\mathcal{D}}$ and $A_w^H = (C_w^H)_{\mathcal{D}}$.*

3. *For each $\lambda \in \mathbf{L}$, let $(A_w)_\lambda = \{a \in A_w \mid r_h(a) = \lambda(h)a\}$. Then $A_w = \bigoplus_{\lambda \in \mathbf{L}} (A_w)_\lambda$ and $(A_w)_\lambda = A_w^H c_{w\lambda}$. Moreover each $(A_w)_\lambda$ is an ad-invariant subspace.*

Proof. Assertion 1 follows from

$$\forall h \in H, \quad r_h(z_f^\pm) = z_f^\pm, \quad r_h(c_{w\lambda}) = \lambda(h) c_{w\lambda}, \quad r_h(\tilde{c}_{w\lambda}) = \lambda(h)^{-1}\tilde{c}_{w\lambda}.$$

Let $\{v_i; f_i\}_i$ be a dual basis for $L(\Lambda)$. Then

$$1 = \epsilon(c_{f_{-\Lambda}, v_\Lambda}) = \sum_i S(c_{f_{-\Lambda}, v_i}) c_{f_i, v_\Lambda} = \sum_i c_{v_i, f_{-\Lambda}} c_{f_i, v_\Lambda}.$$

Multiplying both sides of the equation by d_Λ and using the normality of $c_{w\Lambda}$ and $\tilde{c}_{w\Lambda}$ yields $d_\Lambda = \sum_i a_i z_{v_i}^- z_{f_i}^+$ for some $a_i \in \mathbb{C}$. Thus $\mathcal{D} \subset C_w^H$. Now by Theorem 4.1 any element of A_w can be written in the form $c_{f_1,v_1} c_{f_2,v_2} d_\Lambda^{-1}$ where $v_1 = v_{\Lambda_1}$, $v_2 = v_{-\Lambda_2}$ and $\Lambda_1, \Lambda_2, \Lambda \in \mathbf{L}^+$. This element lies in $(A_w)_\lambda$ if and only if $\Lambda_1 - \Lambda_2 = \lambda$. In this case $c_{f_1,v_1} c_{f_2,v_2} d_\Lambda^{-1}$ is equal, up to a scalar, to the element $z_{f_1}^+ z_{f_2}^- d_{\Lambda+\Lambda_2}^{-1} c_{w\lambda} \in (C_w^H)_{\mathcal{D}} c_{w\lambda}$. Since the adjoint action commutes with the right action of H, $(A_w)_\lambda$ is an ad-invariant subspace. The remaining assertions then follow. □

We now study the adjoint action of $\mathbb{C}_{q,p}[G]$ on A_w. The key result is Theorem 4.12.

Lemma 4.8. *Let $T_\Lambda = \{z_f^+ \mid f \in L(\Lambda)^*\}$. Then $C_w^+ = \bigcup_{\Lambda \in \mathbf{L}} T_\Lambda$.*

Proof. It suffices to prove that if $\Lambda, \Lambda' \in \mathbf{L}^+$ and $f \in L(\Lambda)^*$, then there exists a $g \in L(\Lambda+\Lambda')^*$ such that $z_f^+ = z_g^+$. Clearly we may assume that f is a weight vector. Let $\iota : L(\Lambda+\Lambda') \to L(\Lambda) \otimes L(\Lambda')$ be the canonical map. Then

$$c_{f,v_\Lambda} c_{f_{-w_+\Lambda'},v_{\Lambda'}} = c_{f_{-w_+\Lambda'}\otimes f, v_\Lambda \otimes v_{\Lambda'}} = c_{g,v_{\Lambda+\Lambda'}}$$

where $g = \iota^*(f_{-w_+\Lambda'} \otimes f)$. Multiplying the images of these elements in A_w by the inverse of $c_{w(\Lambda+\Lambda')} \in \mathbb{C}^* c_{w\Lambda} c_{w\Lambda'}$ yields the desired result. □

