ALGEBRAIC STRUCTURE OF MULTI-PARAMETER QUANTUM GROUPS

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Introduction

Let $G$ be a connected semi-simple complex Lie group. We define and study the multi-parameter quantum group $\mathbb{C}_{q,p}[G]$ in the case where $q$ is a complex parameter that is not a root of unity. Using a method of twisting bigraded Hopf algebras by a cocycle, [2], we develop a unified approach to the construction of $\mathbb{C}_{q,p}[G]$ and of the multi-parameter Drinfeld double $D_{q,p}$. Using a general method of deforming bigraded pairs of Hopf algebras, we construct a Hopf pairing between these algebras from which we deduce a Peter-Weyl-type theorem for $\mathbb{C}_{q,p}[G]$. In particular the orbits in $\mathbb{C}_{q,p}[G]$ are indexed, as in the one-parameter case by the elements of the double Weyl group $W$. Unlike the one-parameter case there is not in general a bijection between $\text{Symp} G$ and $\text{Prim} \mathbb{C}_{q,p}[G]$. However in the case when the symplectic leaves are algebraic such a bijection does exist since the orbits corresponding to a given $w \in W \times W$ have the same dimension.

In the first section we discuss the Poisson structures on $G$ defined by classical $r$-matrices of the form $r = a - u$ where $a = \sum_{\alpha \in \mathbb{R}} e_{\alpha} \wedge e_{-\alpha} \in \wedge^2 g$ and $u \in \wedge^2 h$. Given such an $r$ one constructs a Manin triple of Lie groups $(G \times G, G, G_r)$. Unlike the one-parameter case (where $u = 0$), the dual group $G_r$ will generally not be an algebraic subgroup of $G \times G$. In fact this happens if and only if $u \in \wedge^2 h_0$. Since the quantized universal enveloping algebra $U_q(g)$ is a deformation of the algebra of functions on the algebraic group $G_r$ [11], this explains the difficulty in constructing multi-parameter versions of $U_q(g)$. From [22, 30], one has that the symplectic leaves are the connected components of $G \cap G_r x G_r$ where $x \in G$. Since $r$ is $H$-invariant, the symplectic leaves are permuted by $H$ with the orbits being contained in Bruhat cells in $G \times G$ indexed by $W \times W$. In the case where $G_r$ is algebraic, the symplectic leaves are also algebraic and an explicit formula is given for their dimension.

The philosophy of [15, 16] was that, as in the case of enveloping algebras of algebraic solvable Lie algebras, the primitive ideals of $\mathbb{C}_{q}[G]$ should be in bijection with the symplectic leaves of $G$ (in the case $u = 0$). Indeed, since the Lie bracket on $g_r = \text{Lie}(G_r)$ is the linearization of the Poisson structure on $G$, $\text{Prim} \mathbb{C}_{q,p}[G]$ should resemble $\text{Prim} U(g_r)$. The study of the multi-parameter versions $\mathbb{C}_{q,p}[G]$ is similar to the case of enveloping algebras of general solvable Lie algebras. In the general case $\text{Prim} U(g_r)$ is in bijection with the co-adjoint orbits in $g_r^\ast$ under the action of the ‘adjoint algebraic group’ of $g_r$ [12]. It is therefore natural that, only in the case where the symplectic leaves are algebraic, does one expect and obtain a bijection between the symplectic leaves and the primitive ideals.

In section 2 we define the notion of an $\mathbf{L}$-bigraded Hopf $\mathbb{K}$-algebra, where $\mathbf{L}$ is an abelian group. When $A$ is finitely generated such bigradings correspond bijectively to morphisms from the algebraic group $\mathbf{L}^\vee$ to the (algebraic) group $R(A)$ of one-dimensional representations of $A$. For any antisymmetric bicharacter $p$ on $\mathbf{L}$, the multiplication in $A$ may be twisted to give a new Hopf algebra $A_p$. Moreover, given a pair of $\mathbf{L}$-bigraded Hopf algebras $A$ and $U$ equipped with an $\mathbf{L}$-compatible Hopf pairing $A \times U \rightarrow \mathbb{K}$, one can deform the pairing to get a new Hopf pairing between $A_{p^{-1}}$ and $U_p$. This deformation commutes

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with the formation of the Drinfeld double in the following sense. Suppose that \( A \) and \( U \) are bigraded Hopf algebras equipped with a compatible Hopf pairing \( A^{op} \times U \to \mathbb{K} \). Then the Drinfeld double \( A \ltimes U \) inherits a bigrading such that \( (A \ltimes U)_p \cong A_p \ltimes U_p \).

Let \( C_q[G] \) denote the usual one-parameter quantum group (or quantum function algebra) and let \( U_q(\mathfrak{g}) \) the quantized enveloping algebra associated to the lattice \( L \) of weights of \( G \). Let \( U_q(\mathfrak{b}^+) \) and \( U_q(\mathfrak{b}^-) \) be the usual sub-Hopf algebras of \( U_q(\mathfrak{g}) \) corresponding to the Borel subalgebras \( \mathfrak{b}^+ \) and \( \mathfrak{b}^- \) respectively. Let \( D_q(\mathfrak{g}) = U_q(\mathfrak{b}^+) \ltimes U_q(\mathfrak{b}^-) \) be the Drinfeld double. Since the groups of one-dimensional representations of \( U_q(\mathfrak{b}^+) \), \( U_q(\mathfrak{b}^-) \), \( D_q(\mathfrak{g}) \) and \( C_q[G] \) are all isomorphic to \( H = L' \), these algebras are all equipped with \( L \)-bigradings. Moreover the Rosso-Tanisaki pairing is compatible with the bigradings on \( U_q(\mathfrak{b}^+) \) and \( U_q(\mathfrak{b}^-) \). For any anti-symmetric bicharacter \( p \) on \( L \) one may therefore twist simultaneously the Hopf algebras \( U_q(\mathfrak{b}^+) \), \( U_q(\mathfrak{b}^-) \) and \( D_q(\mathfrak{g}) \) in such a way that \( D_{q,p}(\mathfrak{g}) \cong U_{q,p}(\mathfrak{b}^+) \ltimes U_{q,p}(\mathfrak{b}^-) \). The algebra \( D_{q,p}(\mathfrak{g}) \) is the ‘multi-parameter quantized universal enveloping algebra’ constructed by Okado and Yamane [25] and previously in special cases in [9, 32]. The canonical pairing between \( C_q[G] \) and \( U_q(\mathfrak{g}) \) induces a \( L \)-compatible pairing between \( C_q[G] \) and \( D_q(\mathfrak{g}) \). Thus there is an induced pairing between the multi-parameter quantum group \( C_{q,p}[G] \) and the multi-parameter double \( D_{q,p^{-1}}(\mathfrak{g}) \). Recall that the Hopf algebra \( C_{q,p}[G] \) is defined as the restricted dual of \( U_q(\mathfrak{g}) \) with respect to a certain category \( C \) of modules over \( U_q(\mathfrak{g}) \). There is a natural deformation functor from this category to a category \( C_p \) of modules over \( D_{q,p^{-1}}(\mathfrak{g}) \) and \( C_{q,p}[G] \) turns out to be the restricted dual of \( D_{q,p^{-1}}(\mathfrak{g}) \) with respect to this category. This Peter-Weyl theorem for \( C_{q,p}[G] \) was also found by Andruskiewitsch and Enriquez in [1] using a different construction of the quantized universal enveloping algebra and in special cases in [5, 14].

The main theorem describing the primitive spectrum of \( C_{q,p}[G] \) is proved in the final section. Since \( C_{q,p}[G] \) inherits an \( L \)-bigrading, there is a natural action of \( H \) as automorphisms of \( C_{q,p}[G] \). For each \( w \in W \times W \), we construct an algebra \( A_w = (C_{q,p}[G]/I_w)_{L_w} \) which is a localization of a quotient of \( C_{q,p}[G] \). For each prime \( p \in \text{Spec} C_{q,p}[G] \) there is a unique \( w \in W \times W \) such that \( P \supset I_w \) and \( PA_w \) is proper. Thus \( \text{Spec} C_{q,p}[G] \cong \bigsqcup_{w \in W \times W} \text{Spec}_w C_{q,p}[G] \) where \( \text{Spec}_w C_{q,p}[G] \cong \text{Spec} A_w \) is the set of primes of type \( w \). The key results are then Theorems 4.14 and 4.15 which state that an ideal of \( A_w \) is generated by its intersection with the center and that \( H \) acts transitively on the maximal ideals of the center. From this it follows that the primitive ideals of \( C_{q,p}[G] \) of type \( w \) form an orbit under the action of \( H \).

An earlier version of our approach to the proof of Joseph’s theorem is contained in the unpublished article [17]. The approach presented here is a generalization of this proof to the multi-parameter case.

These results are algebraic analogs of results of Levendorskii [20, 21] on the irreducible representations of multi-parameter function algebras and compact quantum groups. The bijection between symplectic leaves of the compact Poisson group and irreducible *-representations of the compact quantum group found by Soibelman in the one-parameter case, breaks down in the multi-parameter case.

After this work was completed, the authors became aware of the work of Constantini and Varagnolo [7, 8] which has some overlap with the results in this paper.

1. Poisson Lie Groups

1.1. Notation. Let \( g \) be a complex semi-simple Lie algebra associated to a Cartan matrix \( [a_{ij}]_{1 \leq i,j \leq n} \). Let \( \{d_i\}_{1 \leq i \leq n} \) be relatively prime positive integers such that \( |d_i a_{ij}|_{1 \leq i,j \leq n} \) is symmetric positive definite.

Let \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{g} \), \( R \) the associated root system, \( B = \{\alpha_1, \ldots, \alpha_n\} \) a basis of \( R \), \( R_+ \) the set of positive roots and \( W \) the Weyl group. We denote by \( P \) and \( Q \) the lattices of weights and roots respectively. The fundamental weights are denoted by \( \varpi_1, \ldots, \varpi_n \) and the set of dominant integral weights by \( P^+ = \sum_{i=1}^{n} N\varpi_i \). Let \( (-,-) \) be a non-degenerate \( \mathfrak{g} \)-invariant symmetric bilinear form on \( \mathfrak{g} \); it will identify \( \mathfrak{g} \), resp. \( \mathfrak{h} \), with its dual \( \mathfrak{g}^* \), resp. \( \mathfrak{h}^* \). The form \( (-,-) \) can be chosen in order to induce a perfect pairing \( P \times Q \to \mathbb{Z} \) such that

\[
(\varpi_i, \alpha_j) = \delta_{ij} d_i, \quad (\alpha_i, \alpha_j) = d_i a_{ij}.
\]
Hence $d_i = (\alpha_i,\alpha_i)/2$ and $(\alpha,\alpha) \in 2\mathbb{Z}$ for all $\alpha \in \mathbb{R}$. For each $\alpha_j$ we denote by $h_j \in \mathfrak{h}$ the corresponding coroot: $\varpi_i(h_j) = \delta_{ij}$. We also set
\[ n^\pm = \oplus_{\alpha \in \mathbb{R}_+} \mathfrak{h}_{\pm \alpha}, \quad \mathfrak{b} = \mathfrak{g} \times \mathfrak{g}, \quad t = \mathfrak{h} \times \mathfrak{h}, \quad u^\pm = n^\pm \times n^\mp.

Let $G$ be a connected complex semi-simple algebraic group such that Lie$(G) = \mathfrak{g}$ and set $D = G \times G$. We identify $(G)$ and (its subgroups) with the diagonal copy inside $D$. We denote by $exp$ the exponential map from $\mathfrak{d}$ to $D$. We shall in general denote a Lie subalgebra of $\mathfrak{d}$ by a gothic symbol and the corresponding connected analytic subgroup of $D$ by a capital letter.

1.2. Poisson Lie group structure on $G$. Let $a = \sum_{\alpha \in \mathbb{R}_+} c_\alpha \in \wedge^2 \mathfrak{g}$ where the $c_\alpha$ are root vectors such that $(\alpha,\alpha) = \delta_{\alpha,-\alpha}$. Let $u \in \wedge^2 \mathfrak{h}$ and set $r = a - u$. Then it is well known that $r$ satisfies the modified Yang-Baxter equation [3, 20] and that therefore the tensor $\pi(g) = (l_g)_* r - (r_g)_* r$ furnishes $G$ with the structure of a Poisson Lie group, see [13, 22, 30] ($(l_g)_*$ and $(r_g)_*$ are the differentials of the left and right translation by $g \in G$).

We may write $u = \sum_{1 \leq i,j \leq n} u_{ij} h_i \otimes h_j$ for a skew-symmetric $n \times n$ matrix $[u_{ij}]$. The element $u$ can be considered either as an alternating form on $\mathfrak{h}^*$ or a linear map $u \in \text{End} \mathfrak{h}$ by the formula
\[ \forall x \in \mathfrak{h}, \quad u(x) = \sum_{i,j} u_{ij}(x,h_i)h_j. \]

The Manin triple associated to the Poisson Lie structure on $G$ given by $r$ is described as follows. Set $u_\pm = u \pm i I \in \text{End} \mathfrak{h}$ and define
\[ \vartheta : \mathfrak{h} \leftrightarrow t, \quad \vartheta(x) = -(u_-(x),u_+(x)), \]
\[ a = \vartheta(\mathfrak{h}), \quad g_r = a + u^+. \]

Following [30] one sees easily that the associated Manin triple is $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}_r)$ where $\mathfrak{g}$ is identified with the diagonal copy inside $\mathfrak{d}$. Then the corresponding triple of Lie groups is $(D, G, G_r)$, where $A = \exp(a)$ is an analytic torus and $G_r = AU^\perp$. Notice that $g_r$ is a solvable, but not in general algebraic, Lie subalgebra of $\mathfrak{d}$.

The following is an easy consequence of the definition of $a$ and the identities $u_+ + u_- = 2u, u_+ - u_- = 2I$:
\[ (1.1) \quad a = \{(x,y) \in t \mid x + y = u(y - x)\} = \{(x,y) \in t \mid u_+(x) = u_-(y)\}. \]

Recall that $\exp : \mathfrak{h} \rightarrow H$ is surjective; let $L_H$ be its kernel. We shall denote by $X(K)$ the group of characters of an algebraic torus $K$. Any $\chi \in X(H)$ is given by $\chi(\exp x) = \exp dx$, $x \in \mathfrak{h}$, where $dx \in \mathfrak{h}^*$ is the differential of $\chi$. Then
\[ X(H) \cong L = L_H^\circ := \{ \xi \in \mathfrak{h}^* \mid \xi(L_H) \subset 2i\pi \mathbb{Z} \}. \]

One can show that $L$ has a basis consisting of dominant weights.

Recall that if $\tilde{G}$ is a connected simply connected algebraic group with Lie algebra $\mathfrak{g}$ and maximal torus $\tilde{H}$, we have
\[ L_{\tilde{H}} = P^\circ = \oplus_{j=1}^n 2\pi \mathbb{Z} h_j, \quad X(\tilde{H}) \cong P, \]
\[ Q \subseteq L \subseteq P, \quad \pi_1(G) = L_H/P^\circ \cong P/L. \]

Notice that $L_H/P^\circ$ is a finite group and $\exp u(L_{\tilde{H}})$ is a subgroup of $H$. We set
\[ \Gamma_0 = \{(a,a) \in T \mid a^2 = 1\}, \quad \Delta = \{(a,a) \in T \mid a^2 \in \exp u(L_H)\}, \]
\[ \Gamma = A \cap H = \{(a,a) \in T \mid a = \exp x = \exp y, \ x + y = u(y - x)\}. \]

It is easily seen that $\Gamma = G \cap G_r$.

**Proposition 1.1.** We have $\Delta = \Gamma \cdot \Gamma_0.$
Proof. We obviously have $\Gamma_0 \subset \Delta$. Let $(\exp h, \exp h) \in \Gamma$, $h \in \mathfrak{h}$. By definition there exist $(x, y) \in \mathfrak{a}$, $\ell_1, \ell_2 \in L_H$ such that

$$x = h + \ell_1, \quad y = h + \ell_2, \quad y + x = u(y - x).$$

Hence $y + x = 2h + \ell_1 + \ell_2 = u(\ell_2 - \ell_1)$ and $(\exp h)^2 = \exp 2h = \exp u(\ell_2 - \ell_1)$. This shows $(\exp h, \exp h) \in \Delta$. Thus $\Gamma_0 \subseteq \Delta$.

Let $(a, a) \in \Delta$, $a \in \exp h$, $h \in \mathfrak{h}$. From $a^2 \in \exp u L_H$ we get $\ell, \ell' \in L_H$ such that $2h = u(\ell') + \ell$. Set $x = h - \ell'/2 - \ell'/2$, $y = h + \ell'/2 - \ell/2$. Then $y + x = u(y - x)$ and we have $\exp(-\ell/2 - \ell'/2) = \exp(\ell'/2 - \ell/2)$, since $\ell' \in L_H$. If $b = \exp(-\ell'/2 + \ell/2)$ we obtain $\exp x = \exp y = ab^{-1}$, hence $(a, a) = (\exp x, \exp y)$. $(b, b) \in \Gamma_0$. Therefore $\Gamma_0 = \Delta$.

Remark. When $u$ is “generic” $\Gamma_0$ is not contained in $\Gamma$. For example, take $G$ to be $SL(3, \mathbb{C})$ and $u = \alpha(h_1 \otimes h_2 - h_2 \otimes h_1)$ with $\alpha \notin \mathbb{Q}$.

