PRIMITIVE IDEALS OF $\mathbb{C}_q[G]$

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Abstract. In [4] Joseph proved the classification of the primitive ideals of the quantum group $\mathbb{C}_q[G]$ conjectured in [2]. We prove this result taking account of Joseph's analysis in [4] and of the methods already developed in [3].

1. Introduction

Let G be a connected, simply connected semi-simple complex Lie group and let $\mathfrak{g}$ be its Lie algebra. Denote by $\mathfrak{h}$ a fixed Cartan subalgebra of $\mathfrak{g}$. The notation used for the weights and roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is as in Bourbaki [1] (with the minor exception that the set of dominant weights will be denoted by $\mathbf{P}^+$). Let q be a non-zero complex number which is not a root of unity and let $\mathcal{Q}$ be the subgroup of $\mathbb{C}^*$ generated by q. Following [5, 7], we define the quantised enveloping algebra $U_q(\mathfrak{g})$ to be the $\mathbb{C}$-algebra generated by $K_i^{\pm 1}, X_i^{\pm}, 1 \leq i \leq n$ with relations

$$K_i^{-1}K_i = K_iK_i^{-1} = 1, \qquad K_iX_j^{\pm} = q^{\pm(\alpha_i,\alpha_j)}X_j^{\pm}K_i$$

$$[X_i^+, X_j^-] = \delta_{ij}\frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}}, \qquad K_iK_j = K_jK_i$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{2d_i}} (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-a_{ij}-k} = 0, \text{ if } i \neq j$$

where $[m]_t = (t-t^{-1})\dots(t^m-t^{-m})$ and $\begin{bmatrix} m \\ k \end{bmatrix}_t = \frac{[m]_t}{[k]_t[m-k]_t}$. Here $[a_{ij}]$ is the Cartan matrix and $d_i = (\alpha_i, \varpi_i) = (\alpha_i, \alpha_i)/2$. The algebra $U_q(\mathfrak{g})$ is a Hopf algebra. The comultiplication Δ is defined by

$$\Delta(X_i^{\pm}) = X_i^{\pm} \otimes K_i^{-1} + K_i \otimes X_i^{\pm}, \quad \Delta(K_i) = K_i \otimes K_i$$

and the counit and antipode by

$$\varepsilon(X_i^{\pm}) = 0, \quad \varepsilon(K_i) = 1, \quad S(K_i) = K_i^{-1}, \quad S(X_i^{\pm}) = -q^{\mp 2d_i}X_i^{\pm}.$$

We also set

$$E_i = K_iX_i^+,\ F_i = X_i^-K_i^{-1},\ V^+ = \mathbb{C}[E_i,\ 1 \leq i \leq n],\ V^- = \mathbb{C}[F_i,\ 1 \leq i \leq n]$$

and $V_+^{\pm} = V^{\pm} \cap \ker\varepsilon$. Notice that $\Delta(E_i) = E_i \otimes 1 + K_i^2 \otimes E_i$, $S(E_i) = -K_i^{-2}E_i$, and $[E_i, F_j] = \delta_{ij}\hat{q}_i^{-1}(K_i^2 - K_i^{-2})$, where $\hat{q}_i = q^{2d_i} - q^{-2d_i}$.

An element m of a $U_q(\mathfrak{g})$-module M is said to have weight $\mu \in \mathbf{P}$ if $K_i m = q^{(\alpha_i,\mu)}m$ for all $i = 1, \dots, n$. For each dominant weight $\Lambda \in \mathbf{P}^+$ there exists a finite dimensional simple module $L(\Lambda)$ of highest weight Λ. Denote by $\mathcal{C}$ the subcategory of $U_q(\mathfrak{g})$-modules consisting of finite direct sums of such modules. We consider the dual of a $U_q(\mathfrak{g})$-module

Date: March, 1993.

The first author is partially supported by a grant from the N.S.A..

M as a left module by the following action: $uf(x) = f(S(u)x)$ for all $x \in M, f \in M^*, u \in U_q(\mathfrak{g})$. The category $\mathcal{C}$ is closed under tensor products and the formation of duals. Hence the corresponding restricted dual (the algebra of coordinate functions associated to objects of $\mathcal{C}$) is a Hopf algebra. This Hopf algebra is denoted by $\mathbb{C}_q[G]$ and is known as the standard quantization of $\mathbb{C}[G]$.

In [2], the authors conjectured that the primitive ideals of $\mathbb{C}_q[G]$ should be in H-equivariant 1-1 correspondence with the symplectic leaves of G equipped with the standard Poisson group structure; that is, the H-orbits inside $Prim\ \mathbb{C}_q[G]$ are parameterized by the double Weyl group $W \times W$. This result was proved by the authors for $G = SL(n)$ in [3] and in the general case by Joseph in [4]. As shown in [2, 3], the key to the main theorem is to understand the adjoint action of $\mathbb{C}_q[G]$ on some algebras denoted by C_w (see section 3 for the definition). In these papers we were able to explicit the calculation of the adjoint action in order to get a complete description of C_w when $G = SL(n)$. In the general case the key idea in the proof of [4, Theorem 9.2] is to use the "self-duality of $U_q(\mathfrak{b}^+)$". We shall deduce Joseph's theorem by combining this idea and the approach of [3]. In [4] Joseph also proves various other results that the present paper does not recover. Although no explicit references are made to the geometry, the proof follows closely the ideas involved in the description of the symplectic leaves given in the appendix of [2]. The original conjectures were inspired by work of Soibelman [10] on the irreducible unitary representations of the corresponding Hopf*-algebra. For further details and background to these results the reader is referred to the introductions of [2] and [6].

2. Parameterization of the Prime Spectrum

Let $M \in \mathcal{C}$ and let $f \in M^*$, $v \in M$. Define $c_{f,v}$ to be the coordinate function given by:

$$\forall u \in U_q(\mathfrak{g}), \quad c_{f,v}(u) = f(uv).$$

Then $\mathbb{C}_q[G] = \oplus_{\Lambda \in P^+} C(\Lambda)$ where $C(\Lambda)$ is the vector space spanned by the $c_{f,v}$ where $f \in L(\Lambda)^*$ and $v \in L(\Lambda)$. Denote by Δ, ε, S the comultiplication, counit and antipode in $\mathbb{C}_q[G]$ (strictly speaking, these should be denoted Δ^*, ε^*, S^*, but no ambiguity arises). Then if $\{v_1, \dots, v_s; f_1, \dots, f_s\}$ is a dual basis for M,

$$\Delta(c_{f,v}) = \sum_i c_{f,v_i} \otimes c_{f_i,v}, \quad \varepsilon(c_{f,v}) = f(v), \quad S(c_{f,v}) = c_{v,f}.$$

For each $\Lambda \in \mathbf{P}^+$, choose $v_\Lambda \in L(\Lambda)_\Lambda$, $v_{w_0\Lambda} \in L(\Lambda)_{w_0\Lambda}$ and $f_{-\Lambda} \in L(\Lambda)^*_{-\Lambda}$, $f_{-w_0\Lambda} \in L(\Lambda)^*_{-w_0\Lambda}$ such that $f_{-\Lambda}(v_\Lambda) = 1$ and $f_{-w_0\Lambda}(v_{w_0\Lambda}) = 1$. Set

$$A^+ = \sum_{\mu \in \mathbf{P}^+} \sum_{f \in L(\mu)^*} \mathbb{C} c_{f,v_\mu}, \quad A^- = \sum_{\mu \in \mathbf{P}^+} \sum_{f \in L(\mu)^*} \mathbb{C} c_{f,v_{w_0\mu}}.$$

The multiplication map $A^+ \otimes A^- \to \mathbb{C}_q[G]$ is surjective, [10, 3.1]. A proof of this fact is given in [4, 3.7].