Proposition 4.9. *Let E be an object of $\mathcal{C}_{q,p}$ and let $\Lambda \in \mathbf{L}^+$. Let $\sigma : L(\Lambda) \to E \otimes L(\Lambda) \otimes E^*$ be the map $(1 \otimes \psi^{-1})(\iota \otimes 1)$ where $\iota : \mathbb{C} \to E \otimes E^*$ is the canonical embedding and $\psi^{-1} : E^* \otimes L(\Lambda) \to L(\Lambda) \otimes E^*$ is the inverse of the braiding map described in §3.5. Then for any $c = c_{g,v} \in C(E)_{-\eta,\gamma}$ and $f \in L(\Lambda)^*$*

$$\mathrm{ad}(c).z_f^+ = q^{(\Phi_+ w_+\Lambda,\eta)}\, z^+_{\sigma^*(v\otimes f\otimes g)}.$$

In particular C_w^+ is a locally finite $\mathbb{C}_{q,p}[G]$-module for the adjoint action.

Proof. Let $\{v_i; g_i\}_i$ be a dual basis of E where $v_i \in E_{\nu_i}$, $g_i \in E^*_{-\nu_i}$. Then $\iota(1) = \sum v_i \otimes g_i$. By (3.5) we have

$$\psi^{-1}(g_i \otimes v_\Lambda) = a_i (v_\Lambda \otimes g_i)$$

where $a_i = q^{-(\Phi_+\Lambda,\nu_i)} = q^{(\Phi_-\nu_i,\Lambda)}$. On the other hand the commutation relations given in Corollary 3.10 imply that $c_{g,v_i} c_{w\Lambda}^{-1} = b a_i c_{w\Lambda}^{-1} c_{g,v_i}$, where $b = q^{(\Phi_+ w_+\Lambda,\eta)}$. Therefore

$$\mathrm{ad}(c).z_f^+ = \sum b a_i c_{w\Lambda}^{-1} c_{g,v_i} c_{f,v_\Lambda} c_{v,g_i} = b c_{w\Lambda}^{-1} c_{v\otimes f\otimes g, \sum a_i v_i \otimes v_\Lambda \otimes g_i} = b c_{w\Lambda}^{-1} c_{v\otimes f\otimes g, \sigma(v_\Lambda)}.$$

Since the map σ is a morphism of $D_{q,p^{-1}}(\mathfrak{g})$-modules it is easy to see that $c_{v\otimes f\otimes g,\sigma(v_\Lambda)} = c_{\sigma^*(v\otimes f\otimes g),v_\Lambda}$. □

Let $\gamma : \mathbb{C}_{q,p}[G] \to U_{q,p^{-1}}(\mathfrak{b}^+)$ be the algebra anti-isomorphism given in Proposition 4.6.

Lemma 4.10. *Let $c = c_{g,v} \in \mathbb{C}_{q,p}[G]_{-\eta,\gamma}$, $f \in L(\Lambda)^*$ be as in the previous theorem and $x \in U_{q,p^{-1}}(\mathfrak{b}^+)$ be such that $\gamma(c) = x$. Then*

$$c_{S^{-1}(x).f,v_\Lambda} = c_{\sigma^*(v\otimes f\otimes g),v_\Lambda}.$$

Proof. Notice that it suffices to show that

$$c_{S^{-1}(x).f,v_\Lambda}(y) = c_{\sigma^*(v\otimes f\otimes g),v_\Lambda}(y)$$

for all $y \in U_{q,p^{-1}}(\mathfrak{b}^-)$. Denote by $\langle\ |\ \rangle$ the Hopf pairing $\langle\ |\ \rangle_{p^{-1}}$ between $U_{q,p^{-1}}(\mathfrak{b}^+)^{\mathrm{op}}$ and $U_{q,p^{-1}}(\mathfrak{b}^-)$ as in §3.4. Let χ be the one dimensional representation of $U_{q,p^{-1}}(\mathfrak{b}^+)$ associated to v_Λ and let $\tilde{\chi} = \chi \cdot \gamma$. Notice that $\chi(x) = \langle x \mid t_{-\Lambda}\rangle$; so $\tilde{\chi}(c) = c(t_{-\Lambda})$. Recalling that γ is a morphism of coalgebras and using the relation (c_{xy}) of §2.3 in the double $U_{q,p^{-1}}(\mathfrak{b}^+) \bowtie U_{q,p^{-1}}(\mathfrak{b}^-)$, we obtain