Considered as a Poisson variety, $G$ decomposes as a disjoint union of symplectic leaves. Denote by $\text{Symp} G$ the set of these symplectic leaves. Since $r$ is $H$-invariant, translation by an element of $H$ is a Poisson morphism and hence there is an induced action of $H$ on $\text{Symp} G$. The key to classifying the symplectic leaves is the following result, cf. [22, 30].

Theorem 1.2. The symplectic leaves of $G$ are exactly the connected components of $G \cap G_r x G_r$ for $x \in G$.

Remark. $A$, $\Gamma$ and $G_r$ are in general not closed subgroups of $D$. This has for consequence that the analysis of Symp $G$ made in [15, Appendix A] in the case $u = 0$ does not apply in the general case.

Set $Q = HG_r = T U^+$. Then $Q$ is a Borel subgroup of $D$ and, recalling that the Weyl group associated to the pair $(G, T)$ is $W \times W$, the corresponding Bruhat decomposition yields $D = \bigsqcup_{w \in W \times W} Q w G_r$. Therefore any symplectic leaf is contained in a Bruhat cell $Q w G_r$ for some $w \in W \times W$.

Definition. A leaf $A$ is said to be of type $w$ if $A \subset Q w G_r$. The set of leaves of type $w$ is denoted by $\text{Symp}_w G$.

For each $w \in W \times W$ set $w = (w_+, w_-), w_{\pm} \in W$, and fix a representative $w$ in the normaliser of $T$. One shows as in [15, Appendix A] that $G \cap Q w G_r \neq \emptyset$, for all $w \in W \times W$; hence $\text{Symp}_w G \neq \emptyset$ and $G \cap G_r w G_r \neq \emptyset$, since $Q w G_r = \bigsqcup_{h \in H} h G_r w G_r$.

The adjoint action of $D$ on itself is denoted by $\text{Ad}$. Set

$$U_w^{-} = \text{Ad} w(U) \cap U^+, \quad A'_w = \{a \in A \mid a w G_r = w G_r\},$$

$$T_w' = \{t \in T \mid t G_r w G_r = G_r w G_r\}, \quad H'_w = H \cap T_w'.$$

Recall that $U_w^{-}$ is isomorphic to $\mathcal{C}(w)$ where $l(w) = l(w_+) + l(w_-)$ is the length of $w$. We set $s(w) = \dim H'_w$.

Lemma 1.3. (i) $A'_w = \text{Ad} w(A) \cap A$ and $T'_w = A \text{Ad} w(A) = AH'_w$.

(ii) We have $\text{Lie}(A'_w) = a'_w = \{\theta(x) \mid x \in \ker(w_- u_+ u_+ - u_+ w_- u_-)\}$ and $\dim a'_w = n - s(w)$.

Proof. (i) The first equality is obvious and the second is an easy consequence of the bijection, induced by multiplication, between $U_w^{-} \times T \times U^+$ and $Q w G_r$.

(ii) By definition we have $a'_w = \{\theta(x) \mid x \in h, w^{-1}(\theta(x)) \in \mathfrak{a}\}$. From (1.1) we deduce that $\theta(x) \in a'_w$ if and only if $u_+ w_-^{-1}(u_- x) = u_+ w_-^{-1}(u_- x)$. It follows from (i) that $\dim T'_w = n + \dim H'_w = 2n - \dim A'_w$, hence $\dim a'_w = n - s(w)$.

Recall that $u \in \text{End} \mathfrak{h}$ is an alternating bilinear form on $\mathfrak{h}^*$. It is easily seen that $\forall \lambda, \mu \in \mathfrak{h}^*$, $u(\lambda, \mu) = -\langle u(\lambda), \mu \rangle$, where $\langle \cdot \rangle u \in \text{End} \mathfrak{h}^*$ is the transpose of $u$.

Notation. Set $t u = -\Phi, \Phi_\pm = \Phi \pm I, \sigma(w) = \Phi_- w^{-1} \Phi_+ - \Phi_+ w_+ \Phi_-$, where $w_{\pm} \in W$ is considered as an element of $\text{End} \mathfrak{h}^*$. 
Observe that \( t^* u_\pm = -\Phi_{\mp} \) and that
\[
(1.2) \quad w(\lambda, \mu) = (\Phi \lambda, \mu), \quad \text{for all } \lambda, \mu \in h^*.
\]
Furthermore, since the transpose of \( w_\pm \in \text{End } h^* \) is \( w^\pm_\pm \in \text{End } h \), we have \( t^* \sigma(w) = u^- w^- u_+ - u_+ w_+ u^- \). Hence by Lemma 1.3
\[
(1.3) \quad s(w) = \text{codim } \ker h^*_\lambda, \sigma(w), \quad \dim A'_w = \dim \ker h^*_\lambda, \sigma(w).
\]

1.3. The algebraic case. As explained in 1.1 the Lie algebra \( g_\gamma \) is in general not algebraic. We now describe its algebraic closure. Recall that a Lie subalgebra \( m \) of \( g \) is said to be algebraic if \( m \) is the Lie algebra of a closed (connected) algebraic subgroup of \( D \).

Definition. Let \( m \) be a Lie subalgebra of \( D \). The smallest algebraic Lie subalgebra of \( D \) containing \( m \) is called the algebraic closure of \( m \) and will be denoted by \( \overline{m} \).

Recall that \( g_\gamma = a \oplus u^+ \). Notice that \( u^+ \) is an algebraic Lie subalgebra of \( D \); hence it follows from [4, Corollary II.7.7] that \( g_\gamma = a \oplus u^+ \). Thus we only need to describe \( a \). Since \( t \) is algebraic we have \( \overline{a} \subseteq t \) and we are reduced to characterize the algebraic closure of a Lie subalgebra of \( t = \text{Lie}(T) \).

The group \( T = H \times H \) is an algebraic torus (of rank \( 2n \)). The map \( \chi \mapsto d\chi \) identifies \( X(T) \) with \( L \times L \).

Let \( t \subset t \) be a subalgebra. We set
\[ t^\perp = \{ \theta \in X(T) \mid t \subset \ker t \} \]
The following proposition is well known. It can for instance be deduced from the results in [4, II.8].

Proposition 1.4. Let \( t \) be a subalgebra of \( t \). Then \( \tilde{t} = \cap_{\theta \in t^\perp} \ker t \theta \) and \( \tilde{t} \) is the Lie algebra of the closed connected algebraic subgroup \( \overline{K} = \cap_{\theta \in t^\perp} \ker t \theta \).

Corollary 1.5. We have
\[
a^\perp = \{ (\lambda, \mu) \in X(T) \mid \Phi_+ \lambda + \Phi_- \mu = 0 \},
\]
\[ \tilde{a} = \cap_{(\lambda, \mu) \in a^\perp} \ker t(\lambda, \mu), \quad \Lambda = \cap_{(\lambda, \mu) \in a^\perp} \ker t(\lambda, \mu). \]

Proof. From the definition of \( a = \phi(h) \) we obtain
\[
(\lambda, \mu) \in a^\perp \iff \forall x \in h, \lambda(-u_-(x)) + \mu(-u_+(x)) = 0.
\]
The first equality then follows from \( t^* u_\pm = -\Phi_{\mp} \). The remaining assertions are consequences of Proposition 1.4.

Set
\[
\begin{align*}
h_Q &= Q \otimes \mathbb{Z} P^\circ = \bigoplus_{i=1}^n Q h_i, \quad h^*_Q = Q \otimes \mathbb{P} = \bigoplus_{i=1}^n Q \omega_i, \\
a_Q^\perp &= Q \otimes \mathbb{Z} a^\perp = \{ (\lambda, \mu) \in h_Q^* \times h_Q^* \mid \Phi_+ \lambda + \Phi_- \mu = 0 \}.
\end{align*}
\]
Observe that \( \dim Q a_Q^\perp = \text{rk} \mathbb{Z} a^\perp \) and that, by Corollary 1.5,
\[
(1.4) \quad \dim \tilde{a} = 2n - \dim Q a_Q^\perp.
\]

Lemma 1.6. \( a_Q^\perp \cong \{ \nu \in h_Q^* \mid \Phi \nu \in h_Q^* \} \).

Proof. Define a \( \mathbb{Q} \)-linear map
\[
\{ \nu \in h_Q^* \mid \Phi \nu \in h_Q^* \} \longrightarrow a_Q^\perp, \quad \nu \mapsto (-\Phi_- \nu, \Phi_+ \nu),
\]
It is easily seen that this provides the desired isomorphism.
Theorem 1.7. The following assertions are equivalent:

(i) $g_v$ is an algebraic Lie subalgebra of $\mathfrak{g}$;
(ii) $u(P \times P) \subset \mathbb{Q}$;
(iii) $\exists m \in \mathbb{N}^*, \Phi(mP) \subset P$;
(iv) $\Gamma$ is a finite subgroup of $T$.

Proof. Recall that $g_v$ is algebraic if and only if $\mathfrak{a} = \mathfrak{a}, i.e. n = \dim \mathfrak{a} = \dim \mathfrak{a}$. By (1.4) and Lemma 1.6 this is equivalent to $\Phi(P) \subset \mathbb{Q} \otimes \mathbb{Z}P$. The equivalence of (i) to (iii) then follows from the definitions, (1.2) and the fact that $t_u = -\Phi$.

To prove the equivalence with (iv) we first observe that, by Proposition 1.1, $\Gamma$ is finite if and only if $\exp u(L_H)$ is finite. Since $L_H/\mathfrak{p}^\circ$ is finite this is also equivalent to $\exp u(P^\circ)$ being finite. This holds if and only if $u(mP^\circ) \subset P^\circ$ for some $m \in \mathbb{N}^*$. Hence the result. $\square$

When the equivalent assertions of Theorem 1.7 hold, we shall say that we are in the algebraic case or that $u$ is algebraic. In this case all the subgroups previously introduced are closed algebraic subgroups of $D$ and we may define the algebraic quotient varieties $D/G_r$ and $G = G/T$. Let $p$ be the projection $G \to \hat{G}$. Observe that $\hat{G}$ is open in in $D/G_r$ and that the Poisson bracket of $G$ passes to $\hat{G}$. We set

$$C_w = G_r w G_r/G_r, \quad C_w = Q w G_r/G_r = \cup_{h \in H} h C_{w}$$

$$B_w = C_w \cap \hat{G}, \quad B_w = C_w \cap \hat{G}, \quad A_w = p^{-1}(B_w).$$

The next theorem summarizes the description of the symplectic leaves in the algebraic case.

Theorem 1.8. 1. $\text{Symp}_w G \neq \emptyset$ for all $w \in W \times W$, $\text{Symp} G = \sqcup_{w \in W \times W} \text{Symp}_w G$.

2. Each symplectic leaf of $\hat{G}$, resp. $G$, is of the form $hS_w$, resp. $hA_w$, for some $h \in H$ and $w \in W \times W$, where $A_w$ denotes a fixed connected component of $p^{-1}(B_w)$.

3. $C_w \cong A_w \times U_w^-$ where $A_w = A/A_w'$ is a torus of rank $s(w)$. Hence $\dim C_w = \dim B_w = \dim A_w = l(w) + s(w)$ and $H/\text{Stab}_H A_w$ is a torus of rank $n - s(w)$.

Proof. The proofs are similar to those given in [15, Appendix A] for the case $u = 0$. $\square$

2. Deformations of Bigraded Hopf Algebras

2.1. Bigraded Hopf Algebras and their deformations. Let $L$ be an (additive) abelian group. We will say that a Hopf algebra $(A, i, m, \epsilon, \Delta, S)$ over a field $\mathbb{k}$ is an $L$-bigraded Hopf algebra if it is equipped with an $L \times L$ grading

$$A = \bigoplus_{(\lambda, \mu) \in L \times L} A_{\lambda, \mu}$$

such that

(1) $\mathbb{k} \subset A_{0,0}$, $A_{\lambda, \mu} A_{\lambda', \mu'} \subset A_{\lambda + \lambda', \mu + \mu'}$ (i.e. $A$ is a graded algebra)

(2) $\Delta(A_{\lambda, \mu}) \subset \sum_{\nu \in L} A_{\lambda, \nu} \otimes A_{-\nu, \mu}$

(3) $\lambda \neq -\mu$ implies $\epsilon(A_{\lambda, \mu}) = 0$

(4) $S(A_{\lambda, \mu}) \subset A_{\mu, \lambda}$.

For sake of simplicity we shall often make the following abuse of notation: If $a \in A_{\lambda, \mu}$ we will write $\Delta(a) = \sum_{\nu \in L} a_{\lambda, \nu} \otimes a_{-\nu, \mu}$. Let $\lambda, \mu \in A_{\lambda, \mu}$.

Let $p : L \times L \to \mathbb{k}^*$ be an antisymmetric bicharacter on $L$ in the sense that $p$ is multiplicative in both entries and that, for all $\lambda, \mu \in L$,

$$\begin{align*}
(1) & \quad p(\mu, \mu) = 1; \\
(2) & \quad p(\lambda, \mu) = p(\mu, -\lambda).
\end{align*}$$

Then the map $\tilde{p} : (L \times L) \times (L \times L) \to \mathbb{k}^*$ given by

$$\tilde{p}(\lambda, \mu, \lambda', \mu') = p(\lambda, \lambda') p(\mu, \mu')^{-1}$$

is a 2-cocycle on $L \times L$ such that $\tilde{p}(0,0) = 1$. 
One may then define a new multiplication, $m_p$, on $A$ by

$$\forall a \in A_{\lambda, \mu}, \ b \in A_{\lambda', \mu'}, \ a \cdot b = p(\lambda, \lambda')p(\mu, \mu')^{-1}ab.$$  

\[\text{(2.1)}\]

**Theorem 2.1.** $A_p := (A, i, m_p, \epsilon, \Delta, S)$ is an $L$-bigraded Hopf algebra.

**Proof.** The proof is a slight generalization of that given in [2]. It is well known that $A_p = (A, i, m_p)$ is an associative algebra. Since $\Delta$ and $\epsilon$ are unchanged, $(A, \Delta, \epsilon)$ is still a coalgebra. Thus it remains to check that $\epsilon, \Delta$ are algebra morphisms and that $S$ is an antipode.

Let $x \in A_{\lambda, \mu}$ and $y \in A_{\lambda', \mu'}$. Then

$$\epsilon(x \cdot y) = p(\lambda, \lambda')p(\mu, \mu')^{-1}\epsilon(xy)$$

$$= p(\lambda, \lambda')p(\mu, \mu')^{-1}\delta_{\lambda, -\lambda'}\delta_{\mu, -\mu'}\epsilon(x)\epsilon(y)$$

$$= p(\lambda, \lambda')p(-\lambda, -\lambda')^{-1}\epsilon(x)\epsilon(y)$$

$$= \epsilon(x)\epsilon(y)$$

So $\epsilon$ is a homomorphism. Now suppose that $\Delta(x) = \sum x_{\lambda, \nu} \otimes x_{-\nu, \mu}$ and $\Delta(y) = \sum y_{\lambda', \nu'} \otimes y_{-\nu', \mu'}$. Then

$$\Delta(x) \cdot \Delta(y) = \left(\sum x_{\lambda, \nu} \otimes x_{-\nu, \mu}\right) \cdot \left(\sum y_{\lambda', \nu'} \otimes y_{-\nu', \mu'}\right)$$

$$= \sum x_{\lambda, \nu} \cdot y_{\lambda', \nu'} \otimes x_{-\nu, \mu} \cdot y_{-\nu', \mu'}$$

$$= p(\lambda, \lambda')p(\mu, \mu')^{-1}\sum p(\nu, \nu')^{-1}p(-\nu, -\nu')x_{\lambda, \nu}y_{\lambda', \nu'} \otimes x_{-\nu, \mu}y_{-\nu', \mu'}$$

$$= p(\lambda, \lambda')p(\mu, \mu')^{-1}\Delta(xy)$$

So $\Delta$ is also a homomorphism. Finally notice that

$$\sum S(x_{(1)}) \cdot x_{(2)} = \sum S(x_{\lambda, \nu}) \cdot x_{-\nu, \mu}$$

$$= \sum p(\nu, -\nu)p(\lambda, \mu)^{-1}S(x_{\lambda, \nu})x_{-\nu, \mu}$$

$$= p(\lambda, \mu)^{-1}\sum S(x_{\lambda, \nu}) \cdot x_{-\nu, \mu}$$

$$= p(\lambda, \mu)^{-1}\epsilon(x)$$

$$= \epsilon(x)$$

A similar calculation shows that $\sum x_{(1)} \cdot S(x_{(2)}) = \epsilon(x)$. Hence $S$ is indeed an antipode. \[\square\]

**Remark.** The isomorphism class of the algebra $A_p$ depends only on the cohomology class $[\hat{p}] \in H^2(L \times L, \mathbb{K}^*)$, [2, §3].

**Remark.** Theorem 2.1 is a particular case of the following more general construction. Let $(A, i, m)$ be a $\mathbb{K}$-algebra. Assume that $F \in GL_{\mathbb{K}}(A \otimes A)$ is given such that (with the usual notation)

1. $F(m \otimes 1) = (m \otimes 1)F_{23}F_{13}$; $F(1 \otimes m) = (1 \otimes m)F_{12}F_{13}$
2. $F(i \otimes 1) = i \otimes 1$; $F(1 \otimes i) = 1 \otimes i$
3. $F_{12}F_{13}F_{23} = F_{23}F_{13}F_{12}$, i.e. $F$ satisfies the Quantum Yang-Baxter Equation.