Consider the algebras $U_{q^{-1}}(\mathfrak{g})$ and $\mathbb{C}_{q^{-1}}[G]$ and use $\hat{}$ to distinguish elements, subalgebras, etc. of $U_{q^{-1}}(\mathfrak{g})$ and $\mathbb{C}_{q^{-1}}[G]$. It is easily verified that the map $\sigma : U_q(\mathfrak{g}) \to U_{q^{-1}}(\mathfrak{g})$ given by $\sigma(X_i^\pm) = \hat{X}_i^\mp$ and $\sigma(K_i) = \hat{K}_i$ is an isomorphism of Hopf algebras. For each $\Lambda \in \mathbf{P}^+$, σ gives a bijection $\sigma : L(-w_0\Lambda) \to \hat{L}(\Lambda)$ which sends $v \in L(-w_0\Lambda)_\mu$ onto $\hat{v} \in \hat{L}(\Lambda)_{-\mu}$. Therefore σ induces an isomorphism $\sigma : \mathbb{C}_{q^{-1}}[G] \to \mathbb{C}_q[G]$ such that

$$\forall f \in L(-w_0\Lambda)^*_{-\lambda},\ \forall v \in L(-w_0\Lambda)_\mu, \quad \sigma(\hat{c}_{\hat{f},\hat{v}}) = c_{f,v}.$$

In particular we have $\sigma(\hat{A}^{\pm}) = A^{\mp}$. This observation allows one to deduce results for A^- from analogous results for A^+.

The isomorphisms $L(\Lambda) \otimes L(\Lambda') \cong L(\Lambda') \otimes L(\Lambda)$ given by the universal R-matrix yield the following commutation relations.

Lemma 2.1. *Let $\Lambda, \Lambda' \in \mathbf{P}^+$ and let $g \in L(\Lambda')^*_{-\eta}$ and $f \in L(\Lambda)^*_{-\mu}$. Then there exists a finite collection of triples $(a_\nu, f_\nu, g_\nu)_{\nu \in \mathbf{Q}^+}$ where $a_\nu \in \mathbb{C}$, $f_\nu \in (U_q(\mathfrak{b}^+)f)_{-\mu+\nu}$ and $g_\nu \in (U_q(\mathfrak{b}^-)g)_{-\eta-\nu}$ such that for any $v \in L(\Lambda)_\gamma$,*

$$c_{g,v}c_{f,v_\Lambda} = q^{2[(\Lambda,\gamma)-(\mu,\eta)]}c_{f,v_\Lambda}c_{g,v} + q^{2(\Lambda,\gamma)} \sum_{\nu \in \mathbf{Q}^+} a_\nu c_{f_\nu,v_\Lambda} c_{g_\nu,v}$$

Define $A_{\lambda,\mu} = \{c_{f,v} \mid f \in M^*_\lambda,\ v \in M_\mu,\ M \in \mathcal{C}\}$. It is easily verified that this defines a $\mathbf{P} \times \mathbf{P}$ grading on $\mathbb{C}_q[G]$. Denote by $R(\mathbb{C}_q[G])$ the set of one dimensional representations of $\mathbb{C}_q[G]$ with the usual group structure. Let H be the maximal torus of G with Lie algebra $\mathfrak{h}$. For each $h \in H$ define $\chi_h \in R(\mathbb{C}_q[G])$ by

$$\forall c_{f,v} \in A_{\lambda,\mu}, \quad \chi_h(c_{f,v}) = \mu(h)\varepsilon(c_{f,v}).$$

The map $h \mapsto \chi_h$ is easily seen to be an injective group homomorphism. The usual map $r : R(\mathbb{C}_q[G]) \to \operatorname{Aut} \mathbb{C}_q[G]$ corresponding to right translation, $r_\chi(c) = \sum c_{(1)}\chi(c_{(2)})$, induces an action of H on $\mathbb{C}_q[G]$ given by

$$\forall c_{f,v} \in A_{\lambda,\mu},\ h \in H, \quad h.c_{f,v} = \mu(h)c_{f,v}.$$

For each $y \in W$ define

$$I^+_y = \langle c_{f,v_\Lambda} \mid f \in (U_q(\mathfrak{b}^+)L(\Lambda)_{y\Lambda})^\perp \rangle, \quad I^-_y = \langle c_{f,v_{w_0\Lambda}} \mid f \in (U_q(\mathfrak{b}^-)L(\Lambda)_{yw_0\Lambda})^\perp \rangle = \sigma(\hat{I}^+_y).$$

For $w = (w_+, w_-) \in W \times W$ set $I_w = I^+_{w_+} + I^-_{w_-}$. For a given w we denote by $c_{w\Lambda}$ and $\tilde{c}_{w\Lambda}$ the images of $c_{f_{-w_+\Lambda},v_\Lambda}$ and $c_{v_{w_-\Lambda},f_{-\Lambda}}$ respectively in $\mathbb{C}_q[G]/I_w$. It follows from Lemma 2.1 that $c_{g,v}c_{w\Lambda} \in \mathcal{Q}c_{w\Lambda}c_{g,v}$ and $c_{g,v}\tilde{c}_{w\Lambda} \in \mathcal{Q}\tilde{c}_{w\Lambda}c_{g,v}$ modulo $I^+_{w_+}$ and $I^-_{w_-}$ respectively. Therefore the sets $\mathcal{E}_{w_+} = \{\alpha c_{w\Lambda} \mid \alpha \in \mathbb{C}^*\}$, $\mathcal{E}_{w_-} = \{\alpha\tilde{c}_{w\Lambda} \mid \alpha \in \mathbb{C}^*\}$ and $\mathcal{E}_w = \mathcal{E}_{w_+}\mathcal{E}_{w_-}$ are multiplicatively closed sets of normal elements. Thus $\mathcal{E}_w$ is an Ore set in $\mathbb{C}_q[G]/I_w$. Define

$$A_w = (\mathbb{C}_q[G]/I_w)_{\mathcal{E}_w}.$$

Notice that the map σ extends to an isomorphism $\sigma : \hat{A}_{\hat{w}} \to A_w$ where $\hat{w} = (w_-, w_+)$.

Theorem 2.2. *For all $w \in W \times W$, $A_w \neq 0$.*

Proof. The proof is as in [2, 2.4]. □

The following lemma is well known.

Lemma 2.3. *Let $w_\Lambda \in W$ for all $\Lambda \in \mathbf{P}^+$. Then the following are equivalent:*
a) $\forall \Lambda, \Lambda' \in \mathbf{P}^+$, $(w_\Lambda\Lambda, w_{\Lambda'}\Lambda') = (\Lambda, \Lambda')$;
b) There exists a unique $w \in W$ such that $w\Lambda = w_\Lambda\Lambda$ for all $\Lambda \in \mathbf{P}^+$.

The proof of the following result was found independently by the authors in [3, 1.2] and Joseph in [4, 6.2].

Theorem 2.4. *Let $P \in Spec\, \mathbb{C}_q[G]$. There exists a unique $w \in W \times W$ such that $P \supset I_w$ and $(P/I_w) \cap \mathcal{E}_w = \emptyset$.*

Proof. Fix a dominant weight Λ. Define an ordering on the weight vectors of $L(\Lambda)^*$ by $f \leq f'$ if $f' \in U_q(\mathfrak{b}^+)f$. This is a preordering which induces a partial ordering on the set of one dimensional weight spaces. Consider the set:

$$\mathcal{D}(\Lambda) = \{f \in L(\Lambda)^*_\mu \mid c_{f,v_\Lambda} \notin P\}.$$

Let f be an element of $\mathcal{D}(\Lambda)$ which is maximal for the above ordering. Suppose that f' has the same property and that f and f' have weights μ and μ' respectively. By Lemma 2.1 the two elements c_{f,v_Λ} and c_{f',v_Λ} are normal modulo P. Therefore we have, modulo P,

$$c_{f,v_\Lambda}c_{f',v_\Lambda} = (q^2)^{(\Lambda,\Lambda)-(\mu,\mu')}c_{f',v_\Lambda}c_{f,v_\Lambda} = (q^4)^{(\Lambda,\Lambda)-(\mu',\mu)}c_{f,v_\Lambda}c_{f',v_\Lambda}.$$