$$\begin{aligned}
c_{S^{-1}(x).f,v_\Lambda}(y) &= f(xyv_\Lambda)\\
&= \sum \langle x_{(1)} \mid y_{(1)}\rangle \langle x_{(3)} \mid S(y_{(3)})\rangle\, f(y_{(2)} x_{(2)} v_\Lambda)\\
&= \sum \langle x_{(1)} \mid y_{(1)}\rangle \langle x_{(3)} \mid S(y_{(3)})\rangle\, \chi(x_{(2)})\, f(y_{(2)} v_\Lambda)\\
&= \sum \langle x_{(1)}\chi(x_{(2)}) \mid y_{(1)}\rangle \langle x_{(3)} \mid S(y_{(3)})\rangle\, f(y_{(2)} v_\Lambda)\\
&= \sum (c_{(1)}\tilde{\chi}(c_{(2)}))(y_{(1)})\, c_{(3)}(S(y_{(3)}))\, f(y_{(2)} v_\Lambda)\\
&= \sum r_{\tilde{\chi}}(c_{(1)})(y_{(1)})\, c_{f,v_\Lambda}(y_{(2)})\, S(c_{(2)})(y_{(3)}).
\end{aligned}$$

Since $r_{\tilde{\chi}}(c_{g,v_i}) = q^{(\Phi_-\nu_i,\Lambda)} c_{g,v_i}$, one shows as in the proof of Proposition 4.9 that

$$\begin{aligned}
c_{S^{-1}(x).f,v_\Lambda}(y) &= \sum r_{\tilde{\chi}}(c_{(1)})(y_{(1)})\, c_{f,v_\Lambda}(y_{(2)})\, S(c_{(2)})(y_{(3)})\\
&= \sum q^{(\Phi_-\nu_i,\Lambda)}\, (c_{g,v_i} c_{f,v_\Lambda} c_{v,g_i})(y)\\
&= c_{\sigma^*(v\otimes f\otimes g),v_\Lambda}(y),
\end{aligned}$$

as required. □

Theorem 4.11. *Consider C_w^+ as a $\mathbb{C}_{q,p}[G]$-module via the adjoint action. Then*

1. $\operatorname{Soc} C_w^+ = \mathbb{C}$.
2. $\operatorname{Ann} C_w^+ \supset I_{(w_0,e)}$.
3. *The elements $c_{f_{-\mu},v_\mu}$, $\mu \in \mathbf{L}^+$, act diagonalizably on C_w^+.*
4. $\operatorname{Soc} C_w^+ = \{z \in C_w^+ \mid \operatorname{Ann} z \supset I_{(e,e)}\}$.

Proof. For $\Lambda \in \mathbf{L}^+$, define a $U_{q,p^{-1}}(\mathfrak{b}^+)$-module by

$$S_\Lambda = (U_{q,p^{-1}}(\mathfrak{b}^+)v_{w_+\Lambda})^* = L(\Lambda)^*/(U_{q,p^{-1}}(\mathfrak{b}^+)v_{w_+\Lambda})^\perp.$$

It is easily checked that $\operatorname{Soc} S_\Lambda = \mathbb{C}f_{-w_+\Lambda}$ (see [18, 7.3]). Let $\delta : S_\Lambda \to T_\Lambda$ be the linear map given by $\bar{f} \mapsto z_f^+$. Denote by ζ the one-dimensional representation of $\mathbb{C}_{q,p}[G]$ given by $\zeta(c) = c(t_{-w_+\Lambda})$. Let $c = c_{g,v} \in C(E)_{-\eta,\gamma}$. Then $l_\zeta(c) = q^{(\Phi_-\eta,w_+\Lambda)}c = q^{-(\Phi_+w_+\Lambda,\eta))}c$. Then, using Proposition 4.9 and Lemma 4.10 we obtain,

$$\operatorname{ad}(l_\zeta(c)).\delta(\bar{f}) = q^{-(\Phi_+w_+\Lambda,\eta)} \operatorname{ad}(c).z_f^+ = z^+_{S^{-1}\gamma(c).f} = \delta(S^{-1}(\gamma(c))\bar{f}).$$