Set $m_F = m \circ F$. Then $(A, i, m_F)$ is a $\mathbb{K}$-algebra.

Assume furthermore that $(A, i, m, \epsilon, \Delta, S)$ is a Hopf algebra and that

1. $F : A \otimes A \to A \otimes A$ is morphism of coalgebras
2. $mF(S \otimes 1)\Delta = m(S \otimes 1)\Delta$; $mF(1 \otimes S)\Delta = m(1 \otimes S)\Delta$.

Then $A_F := (A, i, m_F, \epsilon, \Delta, S)$ is a Hopf algebra. The proofs are straightforward verifications and are left to the interested reader.
When $A$ is an $L$-bigraded Hopf algebra and $p$ is an antisymmetric bicharacter as above, we may define $F \in GL_K(A \otimes A)$ by

$$\forall a \in A_{\lambda, \mu}, \forall b \in A_{\lambda', \mu'}, \quad F(a \otimes b) = p(\lambda, \lambda')p(\mu, \mu')^{-1}a \otimes b.$$ 

It is easily checked that $F$ satisfies the conditions (1) to (5) and that the Hopf algebras $AF$ and $A_p$ coincide.

A related construction of the quantization of a monoidal category is given in [24].

### 2.2. Diagonalizable subgroups of $R(A)$

In the case where $L$ is a finitely generated group and $A$ is a finitely generated algebra (which is the case for the multi-parameter quantum groups considered here), there is a simple geometric interpretation of $L$-bigradings. They correspond to algebraic group maps from the diagonalizable group $L^\vee$ to the group of one dimensional representations of $A$.

Assume that $K$ is algebraically closed. Let $(A, i, m, \epsilon, \Delta, S)$ be a Hopf $K$-algebra. Denote by $R(A)$ the multiplicative group of one dimensional representations of $A$, i.e. the character group of the algebra $A$. Notice that when $A$ is a finitely generated $K$-algebra, $R(A)$ has the structure of an affine algebraic group over $K$, with algebra of regular functions given by $K[R(A)] = A/\mathfrak{J}$ where $J$ is the semi-prime ideal $\cap_{h \in R(A)} \text{Ker } h$. Recall that there are two natural group homomorphisms $l, r : R(A) \rightarrow \text{Aut}_K(A)$ given by

$$l_h(x) = \sum h(S(x_{(1)})) x_{(2)} = \sum h^{-1}(x_{(1)}) x_{(2)}$$

$$r_h(x) = \sum x_{(1)} h(x_{(2)}).$$

**Theorem 2.2.** Let $A$ be a finitely generated Hopf algebra and let $L$ be a finitely generated abelian group. Then there is a natural bijection between:

1. $L$-bigradings on $A$;
2. Hopf algebra maps $A \rightarrow KL$ (where $KL$ denotes the group algebra);
3. morphisms of algebraic groups $L^\vee \rightarrow R(A)$.

**Proof.** The bijection of the last two sets of maps is well-known. Given an $L$-bigrading on $A$, we may define a map $\phi : A \rightarrow KL$ by $\phi(a_{\lambda, \mu}) = \epsilon(a) u_\lambda$. It is easily verified that this is a Hopf algebra map. Conversely, given a map $L^\vee \rightarrow R(A)$ we may construct an $L$ bigrading using the following result.

**Theorem 2.3.** Let $(A, i, m, \epsilon, \Delta, S)$ be a finitely generated Hopf algebra over $K$. Let $H$ be a closed diagonalizable algebraic subgroup of $R(A)$. Denote by $L$ the (additive) group of characters of $H$ and by $\langle -, - \rangle : L \times H \rightarrow K^\ast$ the natural pairing. For $(\lambda, \mu) \in L \times L$ set

$$A_{\lambda, \mu} = \{ x \in A \mid \forall h \in H, \ l_h(x) = \langle \lambda, h \rangle x, \ r_h(x) = \langle \mu, h \rangle x \}.$$ 

Then $(A, i, m, \epsilon, \Delta, S)$ is an $L$-bigraded Hopf algebra.

**Proof.** Recall that any element of $A$ is contained in a finite dimensional subcoalgebra of $A$. Therefore the actions of $H$ via $r$ and $l$ are locally finite. Since they commute and $H$ is diagonalizable, $A$ is $L \times L$ diagonalizable. Thus the decomposition $A = \bigoplus_{(\lambda, \mu) \in L \times L} A_{\lambda, \mu}$ is a grading.

Now let $C$ be a finite dimensional subcoalgebra of $A$ and let $\{ c_1, \ldots, c_n \}$ be a basis of $H \times H$ weight vectors. Suppose that $\Delta(c_i) = \sum t_{ij} \otimes c_j$. Then since $c_i = \sum t_{ij} \epsilon(c_j)$, the $t_{ij}$ span $C$ and it is easily checked that $\Delta(t_{ij}) = \sum t_{ik} \otimes t_{kj}$. Since $l_h(c_i) = \sum h^{-1}(t_{ij}) c_j$ for all $h \in H$ and the $c_i$ are weight vectors, we must have that $h(t_{ij}) = 0$ for $i \neq j$. This implies that

$$l_h(t_{ij}) = h^{-1}(t_{ii}) t_{ij}, \quad r_h(t_{ij}) = h(t_{jj}) t_{ij}$$

and that the map $\lambda(h) = h(t_{ii})$ is a character of $H$. Thus $t_{ij} \in A_{-\lambda_i, \lambda_j}$ and hence

$$\Delta(t_{ij}) = \sum t_{ik} \otimes t_{kj} \in \sum A_{-\lambda_i, \lambda_k} \otimes A_{-\lambda_j, \lambda_j}.$$
This gives the required condition on $\Delta$. If $\lambda + \mu \neq 0$ then there exists $h \in H$ such that $\langle -\lambda, h \rangle \neq (\mu, h)$.

Let $x \in A_{\lambda, \mu}$. Then

$$\langle \mu, h \rangle e(x) = e(r_h(x)) = h(x) = e(l_{h^{-1}}(x)) = \langle -\lambda, h \rangle e(x).$$

Hence $e(x) = 0$. The assertion on $S$ follows similarly. \hfill \qed

Remark. In particular, if $G$ is any algebraic group and $H$ is a diagonalizable subgroup with character group $L$, then we may deform the Hopf algebra $K[G]$ using an antisymmetric bicharacter on $L$. Such deformations are algebraic analogs of the deformations studied by Rieffel in [27].

2.3. Deformations of dual pairs. Let $A$ and $U$ be a dual pair of Hopf algebras. That is, there exists a bilinear pairing $\langle \ | \rangle : A \times U \to K$ such that:

1. $\langle a \mid 1 \rangle = \epsilon(a); \langle 1 \mid u \rangle = \epsilon(u)$
2. $\langle a \mid u_1 u_2 \rangle = \sum (a_{(1)} \langle a_{(2)} \mid u_2 \rangle)$
3. $\langle a_1 a_2 \mid u \rangle = \sum (a_1 \mid u_{(1)} \langle a_2 \mid u_{(2)} \rangle)$
4. $\langle S(a) \mid u \rangle = \langle a \mid S(u) \rangle.$

Assume that $A$ is bigraded by $L$, $U$ is bigraded by an abelian group $Q$ and that there is a homomorphism $\gamma : Q \to L$ such that

$$\langle A_{\lambda, \mu} \mid U_{\gamma, \delta} \rangle \neq 0 \text{ only if } \lambda + \mu = \gamma + \delta.$$

In this case we will call the pair $\{A, U\}$ an $L$-bigraded dual pair. We shall be interested in §3 and §4 in the case where $Q = L$ and $\gamma = Id$.

Remark. Suppose that the bigradings above are induced from subgroups $H$ and $\tilde{H}$ of $R(A)$ and $R(U)$ respectively and that the map $Q \to L$ is induced from a map $h \mapsto \tilde{h}$ from $H$ to $\tilde{H}$. Then the condition on the pairing may be restated as the fact that the form is ad-invariant in the sense that for all $a \in A$, $u \in U$ and $h \in H$,

$$\langle \text{ad}_h a \mid u \rangle = \langle a \mid \text{ad}_h u \rangle$$

where $\text{ad}_h a = r_h b_h(a)$.

Theorem 2.4. Let $\{A, U\}$ be the bigraded dual pair as described above. Let $p$ be an antisymmetric bicharacter on $L$ and let $\tilde{p}$ be the induced bicharacter on $Q$. Define a bilinear form $\langle \ | \rangle_p : A_{\gamma, \delta} \times U_{\gamma, \delta} \to K$ by:

$$\langle a_{\lambda, \mu} \mid u_{\gamma, \delta} \rangle_p = p(\lambda, \gamma)^{-1}p(\mu, \delta)^{-1}\langle a_{\lambda, \mu} \mid u_{\gamma, \delta} \rangle.$$

Then $\langle \ | \rangle_p$ is a Hopf pairing and $\{A_{\gamma, \delta}, U_p\}$ is an $L$-bigraded dual pair.

Proof. Let $a \in A_{\lambda, \mu}$ and let $u_i \in U_{\gamma_i, \delta_i}, i = 1, 2$. Then

$$\langle a \mid u_1 u_2 \rangle_p = p(\gamma_1, \gamma_2)p(\delta_1, \delta_2)^{-1}p(\lambda, \gamma_1 + \gamma_2)^{-1}p(\mu, \delta_1 + \delta_2)^{-1}\langle a \mid u_1 u_2 \rangle.$$

Suppose that $\Delta(a) = \sum a_{\lambda, \mu} \otimes a_{-\nu, \mu}$. Then by the assumption on the pairing, the only possible value of $\nu$ for which $\langle a_{\lambda, \mu} \mid u_1 \rangle_{\nu, \mu} \langle a_{-\nu, \mu} \mid u_2 \rangle$ is non-zero is $\nu = \gamma_1 + \delta_1 - \lambda = \mu - \gamma_2 - \delta_2$. Therefore

$$\langle a_{(1)} \mid u_1 \rangle_{\gamma_1, \delta_1} \langle a_{(2)} \mid u_2 \rangle = p(\lambda, \gamma_1)^{-1}p(\nu, \delta_1)^{-1}p(-\nu, \gamma_2)^{-1}p(\lambda, \delta_1 + \delta_2)^{-1}\langle a_{(1)} \mid u_1 \rangle_{\gamma_1, \delta_1} \langle a_{(2)} \mid u_2 \rangle$$

$$= p(\lambda, \gamma_1)^{-1}p(\mu - \gamma_2 - \delta_2, \delta_1)^{-1}p(\lambda - \gamma_1 - \delta_1, \gamma_2)^{-1}p(\mu, \delta_1 + \delta_2)^{-1}\langle a_{(1)} \mid u_1 \rangle_{\gamma_1, \delta_1} \langle a_{(2)} \mid u_2 \rangle$$

$$= p(\gamma_1, \gamma_2)p(\delta_1, \delta_2)^{-1}p(\lambda, \gamma_1 + \gamma_2)^{-1}p(\mu, \delta_1 + \delta_2)^{-1}\langle a \mid u_1 u_2 \rangle = \langle a \mid u_1 u_2 \rangle_p.$$

This proves the first axiom. The others are verified similarly. \hfill \qed
Corollary 2.5. Let \( \{A, U, p\} \) be as in Theorem 2.4. Let \( M \) be a right \( A \)-comodule with structure map \( \rho : M \to M \otimes A \). Then \( M \) is naturally endowed with \( U \) and \( U_\rho \) left module structures, denoted by \( (u, x) \mapsto ux \) and \( (u, x) \mapsto u \cdot x \) respectively. Assume that \( M = \bigoplus_{\lambda \in \mathbb{L}} M_\lambda \) for some \( \mathbb{K} \)-subspaces such that \( \rho(M_\lambda) \subset \bigoplus_\nu M_{\nu} \otimes A_{\nu, \lambda} \). Then we have \( U_{\gamma, \delta} M_\lambda \subset M_{\lambda - \gamma, \delta} \) and the two structures are related by
\[
\forall u \in U_{\gamma, \delta}, \forall x \in M_\lambda, \quad u \cdot x = p(\lambda, \gamma - \delta) p(\gamma, \delta) ux.
\]

Proof. Notice that the coalgebras \( A \) and \( A_{p^{-1}} \) are the same. Set \( \rho(x) = \sum x_{(0)} \otimes x_{(1)} \) for all \( x \in M \). Then it is easily checked that the following formulas define the desired \( U \) and \( U_\rho \) module structures:
\[
\forall u \in U, \quad ux = \sum x_{(0)}(x_{(1)} \mid u), \quad u \cdot x = \sum x_{(0)}(x_{(1)} \mid u)_p.
\]
When \( x \in M_\lambda \) and \( u \in U_{\gamma, \delta} \) the additional condition yields
\[
u \cdot x = \sum x_{(0)}(\nu, -\gamma)p(\lambda, -\delta)(x_{(1)} \mid u).
\]
But \( \langle x_{(1)} \mid u \rangle \neq 0 \) forces \( -\nu = \lambda - \gamma - \delta \), hence \( u \cdot x = p(\lambda, \gamma - \delta)p(\gamma, \delta)\sum x_{(0)}(x_{(1)} \mid u) = p(\lambda, \gamma - \delta)p(\gamma, \delta)ux \).

Denote by \( A^{\text{op}} \) the opposite algebra of the \( \mathbb{K} \)-algebra \( A \). Let \( \{A^{\text{op}}, U, (\mid \mid)\} \) be a dual pair of Hopf algebras. The double \( A \ltimes U \) is defined as follows, \([10, 3.3]\). Let \( I \) be the ideal of the tensor algebra \( T(A \otimes U) \) generated by elements of type
\begin{align*}
(a) & \quad 1 - 1_A, \quad 1 - 1_U \\
(b) & \quad xx' - x \otimes x', \quad x, x' \in A, \quad yy' - y \otimes y', \quad y, y' \in U \\
(c) & \quad x_{(1)} \otimes y_{(1)}(x_{(2)} \mid y_{(2)}) - (x_{(1)} \mid y_{(1)})y_{(2)} \otimes x_{(2)}, \quad x \in A, \quad y \in U
\end{align*}
Then the algebra \( A \ltimes U := T(A \otimes U)/I \) is called the Drinfeld double of \( \{A, U\} \). It is a Hopf algebra in a natural way:
\[
\Delta(a \otimes u) = (a_{(1)} \otimes u_{(1)}) \otimes (a_{(2)} \otimes u_{(2)}), \quad 
\epsilon(a \otimes u) = \epsilon(a)\epsilon(u), \quad S(a \otimes u) = (S(a) \otimes 1)(1 \otimes S(u)).
\]

Notice for further use that \( A \ltimes U \) can equally be defined by relations of type (a), (b), (c), (e, y, x) or (a), (b), (c, x, y), where we set
\begin{align*}
(c_{x,y}) & \quad x \otimes y = \langle x_{(1)} \mid y_{(1)} \rangle(x_{(3)} \mid S(y_{(3)}))y_{(2)} \otimes x_{(2)}, \quad x \in A, \quad y \in U \\
(c_{y,x}) & \quad y \otimes x = \langle x_{(1)} \mid S(y_{(1)}) \rangle(x_{(3)} \mid y_{(3)})x_{(2)} \otimes y_{(2)}, \quad x \in A, \quad y \in U
\end{align*}

Theorem 2.6. Let \( \{A^{\text{op}}, U\} \) be an \( \mathbb{L} \)-bigraded dual pair, \( p \) be an antisymmetric bicharacter on \( \mathbb{L} \) and \( \bar{p} \) be the induced bicharacter on \( \mathbb{Q} \). Then \( A \ltimes U \) inherits an \( \mathbb{L} \)-bigrading and there is a natural isomorphism of \( \mathbb{L} \)-bigraded Hopf algebras:
\[
(A \ltimes U)_p \cong A_p \ltimes U_{\bar{p}}.
\]

Proof. Recall that as a \( \mathbb{K} \)-vector space \( A \ltimes U \) identifies with \( A \otimes U \). Define an \( \mathbb{L} \)-bigrading on \( A \ltimes U \) by
\[
\forall \alpha, \beta \in \mathbb{L}, \quad (A \ltimes U)_{\alpha, \beta} = \sum_{\lambda - \gamma = \alpha, \mu - \delta = \beta} A_{\lambda, \mu} \otimes U_{\gamma, \delta}.
\]
To verify that this yields a structure of graded algebra on \( A \ltimes U \) it suffices to check that the defining relations of \( A \ltimes U \) are homogeneous. This is clear for relations of type (a) or (b). Let \( x_{\lambda, \mu} \in A_{\lambda, \mu} \) and \( y_{\gamma, \delta} \in U_{\gamma, \delta} \). Then the corresponding relation of type (c) becomes
\[
(*) \quad \sum_{\nu, \xi} x_{\lambda, \mu} y_{\gamma, \xi} \langle x_{-\nu, \mu} \mid y_{-\xi, \delta} \rangle = \langle x_{\lambda, \mu} \mid y_{\gamma, \xi} \rangle y_{-\xi, \delta} x_{-\nu, \mu}.
\]
Then it is clear that the matrix associated to \( \gamma \) is homogeneous. It is easy to see that the conditions (2), (3), (4) of 2.1 hold. Hence \( A \rtimes U \) is an \( L \)-bigraded Hopf algebra.

