EQUIVARIANT D-MODULES ATTACHED TO NILPOTENT ORBITS IN A SEMISIMPLE LIE ALGEBRA

T. LEVASSEUR

Abstract. Let $\mathfrak{g}_u$ be a compact Lie algebra and $\mathfrak{g}$ be its complexification. Let $\zeta^{-\frac{1}{2}}$ be the inverse, on the set of regular elements of $\mathfrak{g}_u$, of a square root of the discriminant of $\mathfrak{g}$. Generalizing a result of W. Lichtenstein in the case $\mathfrak{g}_u = \mathfrak{su}(n,\mathbb{C})$ or $\mathfrak{so}(n,\mathbb{R})$, we prove that $\partial(q).\zeta^{-\frac{1}{2}}$ is non zero for all harmonic polynomials $q \in S(\mathfrak{g})\setminus\{0\}$. This fact is deduced from results about equivariant D-modules supported on the nilpotent cone of $\mathfrak{g}$.

1. Introduction

Let G_u be a connected compact semisimple group with Lie algebra $\mathfrak{g}_u$. Denote by $\mathfrak{g} = \mathfrak{g}_u \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of $\mathfrak{g}_u$ and let G be the adjoint group of $\mathfrak{g}$. Choose a Cartan subalgebra $\mathfrak{h}_u$ of $\mathfrak{g}_u$ and set $\mathfrak{h} = \mathfrak{h}_u \otimes_{\mathbb{R}} \mathbb{C}$. Fix a positive system of roots R^+ for $\mathfrak{g}$ with respect to $\mathfrak{h}_u$.

Let $\mathcal{O}(\mathfrak{g}) = S(\mathfrak{g}^*)$ be the algebra of polynomial functions on $\mathfrak{g}$, and denote by $\mathcal{D}(\mathfrak{g})$ the algebra of differential operators on $\mathfrak{g}$, with coefficients in $\mathcal{O}(\mathfrak{g})$. For $q \in S(\mathfrak{g})$, let $\partial(q) \in \mathcal{D}(\mathfrak{g})$ be the corresponding differential operator with constant coefficients. The group G acts on $\mathfrak{g}$, via the adjoint action, and hence has an induced action on $S(\mathfrak{g}^*)$, $S(\mathfrak{g})$ and $\mathcal{D}(\mathfrak{g})$. Denote the differential of this action by $\tau : \mathfrak{g} \to \mathcal{D}(\mathfrak{g})$. Let $S_+(\mathfrak{g}^*)^G$ be the set of invariant elements without constant term and let $H(\mathfrak{g}^*)$ be the space of harmonic polynomials on $\mathfrak{g}$. Then $H(\mathfrak{g}^*)$ is a G-stable complement of $S_+(\mathfrak{g}^*)^G S(\mathfrak{g}^*)$ in $S(\mathfrak{g}^*)$, see [9]. We adopt a similar notation for polynomial functions on $\mathfrak{g}^*$, hence $S(\mathfrak{g}) = H(\mathfrak{g}) \oplus S_+(\mathfrak{g})^G S(\mathfrak{g})$.

Let $\zeta \in S(\mathfrak{g}^*)$ be the discriminant of $\mathfrak{g}$, see [5, §7] or [18, §4]. Hence $\zeta \in S(\mathfrak{g}^*)^G$ corresponds, through the Chevalley isomorphism, to π^2, where $\pi = \prod_{\alpha\in R^+} \alpha \in S(\mathfrak{h}^*)$. Define the set of generic elements by $\mathfrak{g}' = \{x \in \mathfrak{g} : \zeta(x) \neq 0\}$ and set $\mathfrak{g}'_u = \mathfrak{g}' \cap \mathfrak{g}_u$. Then, one can define a function $\zeta^{\frac{1}{2}}$, real analytic on $\mathfrak{g}'_u$, and one considers the "potential" $\zeta^{-\frac{1}{2}}$ on $\mathfrak{g}'_u$. Clearly, the algebra $\mathcal{A}(\mathfrak{g}'_u)$ of analytic functions (with complex values) on $\mathfrak{g}'_u$ has a natural $\mathcal{D}(\mathfrak{g})$-module structure. W. Lichtenstein has proved the following result:

Theorem 1.1. ([14, Theorem 2]) *Suppose that* $\mathfrak{g}_u = \mathfrak{su}(n,\mathbb{C})$ *or* $\mathfrak{so}(n,\mathbb{R})$. *Then the map*

$$\mathrm{M} : H(\mathfrak{g}) \to \mathcal{A}(\mathfrak{g}'_u), \quad q \mapsto \partial(q).\zeta^{-\frac{1}{2}}$$

is injective.

1991 *Mathematics Subject Classification.* 14L30, 16S32, 17B20, 22E46.

Key words and phrases. semisimple Lie algebra, equivariant D-module, nilpotent orbit, Weyl group representation.

Actually, [14] states the result with a map $\mathrm{M}^*(p) = \partial(p).\zeta^{-\frac{1}{2}}$, $p \in H(\mathfrak{g}^*)$, after identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form B. These elements are called multipole potentials. In [14, §3] it is shown that $\mathrm{M}(p)$ is the term of lowest degree in the asymptotic expansion, when $\lambda \to +\infty$, of the integral

$$\int_{\mathbf{O}} e^{-iB(\lambda x,y)} p(y) d\mu(y),$$

where $\mathbf{O} \subset \mathfrak{g}_u$ is a regular orbit and $d\mu$ is a G_u-invariant measure on $\mathbf{O}$.

One aim of this note is to establish Theorem 1.1 for an arbitrary compact Lie algebra $\mathfrak{g}_u$. We will show in §2 that this result is consequence of the decomposition of the invariant holonomic systems given in [12], which we now recall. Let W be the Weyl group associated to $(\mathfrak{g}, \mathfrak{h})$. Denote by $W\hat{}$ the set of isomorphism classes of irreducible W-modules and, for $\chi \in W\hat{}$, let V_χ be a W-module in the class of χ. Set

$$\mathcal{M} = \mathcal{D}(\mathfrak{g})/\big(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})S_+(\mathfrak{g}^*)^G\big), \quad \mathcal{N} = \mathcal{D}(\mathfrak{g})/\big(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})S_+(\mathfrak{g})^G\big).$$

Then, $\mathcal{M}$ and $\mathcal{N}$ are semisimple holonomic $\mathcal{D}(\mathfrak{g})$-modules and the full subcategory of $\mathcal{D}(\mathfrak{g})$-modules generated by $\mathcal{N}$, or $\mathcal{M}$, is equivalent to the category of W-modules. Let $\mathcal{N}_\chi$ and $\mathcal{M}_\chi$ be the simple submodules of $\mathcal{N}$ and $\mathcal{M}$ which correspond to V_χ through these equivalences (see §2 for details).