Since P is prime and q is not a root of unity we can deduce that $(\Lambda,\Lambda) = (\mu,\mu')$. This forces $\mu = \mu' \in W(-\Lambda)$. In conclusion, we have shown that for all dominant Λ there exists a unique (up to scalar multiplication) maximal element $g_\Lambda \in \mathcal{D}(\Lambda)$ with weight $-w_\Lambda\Lambda$, $w_\Lambda \in W$. Applying the argument above to a pair of such elements, c_{g_Λ,v_Λ} and $c_{g_{\Lambda'},v_{\Lambda'}}$ yields that $(w_\Lambda\Lambda, w_{\Lambda'}\Lambda') = (\Lambda,\Lambda')$. Lemma 2.3 then furnishes a unique element $w_+ \in W$ such that $w_+\Lambda = w_\Lambda\Lambda$ for all $\Lambda \in \mathbf{P}^+$. Thus for each $\Lambda \in \mathbf{P}^+$,

$$c_{g,v_\Lambda} \in P \Leftrightarrow g \not\leq f_{-w_+\Lambda}.$$

Hence $P \supseteq I^+_{w_+}$ and $P \cap \mathcal{E}_{w_+} = \emptyset$. It is easily checked that such a w_+ must be unique. Using σ one deduces the existence and uniqueness of w_-. □

A prime ideal P such that $P \supset I_w$ and $(P/I_w) \cap \mathcal{E}_w = \emptyset$ will be called a prime ideal of type w. Denote by $\mathit{Spec}_w\mathbb{C}_q[G]$ the subset of $\mathit{Spec}\,\mathbb{C}_q[G]$ consisting of prime ideals of type w. Clearly $\mathit{Spec}_w\,\mathbb{C}_q[G] \cong \mathit{Spec}\,A_w$ and $\sigma(\mathit{Spec}_{\hat{w}}\,\mathbb{C}_{q^{-1}}[G]) = \mathit{Spec}_w\,\mathbb{C}_q[G]$.

Corollary 2.5. $\mathit{Spec}\,\mathbb{C}_q[G] = \bigsqcup_{w \in W\times W} \mathit{Spec}_w\mathbb{C}_q[G]$.

Henceforth we fix w and work inside A_w. For each $\Lambda \in \mathbf{P}^+$, set $d_\Lambda = (\tilde{c}_{w\Lambda}c_{w\Lambda})^{-1}$ and $t_\Lambda = \tilde{c}_{w\Lambda}(c_{w\Lambda})^{-1}$. For $f \in L(\Lambda)^*$ and $v \in L(\Lambda)$ set

$$z^+_f = c^{-1}_{w\Lambda}(c_{f,v_\Lambda}) \quad z^-_v = \tilde{c}^{-1}_{w\Lambda}(c_{v,f_{-\Lambda}})$$

Since the generators of I_w and the elements of $\mathcal{E}_w$ are eigenvectors for H, the action of H extends to an action on A_w. Let $\Gamma = \{h \in H \mid h^2 = 1\}$ and let $B_w = A^\Gamma_w$. Let C_w be the subalgebra generated by the elements of the form z^+_f, z^-_v, $t^{\pm 1}_\Lambda$. Then it is clear that $C_w \subset B_w$. It is not difficult to show (e.g. using Lemma 3.4 below and its R^-_q analog) that the subalgebra generated by the q-commuting elements $t^{\pm1}_\Lambda$, $\Lambda \in \mathbf{P}^+$, is equal to $\mathbb{C}[t^{\pm1}_{\varpi_i} \mid 1 \leq i \leq n]$. If $\lambda = \sum_{i=1}^n m_i\varpi_i$ is an element of P we set $t_\lambda = \prod_{i=1}^n t^{m_i}_{\varpi_i}$, so that

$$\mathbb{C}[t^{\pm1}_\Lambda \mid \Lambda \in \mathbf{P}^+] = \mathbb{C}[t_\lambda \mid \lambda \in \mathbf{P}].$$

Theorem 2.6. 1. $C^H_w = \mathbb{C}[z^+_f,\ z^-_v \mid f \in L(\Lambda)^*,\ v \in L(\Lambda),\ \Lambda \in \mathbf{P}^+]$.

2. The set $\mathcal{D} = \{d_\Lambda \mid \Lambda \in \mathbf{P}^+\}$ is an Ore subset of C^H_w and C_w. Furthermore $B_w = (C_w)_\mathcal{D}$ and $A^H_w = B^H_w = (C^H_w)_\mathcal{D}$.

3. The set $\{t_\lambda \mid \lambda \in \mathbf{P}\}$ forms a basis for C_w as a left or right C^H_w-module and for B_w as a left or right B^H_w-module.

Proof. Notice that for all $h \in H$, $h.t_\Lambda = \Lambda(h^{-2})t_\Lambda$ and $h.z^\pm_v = z^\pm_v$. This proves part 1 and the first assertion of part 3. Let $\{v_1,\dots,v_s; f_1,\dots,f_s\}$ be a dual basis for $L(\Lambda)$. Then

$$1 = \varepsilon(c_{f_{-\Lambda},v_\Lambda}) = \sum_i S(c_{f_{-\Lambda},v_i})c_{f_i,v_\Lambda} = \sum_i c_{v_i,f_{-\Lambda}}c_{f_i,v_\Lambda}.$$

Multiplying both sides of the equation by d_Λ and using the normality of $c_{w\Lambda}$ and $\tilde{c}_{w\Lambda}$ yields $d_\Lambda = \sum_i a_i z^-_{v_i} z^+_{f_i}$ for some $a_i \in \mathbb{C}$. Thus $\mathcal{D} \subset C_w^H$ and hence C_w contains all elements of the form $c^{-1}_{w\Lambda_1}\tilde{c}^{-1}_{w\Lambda_2}$ where $\Lambda_1 - \Lambda_2 \in 2\mathbf{P}$. Now it follows from [10, 3.1] that A_w is spanned by elements of the form $c_{f,v_1}c_{v,f_2}d_\Lambda$ where $v_1 = v_{\Lambda_1}$, $f_2 = f_{\Lambda_2}$ and $\Lambda_1, \Lambda_2, \Lambda \in \mathbf{P}^+$. So B_w is spanned by elements of the form $c_{f,v_1}c_{v,f_2}d_\Lambda$ where $\Lambda_1 - \Lambda_2 \in 2\mathbf{P}$. If further $\Lambda - \Lambda_1 - \Lambda_2 \in \mathbf{P}^+$, then, up to a scalar, $c_{f,v_1}c_{v,f_2}d_\Lambda = z^+_f z^-_v c^{-1}_{w\Lambda_2}\tilde{c}^{-1}_{w\Lambda_1} d_{\Lambda-\Lambda_1-\Lambda_2} \in C_w$. It is then clear that $B_w = (C_w)_{\mathcal{D}}$. The remaining assertions then follow easily. $\square$

3. The Adjoint Action

We now consider the adjoint action of $\mathbb{C}_q[G]$ on A_w. Notice that B_w and B_w^H are ad-submodules of A_w.

Proposition 3.1. *Let $g \in L(\Lambda')^*_{-\eta}$, $v \in L(\Lambda')_\gamma$, $f \in L(\Lambda)^*_{-\mu}$. Then*

$$ad\, c_{g,v}.z^+_f = \varepsilon(c_{g,v})q^{2(w_+\Lambda-\mu,\eta)}z^+_f + \sum_\nu q^{2(w_+\Lambda,\eta)}\varepsilon(c_{g_\nu,v})a_\nu z^+_{f_\nu}$$

where $a_\nu \in \mathbb{C}$, $f_\nu \in (U_q(\mathfrak{b}^+)f)_{-\mu+\nu}$ and $g_\nu \in (U_q(\mathfrak{b}^-)g)_{-\eta-\nu}$.