Notice that \( (A_p)^{op} \cong (A^{op})_{p^{-1}} \), so that Theorem 2.4 defines a suitable pairing between \( (A_p)^{op} \) and \( U_p \). Thus \( A_p \rtimes U_p \) is defined. Let \( \phi \) be the natural surjective homomorphism from \( T(A \otimes U) \) onto \( A_p \rtimes U_p \). To check that \( \phi \) induces an isomorphism it again suffices to check that \( \phi \) vanishes on the defining relations of \( (A \rtimes U)_p \). Again, this is easy for relations of type (a) and (b). The relation (c) says that
\[
p(\lambda, \gamma)p(-\nu, \xi)(x_{-\nu, \mu} | y_{-\xi, \delta})x_{\lambda, \nu} \cdot y_{\gamma, \xi} = p(\xi, \nu)p(\delta, -\mu)(x_{\lambda, \mu} | y_{\gamma, \xi})y_{-\xi, \delta} \cdot x_{-\nu, \mu} = 0
\]
in \( (A \rtimes U)_p \). Multiply the left hand side of this equation by \( p(\lambda, -\gamma)p(\mu, -\delta) \) and apply \( \phi \). We obtain the following expression in \( A_p \rtimes U_p \):
\[
p(-\nu, \xi)p(\mu, -\delta)(x_{-\nu, \mu} | y_{-\xi, \delta})x_{\lambda, \nu} \cdot y_{\gamma, \xi} = p(\lambda, -\gamma)p(\nu, -\xi)(x_{\lambda, \mu} | y_{\gamma, \xi})y_{-\xi, \delta} \cdot x_{-\nu, \mu}
\]
which is equal to
\[
(x_{-\nu, \mu} | y_{-\xi, \delta})p \cdot x_{\lambda, \nu} \cdot y_{\gamma, \xi} - (x_{\lambda, \mu} | y_{\gamma, \xi})p \cdot y_{-\xi, \delta} \cdot x_{-\nu, \mu}.
\]
But this is a defining relation of type (c) in \( A_p \rtimes U_p \), hence zero.

It remains to see that \( \phi \) induces an isomorphism of Hopf algebras, which is a straightforward consequence of the definitions. \( \square \)

2.4. Cocycles. Let \( L \) be, in this section, an arbitrary free abelian group with basis \( \{\omega_1, \ldots, \omega_n\} \) and set \( h^* = C \otimes_\mathbb{Z} L \). We freely use the terminology of [2]. Recall that \( H^2(L, \mathbb{C}^*) \) is in bijection with the set \( \mathcal{H} \) of multiplicatively antisymmetric \( n \times n \)-matrices \( \gamma = [\gamma_{ij}] \). This bijection maps the class \([e]\) onto the matrix defined by \( \gamma_{ij} = e(\omega_i, \omega_j)/e(\omega_j, \omega_i) \). Furthermore it is an isomorphism of groups with respect to component-wise multiplication of matrices.

Remark . The notation is as in 2.1. We recalled that the isomorphism class of the algebra \( A_p \) depends only on the cohomology class \([\tilde{p}]\) in \( H^2(L, \mathbb{C}^*) \). Let \( \gamma \in \mathcal{H} \) be the matrix associated to \( p \) and \( \gamma^{-1} \) its inverse in \( \mathcal{H} \). Notice that the multiplicative matrix associated to \([\tilde{p}]\) is then \( \tilde{\gamma} = [\gamma_{ij}^{-1}] \) in the basis given by the \( (\omega_i, 0), (0, \omega_i) \in L \times L \). Therefore the isomorphism class of the algebra \( A_p \) depends only on the cohomology class \([\tilde{p}]\) in \( H^2(L, \mathbb{K}^*) \).

Let \( h \in C^* \). If \( x \in C \) we set \( q^x = \exp(-hx/2) \). In particular \( q = \exp(-h/2) \). Let \( u : L \times L \to C \) be a complex alternating \( Z \)-bilinear form. Define
\[
(2.3) \quad p : L \times L \to C^*, \quad p(\lambda, \mu) = \exp \left( -\frac{h}{4} u(\lambda, \mu) \right) = q^{\frac{1}{4} u(\lambda, \mu)}.
\]
Then it is clear that \( p \) is an antisymmetric bicharacter on \( L \).

Observe that, since \( h^* = C \otimes_\mathbb{Z} L \), there is a natural isomorphism of additive groups between \( \wedge^2 h \) and the group of complex alternating \( Z \)-bilinear forms on \( L \), where \( h \) is the \( C \)-dual of \( h^* \). Set \( Z_h = \{u \in \wedge^2 h \mid u(L \times L) \subseteq \frac{4\pi}{h} \mathbb{Z}\} \).

Theorem 2.7. There are isomorphisms of abelian groups:
\[
H^2(L, \mathbb{C}^*) \cong \mathcal{H} \cong \wedge^2 h / Z_h.
\]

Proof. The first isomorphism has been described above. Let \( \varphi = [\gamma_{ij}] \in \mathcal{H} \) and choose \( u_{ij} \), \( 1 \leq i < j \leq n \) such that \( \gamma_{ij} = \exp(-\frac{h}{4} u_{ij}) \). We can define \( u \in \wedge^2 h \) by setting \( u(\omega_i, \omega_j) = u_{ij} \), \( 1 \leq i < j \leq n \). It is then easily seen that one can define an injective morphism of abelian groups
\[
\varphi : H^2(L, \mathbb{C}^*) \cong \mathcal{H} \longrightarrow \wedge^2 h / Z_h, \quad \varphi(\gamma) = [u]
\]
where \([u]\) is the class of \( u \). If \( u \in \wedge^2 h \), define a 2-cocycle \( p \) by the formula (2.3). Then the multiplicative matrix associated to \([\tilde{p}]\) in \( H^2(L, \mathbb{C}^*) \) is given by
\[
\gamma_{ij} = p(\omega_i, \omega_j)/p(\omega_j, \omega_i) = p(\omega_i, \omega_j)^2 = \exp(-\frac{h}{2} u(\omega_i, \omega_j)).
\]
This shows that \([u] = \varphi([\gamma_0])\); thus \(\varphi\) is an isomorphism. \(\square\)

We list some consequences of Theorem 2.7. We denote by \([u]\) an element of \(\wedge^2 \mathfrak{h}/\mathbb{Z}_h\) and we set \([p] = \varphi^{-1}([u])\). We have seen that we can define a representative \(p\) by the formula (2.3).

1. \([p]\) of finite order in \(H^2(\mathbb{L}, \mathbb{C}^*) \leftrightarrow u(\mathbb{L} \times \mathbb{L}) \subseteq \frac{i\pi}{\hbar} \mathbb{Q}\), and \(q\) root of unity \(\leftrightarrow h \in i\pi \mathbb{Q}\).

2. Notice that \(u = 0\) is algebraic, whether \(q\) is a root of unity or not. Assume that \(q\) is a root of unity; then we get from 1 that

\(\boxed{[p]\text{ of finite order } \leftrightarrow u\text{ is algebraic.}}\)

3. Assume that \(q\) is not a root of unity and that \(u \neq 0\). Then \([p]\) of finite order implies \((0) \neq u(\mathbb{L} \times \mathbb{L}) \subseteq \frac{i\pi}{\hbar} \mathbb{Q}\). This shows that

\(0 \neq u\text{ algebraic } \Rightarrow [p]\text{ is not of finite order.}\)

**Definition.** The bicharacter \(p : (\lambda, \mu) \mapsto q^{\frac{1}{2}u(\lambda,\mu)}\) is called \(q\)-rational if \(u \in \wedge^2 \mathfrak{h}\) is algebraic.

The following technical result will be used in the next section. Recall the definition of \(\Phi_- = \Phi - I\) given in the Section 1.

**Proposition 2.8.** Let \(K = \{\lambda \in \mathbb{L} : (\Phi_- \lambda, \mathbb{L}) \subset 4\pi \mathbb{Z}\}\). If \(q\) is not a root of unity, then \(K = 0\).

**Proof.** Let \(\lambda \in K\). We can define \(z : \mathfrak{h}_Q^* \rightarrow \mathbb{Q}\), by

\(\forall \mu \in \mathfrak{h}_Q^*, \quad (\Phi_- \lambda, \mu) = \frac{4i\pi}{\hbar} z(\mu).\)

The map \(z\) is clearly \(\mathbb{Q}\)-linear. It follows, since \((\ ,\ )\) is non-degenerate on \(\mathfrak{h}_Q^*\), that there exists \(\nu \in \mathfrak{h}_Q^*\) such that \(z(\mu) = (\nu, \mu)\) for all \(\mu \in \mathfrak{h}_Q^*\). Therefore \(\Phi_- \lambda = \frac{4i\pi}{\hbar} \nu\), and so \(\Phi \lambda = \lambda + \frac{4i\pi}{\hbar} \nu\).

Now, \((\Phi \lambda, \lambda) = u(\lambda, \lambda) = 0\) implies that

\(\frac{4i\pi}{\hbar} (\nu, \lambda) = -(\lambda, \lambda)\)

If \((\lambda, \lambda) \neq 0\) then \(h \in i\pi \mathbb{Q}\), contradicting the assumption that \(q\) is not a root of unity. Hence \((\lambda, \lambda) = 0\), which forces \(\lambda = 0\) since \(\lambda \in \mathbb{L} \subseteq \mathfrak{h}_Q^*\). \(\square\)

### 3. Multiparameter Quantum Groups

#### 3.1. One-parameter quantized enveloping algebras

The notation is as in sections 1 and 2. In particular we fix a lattice \(\mathbb{L}\) such that \(\mathbb{Q} \subseteq \mathbb{L} \subseteq \mathbb{P}\) and we denote by \(G\) the connected semi-simple algebraic group with maximal torus \(H\) such that \(\text{Lie}(G) = \mathfrak{g}\) and \(\mathbf{X}(H) \cong \mathbb{L}\).

Let \(q \in \mathbb{C}^*\) and assume that \(q\) is not a root of unity. Let \(h \in \mathbb{C} \setminus i\pi \mathbb{Q}\) such that \(q = \exp(-h/2)\) as in 2.4. We set

\[ q_i = q^{d_i}, \quad \hat{q}_i = (q_i - q_i^{-1})^{-1}, \quad 1 \leq i \leq n. \]

Denote by \(U^0\) the group algebra of \(\mathbf{X}(H)\), hence

\[ U^0 = \mathbb{C}[k_\lambda; \lambda \in \mathbb{L}], \quad k_0 = 1, \quad k_\lambda k_\mu = k_{\lambda + \mu}. \]

Set \(k_i = k_{\alpha_i}, \quad 1 \leq i \leq n\). The one parameter quantized enveloping algebra associated to this data, cf. [33], is the Hopf algebra

\[ U_q(\mathfrak{g}) = U^0[e_i, \hat{f}_i; \quad 1 \leq i \leq n] \]
with defining relations:

\[ k_\lambda e_j k_\lambda^{-1} = q^{(\lambda, \alpha_j)} e_j, \quad k_\lambda f_j k_\lambda^{-1} = q^{-(\lambda, \alpha_j)} f_j \]
\[ e_i f_j - f_j e_i = \delta_{ij} q_i (k_i - k_i^{-1}) \]
\[ \sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1-a_{ij}}{k} \right] q_i^{1-a_{ij}-k} e_j e_i^k = 0, \text{ if } i \neq j \]
\[ \sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1-a_{ij}}{k} \right] q_i^{1-a_{ij}-k} f_i f_j^k = 0, \text{ if } i \neq j \]

where \([m]_i = (t - t^{-1}) \ldots (t^m - t^{-m})\) and \([m]_k = \left[ \frac{m}{k} \right] \). The Hopf algebra structure is given by

\[ \Delta(k_\lambda) = k_\lambda \otimes k_\lambda, \quad \epsilon(k_\lambda) = 1, \quad S(k_\lambda) = k_\lambda^{-1} \]
\[ \Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i \]
\[ \epsilon(e_i) = \epsilon(f_i) = 0, \quad S(e_i) = -k_i^{-1} e_i, \quad S(f_i) = -f_i k_i. \]

We define subalgebras of \( U_q(g) \) as follows

\[ U_q(n^+) = \mathbb{C}[e_i; 1 \leq i \leq n], \quad U_q(n^-) = \mathbb{C}[f_i; 1 \leq i \leq n] \]
\[ U_q(b^+)^o = U^0[e_i; 1 \leq i \leq n], \quad U_q(b^-)^o = U^0[f_i; 1 \leq i \leq n]. \]

For simplicity we shall set \( U^\pm = U_q(n^\pm) \). Notice that \( U^0 \) and \( U_q(b^\pm) \) are Hopf subalgebras of \( U_q(g) \).

Recall [23] that the multiplication in \( U_q(g) \) induces isomorphisms of vector spaces

\[ U_q(g) \cong U^- \otimes U^0 \otimes U^+ \cong U^+ \otimes U^0 \otimes U^- . \]

Set \( Q_+ = \oplus_{i=1}^n \mathbb{N} \alpha_i \) and

\[ \forall \beta \in Q_+, \quad U^\pm_{\beta} = \{ u \in U^\pm : \forall \lambda \in L, k_\lambda u k_\lambda^{-1} = q^{(\lambda, \beta)} u \}. \]

Then one gets: \( U^\pm = \oplus_{\beta \in Q_+} U^\pm_{\beta} \).

3.2. The Rosso-Tanisaki-Killing form. Recall the following result, [28, 33].

**Theorem 3.1.** There exists a unique non-degenerate Hopf pairing

\[ \langle \ , \ \rangle : U_q(b^+)^o \otimes U_q(b^-) \longrightarrow \mathbb{C} \]

satisfying the following conditions:

(i) \( \langle k_\lambda \mid k_\mu \rangle = q^{-(\lambda, \mu)} \); 
(ii) \( \forall \lambda \in L, 1 \leq i \leq n, \langle k_\lambda \mid f_i \rangle = \langle e_i \mid k_\lambda \rangle = 0 \);
(iii) \( \forall 1 \leq i, j \leq n, \langle e_i \mid f_j \rangle = -\delta_{ij} q_i \). 

2. If \( \gamma, \eta \in Q_+ \), \( \langle U^+_{\gamma} \mid U^-_{-\eta} \rangle \neq 0 \) implies \( \gamma = \eta \).

The results of §2.3 then apply and we may define the associated double:

\[ D_q(g) = U_q(b^+) \times U_q(b^-). \]

It is well known, e.g. [10], that

\[ D_q(g) = \mathbb{C}[s_\lambda, t_\lambda; e_i, f_i; \lambda \in L, 1 \leq i \leq n] \]

where \( s_\lambda = k_\lambda \otimes 1, t_\lambda = 1 \otimes k_\lambda, e_i = e_i \otimes 1, f_i = 1 \otimes f_i \). The defining relations of the double given in §2.3 imply that

\[ s_\lambda e_j s_\lambda^{-1} = q^{(\lambda, \alpha_j)} e_j, \quad t_\lambda e_j t_\lambda^{-1} = q^{(\lambda, \alpha_j)} e_j, \quad s_\lambda f_j s_\lambda^{-1} = q^{-(\lambda, \alpha_j)} f_j, \quad t_\lambda f_j t_\lambda^{-1} = q^{-(\lambda, \alpha_j)} f_j. \]
It follows that
\[ D_q(\mathfrak{g})/(s_\lambda - t_\lambda; \lambda \in \mathcal{L}) \sim U_q(\mathfrak{g}), \quad e_i \mapsto e_i, \quad f_i \mapsto f_i, \quad s_\lambda \mapsto k_\lambda, \quad t_\lambda \mapsto k_\lambda. \]
Observe that this yields an isomorphism of Hopf algebras. The next proposition collects some well known elementary facts.

**Proposition 3.2.**
1. Any finite dimensional simple \( U_q(\mathfrak{b}^{\pm}) \)-module is one dimensional and \( R(U_q(\mathfrak{b}^{\pm})) \) identifies with \( H \) via
   \[ \forall h \in H, \quad h(k_\lambda) = \langle \lambda, h \rangle, \quad h(e_i) = 0, \quad h(f_i) = 0. \]
2. \( R(D_q(\mathfrak{g})) \) identifies with \( H \) via
   \[ \forall h \in H, \quad h(s_\lambda) = \langle \lambda, h \rangle, \quad h(t_\lambda) = \langle \lambda, h \rangle^{-1}, \quad h(e_i) = h(f_i) = 0. \]

**Corollary 3.3.**
1. \( \{U_q(\mathfrak{b}^+)^{op}, U_q(\mathfrak{b}^-)\} \) is an \( \mathcal{L} \)-bigraded dual pair. We have
   \[ k_\lambda \in U_q(\mathfrak{b}^{\pm})_{-\lambda, \lambda}, \quad e_i \in U_q(\mathfrak{b}^+)_{-\alpha_i, 0}, \quad f_i \in U_q(\mathfrak{b}^-)_{0, -\alpha_i}. \]
2. \( D_q(\mathfrak{g}) \) is an \( \mathcal{L} \)-bigraded Hopf algebra where
   \[ s_\lambda \in D_q(\mathfrak{g})_{-\lambda, \lambda}, \quad t_\lambda \in D_q(\mathfrak{g})_{\lambda, -\lambda}, \quad e_i \in D_q(\mathfrak{g})_{-\alpha_i, 0}, \quad f_i \in D_q(\mathfrak{g})_{0, -\alpha_i}. \]

**Proof.**
1. Observe that for all \( h \in H, \)
   \[ l_h(k_\lambda) = h^{-1}(k_\lambda) = \langle -\lambda, h \rangle k_\lambda, \quad r_h(k_\lambda) = h(k_\lambda) = \langle \lambda, h \rangle k_\lambda, \]
   \[ l_h(e_i) = h^{-1}(e_i) = (-\alpha_i, h)e_i, \quad r_h(e_i) = e_i, \]
   \[ l_h(f_i) = f_i, \quad r_h(f_i) = h(k_\lambda^{-1})f_i = \langle -\alpha_i, h \rangle f_i. \]
   It is then clear that \( U_q^{+\gamma, 0} = U_q^{+} \) and \( U_q^{-\gamma} = U_q^{-\gamma} \) for all \( \gamma \in \mathcal{Q}_+. \) The claims then follow from these formulas, Theorem 2.3, Theorem 3.1, and the definitions.