Then, in the general case, Theorem 1.1 follows from (and is explained by) the following result.

Theorem 1.2. *Let* triv $\in W\hat{}$ *be the trivial character. Then:*

(i) $\mathcal{N}_{\mathrm{triv}} \cong \mathcal{D}(\mathfrak{g}).\zeta^{-\frac{1}{2}}$;

(ii) *The support of* $\mathcal{M}_{\mathrm{triv}}$ *is the nilpotent cone of* $\mathfrak{g}$.

This theorem is presumably implicit in [6, 18]. Its significance for this paper is to show how the results of [12] can be applied to prove the generalization of Theorem 1.1, see Theorem 2.8.

In order to show that Theorem 1.2 (which deals with the regular nilpotent orbit) is a particular case of more general results, we present in §3 a construction, due to Hotta & Kashiwara [6] and Wallach [18], which associates in a natural way to any nilpotent orbit $\mathbf{O} \subset \mathfrak{g}$ a $\mathcal{D}(\mathfrak{g})$-module $\mathcal{M}_{\chi_{\mathbf{O}}}$ (equivalently, a representation $\chi_{\mathbf{O}}$ of W).

We end this note with some comments regarding the correspondence, via invariant holonomic systems, between nilpotent orbits and irreducible representations of the Weyl group [12, §7]. In particular, we show that the inverse image of $\mathcal{M}_{\chi_{\mathbf{O}}}$ on $\mathbf{O}$ is isomorphic (as a $\mathcal{D}_{\mathbf{O}}$-module) to the standard module $\mathcal{O}_{\mathbf{O}}$; equivalently, $\chi_{\mathbf{O}}$ corresponds to the pair $(\mathbf{O}, \mathrm{triv})$ where triv is the trivial representation of the component group $A(\mathbf{O})$ of the orbit $\mathbf{O}$.

2. The modules $\mathcal{N}_{\mathrm{triv}}$ and $\mathcal{M}_{\mathrm{triv}}$

We begin this section by recalling some facts from [12]. Set $n = \dim \mathfrak{g}$, $\ell = \dim \mathfrak{h}$ and $\nu = \# R^+$, hence $n = 2\nu + \ell$. Let $\{e_i\}_{1 \leqslant i \leqslant n}$ be an orthonormal basis of $\mathfrak{g}$ with respect to B such that $\{e_i\}_{1 \leqslant i \leqslant \ell}$ is a basis of $\mathfrak{h}$. Denote by $x_i \in S(\mathfrak{g}^*)$, $1 \leqslant i \leqslant n$, the associated coordinate functions; thus $\partial(e_i)$ identifies with the partial derivative $\partial_i = \frac{\partial}{\partial x_i}$. Denote the Euler vector fields on $\mathfrak{g}$ and $\mathfrak{h}$ by $\mathtt{E}_{\mathfrak{g}} = \sum_{i=1}^n x_i \partial_i$

and $\mathtt{E}_{\mathfrak{h}} = \sum_{i=1}^{\ell} x_i \partial_i$. Recall that there exists an algebra homomorphism, defined by Harish-Chandra,

$$\delta : \mathcal{D}(\mathfrak{g})^G \longrightarrow \mathcal{D}(\mathfrak{h})^W$$

which extends the Chevalley isomorphisms $S(\mathfrak{g})^G \xrightarrow{\sim} S(\mathfrak{h})^W$ and $S(\mathfrak{g}^*)^G \xrightarrow{\sim} S(\mathfrak{h}^*)^W$. The map δ is surjective and its kernel is $\mathcal{I} = (\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}))^G$, see [11]. This enables one to identify, through δ, modules over $\mathcal{D}(\mathfrak{g})^G/\mathcal{I}$ with $\mathcal{D}(\mathfrak{h})^W$-modules. Let $\mathcal{R}$ be the full subcategory of $\mathcal{D}(\mathfrak{g})$-modules generated by the module $\mathcal{N}$. Then, if $M \in \mathcal{R}$, the $\mathcal{D}(\mathfrak{g})^G$-module M^G can be considered as a $\mathcal{D}(\mathfrak{h})^W$-module. Let

$$\mathcal{H} = \{f \in S(\mathfrak{h}^*) : \forall q \in S_+(\mathfrak{h})^W,\ \partial(q).f = 0\}$$

be the set of harmonic polynomials on $\mathfrak{h}$. The vector space $\mathcal{H}$ is graded by the $\mathcal{H}^j = \mathcal{H} \cap S^j(\mathfrak{h}^*)$ and, as a W-module, $\mathcal{H}$ identifies with the left regular representation.

The following theorem summarizes some of the results contained in [12]. (In the notation of [12], we are only concerned with the case $\lambda = 0$ and so the modules $\mathcal{M}_0$ and $\mathcal{N}_0$ of [12] are denoted by $\mathcal{M}$, respectively $\mathcal{N}$.)

Theorem 2.1. (1) ([6, Theorem 5.3], [12, Theorem A]) $\mathcal{M}$ *and* $\mathcal{N}$ *are semisimple* $(\mathcal{D}(\mathfrak{g}), W)$*-modules and, as such, have the following decompositions*

$$\mathcal{M} \cong \bigoplus\nolimits_{\chi \in W^\wedge} \mathcal{M}_\chi \otimes_{\mathbb{C}} V_\chi, \qquad \mathcal{N} \cong \bigoplus\nolimits_{\chi \in W^\wedge} \mathcal{N}_\chi \otimes_{\mathbb{C}} V_\chi$$

where the $\mathcal{M}_\chi$*, resp.* $\mathcal{N}_\chi$*, are pairwise non-isomorphic simple* $\mathcal{D}(\mathfrak{g})$*-modules.*

(2) ([12, Corollary 6.11]) *The category* $\mathcal{R}$ *is equivalent to* $(W\text{-mod})^{\mathrm{op}}$*, through the functor* $M \to \mathrm{Sol}(M^G) = \mathrm{Hom}_{\mathcal{D}(\mathfrak{h})^W}(M^G, S(\mathfrak{h}^*))$*. Moreover,*

- (i) *when* M *is a quotient of* $\mathcal{N}$, $\mathrm{Sol}(M^G)$ *identifies naturally with a* W*-submodule of* $\mathcal{H}$*;*
- (ii) $\mathrm{Sol}(\mathcal{N}_\chi^G) \cong V_\chi$.