Proof. Let $\{v_1, \dots, v_s; f_1, \dots, f_s\}$ be a dual basis for $L(\Lambda')$ and suppose that $v_i \in L(\Lambda)_{\gamma_i}$. It follows from Lemma 2.1 that

$$ad\, c_{g,v}.z^+_f = \sum_i c_{g,v_i}c^{-1}_{w\Lambda}c_{f,v_\Lambda}S(c_{f_i,v}) = \sum_i q^{-2[(\Lambda,\gamma_i)-(w_+\Lambda,\eta)]}c^{-1}_{w\Lambda}c_{g,v_i}c_{f,v_\Lambda}S(c_{f_i,v})$$

$$= \sum_i q^{-2[(\Lambda,\gamma_i)-(w_+\Lambda,\eta)]}c^{-1}_{w\Lambda}\left(q^{2[(\Lambda,\gamma_i)-(\mu,\eta)]}c_{f,v_\Lambda}c_{g,v_i} + q^{2(\Lambda,\gamma_i)}\sum_\nu a_\nu c_{f_\nu,v_\Lambda}c_{g_\nu,v_i}\right)S(c_{f_i,v})$$

$$= q^{(w_+\Lambda,\eta)}c^{-1}_{w\Lambda}\left(q^{-2(\mu,\eta)}c_{f,v_\Lambda}\sum_i c_{g,v_i}S(c_{f_i,v}) + \sum_\nu a_\nu c_{f_\nu,v_\Lambda}\sum_i c_{g_\nu,v_i}S(c_{f_i,v})\right)$$

$$= \varepsilon(c_{g,v})q^{2(w_+\Lambda-\mu,\eta)}z^+_f + \sum_\nu q^{2(w_+\Lambda,\eta)}\varepsilon(c_{g_\nu,v})a_\nu z^+_{f_\nu}$$

$\square$

Set $C_w^+ = \mathbb{C}[z^+_f \mid f \in L(\Lambda)^*,\ \Lambda \in \mathbf{P}^+]$ and $C_w^- = \mathbb{C}[z^-_v \mid v \in L(\Lambda),\ \Lambda \in \mathbf{P}^+]$. Set $J^+ = I_{(w_0,e)}$ and $J^- = I_{(e,w_0)}$. Recall that $I^+_{w_0} = (0)$, so that J^+ is the ideal generated by the elements of the form $c_{f,v_{w_0\mu}}$ where $\mu \in \mathbf{P}^+$, $f \in L(\mu)^*_\eta$, $\eta \neq -w_0\mu$. Similarly J^- is the ideal generated by the c_{f,v_Λ}, $\Lambda \in \mathbf{P}^+$, $f \in L(\Lambda)^*_\gamma, \gamma \neq -\Lambda$.

Corollary 3.2. *C_w^H is a locally finite ad-$\mathbb{C}_q[G]$ module. Furthermore, $Ann_{ad}C_w^\pm \supset J^\pm$.*

Proof. This follows from Proposition 3.1 together with a similar result for C_w^-. $\square$

Set $\mathbb{C}_q[B^\pm] = \mathbb{C}_q[G]/J^\mp$ and define $\mathbb{C}[B^+/N^+]$ to be the subalgebra of $\mathbb{C}_q[B^+]$ generated by the images of the elements $c_\Lambda := c_{f_{-\Lambda},v_\Lambda}$, $\Lambda \in \mathbf{P}^+$.

Theorem 3.3. 1. $\mathbb{C}_q[G]/I_e \cong \mathbb{C}[H]$.

2. All finite dimensional $\mathbb{C}_q[G]$-modules are one dimensional and are annihilated by I_e.
3. The map $\chi : H \mapsto R(\mathbb{C}_q[G])$ is an isomorphism of groups.

Proof. For each $c \in \mathbb{C}_q[G]$ define $\varphi_c : H \to \mathbb{C}$ by $\varphi_c(h) = \chi_h(c)$. It is easily verified that φ defines an algebra map from $\mathbb{C}_q[G]$ to $\mathbb{C}[H]$ and that $\ker \varphi \supset I_e$. From [10, 3.1] it follows that the kernel is exactly I_e.

Let P be the annihilator of a simple finite dimensional $\mathbb{C}_q[G]$-module. Suppose that P is of type w. Then $\mathbb{C}_q[G]/P$ is a homomorphic image of A_w. Notice from the proof of 2.4, that $z_g^+ \neq 0$ in $\mathbb{C}_q[G]/P$ if $g \leq f_{-w_+\Lambda}$. By Proposition 3.1 the adjoint action of $\mathbb{C}[B^+/N^+]$ on such elements is diagonalisable. Moreover, unless $w_+ = e$, there are infinitely many distinct weight vectors of this form, contradicting the fact that $\mathbb{C}_q[G]/P$ is finite dimensional. Thus $w_+ = e$. Using σ one deduces that $w_- = e$ also. The remaining assertions are then clear. □

As already explained in the introduction, the key to the main theorem is to understand the adjoint action of $\mathbb{C}_q[G]$ on C_w^+. To do this we use the "self-duality of $U_q(\mathfrak{b}^+)$". Since $U_q(\mathfrak{b}^+)$ is not exactly self-dual we are obliged to work with the "simply-connected" version of $U_q(\mathfrak{g})$. What follows is a distillation of results in [4]. Define $\check{U}_q(\mathfrak{g})$ to be the algebra generated by $X_i^\pm,\ i = 1, \dots n$ and $\tau(\lambda),\ \lambda \in \mathbf{P}$ subject to the relations:

$$\forall \lambda, \mu \in \mathbf{P}, \quad \tau(\lambda)\tau(\mu) = \tau(\lambda + \mu), \quad \tau(\lambda)X_i^\pm = q^{\pm(\lambda,\alpha_i)}X_i^\pm\tau(\lambda)$$

$$[X_i^+, X_j^-] = \delta_{ij}\frac{\tau(\alpha_i)^2 - \tau(\alpha_i)^{-2}}{q^{2d_i} - q^{-2d_i}}$$

and the usual Serre relations for the $X_i^\pm$ described in the introduction. Setting $K_i = \tau(\alpha_i)$ then identifies $U_q(\mathfrak{g})$ with a subalgebra of $\check{U}_q(\mathfrak{g})$. The Hopf algebra structure can be extended to $\check{U}_q(\mathfrak{g})$ by defining

$$\Delta(\tau(\lambda)) = \lambda \otimes \lambda, \quad \varepsilon(\tau(\lambda)) = 1, \quad S(\tau(\lambda)) = \tau(-\lambda).$$

Set $\check{U}_q(\mathfrak{b}^\pm) = \mathbb{C}[X_i^\pm, \tau(\lambda) \mid i = 1, \dots, n,\ \lambda \in \mathbf{P}]$. The action of $U_q(\mathfrak{g})$ on the category $\mathcal{C}$ described in the introduction extends to an action of $\check{U}_q(\mathfrak{g})$. Moreover $\mathbb{C}_q[G]$ may equally well be considered as the restricted dual of $\check{U}_q(\mathfrak{g})$ with respect to $\mathcal{C}$. See [5] for further details.

Let $R_q^+ = \bigoplus_{\Lambda \in \mathbf{P}^+} L(\Lambda)^*$ considered as an algebra via the multiplication maps defined by $L(\Lambda_1)^* \otimes L(\Lambda_2)^* \to L(\Lambda_1 + \Lambda_2)^*$. These maps are obtained as follows. Recall that there is an embedding of $\check{U}_q(\mathfrak{g})$-modules $L(\Lambda_1 + \Lambda_2) \to L(\Lambda_2) \otimes L(\Lambda_1)$, mapping $v_{y(\Lambda_1+\Lambda_2)}$ onto $v_{y\Lambda_2} \otimes v_{y\Lambda_1}$ for all $y \in W$. Recall also that we have a canonical isomorphism of left $\check{U}_q(\mathfrak{g})$-modules $(L(\Lambda_2) \otimes L(\Lambda_1))^* \cong L(\Lambda_1)^* \otimes L(\Lambda_2)^*$, such that $(f_1 \otimes f_2)(v_2 \otimes v_1) = f_1(v_1)f_2(v_2)$ when $f_i \in L(\Lambda_i)^*, v_i \in L(\Lambda_i)$ for $i = 1, 2$. The multiplication in R_q^+ is deduced from these two maps of left $\check{U}_q(\mathfrak{g})$-modules. It is then easily seen that R_q^+ is a $\mathbf{P}$-graded $\check{U}_q(\mathfrak{g})$-module algebra and that if $x = \sum x_{(2)} \otimes x_{(1)} \in L(\Lambda_1 + \Lambda_2) \subset L(\Lambda_2) \otimes L(\Lambda_1)$, we have

$$\forall u \in \check{U}_q(\mathfrak{g}), \quad (f_1f_2)(ux) = f_1(u_{(2)}x_{(1)})f_2(u_{(1)}x_{(2)}). \tag{1}$$

The proofs of the next three results are taken from [4].