2. The fact that \( D_q(\mathfrak{g}) \) is an \( \mathcal{L} \)-bigraded Hopf algebra follows from Theorem 2.3. The assertions about the \( \mathcal{L} \times \mathcal{L} \) degree of the generators is proved by direct computation using Proposition 3.2. \( \square \)

**Remark.** We have shown in Theorem 2.6 that, as a double, \( D_q(\mathfrak{g}) \) inherits an \( \mathcal{L} \)-bigrading given by:
\[ D_q(\mathfrak{g})_{\alpha, \beta} = \sum_{\lambda, \gamma = -\alpha, \mu = -\beta} U_q(\mathfrak{b}^+)_{\lambda, \mu} \otimes U_q(\mathfrak{b}^-)_{\gamma, \delta}. \]
It is easily checked that this bigrading coincides with the bigrading obtained in the above corollary by means of Theorem 2.3.

### 3.3. One-parameter quantized function algebras
Let \( M \) be a left \( D_q(\mathfrak{g}) \)-module. The dual \( M^* \) will be considered in the usual way as a left \( D_q(\mathfrak{g}) \)-module by the rule: \((uf)(x) = f(S(u)x), \) \( x \in M, f \in M^*, u \in D_q(\mathfrak{g}). \) Assume that \( M \) is an \( U_q(\mathfrak{g}) \)-module. An element \( x \in M \) is said to have weight \( \mu \in \mathcal{L} \) if \( k_\lambda x = q^{\langle \lambda, \mu \rangle} x \) for all \( \lambda \in \mathcal{L}; \) we denote by \( M_\mu \) the subspace of elements of weight \( \mu. \)

It is known, [13], that the category of finite dimensional (left) \( U_q(\mathfrak{g}) \)-modules is a completely reducible braided rigid monoidal category. Set \( \mathcal{L}^+ = \mathcal{L} \cap \mathcal{P}^+ \) and recall that for each \( \Lambda \in \mathcal{L}^+ \) there exists a finite dimensional simple module of highest weight \( \Lambda, \) denoted by \( L(\Lambda), \) cf. [29] for instance. One has \( L(\Lambda)^* \cong L(u_0 \Lambda) \) where \( u_0 \) is the longest element of \( W. \) (Notice that the results quoted usually cover the case where \( \mathcal{L} = \mathbb{Q}. \) One defines the modules \( L(\Lambda) \) in the general case in the following way. Let us denote temporarily the algebra \( U_q(\mathfrak{g}) \) for a given choice of \( \mathcal{L} \) by \( U_q(\mathfrak{L}(\mathfrak{g})). \) Given a module \( L(\lambda) \) defined on \( U_q(\mathfrak{g}) \) we may define an action of \( U_q(\mathfrak{L}(\mathfrak{g})) \) by setting \( k_\lambda x = q^{\langle \lambda, \mu \rangle} x \) for all elements \( x \) of weight \( \mu, \) where \( q^{\langle \lambda, \mu \rangle} \) is as defined in section 2.4.)

Let \( \mathcal{C}_q \) be the subcategory of finite dimensional \( U_q(\mathfrak{g}) \)-modules consisting of finite direct sums of \( L(\Lambda), \) \( \Lambda \in \mathcal{L}^+. \) The category \( \mathcal{C}_q \) is closed under tensor products and the formation of duals. Notice that \( \mathcal{C}_q \) can...
be considered as a braided rigid monoidal category of $D_q(g)$-modules where $s_\lambda, t_\lambda$ act as $k_\lambda$ on an object of $C_q$.

Let $M \in \text{obj}(C_q)$, then $M = \oplus_{\mu \in L} M_\mu$. For $f \in M^*$, $v \in M$ we define the coordinate function $c_{f,v} \in U_q(g)^*$ by

$$\forall u \in U_q(g), \quad c_{f,v}(u) = \langle f, uv \rangle$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing. Using the standard isomorphism $(M \otimes N)^* \cong N^* \otimes M^*$ one has the following formula for multiplication,

$$c_{f,v}c_{f',v'} = c_{f \otimes f', v \otimes v'}.$$

**Definition.** The quantized function algebra $C_q[G]$ is the restricted dual of $C_q$: that is to say

$$C_q[G] = C[c_{f,v}; v \in M, f \in M^*, M \in \text{obj}(C_q)].$$

The algebra $C_q[G]$ is a Hopf algebra; we denote by $\Delta, \epsilon, S$ the comultiplication, counit and antipode on $C_q[G]$. If $\{v_1, \ldots, v_s; f_1, \ldots, f_t\}$ is a dual basis for $M \in \text{obj}(C_q)$ one has

$$\Delta(c_{f,v}) = \sum_i c_{f,v_i} \otimes c_{f,v_i}, \quad \epsilon(c_{f,v}) = \langle f, v \rangle, \quad S(c_{f,v}) = c_{v,f}.$$

Notice that we may assume that $v_j \in M_{v_j}, f_j \in M_{f_j}^*$. We set

$$C(M) = C(c_{f,v}; f \in M^*, v \in M), \quad C(M)_{\lambda, \mu} = C(c_{f,v}; f \in M_{\lambda}^*, v \in M_{\mu}).$$

Then $C(M)$ is a subalgebra of $C_q[G]$ such that $C(M) = \bigotimes_{(\lambda,\mu) \in L \times L} C(M)_{\lambda,\mu}$. When $M = L(\Lambda)$ we abbreviate the notation to $C(M) = C(\Lambda)$. It is then classical that

$$C_q[G] = \bigoplus_{\Lambda \in L^*} C(\Lambda).$$

Since $C_q[G] \subset U_q(g)^*$ we have a duality pairing

$$\langle \cdot, \cdot \rangle : C_q[G] \times D_q(g) \rightarrow \mathbb{C}.$$

Observe that there is a natural injective morphism of algebraic groups

$$H \rightarrow R(C_q[G]), \quad h(c_{f,v}) = \langle \mu, h \rangle \epsilon(c_{f,v}) \text{ for all } v \in M_\mu, M \in \text{obj}(C_q).$$

The associated automorphisms $r_h, l_h \in \text{Aut}(C_q[G])$ are then described by

$$\forall c_{f,v} \in C(M)_{\lambda,\mu}, \quad r_h(c_{f,v}) = \langle \mu, h \rangle c_{f,v}, \quad l_h(c_{f,v}) = \langle \lambda, h \rangle c_{f,v}.$$ 

Define

$$\forall (\lambda, \mu) \in L \times L, \quad C_q[G]_{\lambda,\mu} = \{ a \in C_q[G] \mid r_h(a) = \langle \mu, h \rangle a, l_h(a) = \langle \lambda, h \rangle a \}.$$

**Theorem 3.4.** The pair of Hopf algebras $\{C_q[G], D_q(g)\}$ is an $L$-bigraded dual pair.

**Proof.** It follows from (3.1) that $C_q[G]$ is an $L$-bigraded Hopf algebra. The axioms (1) to (4) of 2.3 are satisfied by definition of the Hopf algebra $C_q[G]$. We take $\sim$ to be the identity map of $L$. The condition (2.2) is consequence of $D_q(g)_{\gamma,\delta} M_\mu \subset M_{\mu - \gamma - \delta}$ for all $M \in C_q$. To verify this inclusion, notice that

$$e_j \in D_q(g)_{-\alpha_j,0}, f_j \in D_q(g)_{0,\alpha_j}, \quad e_j M_\mu \subset M_{\mu + \alpha_j}, \quad f_j M_\mu \subset M_{\mu - \alpha_j}.$$ 

The result then follows easily. 

Consider the algebras $D_q^{-1}(g)$ and $C_q^{-1}[G]$ and use $\sim$ to distinguish elements, subalgebras, etc. of $D_q^{-1}(g)$ and $C_q^{-1}[G]$. It is easily verified that the map $\sigma : D_q(g) \rightarrow D_q^{-1}(g)$ given by

$$s_\lambda \mapsto \tilde{s}_\lambda, \quad t_\lambda \mapsto \tilde{t}_\lambda, \quad e_i \mapsto q_i^{1/2} \tilde{f}_i \tilde{\alpha}_i, \quad f_i \mapsto q_i^{-1/2} \tilde{e}_i \tilde{\alpha}_i^{-1}$$

is an isomorphism of Hopf algebras.
For each \( \Lambda \in \mathbb{L}^+ \), \( \sigma \) gives a bijection \( \sigma : L(-\lambda_0 \Lambda) \to \hat{L}(\Lambda) \) which sends \( v \in L(-\lambda_0 \Lambda) \) onto \( \hat{v} \in \hat{L}(\Lambda) \). Therefore we obtain an isomorphism \( \sigma : \mathbb{C}_{q^{-1}}[G] \to \mathbb{C}_q[G] \) such that

\[
\forall f \in L(-\lambda_0 \Lambda)_{-\lambda}, \, \forall v \in L(-\lambda_0 \Lambda)_{\mu}, \quad \sigma(f \cdot v) = c_{f,v}.
\]

Notice that

\[
\sigma(D_q(\mathfrak{g})_{\gamma, \delta}) = D_{q^{-1}}(\mathfrak{g})_{-\gamma, -\delta} \quad \text{and} \quad \sigma(C_{q^{-1}}[G]_{\lambda, \mu}) = C_q[G]_{-\lambda, -\mu}.
\]

### 3.4. Deformation of one-parameter quantum groups

We continue with the same notation. Let \([p] \in H^2(\mathbb{L}, \mathbb{C}^*)\). As seen in §2.4 we can, and we do, choose \( p \) to be an antisymmetric bicharacter such that

\[
\forall \lambda, \mu \in \mathbb{L}, \quad p(\lambda, \mu) = q^{\frac{1}{2}u(\lambda, \mu)}
\]

for some \( u \in \Lambda^2 \mathfrak{h} \). Recall that \( \hat{p} \in Z^2(\mathbb{L} \times \mathbb{L}, \mathbb{C}^*) \), cf. 2.1.

We now apply the results of §2.1 to \( D_q(\mathfrak{g}) \) and \( \mathbb{C}_q[G] \). Using Theorem 2.1 we can twist \( D_q(\mathfrak{g}) \) by \( \hat{p}^{-1} \) and \( \mathbb{C}_q[G] \) by \( \hat{p} \). The resulting \( \mathbb{L} \)-bigraded Hopf algebras will be denoted by \( D_{q,p^{-1}}(\mathfrak{g}) \) and \( \mathbb{C}_{q,p}[G] \). The algebra \( \mathbb{C}_{q,p}[G] \) will be referred to as the multi-parameter quantized function algebra. Versions of \( D_{q,p^{-1}}(\mathfrak{g}) \) can be referred to by authors as the multi-parameter quantized enveloping algebra. Alternatively, this name can be applied to the quotient of \( D_{q,p^{-1}}(\mathfrak{g}) \) by the radical of the pairing with \( \mathbb{C}_{q,p}[G] \).

**Theorem 3.5.** Let \( U_{q,p^{-1}}(\mathfrak{b}^+) \) and \( U_{q,p^{-1}}(\mathfrak{b}^-) \) be the deformations by \( p^{-1} \) of \( U_q(\mathfrak{b}^+) \) and \( U_q(\mathfrak{b}^-) \) respectively. Then the deformed pairing

\[
\langle p^{-1} : U_{q,p^{-1}}(\mathfrak{b}^+) \otimes U_{q,p^{-1}}(\mathfrak{b}^-) \to \mathbb{C}
\]

is a non-degenerate Hopf pairing satisfying:

\[
\forall x \in \mathfrak{u}^+, \, y \in \mathfrak{u}^-, \, \lambda, \mu \in \mathbb{L}, \quad \langle x \cdot k_\lambda \mid y \cdot k_\mu \rangle_{p^{-1}} = q^{(\Phi - \lambda, \mu)} \langle x \mid y \rangle.
\]

Moreover,

\[
U_{q,p^{-1}}(\mathfrak{b}^+) \ltimes U_{q,p^{-1}}(\mathfrak{b}^-) \cong (U_q(\mathfrak{b}^+) \ltimes U_q(\mathfrak{b}^-))_{p^{-1}} = D_{q,p^{-1}}(\mathfrak{g}).
\]

**Proof.** By Theorem 2.4 the deformed pairing is given by

\[
\langle a_{\lambda, \mu} \mid u_{\gamma, \delta} \rangle_{p^{-1}} = p(\lambda, \gamma)p(\mu, \delta)(a_{\lambda, \mu} \mid u_{\gamma, \delta}).
\]

To prove (3.4) we can assume that \( x \in U^+_{\gamma, \delta}, \, y \in U^-_{x, -\nu} \). Then we obtain

\[
\langle x \cdot k_\lambda \mid y \cdot k_\mu \rangle_{p^{-1}} = p(\lambda + \gamma, \mu)p(\lambda, \mu - \nu)(x \cdot k_\lambda \mid y \cdot k_\mu) = p(\lambda, 2\mu)p(\lambda - \mu, \gamma - \nu)q_{-\lambda, \mu}^{-1} \langle x \mid y \rangle
\]

by the definition of the product \( \cdot \) and [33, 2.1.3]. But \( (x \mid y) = 0 \) unless \( \gamma = \nu \), hence the result. Observe in particular that \( (x \mid y)_{p^{-1}} = (x \mid y) \). Therefore [33, 2.1.4] shows that \( \langle p^{-1} \rangle \) is non-degenerate on \( U^+_{\gamma} \times U^-_{\gamma} \). It then follows from (3.4) and Proposition 2.8 that \( \langle p^{-1} \rangle \) is non-degenerate. The remaining isomorphism follows from Theorem 2.6.

Many authors have defined multi-parameter quantized enveloping algebras. In [14, 25] a definition is given using explicit generators and relations, and in [1] the construction is made by twisting the comultiplication, following [26]. It can be easily verified that these algebras and the algebras \( D_{q,p^{-1}}(\mathfrak{g}) \) coincide. The construction of a multi-parameter quantized function algebra by twisting the multiplication was first performed in the \( GL(n) \)-case in [2].

The fact that \( D_{q,p^{-1}}(\mathfrak{g}) \) and \( \mathbb{C}_{q,p}[G] \) form a Hopf dual pair has already been observed in particular cases, see e.g. [14]. We will now deduce from the previous results that this phenomenon holds for an arbitrary semi-simple group.

**Theorem 3.6.** \( \{ \mathbb{C}_{q,p}[G], D_{q,p^{-1}}(\mathfrak{g}) \} \) is an \( \mathbb{L} \)-bigraded dual pair. The associated pairing is given by

\[
\forall a \in \mathbb{C}_{q,p}[G]_{\lambda, \mu}, \forall u \in D_{q,p^{-1}}(\mathfrak{g})_{\gamma, \delta}, \quad (a, u)_p = p(\lambda, \gamma)p(\mu, \delta)(a, u).
\]
Proof. This follows from Theorem 2.4 applied to the pair \(\{A, U\} = \{\mathbb{C}q[G], D_q(g)\}\) and the bicharacter \(p^{-1}\) (recall that the map \(\gamma\) is the identity).

Let \(M \in \text{obj}(\mathbb{C}_q)\). The left \(D_q(g)\)-module structure on \(M\) yields a right \(\mathbb{C}q[G]\)-comodule structure in the usual way. Let \(\{v_1, \ldots, v_s; f_1, \ldots, f_s\}\) be a dual basis for \(M\). The structure map \(\rho : M \to M \otimes \mathbb{C}q[G]\), is given by \(\rho(x) = \sum_j v_j \otimes c_{f_j} x\) for \(x \in M\). Using this comodule structure on \(M\), one can check that \(M_{\mu} = \{x \in M \mid \forall h \in H, r_h(x) = (\mu, h)x\}\).

**Proposition 3.7.** Let \(M \in \text{obj}(\mathbb{C}_q)\). Then \(M\) has a natural structure of left \(D_{q,p^{-1}}(g)\) module. Denote by \(M^\ast\) this module and by \((u, x) \mapsto u \cdot x\) the action of \(D_{q,p^{-1}}(g)\). Then \(\forall u \in D_q(g)_{\mu, \delta}, \forall x \in M_{\mu}, \ u \cdot x = p(\lambda, \delta - \gamma)p(\delta, \gamma)ux\).