Corollary 2.2. *Set* $L = \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})S_+(\mathfrak{g})^G + \mathcal{D}(\mathfrak{g})(\mathtt{E}_{\mathfrak{g}} + \nu)$*. Then,*

$$\mathcal{N}_{\mathrm{triv}} \cong \mathcal{D}(\mathfrak{g})/L.$$

Proof. Clearly, $N = \mathcal{D}(\mathfrak{g})/L$ is a quotient of $\mathcal{N}$. Observe that

$$\begin{aligned} N^G &= \mathcal{D}(\mathfrak{g})^G / \big(\mathcal{I} + \mathcal{D}(\mathfrak{g})^G S_+(\mathfrak{g})^G + \mathcal{D}(\mathfrak{g})^G(\mathtt{E}_{\mathfrak{g}} + \nu)\big) \\ &\cong \mathcal{D}(\mathfrak{h})^W / \big(\mathcal{D}(\mathfrak{h})^W S_+(\mathfrak{h})^W + \mathcal{D}(\mathfrak{h})^W(\delta(\mathtt{E}_{\mathfrak{g}}) + \nu)\big). \end{aligned}$$

It is easily checked that $\delta(\mathtt{E}_{\mathfrak{g}}) = \mathtt{E}_{\mathfrak{h}} - \nu$, (see, for example, [18, Proof of Theorem 6.1]). Therefore, as W-module,

$$\mathrm{Sol}(N^G) \cong \{f \in \mathcal{H} : \mathtt{E}_{\mathfrak{h}}.f = 0\} = \mathbb{C}.$$

Thus $\mathrm{Sol}(N^G) = V_{\mathrm{triv}}$ and the corollary follows from Theorem 2.1. □

Corollary 2.3. *The* $\mathcal{D}(\mathfrak{g})$*-modules* $\mathcal{N}_{\mathrm{triv}}$ *and* $\mathcal{D}(\mathfrak{g}).\zeta^{-\frac{1}{2}}$ *are isomorphic.*

Proof. By [14, Theorem 1 and Corollary] $\zeta^{-\frac{1}{2}}$ is a G_u-invariant harmonic function on $\mathfrak{g}_u'$. Hence, $\tau(\mathfrak{g}).\zeta^{-\frac{1}{2}} = S_+(\mathfrak{g})^G.\zeta^{-\frac{1}{2}} = 0$. Observe that, since ζ is homogeneous of degree 2ν, $\mathtt{E}_{\mathfrak{g}}.\zeta^{-\frac{1}{2}} = -\nu\,\zeta^{-\frac{1}{2}}$. The result then follows from Corollary 2.2. □

Remark. Let sgn be the sign representation of W. Recall that the unique copy of V_{sgn} in $\mathcal{H}$ is $\mathcal{H}^\nu = \mathbb{C}\pi$. The modules $\mathcal{N}_{\mathrm{sgn}}$ and $\mathcal{M}_{\mathrm{sgn}}$ are easy to describe:

$$\mathcal{N}_{\mathrm{sgn}} = \mathcal{D}(\mathfrak{g}).1 = \mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\mathfrak{g}, \qquad \mathcal{M}_{\mathrm{sgn}} = \mathcal{D}(\mathfrak{g}).T_0 = \mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\mathfrak{g}^*,$$

where T_0 is the Dirac distribution at 0 on a real form of $\mathfrak{g}$. This can be proved as follows. Set:

$$N = \mathcal{D}(\mathfrak{g})/\big(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})S_+(\mathfrak{g})^G + \mathcal{D}(\mathfrak{g})\mathtt{E}_{\mathfrak{g}}\big)$$

Then, the $\mathcal{D}(\mathfrak{g})^G$-module N^G is isomorphic to the $\mathcal{D}(\mathfrak{h})^W$-module

$$\mathcal{D}(\mathfrak{h})^W/\big(\mathcal{D}(\mathfrak{h})^W S_+(\mathfrak{h})^W + \mathcal{D}(\mathfrak{h})^W(\mathtt{E}_{\mathfrak{h}} - \nu)\big)$$

and it follows that $\operatorname{Sol}(N^G) = \mathbb{C}\pi$. This proves $N \cong \mathcal{N}_{\mathrm{sgn}}$. Clearly, $\mathcal{D}(\mathfrak{g}).1 = \mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g})\mathfrak{g}$ is a factor of N, hence $N = \mathcal{D}(\mathfrak{g}).1$. The description of $\mathcal{M}_{\mathrm{sgn}}$ can be deduced from that of $\mathcal{N}_{\mathrm{sgn}}$ by Fourier transform (see below), or from a general argument (see §3).

Recall [12, §7] that the modules $\mathcal{N}_\chi$ are the Fourier transforms of the modules $\mathcal{M}_\chi$. The Fourier transformation is defined as follows: Use the G-invariant bilinear form B to define an isomorphism $B : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$. Since one has a G-module isomorphism $\mathcal{D}(\mathfrak{g}) \cong S(\mathfrak{g}^*) \otimes S(\mathfrak{g})$, B induces an algebra automorphism F of $\mathcal{D}(\mathfrak{g})$ by $F(f) = -B^{-1}(f)$ and $F(x) = B(x)$ for $f \in \mathfrak{g}^*$ and $x \in \mathfrak{g}$. Moreover, F is a G-automorphism of $\mathcal{D}(\mathfrak{g})$ such that $F(\tau(x)) = \tau(x)$ for all $x \in \mathfrak{g}$. Given a $\mathcal{D}(\mathfrak{g})$-module M, define the Fourier transform M^F of M to be the abelian group M with multiplication given by $a \centerdot m = F(a)m$, for $a \in \mathcal{D}(\mathfrak{g})$ and $m \in M$. Then, as in [6, §6], $\mathcal{N} = \mathcal{M}^F$. Notice that, by definition, $\mathcal{N}_\chi = \mathcal{M}_\chi^F$. For example, one deduces from Corollary 2.2 that

$$\mathcal{M}_{\mathrm{triv}} \cong \mathcal{D}(\mathfrak{g})/\big(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})S_+(\mathfrak{g}^*)^G + \mathcal{D}(\mathfrak{g})(\mathtt{E}_{\mathfrak{g}} + \nu + \ell)\big).$$

Let $\chi \in W\hat{}$. Set

$$b(\chi) = \inf\big\{j : [S^j(\mathfrak{h}^*) : V_\chi] \neq 0\big\} = \inf\big\{j : [\mathcal{H}^j : V_\chi] \neq 0\big\}.$$

Lemma 2.4. ([18, Lemma 6.5]) *The highest eigenvalue of* $\mathtt{E}_{\mathfrak{g}}$ *on* $\mathcal{M}_\chi^G$ *is* $\nu - n - b(\chi)$*; this eigenvalue occurs with multiplicity* $[\mathcal{H}^{b(\chi)} : V_\chi]$.