Hence, $\operatorname{ad}(l_\zeta(c)).\delta(\bar{f}) = \delta(S^{-1}(\gamma(c))\bar{f})$ for all $c \in A$. This immediately implies part (2) since $\operatorname{Ker}\gamma \supset I_{(w_0,e)}$ and $l_\zeta(I_{(w_0,e)}) = I_{(w_0,e)}$. If S_Λ is given the structure of an A-module via $S^{-1}\gamma$, then δ is a homomorphism from S_Λ to the module T_Λ twisted by the automorphism l_ζ. Since $\delta(f_{-w_+\Lambda}) = 1$ it follows that δ is bijective and that $\operatorname{Soc} T_\Lambda = \delta(\operatorname{Soc} S_\Lambda) = \mathbb{C}$. Part (1) then follows from Lemma 4.8. Part (3) follows from the above formula and the fact that $\gamma(c_{f_{-\mu},v_\mu}) = s_{-\mu}$. Since $A/I_{(e,e)}$ is generated by the images of the elements $c_{f_{-\mu},v_\mu}$, (4) is a consequence of the definitions. □

Theorem 4.12. *Consider C_w^H as a $\mathbb{C}_{q,p}[G]$-module via the adjoint action. Then*

$$\operatorname{Soc} C_w^H = \mathbb{C}.$$

Proof. By Theorem 4.11 we have that $\operatorname{Soc} C_w^+ = \mathbb{C}$. Using the map σ, one obtains analogous results for C_w^-. The map $C_w^+ \otimes C_w^- \to C_w^H$ is a module map for the adjoint action which is surjective by Theorem 4.1. So it suffices to show that $\operatorname{Soc} C_w^+ \otimes C_w^- = \mathbb{C}$. The following argument is taken from [18].

By the analog of Theorem 4.11 for C_w^- we have that the elements $c_{f_{-\Lambda},v_\Lambda}$ act as commuting diagonalizable operators on C_w^-. Therefore an element of $C_w^+ \otimes C_w^-$ may be written as $\sum a_i \otimes b_i$ where the b_i are linearly independent weight vectors. Let c_{f,v_Λ} be a generator of I_e^+. Suppose that $\sum a_i \otimes b_i \in \operatorname{Soc}(C_w^+ \otimes C_w^-)$. Then

$$\begin{aligned} 0 = \operatorname{ad}(c_{f,v_\Lambda}).(\sum_i a_i \otimes b_i) &= \sum_{i,j} \operatorname{ad}(c_{f,v_j}).a_i \otimes \operatorname{ad}(c_{f_j,v_\Lambda}).b_i \\ &= \sum_i \operatorname{ad}(c_{f,v_\Lambda}).a_i \otimes \operatorname{ad}(c_{f_{-\Lambda},v_\Lambda}).b_i \\ &= \sum_i \operatorname{ad}(c_{f,v_\Lambda}).a_i \otimes \alpha_i b_i \end{aligned}$$

for some $\alpha_i \in \mathbb{C}^*$. Thus $\operatorname{ad}(c_{f,v_\Lambda}).a_i = 0$ for all i. Thus the a_i are annihilated by the left ideal generated by the c_{f,v_Λ}. But this left ideal is two-sided modulo $I_{(w_0,e)}$ and $\operatorname{Ann} C_w^+ \supset I_{(w_0,e)}$. Thus the a_i are annihilated by $I_{(e,e)}$ and so lie in $\operatorname{Soc} C_w^+$ by Theorem 4.11. Thus $\sum a_i \otimes b_i \in \operatorname{Soc}(\mathbb{C} \otimes C_w^-) = \mathbb{C} \otimes \mathbb{C}$. □

Corollary 4.13. *The algebra A_w^H contains no nontrivial ad-invariant ideals. Furthermore, $(A_w^H)^{\mathrm{ad}} = \mathbb{C}$.*

Proof. Notice that Theorem 4.12 implies that C_w^H contains no nontrivial ad-invariant ideals. Since A_w^H is a localization of C_w^H the same must be true for A_w^H. Let $a \in (A_w^H)^{\mathrm{ad}} \backslash \mathbb{C}$. Then a is central and so for any $\alpha \in \mathbb{C}$, $(a - \alpha)$ is a non-zero ad-invariant ideal of A_w^H. This implies that $a - \alpha$ is invertible in A_w^H for any $\alpha \in \mathbb{C}$. This contradicts the fact that A_w^H has countable dimension over $\mathbb{C}$. □