Lemma 3.4. *The map $\psi : R_q^+ \to \mathbb{C}_q[G]$ given by $\psi(f) = c_{f,v_\Lambda}$ for all $f \in L(\Lambda)^*$, is an injective anti-algebra map with image A^+.*

Proof. In $\mathbb{C}_q[G]$ we have $c_{f_2,v_{\Lambda_2}}c_{f_1,v_{\Lambda_1}} = c_{f_2\otimes f_1,v_{\Lambda_2}\otimes v_{\Lambda_1}} = \psi(f_1f_2)$, since f_1f_2 is the image of $f_2 \otimes f_1$ by the multiplication map previously described. The other assertions are clear. □

For each $y \in W$, define $F(y\Lambda) = \check{U}_q(\mathfrak{b}^+)v_{y\Lambda}$ and set

$$Q_y = \bigoplus_{\Lambda \in \mathbf{P}^+} F(y\Lambda)^\perp.$$

Proposition 3.5. *Q_y is a graded $\check{U}_q(\mathfrak{b}^+)$-invariant two-sided ideal of R_q^+. Furthermore*

1. $Soc_{\check{U}_q(\mathfrak{b}^+)} R_q^+/Q_y = \bigoplus_{\Lambda\in\mathbf{P}^+} \mathbb{C}f_{-y\Lambda}$.

2. The set $\mathcal{F} = \{af_{-y\Lambda} \mid a \in \mathbb{C}^*,\ \Lambda \in \mathbf{P}^+\}$ is a multiplicatively closed $\check{U}_q(\mathfrak{b}^+)$-invariant set of homogeneous regular normal elements of R_q^+/Q_y.

Proof. It is clear that Q_y is $\check{U}_q(\mathfrak{b}^+)$-invariant. Let $f \in F(y\Lambda)^\perp$, $g \in L(\Lambda')^*$. Then if $u \in \check{U}_q(\mathfrak{b}^+)$, we have by (1): $fg(uv_{y(\Lambda+\Lambda')}) = f(u_{(2)}v_{y\Lambda})g(u_{(1)}v_{y\Lambda'}) = 0$. Thus $fg \in F(y(\Lambda+\Lambda'))^\perp$. Hence Q_y is a right ideal. A similar argument shows that Q_y is a left ideal.

It is easily seen that $Soc_{\check{U}_q(\mathfrak{b}^+)} R_q^+/Q_y \supset \bigoplus_{\Lambda\in\mathbf{P}^+} \mathbb{C}f_{-y\Lambda}$. To check the converse notice that the natural map: $L(\Lambda)^*/F(y\Lambda)^\perp \to F(y\Lambda)^*$ is an isomorphism of $\check{U}_q(\mathfrak{b}^+)$-modules. Furthermore, $F(y\Lambda) = \mathbb{C}v_{y\Lambda} \oplus \sum_i X_i^+ F(y\Lambda)$. Now all finite dimensional simple $\check{U}_q(\mathfrak{b}^+)$-modules are one dimensional and are annihilated by all the X_i^+. Thus if $f \in Soc_{\check{U}_q(\mathfrak{b}^+)} F(y\Lambda)^*$, then $X_i^+ f = 0$ for all i. Hence $f(\sum X_i^+ F(y\Lambda)) = 0$. Therefore $Soc_{\check{U}_q(\mathfrak{b}^+)} F(y\Lambda)^* = \mathbb{C}f_{-y\Lambda}$.

Notice that $\psi(Q_y) \subset \operatorname{im}\psi$ is the ideal generated by the c_{f,v_Λ}, $f \in F(y\Lambda)^\perp$. By Lemma 2.1, $\psi(f_{-y\Lambda}) \in \operatorname{im}\psi$ is normal modulo $\psi(Q_y)$. Thus $f_{-y\Lambda}$ is normal modulo Q_y.

Suppose that $g \in L(\Lambda')^*$ is such that $gf_{-y\Lambda} \in Q_y$. From the above there exists a $\chi \in R(\check{U}_q(\mathfrak{b}^+))$ such that $uf_{-y\Lambda} = \chi(u)f_{-y\Lambda}$ for all $u \in \check{U}_q(\mathfrak{b}^+)$. Then we obtain from (1)

$$\begin{aligned} 0 &= gf_{-y\Lambda}(S(u)v_{y(\Lambda'+\Lambda)}) = g(S(u_{(1)})v_{y\Lambda'})f_{-y\Lambda}(S(u_{(2)})v_{y\Lambda}) \\ &= g(\chi(u_{(2)})S(u_{(1)})v_{y\Lambda'})f_{-y\Lambda}(v_{y\Lambda}) = g(S(r_\chi(u))v_{y\Lambda'}) \end{aligned}$$

Hence $g \in Q_y$. This proves the regularity of $f_{-y\Lambda}$. □

Theorem 3.6. *The localization $(R_q^+/Q_y)_{\mathcal{F}}$ is a $\mathbf{P}$-graded $\check{U}_q(\mathfrak{b}^+)$-module algebra. The zero-th degree part $T_y = ((R_q^+/Q_y)_{\mathcal{F}})_0$ is a submodule algebra and $Soc_{\check{U}_q(\mathfrak{b}^+)} T_y = \mathbb{C}$.*

Proof. The first comment about localizations is a consequence of a general result concerning localizations of module algebras. The action is defined in the following way. Let $f \in \mathcal{F}$ such that $uf = \chi_f(u)f$ and let l_f be the automorphism $l_f(u) = \chi_f(u_{(1)})u_{(2)}$. Then for any $x \in R_q^+/Q_y$, the action is given by $u(f^{-1}x) = f^{-1}(l_f^{-1}(u)x)$. It is then clear that $(R_q^+/Q_y)_{\mathcal{F}}$ is a $\mathbf{P}$-graded $\check{U}_q(\mathfrak{b}^+)$-algebra and that T_y is a submodule algebra. It also follows immediately that $Soc_{\check{U}_q(\mathfrak{b}^+)}(R_q^+/Q_y)_{\mathcal{F}} = \mathcal{F}^{-1}(Soc_{\check{U}_q(\mathfrak{b}^+)} R_q^+/Q_y)$. Hence

$$Soc_{\check{U}_q(\mathfrak{b}^+)} T_y = [\mathcal{F}^{-1}(Soc_{\check{U}_q(\mathfrak{b}^+)} R_q^+/Q_y)]_0 = \mathbb{C}.$$

□

We shall apply these results to the element $y = w_+$.

We aim to show that $Soc_{ad}\, C_w^+$, the socle under the adjoint action, reduces to $\mathbb{C}$. Following an idea of [4], we shall deduce this result from a suitable anti-isomorphism between a subalgebra of $\check{U}_q(\mathfrak{b}^-)$, denoted by $U_q(\mathfrak{c}^+)$, and $\mathbb{C}_q[B^-]$. Our approach to this fact differs from [4]. We shall obtain it as a consequence of the existence of a nondegenerate pairing between $\check{U}_q(\mathfrak{b}^+)$ and $\check{U}_q(\mathfrak{b}^-)$, studied in [9, 11]. The properties of this pairing enables us to relate the adjoint actions of $U_q(\mathfrak{c}^+)$ and $\mathbb{C}_q[B^-]$, see Proposition 3.11. For convenience we follow the presentation given in [11]. The following theorem summarizes the results from [11, section 2] that we need. Recall that if $\check{U}^0 = \mathbb{C}[\tau(\lambda) \mid \lambda \in \mathbf{P}]$, there are triangular decompositions

$$\check{U}_q(\mathfrak{b}^\pm) = \check{U}^0 V^\pm, \quad \check{U}_q(\mathfrak{g}) = \check{U}_q(\mathfrak{b}^-) \oplus \check{U}_q(\mathfrak{g}) V_+^+.$$

Theorem 3.7. 1. There exists a unique nondegenerate bilinear form

$$(\ \ /\ \ ) : \check{U}_q(\mathfrak{b}^+) \times \check{U}_q(\mathfrak{b}^-) \longrightarrow \mathbb{C}$$

satisfying the following properties:
(i) $(u^+/u_1^- u_2^-) = (\Delta(u^+)/u_1^- \otimes u_2^-)$, $\quad (u_1^+ u_2^+/u^-) = (u_2^+ \otimes u_1^+/\Delta(u^-))$ for all $u^\pm$, $u_i^\pm \in \check{U}_q(\mathfrak{b}^\pm)$, $i = 1, 2$;
(ii) $\forall \lambda, \mu \in \mathbf{P}$, $\quad (\tau(\lambda)/\tau(\mu)) = q^{-(\lambda,\mu)/2}$;
(iii) $\forall \lambda \in \mathbf{P}$, $\forall i \in \{1, \dots, n\}$, $\quad (\tau(\lambda)/F_i) = (E_i/\tau(\lambda)) = 0$;
(iv) $\forall i, j \in \{1, \dots, n\}$, $\quad (E_i/F_j) = -\delta_{ij}\hat{q}_i^{-1}$.