Proof. Recall [12] that as a $\mathcal{D}(\mathfrak{h})^W$-module, $\mathcal{N}_\chi^G \cong V^\chi$, where

$$V^\chi = \operatorname{Hom}_W(V_\chi, S(\mathfrak{h}^*)).$$

Since $\mathtt{E}_{\mathfrak{h}}$ acts semisimply on $S(\mathfrak{h}^*)$, the same is true on V^χ. Moreover, the smallest eigenvalue of $\mathtt{E}_{\mathfrak{h}}$ on V^χ is $b(\chi)$. Indeed: realize V^χ as a $\mathcal{D}(\mathfrak{h})^W$-submodule of $S(\mathfrak{h}^*)$, and note that $\mathbb{C}W.u \cong V_\chi$ for all $u \in V^\chi$; this implies that the degree of each u is $\geqslant b(\chi)$. Observe also that the simple module V^χ occurs in $S(\mathfrak{h}^*)$ with multiplicity $\dim V_\chi$, hence the multiplicity of the eigenvalue $b(\chi)$ is $[\mathcal{H}^{b(\chi)} : V_\chi]$.

Now, since $\delta(\mathtt{E}_{\mathfrak{g}}) = \mathtt{E}_{\mathfrak{h}} - \nu$, the smallest eigenvalue of $\mathtt{E}_{\mathfrak{g}}$ on $\mathcal{N}_\chi^G$ is $b(\chi) - \nu$. The lemma is then consequence of $F(\mathtt{E}_{\mathfrak{g}}) = -\mathtt{E}_{\mathfrak{g}} - n$ and $\mathcal{N}_\chi = \mathcal{M}_\chi^F$. □

Let $\mathbf{N}(\mathfrak{g})$ be the set of nilpotent elements in $\mathfrak{g}$. Then, $\mathbf{N}(\mathfrak{g})$ is a finite union of G-orbits and its defining ideal is generated by $S_+(\mathfrak{g}^*)^G$, see [9]. If $\mathbf{O} = G.x \subset \mathfrak{g}$ is a nilpotent orbit, one sets [6, §7]

$$\lambda_{\mathbf{O}} = \frac{1}{2}\dim \mathbf{O} - \dim \mathfrak{g} = -\frac{1}{2}(\dim \mathfrak{g} + \dim \mathfrak{g}^x),$$

where $\mathfrak{g}^x$ is the stabilizer of x. Note that if $\overline{\mathbf{O}_1} \subset \overline{\mathbf{O}_2}$, then $\lambda_{\mathbf{O}_1} \leqslant \lambda_{\mathbf{O}_2}$, with equality if and only if $\mathbf{O}_1 = \mathbf{O}_2$.

The support of a coherent $\mathcal{D}(\mathfrak{g})$-module M will be denoted by $\operatorname{Supp} M$. (We refer to [4] for definitions related to D-modules.) The next proposition provides an analogue for G-equivariant $\mathcal{D}(\mathfrak{g})$-modules of [1, Corollary 3.9] and [18, Lemma 6.2]. A version of this result is proved, in slightly more generality, in [13].

Proposition 2.5. *Let M be a finitely generated G-equivariant $\mathcal{D}(\mathfrak{g})$-module such that* $\operatorname{Supp} M \subset \mathbf{N}(\mathfrak{g})$.

(i) *Set* $\lambda_M = \max\{\lambda_{\mathbf{O}} : \mathbf{O}$ *nilpotent orbit contained in* $\operatorname{Supp} M\}$*; then every eigenvalue of* $\mathtt{E}_{\mathfrak{g}}$ *on* M^G *is* $\leqslant \lambda_M$.

(ii) *Assume that:* $\operatorname{Supp} M = \overline{\mathbf{O}}$, $\mathbf{O}$ *nilpotent orbit, and* $M = \mathcal{D}(\mathfrak{g}).v$ *with* $0 \neq v \in M^G$ *such that* $\mathtt{E}_{\mathfrak{g}}.v = \lambda_{\mathbf{O}} v$. *Then, the* $\lambda_{\mathbf{O}}$*-eigenspace of* $\mathtt{E}_{\mathfrak{g}}$ *in* M^G *has dimension one.*

Proof. Recall first that, since M is G-equivariant, the differential of the G-action on M coincides with the multiplication by the elements $\tau(\xi)$, $\xi \in \mathfrak{g}$ (see, for example, [17, Proposition 2.6] and Appendix A). In particular, $\tau(\mathfrak{g}).v = 0$ for all $v \in M^G$; it then follows from [16, §6], or [12, Lemma 6.1], that any subquotient of $\mathcal{D}(\mathfrak{g}).v$ is G-equivariant.

(i) Let $0 \neq v \in M^G$ be an eigenvector of $\mathtt{E}_{\mathfrak{g}}$. Set $N = \mathcal{D}(\mathfrak{g}).v$. By the previous remark and the inclusion $\operatorname{Supp} N \subset \mathbf{N}(\mathfrak{g})$, the annihilator of v in $\mathcal{D}(\mathfrak{g})$ contains $\tau(\mathfrak{g})$ and a power of $S_+(\mathfrak{g}^*)^G$. Then, [13, Remark 3.6] applied to N implies that the eigenvalue of v is $\leqslant \lambda_N \leqslant \lambda_M$ (see [13, Remark following Lemma 3.4]).

(ii) Let $v' \in M^G$ be such that $\mathtt{E}_{\mathfrak{g}}.v' = \lambda_{\mathbf{O}} v'$. By [13, Lemma 3.4], there exists $c \in \mathbb{C}$ such that $M' = \mathcal{D}(\mathfrak{g}).(v' - cv)$ satisfies $\operatorname{Supp} M' \subsetneq \overline{\mathbf{O}}$. Since $\operatorname{Supp} M'$ is a finite union of nilpotent orbits, (i) yields $\lambda_{M'} < \lambda_{\mathbf{O}}$. Suppose that $v' - cv \neq 0$, so that $v' - cv$ is a non-zero eigenvector for $\lambda_{\mathbf{O}}$. Then, by (i) again, $\lambda_{\mathbf{O}} \leqslant \lambda_{M'}$ and a contradiction. □

As explained in [12, §7], $\mathcal{M}_\chi$ is a G-equivariant $\mathcal{D}(\mathfrak{g})$-module, i.e. compatible in the sense of [12, §6], and $\operatorname{Supp} \mathcal{M}_\chi = \overline{\mathbf{O}(\chi)}$ is the closure of some nilpotent orbit.