**Proof.** The proposition is a translation in this particular setting of Corollary 2.5.

Denote by \(\mathbb{C}_{q,p}\) the subcategory of finite dimensional left \(D_{q,p^{-1}}(g)\)-modules whose objects are the \(M^\ast, M \in \text{obj}(\mathbb{C}_q)\). It follows from Proposition 3.7 that if \(M \in \text{obj}(\mathbb{C}_q)\), then \(M^\ast = \mathbb{C}q[y]d_{\mu, \mu}\), where \(M_{\mu} = \{x \in M \mid \forall \alpha \in \mathbb{L}, s_{\alpha} \cdot x = p(\mu, 2\alpha)q^{(\mu, \alpha)}x, t_{\alpha} \cdot x = p(\mu, -2\alpha)q^{(\mu, \alpha)}x\}\).

Notice that \(p(\mu, \pm 2\alpha)q^{(\mu, \alpha)} = q^{(\Phi_{\mu}, \mu)}\).

**Theorem 3.8.** 1. The functor \(M \to M^\ast\) from \(\mathbb{C}_q\) to \(\mathbb{C}_{q,p}\) is an equivalence of rigid monoidal categories.

2. The Hopf pairing \(\langle \cdot, \cdot \rangle_p\) identifies the Hopf algebra \(\mathbb{C}_{q,p}[G]\) with the restricted dual of \(\mathbb{C}_{q,p}\), i.e. the Hopf algebra of coordinate functions on the objects of \(\mathbb{C}_{q,p}\).

**Proof.** 1. One needs in particular to prove that, for \(M, N \in \text{obj}(\mathbb{C}_q)\), \(\varphi_{M,N} : (M \otimes N)^\ast \to M^\ast \otimes N^\ast\) are natural isomorphisms of \(D_{q,p^{-1}}(g)\)-modules: \(\varphi_{M,N} : (M \otimes N)^\ast \to M^\ast \otimes N^\ast\). These isomorphisms are given by \(x \otimes y \mapsto p(\lambda, \mu)x \otimes y\) for all \(x \in M_{\lambda}, y \in N_{\mu}\). The other verifications are elementary.

2. We have to show that if \(M \in \text{obj}(\mathbb{C}_q)\), \(f \in M^\ast, v \in M\) and \(u \in D_{q,p^{-1}}(g)\), then \(\langle c_{f,v}, u \rangle_p = \langle f, u \cdot v \rangle\). It suffices to prove the result in the case where \(f \in M_{\mu}^\ast, v \in M_{\mu}\) and \(u \in D_{q,p^{-1}}(g)\) and \(\gamma\).

Then \(\langle f, u \cdot v \rangle = p(\mu, \delta - \gamma)p(\delta, \gamma)\langle f, uv \rangle = \delta_{-\lambda + \gamma + \delta, \mu}p(-\lambda + \gamma + \delta, \delta - \gamma)p(\delta, \gamma)\langle f, uv \rangle = p(\lambda, \gamma)p(\mu, \delta)\langle f, uv \rangle = \langle c_{f,v}, u \rangle_p\) by Theorem 3.6.

Recall that we introduced in §3.3 isomorphisms \(\sigma : D_q(g) \to D_{q^{-1}}(g)\) and \(\sigma : \mathbb{C}q[G] \to \mathbb{C}_{q^{-1}}[G]\). From (3.3) it follows that, after twisting by \(\tilde{p}^{-1}\) or \(\tilde{p}\), \(\sigma\) induces isomorphisms

\(D_{q,p^{-1}}(g) \xrightarrow{\sim} D_{q^{-1},p^{-1}}(g), \quad \mathbb{C}_{q^{-1},p}[G] \xrightarrow{\sim} \mathbb{C}_{q,p}[G]\)

which satisfy (3.2).

### 3.5. Braiding isomorphisms

We remarked above that the categories \(\mathbb{C}_{q,p}\) are braided. In the one parameter case this braiding is well-known. Let \(M\) and \(N\) be objects of \(\mathbb{C}_q\). Let \(E : M \otimes N \to M \otimes N\) be the operator given by

\[E(m \otimes n) = q^{(\lambda, \mu)}m \otimes n\]

for \(m \in M_{\lambda}\) and \(n \in N_{\mu}\). Let \(\tau : M \otimes N \to N \otimes M\) be the usual twist operator. Finally let \(C\) be the operator given by left multiplication by

\[C = \sum_{\beta \in \mathbb{Q}_p} C_{\beta}\]
where $C_\beta$ is the canonical element of $D_q(\mathfrak{g})$ associated to the non-degenerate pairing $U^+_{\beta} \otimes U^-_{-\beta} \rightarrow \mathbb{C}$ described above. Then one deduces from [33, 4.3] that the operators

$$\theta_{M,N} = \tau \circ C \circ E^{-1} : M \otimes N \rightarrow N \otimes M$$

define the braiding on $\mathcal{C}_q$.

As mentioned above, the category $\mathcal{C}_{q,p}$ inherits a braiding given by

$$\psi_{M,N} = \varphi_{N,M} \circ \theta_{M,N} \circ \varphi_{M,N}^{-1}$$

where $\varphi_{M,N}$ is the isomorphism $(M \otimes N)^{-} \xrightarrow{\sim} M^{-} \otimes N^{-}$ introduced in the proof of Theorem 3.8 (the same formula can be found in [1, §10] and in a more general situation in [24]). We now note that these general operators are of the same form as those in the one parameter case. Let $M$ and $N$ be objects of $\mathcal{C}_{q,p}$ and let $E : M \otimes N \rightarrow M \otimes N$ be the operator given by

$$E(m \otimes n) = q^{(\Phi_+ \lambda, \mu)}_{m \otimes n}$$

for $m \in M_\lambda$ and $n \in N_\mu$. Denote by $C_\beta$ the canonical element of $D_{q,p^{-1}}(\mathfrak{g})$ associated to the nondegenerate pairing $U_{q,p^{-1}}(\mathfrak{b}^+)_{-\beta} \otimes U_{q,p^{-1}}(\mathfrak{b}^-)_{0,-\beta} \rightarrow \mathbb{C}$ and let $C : M \otimes N \rightarrow M \otimes N$ be the operator given by left multiplication by

$$C = \sum_{\beta \in \mathfrak{g}_+} C_\beta.$$

**Theorem 3.9.** The braiding operators $\psi_{M,N}$ are given by

$$\psi_{M,N} = \tau \circ C \circ E^{-1}.$$  

Moreover $(\psi_{M,N})^* = \psi_{M^*,N^*}$.  

**Proof.** The assertions follow easily from the analogous assertions for $\theta_{M,N}$.  

The following commutation relations are well known [31], [21, 4.2.2]. We include a proof for completeness.

**Corollary 3.10.** Let $\Lambda, \Lambda' \in \mathbf{L}^+$, let $g \in L(\Lambda')^*_{-\mu}$ and $f \in L(\Lambda)^*_{-\mu}$ and let $v_{\Lambda} \in L(\Lambda)_{\Lambda}$. Then for any $v \in L(\Lambda)_{\gamma}$,

$$c_{g,v} \cdot c_{f,v} = q^{(\Phi_+ \lambda, \gamma)-(\Phi_+ \mu, \eta)} c_{f,v} \cdot c_{g,v} + q^{(\Phi_+ \lambda, \gamma)-(\Phi_+ \mu, \eta)} \sum_{v' \in \mathfrak{g}_+} c_{f,v} \cdot c_{g,v}$$

where $f_v \in (U_{q,p^{-1}}(\mathfrak{b}^+)_{-\mu+v} \otimes U_{q,p^{-1}}(\mathfrak{b}^-)_{-\eta-v}$ are such that $\sum_{\nu} f_{\nu} \otimes g_{v} = \sum_{\beta \in \mathfrak{g}_+} C_\beta(f \otimes g)$.  

**Proof.** Let $\psi = \psi_{L(\Lambda), L(\Lambda')}$. Notice that

$$c_{f \otimes g, \psi(v_{\Lambda} \otimes v)} = c_{\psi^*(f \otimes g), v_{\Lambda} \otimes v}.$$  

Using the theorem above we obtain

$$\psi^*(f \otimes g) = q^{-\Phi_+ \lambda, \eta}(g \otimes f + \sum g_{\nu} \otimes f_{\nu})$$

and

$$\psi(v_{\Lambda} \otimes v) = q^{-\Phi_+ \lambda, \eta}(v \otimes v_{\Lambda}).$$

Combining these formulae yields the required relations.  

$\square$
4. Prime and Primitive Spectrum of $\mathbb{C}_{q,p}[G]$

In this section we prove our main result on the primitive spectrum of $\mathbb{C}_{q,p}[G]$; namely that the $H$ orbits inside $\text{Prim}_q \mathbb{C}_{q,p}[G]$ are parameterized by the double Weyl group. For completeness we have attempted to make the proof more or less self-contained. The overall structure of the proof is similar to that used in [16] except that the proof of the key 4.12 (and the lemmas leading up to it) form a modified and abbreviated version of Joseph’s proof of this result in the one-parameter case [18]. One of the main differences with the approach of [18] is the use of the Rosso-Tanisaki form introduced in 3.2 which simplifies the analysis of the adjoint action of $\mathbb{C}_{q,p}[G]$. The ideas behind the first few results of this section go back to Soibelman’s work in the one-parameter ‘compact’ case [31]. These ideas were adapted to the multi-parameter case by Levendorskii [20].

4.1. Parameterization of the prime spectrum. Let $q, p$ be as in §3.4. For simplicity we set

$$A = \mathbb{C}_{q,p}[G]$$

and the product $a \cdot b$ as defined in (2.1) will be denoted by $ab$.

For each $\Lambda \in \mathbf{L}^+$ choose weight vectors

$$v_{\Lambda} \in L(\Lambda)_{\Lambda}, \quad v_{\Lambda}w_{\Lambda} = L(\Lambda)_{w_{\Lambda}}, \quad f_{-\Lambda} \in L(\Lambda)_{-\Lambda}, \quad f_{-w_{\Lambda}} \in L(\Lambda)_{-w_{\Lambda}}$$

such that $\langle f_{-\Lambda}, v_{\Lambda} \rangle = \langle f_{-w_{\Lambda}}, v_{w_{\Lambda}} \rangle = 1$. Set

$$A^+ = \sum_{\mu \in \mathbf{L}^+} \sum_{f \in L(\mu)^*} C_{f,v_{\mu}} \quad A^- = \sum_{\mu \in \mathbf{L}^+} \sum_{f \in L(\mu)^*} C_{f,v_{\mu}}.$$

Recall the following result.

**Theorem 4.1.** The multiplication map $A^+ \otimes A^- \to A$ is surjective.

*Proof.* Clearly it is enough to prove the theorem in the one-parameter case. When $\mathbf{L} = \mathbf{P}$ the result is proved in [31, 3.1] and [18, Theorem 3.7].

The general case can be deduced from the simply-connected case as follows. One first observes that $\mathcal{C}_q[G] \subset \mathbb{C}_q[G] = \bigoplus_{\Lambda \in \mathbf{P}^+} C(\Lambda)$. Therefore any $a \in \mathcal{C}_q[G]$ can be written in the form $a = \sum_{\Lambda, \Lambda' \in \mathbf{P}^+} c_{f,v_{\Lambda},v'_{\Lambda'}}$ where $\Lambda' - \Lambda'' \in \mathbf{L}$. Let $\Lambda \in \mathbf{P}$ and $\{v_i; f_i\}$ be a dual basis of $L(\Lambda)$. Then we have

$$1 = e(c_{v_{\Lambda}, f_{-\Lambda}}) = \sum_{i} c_{f_i, v_{\Lambda}} c_{v_i, f_{-\Lambda}}.$$

Let $\Lambda' = \Lambda + \Lambda'' \in \mathbf{L}^+$. Then, for all $i$, $c_{f_i, v_{\Lambda}} c_{f_i, v_{\Lambda}} \in C(\Lambda + \Lambda') \cap A^+$ and $c_{v_i, f_{-\Lambda'}} c_{v_i, f_{-\Lambda''}} \in C(-w_{\Lambda}(\Lambda + \Lambda'')) \cap A^-$. The result then follows by inserting 1 between the terms $c_{f_i, v_{\Lambda'}}$ and $c_{v_i, f_{-\Lambda''}}$.

*Remark.* The algebra $A$ is a Noetherian domain (this result will not be used in the sequel). The fact that $A$ is a domain follows from the same result in [18, Lemma 3.1]. The fact that $A$ is Noetherian is consequence of [18, Proposition 4.1] and [6, Theorem 3.7].

For each $y \in W$ define the following ideals of $A$

$$I^+_y = \langle c_{f,v_{\Lambda}} \mid f \in (U_{q,p}^{-1}(b^+)L(\Lambda)_{y\Lambda})^{\perp}, \Lambda \in \mathbf{L}^+ \rangle,$$

$$I^-_y = \langle c_{f,v_{\Lambda}} \mid f \in (U_{q,p}^{-1}(b^-)L(\Lambda)_{y\Lambda})^{\perp}, \Lambda \in \mathbf{L}^+ \rangle$$

where $(\cdot)^\perp$ denotes the orthogonal in $L(\Lambda)^*$. Notice that $I^+_y = \sigma(I^-_y)$, $\sigma$ as in §3.4, and that $I^+_y$ is an $\mathbf{L} \times \mathbf{L}$ homogeneous ideal of $A$.

*Notation.* For $w = (w_+, w_-) \in W \times W$ set $I_w = I^+_{w_+} + I^-_{w_-}$. For $\Lambda \in \mathbf{L}^+$ set $c_{w_{\Lambda}} = c_{f_{-w_{\Lambda}}; v_{\Lambda}} \in C(\Lambda)_{w_{\Lambda}}$ and $\tilde{c}_{w_{\Lambda}} = c_{v_{\Lambda}; f_{-\Lambda}} \in C(-w_{\Lambda})_{w_{\Lambda}}$.
Lemma 4.2. Let $\Lambda \in \mathbf{L}^+$ and $a \in A_{-\eta, \gamma}$. Then
\[
c_{w\Lambda} a \equiv q^{(\Phi_+, \Lambda, \eta, \gamma)_{a}} c_{w\Lambda} \mod I^+_w,
\]
\[
\tilde{c}_{w\Lambda} a \equiv q^{(\Phi_-, \Lambda, \gamma, \eta)_{a}} \tilde{c}_{w\Lambda} \mod I^-_w.
\]

Proof. The first identity follows from Corollary 3.10 and the definition of $I^+_w$. The second identity can be deduced from the first one by applying $\sigma$.

We continue to denote by $c_{w\Lambda}$ and $\tilde{c}_{w\Lambda}$ the images of these elements in $A/I_w$. It follows from Lemma 4.2 that the sets
\[
\mathcal{E}_{w, +} = \{ \alpha c_{w\Lambda} \mid \alpha \in \mathbb{C}^+, \Lambda \in \mathbf{L}^+ \}, \quad \mathcal{E}_{w, -} = \{ \alpha \tilde{c}_{w\Lambda} \mid \alpha \in \mathbb{C}^+, \Lambda \in \mathbf{L}^+ \},
\]
are multiplicatively closed sets of normal elements in $A/I_w$. Thus $\mathcal{E}_w$ is an Ore set in $A/I_w$. Define\[
A_w = (A/I_w)_{\mathcal{E}_w}.
\]

Notice that $\sigma$ extends to an isomorphism $\hat{A}_w : A_w \to A_w$, where $\hat{w} = (w_-, w_+)$.\[\]

Proposition 4.3. For all $w \in W \times W$, $A_w \neq (0)$.\[\]

Proof. Notice first that since the generators of $A_w$ and the elements of $\mathcal{E}_w$ are $\mathbf{L} \times \mathbf{L}$ homogeneous, it suffices to work in the one-parameter case. The proof is then similar to that of [15, Theorem 2.2.3] (written in the $SL(n)$-case). For completeness we recall the steps of this proof. The technical details are straightforward generalizations to the general case of [15, loc. cit.].

For $1 \leq i \leq n$ denote by $U_q(\mathfrak{sl}(2))$ the Hopf subalgebra of $U_q(\mathfrak{g})$ generated by $e_i, f_i, k^\pm 1$. The associated quantized function algebra $A_q \cong \mathbb{C}_q[SL(2)]$ is naturally a quotient of $A$. Let $\sigma_i$ be the reflection associated to the root $\alpha_i$. It is easily seen that there exist $M^+_i$ and $M^-_i$, non-zero $(A_i)_{(\sigma, \epsilon)}$ and $(A_i)_{(\epsilon, \sigma)}$ modules respectively. These modules can then be viewed as non-zero $A$-modules.

Let $w_+ = \sigma_{i_1} \ldots \sigma_{i_k}$ and $w_- = \sigma_{j_1} \ldots \sigma_{j_m}$ be reduced expressions for $w_\pm$. Then\[
M^+_{i_1} \otimes \cdots \otimes M^+_{i_k} \otimes M^-_{j_1} \otimes \cdots \otimes M^-_{j_m}
\]
is a non-zero $A_w$-module. □

In the one-parameter case the proof of the following result was found independently by the authors in [16, 1.2] and Joseph in [18, 6.2].

Theorem 4.4. Let $P \in \text{Spec} \mathbb{C}_q[\mu](G)$. There exists a unique $w \in W \times W$ such that $P \supset I_w$ and $(P/I_w) \cap \mathcal{E}_w = \emptyset$.