Theorem 4.14. *Let Z_w be the center of A_w. Then*

1. $Z_w = A_w^{ad}$;
2. $Z_w = \bigoplus_{\lambda\in\mathbf{L}} Z_\lambda$ *where* $Z_\lambda = Z_w \cap A_w^H c_{w\lambda}$;
3. *If* $Z_\lambda \neq (0)$, *then* $Z_\lambda = \mathbb{C}u_\lambda$ *for some unit* u_λ;
4. *The group H acts transitively on the maximal ideals of Z_w.*

Proof. The proof of (1) is standard. Assertion (2) follows from Theorem 4.7. Let u_λ be a non-zero element of Z_λ. Then $u_\lambda = ac_{w\lambda}$, for some $a \in A_w^H$. This implies that a is normal and hence a generates an ad-invariant ideal of A_w^H. Thus a (and hence also u_λ) is a unit by Theorem 4.13. Since $Z_0 = \mathbb{C}$, it follows that $Z_\lambda = \mathbb{C}u_\lambda$. Since the action of H is given by $r_h(u_\lambda) = \lambda(h)u_\lambda$, it is clear that H acts transitively on the maximal ideals of Z_w. □

Theorem 4.15. *The ideals of A_w are generated by their intersection with the center, Z_w.*

Proof. Any element $f \in A_w$ may be written uniquely in the form $f = \sum a_\lambda c_{w\lambda}$ where $a_\lambda \in A_w^H$. Define $\pi : A_w \to A_w^H$ to be the projection given by $\pi(\sum a_\lambda c_{w\lambda}) = a_0$ and notice that π is a module map for the adjoint action. Define the support of f to be $Supp(f) = \{\lambda \in \mathbf{L} \mid a_\lambda \neq 0\}$. Let I be an ideal of A_w. For any set $Y \subseteq \mathbf{L}$ such that $0 \in Y$ define

$$I_Y = \{b \in A_w^H \mid b = \pi(f) \text{ for some } f \in I \text{ such that } Supp(f) \subseteq Y\}$$

If I is ad-invariant then I_Y is an ad-invariant ideal of A_w^H and hence is either (0) or A_w^H.

Now let $I' = (I \cap Z_w)A_w$ and suppose that $I \neq I'$. Choose an element $f = \sum a_\lambda c_{w\lambda} \in I \backslash I'$ whose support S has the smallest cardinality. We may assume without loss of generality that $0 \in S$. Suppose that there exists $g \in I'$ with $Supp(g) \subset S$. Then there exists a $g' \in I'$ with $Supp(g') \subset S$ and $\pi(g') = 1$. But then $f - a_0 g'$ is an element of I' with smaller support than F. Thus there can be no elements in I' whose support is contained in S. So we may assume that $\pi(f) = a_0 = 1$. For any $c \in \mathbb{C}_{q,p}[G]$, set $f_c = \mathrm{ad}(c).f - \epsilon(c)f$. Since $\pi(f_c) = 0$ it follows that $|Supp(f_c)| < |Supp(f)|$ and hence that $f_c = 0$. Thus $f \in I \cap A_w^{\mathrm{ad}} = I \cap Z_w$, a contradiction. □

Putting these results together yields the main theorem of this section, which completes Corollary 4.5 by describing the set of primitive ideals of type w.

Theorem 4.16. *For $w \in W\times W$ the subsets $\mathrm{Prim}_w\, \mathbb{C}_{q,p}[G]$ are precisely the H-orbits inside $\mathrm{Prim}\, \mathbb{C}_{q,p}[G]$.*

Finally we calculate the size of these orbits in the algebraic case. Set $\mathbf{L}_w = \{\lambda \in \mathbf{L} \mid Z_\lambda \neq (0)\}$. Recall the definition of $s(w)$ from (1.3) and that p is called q-rational if u is algebraic. In this case we know by Theorem 1.7 that there exists $m \in \mathbb{N}^*$ such that $\Phi(m\mathbf{L}) \subset \mathbf{L}$.