Corollary 2.6. *One has:* $b(\chi) + \frac{1}{2} \dim \mathbf{O}(\chi) \geqslant \nu$.

Proof. By Lemma 2.4 and Proposition 2.5, $\nu - n - b(\chi) \leqslant \lambda_{\mathbf{O}(\chi)} = \frac{1}{2} \dim \mathbf{O}(\chi) - n$, as desired. □

Recall [9] that $\mathbf{N}(\mathfrak{g})$ is the closure of a nilpotent orbit, $\mathbf{O}^{reg}$, of dimension 2ν. As observed in [6, §5] one has the following result, which proves Theorem 1.2(ii).

Proposition 2.7. $\operatorname{Supp} \mathcal{M}_{\mathrm{triv}} = \mathbf{N}(\mathfrak{g})$.

Proof. Since $b(\mathrm{triv}) = 0$, Corollary 2.6 implies that $\dim \mathbf{O}(\mathrm{triv}) \geqslant 2\nu$, forcing $\dim \mathbf{O}(\mathrm{triv}) = 2\nu$ and $\mathbf{O}(\mathrm{triv}) = \mathbf{O}^{reg}$. □

We can now complete the proof of Theorem 1.1 in the general case.

Theorem 2.8. *The multipole mapping*

$$\mathrm{M} : H(\mathfrak{g}^*) \to \mathcal{A}(\mathfrak{g}'_u), \quad p \mapsto \partial(F^{-1}(p)).\zeta^{-\frac{1}{2}}$$

is injective.

Proof. Let $L = \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})S_+(\mathfrak{g})^G + \mathcal{D}(\mathfrak{g})(\mathtt{E}_{\mathfrak{g}} + \nu)$ be as in Corollary 2.2. Assume that $\mathrm{M}(p) = 0$ for some $p \in H(\mathfrak{g}^*)$; by Corollary 2.3, this is equivalent to saying that $\partial(F^{-1}(p)) \in L$. Thus $p = F(\partial(F^{-1}(p))) \in F(L)$. Note that $\mathcal{M}_{\mathrm{triv}} = \mathcal{N}_{\mathrm{triv}}^{F^{-1}} \cong \mathcal{D}(\mathfrak{g})/F(L)$. Therefore, by definition of support, $\operatorname{Supp} \mathcal{M}_{\mathrm{triv}} \subset p^{-1}(\{0\}) \subset \mathfrak{g}$ and Proposition 2.7 then implies that $\mathbf{N}(\mathfrak{g}) \subset p^{-1}(\{0\})$. Since the defining ideal of the variety $\mathbf{N}(\mathfrak{g})$ is the prime ideal generated by $S_+(\mathfrak{g}^*)^G$ (see [9]) it follows that $p \in S_+(\mathfrak{g}^*)^G S(\mathfrak{g}^*)$. Hence $p \in H(\mathfrak{g}^*) \cap S_+(\mathfrak{g}^*)^G S(\mathfrak{g}^*) = 0$. □

Remark. In the case when $\mathfrak{g}_u = \mathfrak{so}(3, \mathbb{R})$ the previous result is due to J. C. Maxwell, who used it in his work on electro-magnetism. We would like to thank V. Ginzburg for this historical remark.

3. Additional results

This section collects various results from [6, 18] which make more precise the correspondence (implied by Theorem 2.1) between nilpotent orbits and representations of the Weyl group.

We begin with a construction, using the results of [18], of some simple summands of $\mathcal{M}$. Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$ with adjoint group G_0. Let $x \in \mathfrak{g}_0$ be a nilpotent element and set $\mathbf{O}_x = G.x$. Define [18, §6] a distribution T_x on $\mathfrak{g}_0$ by the formula

$$T_x(f) = \int_{G_0.x} f d\mu_x, \quad f \in \mathcal{C}_c^\infty(\mathfrak{g}_0)$$

($d\mu_x$ being the canonical G_0-invariant measure on $G_0.x$). Then, T_x is G-invariant and $\varphi.T_x = 0$ if $\varphi \in \mathcal{O}(\mathfrak{g})$ vanishes on $\mathbf{O}_x$, thus $\operatorname{Supp} \mathcal{D}(\mathfrak{g}).T_x \subset \overline{\mathbf{O}_x}$. Notice that $\mathtt{E}_{\mathfrak{g}}.T_x = \lambda_{\mathbf{O}_x} T_x$, see [18, Lemma 6.4]. The next result should be compared to [6, Proposition 8.4.1].

Proposition 3.1. *There exists $\chi_x \in W^\wedge$ such that $\mathcal{D}(\mathfrak{g}).T_x \cong \mathcal{M}_{\chi_x}$. Furthermore, the highest eigenvalue of $\mathtt{E}_{\mathfrak{g}}$ on $\mathcal{M}_{\chi_x}^G$ is $\lambda_{\mathbf{O}_x}$, and* $\operatorname{Supp} \mathcal{D}(\mathfrak{g}).T_x = \overline{\mathbf{O}_x}$.

Proof. Set $M = \mathcal{D}(\mathfrak{g}).T_x$ and observe that M is a quotient of $\mathcal{M}$. By [18, Theorem 6.3] and Theorem 2.1, $M \cong \mathcal{M}_{\chi_x}$ for some $\chi_x \in W^\wedge$. Note also that, by [18, Theorem 6.3] again, $b(\chi_x) = \frac{1}{2}(\dim \mathfrak{g}^x - \ell)$. Thus the second assertion follows from Lemma 2.4. Let $\overline{\mathbf{O}}$ be the support of M. By Proposition 2.5, $\lambda_{\mathbf{O}_x} \leqslant \lambda_{\mathbf{O}}$; therefore $\overline{\mathbf{O}} \subset \overline{\mathbf{O}_x}$ yields the third assertion. □