Proof. Fix a dominant weight $\Lambda$. Define an ordering on the weight vectors of $L(\Lambda)^*$ by $f \preceq f'$ if $f' \in U_q, b^{-1} (b^+)^* f$. This is a preordering which induces a partial ordering on the set of one dimensional weight spaces. Consider the set:
\[
\mathcal{F}(\Lambda) = \{ f \in L(\Lambda)^*_\mu \mid c_{f, v, \Lambda} \notin P \}.
\]

Let $f$ be an element of $\mathcal{F}(\Lambda)$ which is maximal for the above ordering. Suppose that $f'$ has the same property and that $f$ and $f'$ have weights $\mu$ and $\mu'$ respectively. By Corollary 3.10 the two elements $c_{f, v, \Lambda}$ and $c_{f', v, \Lambda}$ are normal modulo $P$. Therefore we have, modulo $P$,
\[
c_{f, v, \Lambda} c_{f', v, \Lambda} = q^{(\Phi_+, \Lambda, \mu, \gamma)_{c_{f, v, \Lambda}}} c_{f', v, \Lambda} c_{f, v, \Lambda} = q^{2(\Phi_+, \Lambda, \gamma)_{c_{f, v, \Lambda}}} c_{f, v, \Lambda} c_{f', v, \Lambda}.
\]

But, since $w$ is alternating, $2(\Phi_+, \Lambda, \mu, \gamma) = 2(\Phi_+, \mu, \gamma, \mu) = 2(\Phi_+, \Lambda, \gamma, \mu) = 2(\Lambda, \Lambda, \gamma) = 2(\mu, \mu')$. Since $P$ is prime and $q$ is not a root of unity we can deduce that $\langle A, \Lambda \rangle = \langle A, \Lambda \rangle$. This forces $\mu = \mu' \in W(-\Lambda)$. In conclusion, we have shown that for all dominant $\Lambda$ there exists a unique (up to scalar multiplication) maximal element $g_{\Lambda} \in \mathcal{F}(\Lambda)$ with weight $-\omega_\Lambda$, $\omega_\Lambda \in W$. Applying the argument above to a pair of such elements, $c_{g_{\Lambda}, v, \Lambda}$
and $c_{\mu,\nu}/c_{\mu,\nu}$ yields that $(w_\Lambda, w_\Lambda') = (\Lambda, \Lambda')$ for all $\Lambda, \Lambda' \in L^+$. Then it is not difficult to show that this furnishes a unique $w_+ \in W$ such that $w_+ \Lambda = w_\Lambda \Lambda$ for all $\Lambda \in L^+$. Thus for each $\Lambda \in L^+$,
\[ c_{\nu,v} \in P \iff g \notin \mathcal{I}_{w_+ \Lambda}. \]
Hence $P \supset \mathcal{I}_{w_+}$ and $P \cap \mathcal{E}_{w_+} = \emptyset$. It is easily checked that such a $w_+$ must be unique. Using $\sigma$ one deduces the existence and uniqueness of $w_-$.

**Definition.** A prime ideal $P$ such that $P \supset \mathcal{I}_{w}$ and $P \cap \mathcal{E}_{w} = \emptyset$ will be called a prime ideal of type $w$. We denote by $\text{Spec}_w C_{q,p}[G]$, resp. $\text{Prim}_w C_{q,p}[G]$, the subset of $\text{Spec} C_{q,p}[G]$ consisting of prime, resp. primitive, ideals of type $w$.

Clearly $\text{Spec}_w C_{q,p}[G] \cong \text{Spec} A_w$ and $\sigma(\text{Spec} \overline{C_{q^{-1},p}}[G]) = \text{Spec}_w C_{q,p}[G]$. The following corollary is therefore clear.

**Corollary 4.5.** One has
\[ \text{Spec} C_{q,p}[G] = \sqcup_{w \in W \times W} \text{Spec}_w C_{q,p}[G], \quad \text{Prim} C_{q,p}[G] = \sqcup_{w \in W \times W} \text{Prim}_w C_{q,p}[G]. \]

We end this section by a result which is the key idea in [18] for analyzing the adjoint action of $A$ on $A_w$. It says that in the one parameter case the quantized function algebra $C_q[B^-]$ identifies with $U_q(b^*)$ through the Rosso-Tanisaki-Killing form, [10, 17, 18]. Evidently this continues to hold in the multi-parameter case. For completeness we include a proof of that result.

Set $C_{q,p}[B^-] = A/I_{(w_0,c)}$. The embedding $U_{q,p-1}(b^-) \to D_{q,p-1}(g)$ induces a Hopf algebra map $\phi : A \to U_{q,p-1}(b^-)^\text{op}$, where $U_{q,p-1}(b^-)^\text{op}$ denotes the cofinite dual. On the other hand the non-degenerate Hopf algebra pairing $(\langle, \rangle)_{p-1}$ furnishes an injective morphism $\theta : U_{q,p-1}(b^+)^\text{op} \to U_{q,p-1}(b^-)^*$. 

**Proposition 4.6.** 1. $C_{q,p}[B^-]$ is an $L$-bigraded Hopf algebra.
2. The map $\gamma = \theta^{-1}\phi : C_{q,p}[B^-] \to U_{q,p-1}(b^+)^\text{op}$ is an isomorphism of Hopf algebras.

**Proof.** 1. It is easy to check that $I_{(w_0,c)}$ is an $L \times L$ graded bi-ideal of the bialgebra $A$. Let $\mu \in L^+$ and fix a dual basis $\{v_{\nu}; f_{-\nu}\}_\nu$ of $L(\mu)$ (with the usual abuse of notation). Then
\[ \sum_{\nu} c_{\nu,f_{-\nu}} c_{f_{-\nu},v_{\nu}} = \sum_{\nu} S(c_{f_{-\nu},v_{\nu}}) c_{f_{-\nu},v_{\nu}} = c(\gamma_{\mu,\nu}). \]
Taking $\gamma = \eta = \mu$ yields $c_{\mu,\nu} = 1$ modulo $I_{(w_0,c)}$. If $\gamma = w_0\mu$ and $\eta \neq w_0\mu$, the above relation shows that $S(c_{f_{-\nu},v_{\mu}}) c_{\mu,\nu} \in I_{(w_0,c)}$. Thus $I_{(w_0,c)}$ is a Hopf ideal.

2. We first show that
\[ \forall \Lambda \in L^+, c_{f_{-\nu},v_{\Lambda}} \in C(\Lambda)_{-\Lambda}, \exists! x_\Lambda \in U_{\Lambda}^+, \phi(c_{f_{-\nu},v_{\Lambda}}) = \theta(x_\Lambda \cdot k_{-\Lambda}). \]
Set $c = c_{f_{-\nu},v_{\Lambda}}$. Then $c(U_{-\eta}^+ \eta^*) = 0$ unless $\eta = \Lambda - \Lambda$; denote by $\bar{c}$ the restriction of $c$ to $U^-$. By the non-degeneracy of the pairing on $U_{\Lambda}^+ \times U_{-\Lambda}$ we know that there exists a unique $x_\Lambda \in U_{\Lambda}^+$ such that $\bar{c} = \theta(x_\Lambda)$. Then, for all $g \in U_{-\Lambda}^+$, we have
\[ c(y \cdot k_{\mu}) = \langle f_{-\mu} \cdot x_\Lambda \cdot y \rangle = q^{-\langle \Phi_{-\Lambda} \mu \rangle} \bar{c}(y) = q^{-\langle \Phi_{-\Lambda} \mu \rangle} \langle x_\Lambda \cdot f_{-\mu} \cdot y \rangle = \langle x_\Lambda \cdot k_{-\Lambda} \cdot f_{-\mu} \cdot y \rangle \]
by (3.4). This proves (4.1).

We now show that $\phi$ is injective on $A^+$. Suppose that $c = c_{f_{-\nu},v_{\Lambda}} \in C(\Lambda)_{-\Lambda} \cap \text{Ker} \phi$, hence $c = 0$ on $U_{q,p-1}(b^-)$. Since $L(\Lambda) = U_{q,p-1}(b^-) v_{\Lambda} = D_{q,p-1}(g) v_{\Lambda}$ it follows that $c = 0$. An easy weight argument using (4.1) shows then that $\phi$ is injective on $A^+$.

It is clear that $\text{Ker} \phi \supset I_{(w_0,c)}$, and that $A^+A^* = A$ implies $\phi(A) = \phi(A^+\bar{c}_\mu)$, where $c_\mu \in L^+$. Since $\bar{c}_\mu = c_\mu^\text{op}$ modulo $I_{(w_0,c)}$ by part 1, if $a \in A$ there exists $\nu \in L^+$ such that $\phi(\bar{c}_\mu) \phi(a) = \phi(a^+)$.

The inclusion $\text{Ker} \phi \subset I_{(w_0,c)}$ follows easily. Therefore $\gamma$ is a well defined Hopf algebra morphism.
If \( \alpha_j \in B \), there exists \( \Lambda \in \mathbf{L}^+ \) such that \( L(\Lambda)_{-\alpha_j} \neq 0 \). Pick \( 0 \neq f \in L(\Lambda)_{-\alpha_j}^* \). Then (4.1) shows that, up to some scalar, \( \phi(cf,v) = \theta(e_j, k_{-\Lambda}) \). If \( \lambda \in \mathbf{L} \), there exists \( \Lambda \in \mathbf{W} \cap \mathbf{L}^+ \); in particular \( L(\Lambda)_{\lambda} \neq 0 \). Let \( v \in L(\Lambda)_{\lambda} \) and \( f \in L(\Lambda)_{-\lambda}^* \) such that \( (f,v) = 1 \). Then it is easily verified that \( \phi(cf,v) = \theta(1-\Lambda) \). This proves that \( \gamma \) is surjective, and the proposition.

4.2. The adjoint action. Recall that if \( M \) is an arbitrary \( A \)-bimodule one defines the adjoint action of \( A \) on \( M \) by

\[
\forall a \in A, \ x \in M, \ \text{ad}(a)x = \sum a_{(1)}xS(a_{(2)}).
\]

Then it is well known that the subspace of ad-invariant elements \( M^{\text{ad}} = \{ x \in M \ | \ \forall a \in A, \ \text{ad}(a)x = \epsilon(a)x \} \) is equal to \( \{ x \in M \ | \ \forall a \in A, \ ax = xa \} \). Henceforth we fix \( w \in W \times W \) and work inside \( A_w \). For \( \Lambda \in \mathbf{L}^+, \ f \in L(\Lambda)^* \) and \( v \in L(\Lambda) \) we set

\[
z_f^w = c_{w^\Lambda}f, z_w^\Lambda = \overline{c_{w^\Lambda}}v, \ d_\Lambda = (c_{w^\Lambda}c_{w^\Lambda})^{-1}.
\]

Notice then that, for \( \Lambda \in \mathbf{L}^+ \), the "new" \( c_{w^\Lambda} \) belongs to \( C^*c_{\Lambda,w^\Lambda} \) (similarly for \( \overline{c_{w^\Lambda}} \)). Define subalgebras of \( A_w \) by

\[
C_w = \mathbb{C}[z_f^w, z_v^\Lambda, c_{w^\Lambda} ; f \in L(\Lambda)^*, v \in L(\Lambda), \Lambda \in \mathbf{L}^+, \Lambda \in \mathbf{L}] \quad \text{and} \quad C_w^+ = \mathbb{C}[z_f^w ; f \in L(\Lambda)^*, \Lambda \in \mathbf{L}^+].
\]

Recall that the torus \( H \) acts on \( A_{\Lambda,\mu} \) by \( r_h(a) = \mu(h)a \), where \( \mu(h) = \langle \mu, h \rangle \). Since the generators of \( I_w \) and the elements of \( E_w \) are eigenvectors for \( H \), the action of \( H \) extends to an action on \( A_w \). The algebras \( C_w \) and \( C_w^+ \) are obviously \( H \)-stable.

**Theorem 4.7.** 1. \( C_w^H = \mathbb{C}[z_f^w, z_v^\Lambda ; f \in L(\Lambda)^*, v \in L(\Lambda), \Lambda \in \mathbf{L}^+] \).

2. The set \( \mathcal{D} = \{ d_\Lambda ; \Lambda \in \mathbf{L}^+ \} \) is an Ore subset of \( C_w^H \). Furthermore \( A_w = (C_w)_{\mathcal{D}} \) and \( A_w^H = (C_w^H)_{\mathcal{D}} \).

3. For each \( \Lambda \in \mathbf{L} \), let \( (A_w)_{\Lambda} = \{ a \in A_w \ | \ r_h(a) = \lambda(h)a \} \). Then \( A_w = \bigoplus_{\Lambda \in \mathbf{L}} (A_w)_{\Lambda} \) and \( (A_w)_{\Lambda} = A_w c_{w^\Lambda} \). Moreover each \( (A_w)_{\Lambda} \) is an ad-invariant subspace.

**Proof.** Assertion 1 follows from

\[
\forall h \in H, \ r_h(z_f^w) = z_{f^w h}, \ r_h(c_{w^\Lambda}) = \lambda(h)c_{w^\Lambda}, \ r_h(\overline{c_{w^\Lambda}}) = \lambda(h)^{-1}\overline{c_{w^\Lambda}}.
\]

Let \( \{ v_i ; f_i \} \) be a dual basis for \( L(\Lambda) \). Then

\[
1 = \epsilon(c_{\Lambda-w^\Lambda}, v) = \sum S(c_{\Lambda-w^\Lambda}, v) c_{f_i, v} = \sum_\Lambda c_{v_i, f_i} c_{f_i, v}.
\]

Multiplying both sides of the equation by \( d_\Lambda \) and using the normality of \( c_{w^\Lambda} \) and \( \overline{c_{w^\Lambda}} \) yields \( d_\Lambda = \sum a_i z_{f_i}^w \) for some \( a_i \in \mathbb{C} \). Thus \( \mathcal{D} \subset C_w^H \). Now by Theorem 4.1 any element of \( A_w \) can be written in the form \( c_{f_1, v_1} c_{f_2, v_2} d_\Lambda^{-1} \) where \( v_1 = v_{\Lambda_1}, \ v_2 = v_{\Lambda_2} \) and \( \Lambda_1, \Lambda_2, \Lambda \in \mathbf{L}^+ \). This element lies in \( (A_w)_{\Lambda} \) if and only if \( \Lambda_1 - \Lambda_2 = \lambda \). In this case \( c_{f_1, v_1} c_{f_2, v_2} d_\Lambda^{-1} \) is equal, up to a scalar, to the element \( z_{f_i}^w z_{f_i}^w d_\Lambda^{-1} c_{w^\Lambda} \in (C_w^H)_{\mathcal{D}} c_{w^\Lambda} \). Since the adjoint action commutes with the right action of \( H, (A_w)_{\Lambda} \) is an ad-invariant subspace. The remaining assertions then follow.

We now study the adjoint action of \( \mathbb{C}_{q,p}[G] \) on \( A_w \). The key result is Theorem 4.12.

**Lemma 4.8.** Let \( T_\Lambda = \{ z_f^w \ | \ f \in L(\Lambda)^* \} \). Then \( C_w^+ = \bigcup_{\Lambda \in \mathbf{L}} T_\Lambda \).
Proof. It suffices to prove that if $\Lambda, \Lambda' \in \mathbf{L}^+$ and $f \in L(\Lambda)^*$, then there exists a $g \in L(\Lambda + \Lambda')^*$ such that $z_g^+ = z_f^+$. Clearly we may assume that $f$ is a weight vector. Let $i : L(\Lambda + \Lambda') \to L(\Lambda) \otimes L(\Lambda')$ be the canonical map. Then

$$c_{f,v_A}c_{f_{-w_A},v_A'} = c_{f_{-w_A},v_A} \otimes f_{-w_A} = c_{f,v_A}$$

where $g = i^*(f_{-w_A} \otimes f)$. Multiplying the images of these elements in $A_w$ by the inverse of $c_{w(\Lambda + \Lambda')} \in \mathbb{C}^*$ yields the desired result. \hfill $\square$

**Proposition 4.9.** Let $E$ be an object of $\mathcal{C}_{q,p}$ and let $\Lambda \in \mathbf{L}^+$. Let $\sigma : L(\Lambda) \to E \otimes L(\Lambda) \otimes E^*$ be the map $(1 \otimes \psi^{-1})(i \otimes 1)$ where $i : C \to E \otimes E^*$ is the canonical embedding and $\psi^{-1} : E^* \otimes L(\Lambda) \to L(\Lambda) \otimes E^*$ is the inverse of the braiding map described in §3.5. Then for any $c = c_{g,v} \in C(E)_{-\gamma}$ and $f \in L(\Lambda)^*$

$$\text{ad}(c).z_f^+ = q(\Phi_{w,A,n})z_{g,v}^+(v \otimes f \otimes g)$$

In particular $C_w^+$ is a locally finite $\mathcal{C}_{q,p}[G]$-module for the adjoint action.