2. For all $u^\pm \in \check{U}_q(\mathfrak{b}^\pm)$, $a^\pm \in V^\pm$, $\lambda, \mu \in \mathbf{P}$ we have

$$(S(u^+)/S(u^-)) = (u^+/u^-), \quad (a^+\tau(\lambda)/a^-\tau(\mu)) = q^{-(\lambda,\mu)/2}(a^+/a^-).$$

3. $(V_\gamma^+/V_{-\eta}^-) = 0$ for any $\gamma \neq \eta \in \mathbf{Q}^+$, where $V_{\pm\eta}^\pm$ denotes the space of elements of weight $\pm\eta$ under the adjoint action of $\check{U}^0$.

4. For all $x \in \check{U}_q(\mathfrak{b}^+)$, $y \in \check{U}_q(\mathfrak{b}^-)$ one has

$$xy = \sum (x_{(1)}/y_{(1)})(x_{(3)}/S(y_{(3)}))y_{(2)}x_{(2)}$$

where as usual $(I \otimes \Delta)\Delta(x) = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$.

We denote by $U_q(\mathfrak{c}^+)$ the following Hopf subalgebra of $\check{U}_q(\mathfrak{b}^-)$:

$$U_q(\mathfrak{c}^+) = \mathbb{C}[E_i,\ \tau(2\lambda) \mid 1 \leq i \leq n,\ \lambda \in \mathbf{P}].$$

The embedding $\check{U}_q(\mathfrak{b}^-) \to \check{U}_q(\mathfrak{g})$ yields a Hopf algebra map $\phi : \mathbb{C}_q[G] \to \check{U}_q(\mathfrak{b}^-)^\circ$ where $\check{U}_q(\mathfrak{b}^-)^\circ$ denotes the cofinite dual. On the other hand the bilinear form $(\ /\ )$ yields an injective map $\theta : \check{U}_q(\mathfrak{b}^+) \to \check{U}_q(\mathfrak{b}^-)^*$.

Theorem 3.8. 1. $\ker\phi = J^+$.

2. $\phi(\mathbb{C}_q[G]) = \theta(U_q(\mathfrak{c}^+))$.

3. The map $\beta = \phi^{-1}\theta : U_q(\mathfrak{c}^+) \to \mathbb{C}_q[B^-]$ is an isomorphism of coalgebras and an anti-isomorphism of algebras.

4. Let $f \in L(\Lambda)^*_{-\lambda}$. There exists a unique $x_\lambda \in V_{\Lambda-\lambda}^+$ such that $\beta(x_\lambda\tau(-2\Lambda)) = \phi(c_{f,v_\Lambda}) \in \mathbb{C}_q[B^-]$. In particular $\beta(\tau(-2\Lambda)) = \phi(c_\Lambda)$.

Proof. Assertion 1 is proved in [4, 3.10]. For completeness we give a proof using our notation. Notice that $A^\pm = \{c \in \mathbb{C}_q[G] \mid c(\check{U}_q(\mathfrak{g})V^\pm) = 0\}$. and recall that $\mathbb{C}_q[G] = A^+A^-$. It is clear that $J^+ \subset \ker\phi$, $\phi(\mathbb{C}_q[G]) = \phi(\mathbb{C}_q[G]/J^+) = \phi(A^+[c_{f_{-w_0\mu}, v_{w_0\mu}},\ \mu \in \mathbf{P}^+])$. Futhermore $\check{U}_q(\mathfrak{b}^-)^*$ can be identified with $\{\varphi \in \check{U}_q(\mathfrak{g})^* \mid \varphi(\check{U}_q(\mathfrak{g})V_+^+) = 0\}$. It follows that ϕ is injective on A^+. Fix a dual basis $\{v_\gamma,\ f_{-\gamma}\}$ of $L(\mu)$. Then

$$\sum_\gamma c_{v_\gamma, f_{-\mu}} c_{f_{-\gamma}, v_\mu} = \sum_\gamma S(c_{f_{-\mu}, v_\gamma}) c_{f_{-\gamma}, v_\mu} = \varepsilon(c_{f_{-\mu}, v_\mu}) = 1.$$

Thus $\tilde{c}_\mu c_\mu = 1$ modulo J^+. Therefore $\phi(\mathbb{C}_q[G]) = A^+[\phi(c_\mu^{-1}) \mid \mu \in \mathbf{P}^+]$ and the claim follows easily.

We now prove part 4. The other parts will then follow from 1 and Theorem 3.7. Let $f \in L(\Lambda)^*_{-\lambda}$ and set $c = \phi(c_{f,v_\Lambda})$. Notice first that $c(V^-_{-\eta}) = 0$ unless $\eta = \Lambda - \lambda$. Denote still by c the induced element on $V^-_{\lambda-\Lambda}$. By Theorem 3.7 there exists a unique $x_\lambda \in V^+_{\Lambda-\lambda}$ such that $c = \theta(x_\lambda)$. Let $\nu \in \mathbf{P}, u \in V^-$; observe that $\theta(x_\lambda\tau(-2\Lambda))(u\tau(\nu)) = 0$ unless $u \in V^-_{\lambda-\Lambda}$ and that in this case we have

$$c(u\tau(\nu)) = f(u\tau(\nu)v_\Lambda) = q^{(\Lambda,\nu)}c(u) = q^{(\Lambda,\nu)}\theta(x_\lambda)(u) = \theta(x_\lambda\tau(-2\Lambda))(u\tau(\nu)).$$

Thus $c = \theta(x_\lambda\tau(-2\Lambda))$. The above calculation also shows that $\phi(c_\Lambda) = \theta(\tau(-2\Lambda))$. Hence the result. □

Under the hypothesis in 4 of Theorem 3.8 we shall now simply write $\beta(x_\lambda\tau(-2\Lambda)) = c_{f,v_\Lambda} \in \mathbb{C}_q[B^-]$. We then have $c_{f,v_\Lambda} = c_\Lambda\beta(x_\lambda),\ x_\lambda \in V^+_{\Lambda-\lambda}$.

Since $I_w \subset I^+_{w_+} + J^+ = I^+_{w_+} + I^-_e$, we have a surjective map

$$(\mathbb{C}_q[G]/I_w)_{\mathcal{E}_{w_+}} \longrightarrow (\mathbb{C}_q[G]/(J^+ + I^+_{w_+}))_{\mathcal{E}_{w_+}} = (\mathbb{C}_q[B^-]/(J^+ \cap I^+_{w_+}))_{\mathcal{E}_{w_+}}.$$

We denote by $\bar{C}^+_w$ the image of C^+_w under this map. We continue to denote by z^+_f the image of $z^+_f \in C^+_w$. This will be justified by the proof of Theorem 3.12 below.

Lemma 3.9. *Let $f \in L(\Lambda)^*_{-\lambda}$. Set $c_{f,v_\Lambda} = \beta(x_\lambda\tau(-2\Lambda))$, $x_\lambda \in V^+_{\Lambda-\lambda}$. Then in $\mathbb{C}_q[B^-]$ we have, for all $i \in \{1,\dots,n\}$*

$$c_{E_if,v_\Lambda} = c_\Lambda\beta(ad\,E_i.x_\lambda + q^{-(2\alpha_i,\lambda)}q_{i,\Lambda}x_\lambda E_i)$$

where $q_{i,\Lambda} = q^{(2\alpha_i,\Lambda)} - q^{-(2\alpha_i,\Lambda)}$.