Corollary 3.2. *Let $\mathfrak{g}_0$ be a (split) real form of $\mathfrak{g}$ and $x \in \mathfrak{g}_0 \cap \mathbf{O}^{reg}$. Then,*

$$\mathcal{M}_{\mathrm{triv}} \cong \mathcal{D}(\mathfrak{g}).T_x.$$

Proof. By Proposition 3.1, $\mathcal{D}(\mathfrak{g}).T_x \cong \mathcal{M}_{\chi_x}$. Since $\lambda_{\mathbf{O}^{reg}} = -\nu - \ell$, we obtain from the same proposition and Lemma 2.4 that $b(\chi_x) = 0$. Hence $\chi_x = \mathrm{triv}$. □

Remark. Let $x \in \mathfrak{g}_0$ be nilpotent. Denote by $\widehat{T_x}$ the Fourier transform of the distribution T_x. Let $\mathfrak{h}_0 \subset \mathfrak{g}_0$ be a real form of $\mathfrak{h}$. Then, by [18, Theorem 6.7], the restriction of $\widehat{T_x}$ to each connected component C of $\mathfrak{h}_0 \cap \mathfrak{g}'$ writes p_C/π for some polynomial $p_C \in \mathcal{H}^{b(\chi_x)}$. In particular, when x is regular, $b(\chi_x) = 0$ implies that $\widehat{T_x}_{|C} = p_C/\pi$ for some constant p_C. (Recall that $\zeta^{-\frac{1}{2}} = \pm 1/\sqrt{\pi^2}$ on each connected component of $\mathfrak{h}_u \cap \mathfrak{g}'$.)

We now recall how one can attach, in a natural way, to any nilpotent orbit $\mathbf{O}$ a representation $\chi_{\mathbf{O}}$ of W. This result is due to Hotta & Kashiwara [6, Corollary 7.1.5] but we include a proof for completeness. If $\mathbf{O} \subset \mathfrak{g}$ is a nilpotent orbit, we denote by $J(\mathbf{O}) \subset \mathcal{O}(\mathfrak{g})$ the prime ideal defining $\overline{\mathbf{O}}$.

Theorem 3.3. *Let* $\mathbf{O}$ *be a nilpotent orbit in* $\mathfrak{g}$.

(1) *There exists* $\chi_{\mathbf{O}} \in W^\wedge$ *such that*

$$\mathcal{M}_{\chi_{\mathbf{O}}} \cong \mathcal{D}(\mathfrak{g})/\big(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})S_+(\mathfrak{g}^*)^G + \mathcal{D}(\mathfrak{g})J(\mathbf{O})^k + \mathcal{D}(\mathfrak{g})(\mathtt{E}_{\mathfrak{g}} - \lambda_{\mathbf{O}})\big)$$

for all $k \in \mathbb{N}^*$.

(2) *Let* $\mathfrak{g}_0$ *be a real form of* $\mathfrak{g}$ *and* $x \in \mathfrak{g}_0$ *be a nilpotent element. Set* $\mathbf{O}_x = G.x$. *Then,* $\chi_x = \chi_{\mathbf{O}_x}$*, i.e.* $\mathcal{D}(\mathfrak{g}).T_x \cong \mathcal{M}_{\chi_{\mathbf{O}_x}}$.

Proof. Denote by M_k the $\mathcal{D}(\mathfrak{g})$-module on the right hand side of the isomorphism in (1). Observe that M_k is a quotient of M_{k+1} and

$$M_1 = \mathcal{D}(\mathfrak{g})/\big(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})J(\mathbf{O}) + \mathcal{D}(\mathfrak{g})(\mathtt{E}_{\mathfrak{g}} - \lambda_{\mathbf{O}})\big) \tag{3.1}$$

Assume that $M_k \neq 0$ and denote by v its canonical generator (the class of 1). Notice that $\mathtt{E}_{\mathfrak{g}}.v = \lambda_{\mathbf{O}}v$. It is clear that M_k is a quotient of $\mathcal{M}$ with support contained in $\overline{\mathbf{O}}$. Therefore, by Theorem 2.1, $M_k \cong \bigoplus_{\chi\in W^\wedge} \mathcal{M}_\chi^{(m_\chi)}$ for some integers m_χ. Recall that $\operatorname{Supp}\mathcal{M}_\chi = \overline{\mathbf{O}(\chi)}$ and that the eigenvalues of $\mathtt{E}_{\mathfrak{g}}$ on $\mathcal{M}_\chi^G$ are $\leqslant \lambda_{\mathbf{O}(\chi)}$, see Proposition 2.5(i). Suppose that $m_\chi \neq 0$. Then M_k surjects onto $\mathcal{M}_\chi$; therefore $\mathcal{M}_\chi = \mathcal{D}(\mathfrak{g}).\bar{v}$ with $\mathtt{E}_{\mathfrak{g}}.\bar{v} = \lambda_{\mathbf{O}}\bar{v}$. Hence $\lambda_{\mathbf{O}} \leqslant \lambda_{\mathbf{O}(\chi)}$ and, since $\overline{\mathbf{O}(\chi)} \subset \overline{\mathbf{O}}$, we obtain that $\mathbf{O}(\chi) = \mathbf{O}$. It follows that $\operatorname{Supp} M_k = \overline{\mathbf{O}}$. But, by Proposition 2.5(ii), the $\lambda_{\mathbf{O}}$-eigenspace of $\mathtt{E}_{\mathfrak{g}}$ in M_k^G has dimension 1, thus $M_k \cong \mathcal{M}_\chi$ for some $\chi = \chi_{\mathbf{O}}$.

To complete the proof of the theorem it suffices to show that, if $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$ such that $\mathbf{O} \cap \mathfrak{g}_0 \neq \emptyset$, then $\mathcal{D}(\mathfrak{g}).T_x \cong M_1$ for all $x \in \mathbf{O} \cap \mathfrak{g}_0$. Observe that

$$\tau(\mathfrak{g}).T_x = J(\mathbf{O}).T_x = (\mathtt{E}_{\mathfrak{g}} - \lambda_{\mathbf{O}}).T_x = 0.$$

Therefore, $\mathcal{D}(\mathfrak{g}).T_x$ is a quotient of M_1. This proves that $M_1 \neq 0$ and, by the first part of the proof, $\mathcal{D}(\mathfrak{g}).T_x \cong \mathcal{M}_\chi \cong M_k$ for all $k \in \mathbb{N}^*$. □