Proof. Let $\{v_i; g_i\}$ be a dual basis of $E$ where $v_i \in E_{v_i}$, $g_i \in E^*_{v_i}$. Then $i(1) = \sum v_i \otimes g_i$. By (3.5) we have

$$\psi^{-1}(g_i \otimes v_{A}) = a_i(v_{A} \otimes g_i)$$

where $a_i = q^{-}(\Phi_{w,A}) = q^{(\Phi_{w,A})}$. On the other hand the commutation relations given in Corollary 3.10 imply that $c_{g,v}c_{v_A}^{-1} = b_{c_{w}(C_{g,v})}$, where $b = q^{(\Phi_{w,A})}$. Therefore

$$\text{ad}(c).z_f^+ = \sum b_{c_{w}(C_{g,v})}c_{f,v_A}c_{v,g,i} = bc_{w}(C_{g,v},c_{v,g,i}) = bc_{w}(C_{g,v},c_{v,g,i})$$

Since the map $\sigma$ is a morphism of $D_{q,p-1}(G)$-modules it is easy to see that $c_{v \otimes g,\sigma(v)} = c_{\sigma(v) \otimes g,\sigma(v)}$. \hfill $\square$

Let $\gamma : \mathcal{C}_{q,p}[G] \to U_{q,p-1}(\mathbf{b}^+)$ be the algebra anti-isomorphism given in Proposition 4.6.

**Lemma 4.10.** Let $c = c_{g,v} \in \mathcal{C}_{q,p}[G]_{-\gamma}$, $f \in L(\Lambda)^*$ be as in the previous theorem and $x \in U_{q,p-1}(\mathbf{b}^+)$ be such that $\gamma(c) = x$. Then

$$c_{S^{-1}(x),f,v_A} = c_{\sigma^*(v) \otimes g,\sigma^*(v)}$$

Proof. Notice that it suffices to show that

$$c_{S^{-1}(x),f,v_A}(y) = c_{\sigma^*(v) \otimes g,\sigma^*(v)}(y)$$

for all $y \in U_{q,p-1}(\mathbf{b}^-)$. Denote by $\langle \cdot | \cdot \rangle$ the Hopf pairing $\langle \cdot | \cdot \rangle_{p^{-1}}$ between $U_{q,p-1}(\mathbf{b}^+)^{op}$ and $U_{q,p-1}(\mathbf{b}^-)$ as in §3.4. Let $\chi$ be the one dimensional representation of $U_{q,p-1}(\mathbf{b}^+)$ associated to $v$, and let $\tilde{\chi} = \chi \cdot \gamma$. Notice that $\chi(x) = \langle x | \cdot \rangle_{p^{-1}}$; so $\tilde{\chi}(c) = c(\cdot _{\Lambda})$. Recalling that $\gamma$ is a morphism of coalgebras and using the relation (cxy) of §2.3 in the double $U_{q,p-1}(\mathbf{b}^+)$ $\times$ $U_{q,p-1}(\mathbf{b}^-)$, we obtain

$$c_{S^{-1}(x),f,v_A}(y) = \langle f(xyv) \rangle = \sum \langle x_{(1)} | y_{(1)} \rangle | x_{(2)} \rangle S(y_{(3)}) f(y_{(2)}v_{A})$$

$$= \sum \langle x_{(1)} | y_{(1)} \rangle | x_{(2)} \rangle S(y_{(3)}) \chi(x_{(2)}) f(y_{(2)}v_{A})$$

$$= \sum \langle x_{(1)} \chi(x_{(2)}) | y_{(1)} \rangle | x_{(2)} \rangle S(y_{(3)}) f(y_{(2)}v_{A})$$

$$= \sum \langle c_{1} \chi(c_{2}) \rangle | y_{(1)} \rangle c_{3} S(y_{(3)}) f(y_{(2)}v_{A})$$

$$= \sum r_{\tilde{\chi}}(c_{1}) | y_{(1)} \rangle c_{f,v_{A}}(y_{(2)}) S(c_{2}) | y_{(3)} \rangle.$$

Since $r_{\tilde{\chi}}(c_{g,v}) = q^{(\Phi_{-\nu} - \Lambda)}c_{g,v}$, one shows as in the proof of Proposition 4.9 that

$$c_{S^{-1}(x),f,v_A}(y) = \sum r_{\tilde{\chi}}(c_{1}) | y_{(1)} \rangle c_{f,v_{A}}(y_{(2)}) S(c_{2}) | y_{(3)} \rangle = \sum q^{(\Phi_{-\nu} - \Lambda)}(c_{g,v}c_{f,v_{A}}c_{x,g})(y) = c_{\sigma^*(v) \otimes g,\sigma^*(v)}(y),$$
Theorem 4.11. Consider $C_w^+$ as a $C_{q,p}[G]$-module via the adjoint action. Then

1. $\text{Soc } C_w^+ = \mathbb{C}$.
2. $\text{Ann } C_w^+ \supset I_{(w_0,e)}$.
3. The elements $c_{f-\mu,v_\mu}$, $\mu \in \mathbf{L}^+$, act diagonalizably on $C_w^+$.
4. $\text{Soc } C_w^+ = \{ z \in C_w^+ \mid \text{Ann } z \supset I_{(e,e)} \}$.

Proof. For $\Lambda \in \mathbf{L}^+$, define a $U_{q,p}^{-1}(b^+)$-module by

$$S_\Lambda = (U_{q,p}^{-1}(b^+)v_{w_+\Lambda})^* = L(\Lambda)^*/((U_{q,p}^{-1}(b^+)v_{w_+\Lambda})^\perp).$$

It is easily checked that $\text{Soc } S_\Lambda = \mathbb{C}f_{-w_+\Lambda}$ (see [18, 7.3]). Let $\delta : S_\Lambda \to T_\Lambda$ be the linear map given by $f \mapsto z_f^\perp$. Denote by $\zeta$ the one-dimensional representation of $C_{q,p}[G]$ given by $\zeta(c) = c(t_{-w_+\Lambda})$. Let $e = e_{g,v} \in C(E)_{-\eta}$. Then $l_\zeta(c) = q^{(\Phi_{-\eta}, w_4, \Lambda)}c = q^{-(\Phi_{+w_4, \eta, \Lambda})}c$. Then, using Proposition 4.9 and Lemma 4.10 we obtain,

$$\text{ad}(l_\zeta(c)) \delta(f) = q^{-\Phi_{-\eta, w_4, \Lambda}} \text{ad}(c). z_f^\perp = z_{S^\perp(z(c))}^\perp = \delta(S^{-1}(\gamma(c))f).$$

Hence, $\text{ad}(l_\zeta(c)) \delta(f) = \delta(S^{-1}(\gamma(c))f)$ for all $c \in A$. This immediately implies part (2) since $\text{Ker } \gamma \supset I_{(w_0,e)}$ and $l_\zeta(I_{(w_0,e)}) = I_{(w_0,e)}$. If $S_\Lambda$ is given the structure of an $A$-module via $S^{-1}\gamma$, then $\delta$ is a homomorphism from $S_\Lambda$ to the module $T_\Lambda$ twisted by the automorphism $l_\zeta$. Since $\delta(f_{-w_+\Lambda}) = 1$ it follows that $\delta$ is bijective and that $\text{Soc } T_\Lambda = \delta(\text{Soc } S_\Lambda) = \mathbb{C}$. Part (1) then follows from Lemma 4.8. Part (3) follows from the above formula and the fact that $\gamma(c_{f-\mu,v_\mu}) = s_{-\mu}$. Since $A/I_{(e,e)}$ is generated by the images of the elements $c_{f-\mu,v_\mu}$, (4) is a consequence of the definitions.

Theorem 4.12. Consider $C_w^H$ as a $C_{q,p}[G]$-module via the adjoint action. Then

$$\text{Soc } C_w^H = \mathbb{C}.$$

Proof. By Theorem 4.11 we have that $\text{Soc } C_w^H = \mathbb{C}$. Using the map $\sigma$, one obtains analogous results for $C_w^-$. The map $C_w^+ \otimes C_w^- \to C_w^H$ is a module map for the adjoint action which is surjective by Theorem 4.1. So it suffices to show that $\text{Soc } C_w^+ \otimes C_w^- = \mathbb{C}$. The following argument is taken from [18].

By the analog of Theorem 4.11 for $C_w^-$ we have that the elements $c_{f-\Lambda,v_\Lambda}$ act as commuting diagonalizable operators on $C_w^-$. Therefore an element of $C_w^+ \otimes C_w^-$ may be written as $\sum a_i \otimes b_i$ where the $b_i$ are linearly independent weight vectors. Let $c_{f,v_\Lambda}$ be a generator of $I_{ei}^+$. Suppose that $\sum a_i \otimes b_i \in \text{Soc } (C_w^+ \otimes C_w^-)$. Then

$$0 = \text{ad}(c_{f,v_\Lambda})(\sum a_i \otimes b_i) = \sum_{i,j} \text{ad}(c_{f,v_\Lambda}). a_i \otimes \text{ad}(c_{f,v_\Lambda}). b_i = \sum_i \text{ad}(c_{f,v_\Lambda}). a_i \otimes \text{ad}(c_{f,v_\Lambda}). b_i = \sum_i \text{ad}(c_{f,v_\Lambda}). a_i \otimes a_i b_i$$

for some $a_i \in \mathbb{C}^*$. Thus $\text{ad}(c_{f,v_\Lambda}). a_i = 0$ for all $i$. Thus the $a_i$ are annihilated by the left ideal generated by the $c_{f,v_\Lambda}$. But this left ideal is two-sided modulo $I_{(w_0,e)}$ and $\text{Ann } C_w^+ \supset I_{(w_0,e)}$. Thus the $a_i$ are annihilated by $I_{(e,e)}$ and so lie in $\text{Soc } C_w^H$ by Theorem 4.11. Thus $\sum a_i \otimes b_i \in \text{Soc } (C \otimes C^-) = \mathbb{C} \otimes \mathbb{C}$. □

Corollary 4.13. The algebra $A_w^H$ contains no nontrivial ad-invariant ideals. Furthermore, $(A_w^H)^{ad} = \mathbb{C}$.

Proof. Notice that Theorem 4.12 implies that $C_w^H$ contains no nontrivial ad-invariant ideals. Since $A_w^H$ is a localization of $C_w^H$ the same must be true for $A_w^H$. Let $a \in (A_w^H)^{ad} \setminus \mathbb{C}$. Then $a$ is central and so for any $\alpha \in \mathbb{C}$, $(a - \alpha)$ is a non-zero ad-invariant ideal of $A_w^H$. This implies that $a - \alpha$ is invertible in $A_w^H$ for any $\alpha \in \mathbb{C}$. This contradicts the fact that $A_w^H$ has countable dimension over $\mathbb{C}$. □
Theorem 4.14. Let $Z_w$ be the center of $A_w$. Then

1. $Z_w = A_w^{ad}$;
2. $Z_w = \bigoplus_{\lambda \in \text{L}} Z_{\lambda}$ where $Z_{\lambda} = Z_w \cap A_w^{H} c_{w,\lambda}$;
3. If $Z_{\lambda} \neq (0)$, then $Z_{\lambda} = Cu_{\lambda}$ for some unit $u_{\lambda}$;
4. The group $H$ acts transitively on the maximal ideals of $Z_w$.

Proof. The proof of (1) is standard. Assertion (2) follows from Theorem 4.7. Let $u_{\lambda}$ be a non-zero element of $Z_{\lambda}$. Then $u_{\lambda} = ac_{w,\lambda}$ for some $a \in A_w$. This implies that $a$ is normal and hence $a$ generates an ad-invariant ideal of $A_w^{H}$. Thus $a$ (and hence also $u_{\lambda}$) is a unit by Theorem 4.13. Since $Z_0 = \mathbb{C}$, it follows that $Z_{\lambda} = Cu_{\lambda}$. Since the action of $H$ is given by $\eta(h)u_{\lambda} = \lambda(h)u_{\lambda}$, it is clear that $H$ acts transitively on the maximal ideals of $Z_w$. \qed

Theorem 4.15. The ideals of $A_w$ are generated by their intersection with the center, $Z_w$.

Proof. Any element $f \in A_w$ may be written uniquely in the form $f = \sum a_{\lambda} c_{w,\lambda}$ where $a_{\lambda} \in A_w^{H}$. Define $\pi : A_w \to A_w^{H}$ to be the projection given by $\pi(\sum a_{\lambda} c_{w,\lambda}) = a_0$ and notice that $\pi$ is a module map for the adjoint action. Define the support of $f$ to be $\text{Supp}(f) = \{\lambda \in \text{L} \mid a_\lambda \neq 0\}$. Let $I$ be an ideal of $A_w$. For any set $Y \subseteq \text{L}$ such that $0 \in Y$ define

$I_Y = \{b \in A_w^{H} \mid b = \pi(f) \text{ for some } f \in I \text{ such that } \text{Supp}(f) \subseteq Y\}$

If $I$ is ad-invariant then $I_Y$ is an ad-invariant ideal of $A_w^{H}$ and hence is either (0) or $A_w^{H}$.

Now let $I' = (I \cap Z_w)A_w$ and suppose that $I \neq I'$. Choose an element $f = \sum a_{\lambda} c_{w,\lambda} \in I' \setminus I'$ whose support $S$ has the smallest cardinality. We may assume without loss of generality that $0 \in S$. Suppose that there exists $g \in I'$ with $\text{Supp}(g) \subseteq S$. Then there exists a $g' \in I'$ with $\text{Supp}(g') \subseteq S$ and $\pi(g') = 1$. But then $f - a_0 g'$ is an element of $I'$ with smaller support than $F$. Thus there can be no elements in $I'$ whose support is contained in $S$. So we may assume that $\pi(f) = a_0 = 1$. For any $c \in \mathbb{C}_{q,p}[G]$, set $f_c = \text{ad}(c).f - \epsilon(c).f$. Since $\pi(f_c) = 0$ it follows that $|\text{Supp}(f_c)| < |\text{Supp}(f)|$ and hence that $f_c = 0$. Thus $f \in I \cap A_w^{ad} = I \cap Z_w$, a contradiction. \qed

Putting these results together yields the main theorem of this section, which completes Corollary 4.5 by describing the set of primitive ideals of type $w$.

Theorem 4.16. For $w \in W \times W$ the subsets $\text{Prim}_w \mathbb{C}_{q,p}[G]$ are precisely the $H$-orbits inside $\text{Prim} \mathbb{C}_{q,p}[G]$.

Finally we calculate the size of these orbits in the algebraic case. Set $L_w = \{\lambda \in \text{L} \mid Z_{\lambda} \neq (0)\}$. Recall the definition of $s(w)$ from (1.3) and that $p$ is called $q$-rational if $u$ is algebraic. In this case we know by Theorem 1.7 that there exists $m \in \mathbb{N}^*$ such that $\Phi(mL) \subseteq L$.

Proposition 4.17. Suppose that $p$ is $q$-rational. Let $\lambda \in \text{L}$ and $y_{\lambda} = c_{w,\Phi_{-} m,\lambda} c_{w,\Phi_{+} m,\lambda}$. Then

1. $y_{\lambda}$ is ad-semi-invariant. In fact, for any $c \in A_{-\eta,\gamma}$,
   $$\text{ad}(c).y_{\lambda} = q^{(m \sigma(w)\lambda, \eta)} \epsilon(c)y_{\lambda}.$$  
   where $\sigma(w) = \Phi_{-} w, \Phi_{+} - \Phi_{-} w, \Phi_{-}$
2. $L_w \cap 2mL = 2 \ker \sigma(w) \cap mL$
3. $\dim Z_{\lambda} = n - s(w)$

Proof. Using Lemma 4.2, we have that for $c \in A_{-\eta,\gamma}$

$$cy_{\lambda} = q^{(\Phi_{+} w, \Phi_{-} m \lambda, -\gamma)}q^{(\Phi_{+} \Phi_{-} m \lambda, \eta)}q^{(\Phi_{-} w, \Phi_{+} m \lambda, -\gamma)}y_{\lambda}c = q^{(m \sigma(w)\lambda, \eta)} y_{\lambda}c.$$

From this it follows easily that

$$\text{ad}(c).y_{\lambda} = q^{(m \sigma(w)\lambda, \eta)} \epsilon(c)y_{\lambda}.$$
Since (up to some scalar) \( y_{\lambda} = d^{-1}e^{-1}x_{m\lambda}^{-2} \), it follows from Theorem 4.7 that \( y_{\lambda} \in (A_w)_{2m\lambda} \). However, as a \( (q,p)[G] \)-module via the adjoint action, \( A_w y_{\lambda} \cong A_w^H \cong C y_{\lambda} \) and hence \( \text{Soc} A_w^H y_{\lambda} = C y_{\lambda} \). Thus \( Z_{2m\lambda} \neq \{0\} \) if and only if \( y_{\lambda} \) is ad-invariant; that is, if and only if \( m\sigma(w)\lambda = 0 \). Hence

\[
\dim Z_w = \text{rk} L_w = \text{rk}(L_w \cap 2mL) = \text{rk} \ker_n \sigma(w) = \dim \ker_{n} \sigma(w) = n - s(w)
\]
as required.

Finally, we may deduce that in the algebraic case the size of the of the \( H \)-orbits \( \text{Soc}_{\mu} G \) and \( \text{Prim}_{\mu} C_{q,p}[G] \) are the same, cf. Theorem 1.8.

**Theorem 4.18.** Suppose that \( p \) is \( q \)-rational and let \( w \in W \times W \). Then

\[
\forall P \in \text{Prim}_{\mu} C_{q,p}[G], \quad \dim(H/\text{Stab}_H P) = n - s(w).
\]

**Proof.** This follows easily from theorems 4.15, 4.16 and Proposition 4.17.

**References**