Proposition 4.17. *Suppose that p is q-rational. Let $\lambda \in \mathbf{L}$ and $y_\lambda = c_{w\Phi_- m\lambda}\tilde{c}_{w\Phi_+ m\lambda}$. Then*

1. *y_λ is ad-semi-invariant. In fact, for any $c \in A_{-\eta,\gamma}$,*
$$\mathrm{ad}(c).y_\lambda = q^{(m\sigma(w)\lambda,\eta)}\epsilon(c)y_\lambda.$$
where $\sigma(w) = \Phi_- w_- \Phi_+ - \Phi_+ w_+ \Phi_-$
2. $\mathbf{L}_w \cap 2m\mathbf{L} = 2\,\mathrm{Ker}\,\sigma(w) \cap m\mathbf{L}$
3. $\dim Z_w = n - s(w)$

Proof. Using Lemma 4.2, we have that for $c \in A_{-\eta,\gamma}$

$$\begin{aligned} cy_\lambda &= q^{(\Phi_+ w_+ \Phi_- m\lambda, -\eta)} q^{(\Phi_+\Phi_- m\lambda,\gamma)} q^{(\Phi_- w_- \Phi_+ m\lambda,\eta)} q^{(\Phi_-\Phi_+ m\lambda,-\gamma)} y_\lambda c \\ &= q^{(m\sigma(w)\lambda,\eta)} y_\lambda c. \end{aligned}$$

From this it follows easily that

$$\mathrm{ad}(c).y_\lambda = q^{(m\sigma(w)\lambda,\eta)}\epsilon(c)y_\lambda.$$

Since (up to some scalar) $y_\lambda = d_{\Phi m\lambda}^{-1} d_{m\lambda}^{-1} c_{wm\lambda}^{-2}$ it follows from Theorem 4.7 that $y_\lambda \in (A_w)_{-2m\lambda}$. However, as a $\mathbb{C}_{q,p}[G]$-module via the adjoint action, $A_w^H y_\lambda \cong A_w^H \otimes \mathbb{C} y_\lambda$ and hence $\operatorname{Soc} A_w^H y_\lambda = \mathbb{C} y_\lambda$. Thus $Z_{-2m\lambda} \neq (0)$ if and only if y_λ is ad-invariant; that is, if and only if $m\sigma(w)\lambda = 0$. Hence

$$\begin{aligned} \dim Z_w &= \operatorname{rk} \mathbf{L}_w = \operatorname{rk}(\mathbf{L}_w \cap 2m\mathbf{L}) = \operatorname{rk} \operatorname{Ker}_{m\mathbf{L}} \sigma(w) \\ &= \dim \operatorname{Ker}_{\mathfrak{h}^*} \sigma(w) = n - s(w) \end{aligned}$$

as required. □

Finally, we may deduce that in the algebraic case the size of the of the H-orbits $\operatorname{Symp}_w G$ and $\operatorname{Prim}_w \mathbb{C}_{q,p}[G]$ are the same, cf. Theorem 1.8.

Theorem 4.18. *Suppose that p is q-rational and let $w \in W \times W$. Then*

$$\forall P \in \operatorname{Prim}_w \mathbb{C}_{q,p}[G], \quad \dim(H/\operatorname{Stab}_H P) = n - s(w).$$

Proof. This follows easily from theorems 4.15, 4.16 and Proposition 4.17. □

References

[1] N. Andruskiewitsch and B. Enriquez, Examples of compact matrix pseudogroups arising from the twisting operation, *Comm. Math. Phys.*, 149 (1992), 195-207.

[2] M. Artin, W. Schelter and J. Tate, Quantum deformations of GL_n, *Comm. Pure Appl. Math.*, 44 (1991), 879-895.

[3] A. A. Belavin and V. G. Drinfeld, Triangle equations and simple Lie algebras, Mathematical Physics reviews (S. P. Novikov, ed.), Harwood, New York 1984, 93-166.

[4] A. Borel, *Linear algebraic groups,* W.A. Benjamin, New York 1969.

[5] W. Chin and I. Musson, Multi-parameter quantum enveloping algebras, preprint 1994.

[6] W. Chin and D. Quinn, Rings graded by polycyclic-by-finite groups, *Proc. Amer. Math. Soc.,* 102 (1988), 235-241.

[7] M. Constantini and M. Varagnolo, Quantum double and multi-parameter quantum group, *Comm. Alg.*, 22 (1994), 6305-6321.

[8] M. Constantini and M. Varagnolo, Multi-parameter quantum function algebra at roots of 1, preprint.