Remark. Set $L(\mathbf{O}) = \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})J(\mathbf{O}) + \mathcal{D}(\mathfrak{g})(\mathtt{E}_{\mathfrak{g}} - \lambda_{\mathbf{O}})$. From (3.1) we get that $\mathcal{M}_{\chi_{\mathbf{O}}} = \mathcal{D}(\mathfrak{g})/L(\mathbf{O})$; thus, $\mathcal{N}_{\chi_{\mathbf{O}}} = \mathcal{M}_{\chi_{\mathbf{O}}}^F = \mathcal{D}(\mathfrak{g})/F^{-1}(L(\mathbf{O}))$ with $F^{-1}(L(\mathbf{O})) = \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})F^{-1}(J(\mathbf{O})) + \mathcal{D}(\mathfrak{g})(F^{-1}(\mathtt{E}_{\mathfrak{g}}) - \lambda_{\mathbf{O}})$. By definition of F, the defining ideal of the closure of the (nilpotent) orbit $B(\mathbf{O}) \subset \mathfrak{g}^*$ is $I(\mathbf{O}) = F^{-1}(J(\mathbf{O}))$. Hence,

$$\mathcal{N}_{\chi_{\mathbf{O}}} = \mathcal{D}(\mathfrak{g})/\big(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})I(\mathbf{O}) + \mathcal{D}(\mathfrak{g})(\mathtt{E}_{\mathfrak{g}} + \tfrac{1}{2}\dim\mathbf{O})\big).$$

Setting $K(\mathbf{O}) = \delta\big((\mathcal{D}(\mathfrak{g})I(\mathbf{O}))^G\big)$ and $d_{\mathbf{O}} = \frac{1}{2}\operatorname{codim}_{\mathbf{N}(\mathfrak{g})}\overline{\mathbf{O}}$, we obtain that

$$\mathcal{N}_{\chi_{\mathbf{O}}}^G \cong \mathcal{D}(\mathfrak{h})^W/\big(K(\mathbf{O}) + \mathcal{D}(\mathfrak{h})^W(\mathtt{E}_{\mathfrak{h}} - d_{\mathbf{O}})\big)$$

as a $\mathcal{D}(\mathfrak{h})^W$-module. Therefore, the representation $\chi_{\mathbf{O}}$ is defined by

$$\begin{aligned} V_{\chi_{\mathbf{O}}} &\cong \operatorname{Sol}(\mathcal{N}_{\chi_{\mathbf{O}}}^G) \\ &= \{\varphi \in S^{d_{\mathbf{O}}}(\mathfrak{h}^*) : \forall D \in K(\mathbf{O}),\ D.\varphi = 0\} \\ &= \{\varphi \in \mathcal{H}^{d_{\mathbf{O}}} : \forall D \in K(\mathbf{O}),\ D.\varphi = 0\}. \end{aligned}$$

One can also observe that (in particular cases) the description of the left ideal $L(\mathbf{O})$ can be simpler than the one given above. For example, when $\mathbf{O} = \{0\}$, i.e. $\chi_{\mathbf{O}} = \mathrm{sgn}$, the previous definition gives $L(\{0\}) = \mathcal{D}(\mathfrak{g})\mathfrak{g}^*$ but we have seen in §2 that $L(\{0\}) = \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})S_+(\mathfrak{g}^*)^G + \mathcal{D}(\mathfrak{g})(\mathtt{E}_{\mathfrak{g}} + n)$.

We end this section by showing how the module $\mathcal{M}_{\chi_{\mathbf{O}}}$ is related to the trivial representation of the component group of $\mathbf{O}$ via the “Springer correspondence”.

We first need to recall a few facts from [12, §7]. Let $\chi \in W^\frown$ and consider the module $\mathcal{M}_\chi$. Set $\mathbf{O} = \mathbf{O}(\chi)$ and denote by $\imath : \mathbf{O} \hookrightarrow \mathfrak{g}$ the natural inclusion. Fix $x \in \mathbf{O}$ and set $A(\mathbf{O}) = G^x/G_0^x$, where G_0^x is the identity component of the stabilizer G^x of x in G. We denote by $\mathbf{m}_x \subset \mathcal{O}(\mathfrak{g})$ the maximal ideal associated to x. Then (see [4, 10.4 & 10.6]) $\mathcal{M}_\chi$, or equivalently χ, is uniquely determined by the coherent connection $\mathcal{L}_\chi = \imath^! \mathcal{M}_\chi$. This connection is, in turn, defined by an irreducible representation $\psi \in A(\mathbf{O})^\frown$ on the geometric fibre $\mathcal{L}_\chi(x) = (\mathcal{L}_\chi)_x/\mathbf{m}_x(\mathcal{L}_\chi)_x$ [7, Proposition 4.11.1]. We express these facts by setting $\chi = \sigma(\mathbf{O}, \psi)$. Thus, we have a correspondence $(\mathbf{O}, \psi) \to \sigma(\mathbf{O}, \psi)$ from a subset of $\{\text{nilpotent orbits } \mathbf{O}\} \times \{\psi \in A(\mathbf{O})^\frown\}$ onto $W^\frown$. This correspondence is related to the Springer correspondence, see [6, 18]. The results of Appendix A show how the representation $\psi \in A(\mathbf{O})^\frown$ can be computed. We summarize below some of these results.

When $\mathbf{O} = \{0\}$, $A(\mathbf{O})$ is trivial and we have seen in §2 that $\mathrm{sgn} = \sigma(\{0\}, \mathrm{triv})$. In the rest of this section we assume that $x \neq 0$. Then, there exists $y \in \mathbf{N}(\mathfrak{g})$, $z \in \mathfrak{g}$ such that $[z, x] = 2x$, $[z, y] = -2y$, $[x, y] = z$. Let $G^\phi = \{g \in G : g.x = x, g.y = y\}$ be the centralizer of $\mathbb{C}x + \mathbb{C}y + \mathbb{C}z \cong \mathfrak{sl}(2, \mathbb{C})$ and let G_0^ϕ be its identity component. By [2, Proposition 2.4], $A(\mathbf{O}) = G^\phi/G_0^\phi$ and G^ϕ is reductive. Note that we have the G^ϕ-stable decomposition $\mathfrak{g} = \mathfrak{g}^y \oplus [\mathfrak{g}, x]$. Let $\{y_1, \dots, y_m\}$ and $\{y_{m+1}, \dots, y_n\}$ be coordinate functions on $\mathfrak{g}^y$ and $[\mathfrak{g}, x]$ respectively. Define G^ϕ-stable ideals of $\mathcal{O}(\mathfrak{g})$ by $J_x = (y_{m+1} - y_{m+1}(x), \dots, y_n - y_n(x))$ and $\mathbf{n}_x = (y_1 - y_1(x), \dots, y_m - y_m(x))$. Set:

$$T = \left\{ \overline{v} \in \mathcal{M}_\chi/J_x\mathcal{M}_\chi : v \in \mathcal{M}_\chi \text{ and } \mathbf{n}_x.\overline{v} = 0 \right\}$$

Observe that G^ϕ acts naturally on T by $g.\overline{v} = \overline{g.v}$.