Proof. An easy computation yields

$$(ad\,E_i.x_\lambda + q^{-(2\alpha_i,\lambda)}q_{i,\Lambda}x_\lambda E_i)\tau(-2\Lambda) = E_ix_\lambda\tau(-2\Lambda) + q^{-(2\alpha_i,\Lambda)}K_i^2x_\lambda\tau(-2\Lambda)S(E_i).$$

Therefore for all $y \in \check{U}_q(\mathfrak{b}^-)$ we have

$$c_\Lambda\beta(ad\,E_i.x_\lambda + q^{-(2\alpha_i,\lambda)}q_{i,\Lambda}x_\lambda S(E_i))(y) = (E_ix_\lambda\tau(-2\Lambda) + q^{-(2\alpha_i,\Lambda)}K_i^2x_\lambda\tau(-2\Lambda)S(E_i)/y).$$

But by 1 of Theorem 3.7 this is also

$$(1/y_{(1)})(x_\lambda\tau(-2\Lambda)/y_{(2)})(E_i/y_{(3)}) + q^{-(2\alpha_i,\Lambda)}(S(E_i)/y_{(1)})(x_\lambda\tau(-2\Lambda)/y_{(2)})(\tau(2\alpha_i)/y_{(3)}).$$

By 4 of Theorem 3.7 we know that $S(E_i)y$ is equal to

$$(1/y_{(1)})(E_i/y_{(3)})y_{(2)}+(1/y_{(1)})(\tau(2\alpha_i)/y_{(3)})y_{(2)}S(E_i)+(S(E_i)/y_{(1)})(\tau(2\alpha_i)/y_{(3)})y_{(2)}\tau(-2\alpha_i).$$

The result then follows from the above and the following facts: $c_{E_if,v_\Lambda}(y) = c_{f,v_\Lambda}(S(E_i)y)$, $f(y_{(2)}v_\Lambda) = c_{f,v_\Lambda}(y_{(2)}) = (x_\lambda\tau(-2\Lambda)/y_{(2)})$, $S(E_i)v_\Lambda = 0$, $\tau(-2\alpha_i)v_\Lambda = q^{-(2\alpha_i,\Lambda)}v_\Lambda$. □

Recall that $z^+_f = (c_{w\Lambda})^{-1}c_{f,v_\Lambda}$. Write $c_{w\Lambda} = \beta(x_{w_+\Lambda}\tau(-2\Lambda)))$, with $x_{w_+\Lambda} \in V^+_{\Lambda-w_+\Lambda}$. Then

$$\beta(x_{w_+\Lambda})^{-1} \in (\mathbb{C}_q[B^-]/J^+ \cap I^+_{w_+})_{\mathcal{E}_{w_+}}$$

and we may write $c^{-1}_{w\Lambda} = \beta(x_{w_+\Lambda})^{-1}{c_\Lambda}^{-1}$.

From Lemma 3.9 one deduces the following result.

Corollary 3.10. *Under the notation of Lemma 3.9 we have*

$$z^+_{E_i f} = \beta(x_{w_+\Lambda})^{-1}\beta(ad\, E_i.x_\lambda + q^{-(2\alpha_i,\lambda)}q_{i,\Lambda}x_\lambda E_i).$$

The next result computes the adjoint action of $\mathbb{C}_q[B^-]$ on $\bar{C}^+_w$.

Proposition 3.11. 1. $\forall u, x \in U_q(\mathfrak{c}^+), \quad ad\,\beta(S(u)).\beta(x) = \beta(ad\, u.x)$.
2. $\forall i \in \{1,\dots,n\}, \quad ad\,\beta(S(E_i)).z^+_f = z^+_{E_i f}$.
3. $\forall \nu \in \mathbf{P}, \quad ad\,\beta(S(\tau(\nu))).z^+_f = q^{(\nu, w_+\Lambda-\lambda)}z^+_f$, if $f \in L(\Lambda)^*_{-\lambda}$.

Proof. 1. Recall that β is an anti-homomorphism of algebras and an homomorphism of coalgebras. Futhermore by 2 of Theorem 3.7, $S^*(\beta(a)) = \beta(S^{-1}(a))$ if S^* denotes the antipode in $\mathbb{C}_q[B^-]$ and $a \in \check{U}_q(\mathfrak{b}^+)$. Hence we have

$$ad\,\beta(S(u)).\beta(x) = \beta(S(u_{(2)}))\beta(x)S^*(\beta(S(u_{(1)}))) = \beta(u_{(1)}xS(u_{(2)})) = \beta(ad\, u.x).$$

This proves part 1.

Let $f \in L(\Lambda)^*_{-\lambda}$ and set $c_{f,v_\lambda} = \beta(x_\lambda\tau(-2\Lambda))$, $x_\lambda \in V^+_{\Lambda-\lambda}$, $c_{w\Lambda} = \beta(\tau(-2\Lambda))\beta(x_{w_+\Lambda})$ as above. It follows from 1 and the properties of the adjoint action that for all $\nu \in \mathbf{P}$, $ad\,\beta(S(\tau(\nu))).\beta(x_\lambda) = q^{(\nu,\Lambda-\lambda)}\beta(x_\lambda)$ and

$$\begin{aligned} (2) \quad ad\,\beta(S(E_i)).z^+_f &= ad\,\beta(S(E_i)).[\beta(x_{w_+\Lambda})^{-1}\beta(x_\lambda)] \\ &= \beta(x_{w_+\Lambda})^{-1}[ad\,\beta(S(E_i)).\beta(x_\lambda)] + q^{(2\alpha_i,\Lambda-\lambda)}[ad\,\beta(S(E_i)).\beta(x_{w_+\Lambda})^{-1}]\beta(x_\lambda). \end{aligned}$$

2. Since $c_{E_i f_{-w_+\Lambda}, v_\Lambda} = 0$ modulo $I^+_{w_+}$, we obtain from Lemma 3.9 that $\beta(ad\, E_i.x_{w_+\Lambda}) = -q^{-(2\alpha_i, w_+\Lambda)}q_{i,\Lambda}\beta(x_{w_+\Lambda}E_i)$. From $ad\,\beta(S(E_i)).1 = 0 = ad\,\beta(S(E_i)).[\beta(x_{w_+\Lambda})^{-1}\beta(x_{w_+\Lambda})]$ we deduce that

$$ad\,\beta(S(E_i)).\beta(x_{w_+\Lambda})^{-1} = q^{-(2\alpha_i,\Lambda)}q_{i,\Lambda}\beta(x_{w_+\Lambda})^{-1}\beta(E_i).$$

Therefore we obtain from (2) that

$$ad\,\beta(S(E_i)).z^+_f = \beta(x_{w_+\Lambda})^{-1}\beta(ad\, E_i.x_\lambda + q^{-(2\alpha_i,\lambda)}q_{i,\Lambda}x_\lambda E_i).$$

Hence the result by Corollary 3.10.

3. Using 1, we obtain

$$ad\,\beta(S(\tau(\nu)).z^+_f = [ad\,\beta(S(\tau(\nu)).c^{-1}_{w\Lambda}][ad\,\beta(S(\tau(\nu)).c_{f,v_\Lambda}] = q^{(\nu,\Lambda-\lambda+w_+\Lambda-\Lambda)}z^+_f,$$

hence the result. $\square$

We can now prove the main theorem of this section (cf. [4, Proposition 7.5]).

Theorem 3.12. *The anti-algebra map $\psi : R^+_q \to \mathbb{C}_q[G]$ extends to an anti-isomorphism of algebras $\psi : T_{w_+} \to C^+_w$ such that*

$$\forall t \in T_{w_+},\ \forall u \in U_q(\mathfrak{c}^+), \quad \psi(ut) = ad\,\beta(S(u)).\psi(t).$$

Hence in particular, $Soc_{\,ad}C^+_w = \mathbb{C}$.