[9] P. Cotta-Ramusino and M. Rinaldi, Multi-parameter quantum groups related to link diagrams, *Comm. Math. Phys.* 142 (1991), 589-604.

[10] C. De Concini and V. Lyubashenko, Quantum function algebra at roots of 1, *Adv. Math.*, 108 (1994), 205-262.

[11] C. De Concini and C. Procesi, Quantum groups, *in* D-modules, representation theory and quantum groups, Lecture Notes in Mathematics 1565, Springer 1994, 31-140.

[12] J. Dixmier, *Algèbres Enveloppantes*, Gauthier-Villars, Paris 1974.

[13] V.G.Drinfeld, Quantum groups, *in* Proceedings ICM, Berkeley, AMS, 1987, 798-820.

[14] T. Hayashi, Quantum groups and quantum determinants, *J. Algebra*, 152 (1992), 146-165.

[15] T. J. Hodges and T. Levasseur, Primitive ideals of $\mathbb{C}_q[SL(3)]$, *Comm. Math. Phys.*, 156 (1993), 581-605.

[16] T. J. Hodges and T. Levasseur, Primitive ideals of $\mathbb{C}_q[SL(n)]$, *J. Algebra*, 168 (1994), 455-468.

[17] T. J. Hodges and T. Levasseur, Primitive ideals of $\mathbb{C}_q[G]$, University of Cincinnati 1993 (unpublished).

[18] A. Joseph, On the prime and primitive spectrum of the algebra of functions on a quantum group, *J. Algebra*, 169 (1994), 441-511.

[19] A. Joseph, Idéaux premiers et primitifs de l'algèbre des fonctions sur un groupe quantique, *C. R. Acad. Sci. Paris*, 316 I (1993), 1139-1142.

[20] S. Z. Levendorskii, Twisted function algebras on a compact quantum group and their representations, *St. Petersburg Math J.* 3 (1992), 405-423.

[21] S. Z. Levendorskii and Y. S. Soibelman, Algebras of functions on compact quantum groups, Schubert cells and quantum tori, *Comm. Math. Phys.* 139 (1991), 141-170.

[22] J.-H. Lu and A. Weinstein, Poisson Lie groups, dressing transformations and Bruhat decompositions, *J. Differential Geometry*, 31 (1990), 501–526.

[23] G. Lusztig, Quantum deformations of certain simple modules over enveloping algebras, *Advances in Math.,* 70 (1988), 237-249.

[24] V. Lychagin, Braided differential operators and quantizations in ABC-categories, *C. R. Acad. Sci. Paris*, 318 I (1994), 857-862.

[25] M. Okado and H. Yamane, R-matrices with gauge parameters and multi-parameter quantized enveloping algebras, *in* Special functions (Okayama 1990), ICM-90 Satell. Conf. Proc., Springer 1991, 289-293.

[26] N. Y. Reshetikhin, Multi-parameter quantum groups and twisted quasi-triangular Hopf algebras, *Lett. Math. Phys.* 20 (1990), 331-335.

[27] M. Rieffel, Compact quantum groups associated with toral subgroups, *in* Representation Theory of Groups and Algebras, *Contemporary Math.*, 145 (1993), 465-491.
[28] M. Rosso, Analogues de la forme de Killing et du théorème de Harish-Chandra pour les groupes quantiques, *Ann. Sci. Ec. Norm. Sup.*, 23 (1990), 445-467.
[29] M. Rosso, Représentations des groupes quantiques, *in* Séminaire Bourbaki, Exposé 744, Soc. Math. France 1991, 443-483.
[30] M. A. Semenov-Tian-Shansky, Dressing transformations and Poisson group actions, *Publ. RIMS, Kyoto Univ.*, 21 (1985), 1237–1260.
[31] Ya. S. Soibelman, The algebra of functions on a compact quantum group, and its representations, *Leningrad Math. J.* 2 (1991), 161–178.
[32] A. Sudberry, Consistent multi-parameter quantization of GL(n), *J. Phys.*, A 23 (1990), L697-L704.
[33] T. Tanisaki, Killing forms, Harish-Chandra isomorphisms, and universal R-Matrices for Quantum Algebras, in *Infinite Analysis, Proceedings of the RIMS research project 1991, Part B*, World Scientific, 1992, 941-961.

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