Proof. We already noticed that $\psi(Q_{w_+}) \subset I^+_{w_+}$, $\psi(f_{-w_+\Lambda}) = c_{w\Lambda}$. Therefore we can extend ψ to a surjective anti-algebra map

$$\psi : T_{w_+} \longrightarrow C^+_w, \quad \psi(f(f_{-w_+\Lambda})^{-1}) = z^+_f = c^{-1}_{w\Lambda}c_{f,v_\Lambda}.$$

As seen in Theorem 3.6, T_{w_+} is a $\check{U}_q(\mathfrak{b}^+)$-module algebra and we have $Soc_{\,U_q(\mathfrak{c}^+)}T_{w_+} = Soc_{\,\check{U}_q(\mathfrak{b}^+)}T_{w_+} = \mathbb{C}$. We can compose the map ψ with the projection $C^+_w \to \bar{C}^+_w$ to obtain a surjective anti-algebra map, $\bar{\psi}: T_{w_+} \to \bar{C}_w^+$, mapping $f(f_{-w_+\Lambda})^{-1}$ onto $z_f^+ \in \bar{C}_w^+$. Now observe that for all $f \in L(\Lambda)^*_{-\lambda}$ we have in T_{w_+},

$$E_i(f(f_{-w_+\Lambda})^{-1}) = (E_i f)(f_{-w_+\Lambda})^{-1} + (K_i^2 f)(E_i(f_{-w_+\Lambda})^{-1}) = (E_i f)(f_{-w_+\Lambda})^{-1}$$

and

$$\tau(\nu)(f(f_{-w_+\Lambda})^{-1}) = (\tau(\nu)f)(\tau(\nu)(f_{-w_+\Lambda})^{-1}) = q^{(\nu, w_+\Lambda-\lambda)}(f(f_{-w_+\Lambda})^{-1}) \text{ for all } \nu \in \mathbf{P}.$$

Notice that $\bar{C}_w^+$ is a $U_q(\mathfrak{c}^+)$-module algebra by $a.z = ad\,\beta(S(a)).z$ for all $a \in U_q(\mathfrak{c}^+)$ and $z \in \bar{C}_w^+$. From Proposition 3.11 it follows that $\bar{\psi}$ is a $U_q(\mathfrak{c}^+)$-module map. Hence $\ker\bar{\psi}$ is a $U_q(\mathfrak{c}^+)$-submodule. Since $\bar{\psi}(1) = 1$ and $Soc_{U_q(\mathfrak{c}^+)}T_{w_+} = \mathbb{C}$, we have $\ker\bar{\psi} = 0$. Therefore both ψ and $\bar{\psi}$ are isomorphisms of $U_q(\mathfrak{c}^+)$-modules. This proves the theorem. □

4. The main theorem

In this section we shall prove that $Spec_w\mathbb{C}_q[G]$ is an H-orbit. We first begin with a consequence of Theorem 3.12.

Theorem 4.1. *The socle of C_w^H considered as a $\mathbb{C}_q[G]$-module via the adjoint action is $\mathbb{C}$.*

Proof. By Theorem 3.12 we have that $Soc_{ad}C_w^+ = \mathbb{C}$. Now the map $\sigma: \mathbb{C}_{q^{-1}}[G] \to \mathbb{C}_q[G]$ induces a map $\sigma: \hat{C}_{\hat{w}}^+ \to C_w^-$ such that $\sigma(ad\,\hat{c}.d) = ad\,\sigma(\hat{c}).\sigma(d)$. Hence it follows immediately that $Soc_{ad}C_w^- = \mathbb{C}$ also. Since the map $C_w^+ \otimes C_w^- \to C_w^+ C_w^- = C_w^H$ is a module map, it suffices to show that $Soc_{ad}(C_w^+ \otimes C_w^-) = \mathbb{C}$. Recall that C_w^- is a module over $\mathbb{C}_q[B^+]$ and is diagonalisable as a module over the subalgebra $\mathbb{C}[B^+/N^+]$. An element of $C_w^+ \otimes C_w^-$ can therefore be written in the form $\sum a_i \otimes b_i$ where the b_i are linearly independent $\mathbb{C}_q[B^+/N^+]$-weight vectors. Suppose that $\sum a_i \otimes b_i \in Soc_{ad}(C_w^+ \otimes C_w^-)$ and let c_{f,v_Λ} be a generator of $J^- = I_e^+$. Then

$$\begin{aligned} c_{f,v_\Lambda}(\sum_i a_i \otimes b_i) &= \sum_{i,j} c_{f,v_j}(a_i) \otimes c_{f_j,v_\Lambda}(b_i) = \sum_i c_{f,v_\Lambda}(a_i) \otimes c_{f_{-\Lambda},v_\Lambda}(b_i) \\ &= \sum_i c_{f,v_\Lambda}(a_i) \otimes \alpha_i b_i \end{aligned}$$

for some $\alpha_i \in \mathbb{C}^*$. Thus $c_{f,v_\Lambda}(a_i) = 0$ for all i. Now modulo $J^+ = I_e^-$ it is easily seen that the left ideal generated by the elements c_{f,v_Λ} is two-sided. Since $Ann_{ad}C_w^+ \supset J^+$, each a_i is annihilated by I_e. It follows that $a_i \in Soc_{ad}C_w^+ = \mathbb{C}$. Thus $\sum a_i \otimes b_i \in Soc_{ad}(\mathbb{C} \otimes C_w^-) = \mathbb{C} \otimes \mathbb{C}$. □

Proposition 4.2. *Let $c \in A_{\gamma,\mu}$. Then, if $\lambda \in \mathbf{P}$, we have $ad\,c.t_\lambda = \varepsilon(c)q^{2((w_+-w_-)\lambda,\mu)}t_\lambda$.*

Proof. This follows easily from Lemma 2.1. □

Denote by $\Psi: \mathbf{P} \to \mathbf{P}$ the map given by $\Psi(\lambda) = (w_+ - w_-)\lambda$.

Corollary 4.3. *The algebra of ad-invariant elements of C_w is given by*

$$C_w^{ad} = \mathbb{C}[t_\lambda \mid \lambda \in \ker\Psi].$$

Proof. This is consequence of Theorems 2.6, 4.1 and the proposition above. □

Recall the following notation from [2, Appendix A]: for each $w = (w_+, w_-) \in W \times W$ we set $s(w) = \operatorname{codim}_{\mathfrak{h}^*} \ker(w_+ w_-^{-1} - I)$. Denote by $Prim\ \mathbb{C}_q[G]$ the primitive spectrum of $\mathbb{C}_q[G]$ and set $Prim_w\ \mathbb{C}_q[G] = Prim\ \mathbb{C}_q[G] \cap Spec_w\ \mathbb{C}_q[G]$. We are now able to deduce Joseph's theorem [4, Theorem 9.2].

Theorem 4.4. 1. $\mathrm{Prim}\ \mathbb{C}_q[G] \cong \bigsqcup_{w \in W \times W} \mathrm{Prim}_w \mathbb{C}_q[G]$.

2. For each $w \in W \times W$, $\mathrm{Prim}_w \mathbb{C}_q[G]$ is a non-empty H-orbit. If $P \in \mathrm{Prim}_w \mathbb{C}_q[G]$, then $H/\mathrm{Stab}_H P$ is a torus of rank equal to rank $G - s(w)$.

Proof. Part 1 is an immediate consequence of Corollary 2.5. Fiw $w \in W \times W$ and let P be a primitive ideal of type w. Then PA_w is a primitive ideal of A_w. Let K be a subgroup of $\mathbf{P}$ such that $\mathbf{P} = \ker \Psi \oplus K$. Set $C_{\dot{w}} = C_w^H[t_\lambda \mid \lambda \in K]$. Then clearly $C_w \cong C_{\dot{w}} \otimes C_w^{ad}$ as $\mathbb{C}_q[G]$-module algebras. Notice that the socle of $C_{\dot{w}}$ is $\bigoplus_{\lambda \in K} \mathbb{C} t_\lambda$. Therefore any ad-invariant subspace of $C_{\dot{w}}$ contains some t_λ and hence $C_{\dot{w}}$ contains no nontrivial ad-invariant ideals. If I is a maximal ideal of C_w^{ad}, then $C_w/IC_w \cong C_{\dot{w}}$ as $\mathbb{C}_q[G]$-module algebras. Thus IC_w is a maximal ad-invariant ideal of C_w. Similarly IB_w is a maximal ad-invariant ideal of B_w.

Since C_w^{ad} is contained in the center of A_w, it follows from the nullstellensatz [8, 9.1.7] that $I = PA_w \cap C_w^{ad}$ is a maximal ideal of C_w^{ad}. Thus $PA_w \cap B_w = IB_w$. Let Q be another such primitive ideal of type w. Then $QA_w \cap B_w = JB_w$ for some maximal ideal J of C_w^{ad}. It is clear from the action of H on the t_λ that there exists an $h \in H$ such that $I = J^h$. This implies that $Q^h A_w \cap B_w = PA_w \cap B_w$. Since $B_w = A_w^\Gamma$, there must exist a $g \in \Gamma$ such that $PA_w = (Q^h A_w)^g = Q^{hg} A_w$. Hence $Q^{hg} = P$. Thus the primitive ideals of type w form an H-orbit.

Since the action of H is algebraic and Γ is finite, $H/Stab_H P$ must be a torus of the same dimension as $H/Stab_H I = \dim C_w^{ad} = \dim H - s(w)$. □

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