Theorem 3.4. (1) *The natural action of G^ϕ on T yields the irreducible representation $\psi : A(\mathbf{O}) \to \mathrm{GL}(T)$.*

(2) *Write $\mathcal{M}_\chi = \mathcal{D}(\mathfrak{g}).v$ with $0 \neq v \in \mathcal{M}_\chi^G$. Assume that $\mathtt{E}_\mathfrak{g}.v = \lambda_\mathbf{O} v$. Then,*

$$T = \mathbb{C}\,\overline{v} \quad \textit{and} \quad \chi = \chi_\mathbf{O} = \sigma(\mathbf{O}, \mathrm{triv}).$$

Proof. (1) follows from Theorem A.4.

(2) By Corollary A.5 and the proof of [13, Lemma 3.4(i)] we obtain that $\mathbf{n}_x.\overline{v} = 0$ and $T = \mathbb{C}\,\overline{v}$. Since $\tau(\mathfrak{g}).v = 0$, $\mathtt{E}_\mathfrak{g}.v = \lambda_\mathbf{O} v$ and $\mathrm{Supp}\,\mathcal{M}_\chi = \overline{\mathbf{O}}$, it follows from Theorem 3.3 that $\mathcal{M}_\chi = \mathcal{M}_{\chi_\mathbf{O}}$. Thus $\chi = \chi_\mathbf{O}$ and $\chi_\mathbf{O} = \sigma(\mathbf{O}, \mathrm{triv})$ is consequence of (1) and $v \in \mathcal{M}_\chi^G$. ☐

Remark. As pointed out by a referee and M. Van den Bergh, Theorem 3.4(2) can be proved in the following alternative way. Let Y be a non empty affine open subset of $\mathbf{O}$ and let $\imath : Y \hookrightarrow \mathfrak{g}$ be the inclusion. We aim to show that $\mathcal{M}_{\chi_\mathbf{O}}$ is isomorphic to the unique irreducible submodule of $\imath_+(\mathcal{O}_Y)$, see [4, VII.10.5, VII.10.6]. Set $X = \mathfrak{g} \setminus (\overline{Y} \setminus Y)$. Then $\imath' : Y \hookrightarrow X$ is a closed embedding, $\jmath : X \hookrightarrow \mathfrak{g}$ is open and $\imath_+(\mathcal{O}_Y) = \jmath_+(\imath'_+(\mathcal{O}_Y))$. Recall [4, VI.7.1, VI.7.8] that $\imath'_+(\mathcal{O}_Y)$ contains $\omega_{Y/X} := \omega_Y \otimes_{\mathcal{O}_X} \omega_X^{-1}$. The orbit $\mathbf{O}$ carries a G-invariant symplectic form $\beta_\mathbf{O}$, thus $\beta_Y^{\frac{1}{2}\dim \mathbf{O}}$ is a non zero section of ω_Y. On the other hand, since $\mathfrak{g}$ is unimodular, $\omega_\mathfrak{g}$ is generated by a G-invariant form dx. Hence,

$$\gamma = \beta_Y^{\frac{1}{2}\dim \mathbf{O}} \otimes dx_{|X}^{-1}$$

gives a non zero section of $\omega_{Y/X}$. It is not difficult to see that $\gamma \in \Gamma(X, \imath'_+(\mathcal{O}_Y))$ extends to a section of $\imath_+(\mathcal{O}_Y)$ and satisfies $\tau(\mathfrak{g}).\gamma = (\mathtt{E}_\mathfrak{g} - \lambda_\mathbf{O}).\gamma = J(\mathbf{O}).\gamma = 0$.

It follows then from Theorem 3.3 that $\mathcal{M}_{\chi_{\mathbf{O}}} \cong \mathcal{D}(\mathfrak{g}).\gamma$ is the socle of the $\mathcal{D}(\mathfrak{g})$-module $\imath_+(\mathcal{O}_Y)$.

Appendix A

The material of this section is taken from [10]. A lot of results stated here are probably well known but do not appear in the literature in the form we need. The proofs are very often straightforward (but lengthy) verifications and we will sometimes only give a sketch of them.

Let G be a linear algebraic group and $\mathfrak{g} = \mathrm{Lie}(G)$. We first recall a few definitions about G-equivariant D-modules, see [3, 4, 7, 8, 17]. All the varieties considered are quasi-projective algebraic varieties defined over $\mathbb{C}$. If X is a smooth algebraic variety, $\mathcal{O}_X$ is the sheaf of regular functions on X and $\mathcal{D}_X$ the sheaf of differential operators. We refer to [4] for the basic properties of $\mathcal{D}_X$-modules; all the $\mathcal{D}_X$-modules encountered in the sequel will be quasi-coherent and the category of quasi-coherent $\mathcal{D}_X$-modules is denoted by $\mathrm{Mod}\,\mathcal{D}_X$. Suppose that $Z = X \times Y$ is the product of two varieties. Let M be an $\mathcal{O}_X$-module and N be an $\mathcal{O}_Y$-module, we will write $M \boxtimes N$ for the $\mathcal{O}_Z$-module $M \otimes_{\mathbb{C}} N$.

Let $F : Y \to X$ be a morphism between smooth varieties. The comorphism of F is denoted by $F^\#$ and the inverse image of an $\mathcal{O}_X$-module M by $F^*M = \mathcal{O}_Y \otimes_{F^\#} M$. When X and Y are affine, there exists a natural map $F^\# : M \to F^*M$, $v \mapsto F^\#(v) = 1_Y \otimes_{F^\#} v$. Recall that if $M \in \mathrm{Mod}\,\mathcal{D}_X$ one defines the inverse image of M by setting

$$F^!M = \big(\mathcal{D}_{Y\to X} \otimes^L_{\mathcal{D}_X} M\big)[d_{Y,X}]$$

where $\mathcal{D}_{Y\to X} = F^*\mathcal{D}_X = \mathcal{O}_Y \otimes_{F^\#} \mathcal{D}_X$ and $d_{Y,X} = \dim Y - \dim X$. The inverse image is an object of the derived category $\mathrm{D}^b(\mathcal{D}_Y)$ of $\mathrm{Mod}\,\mathcal{D}_Y$, and the construction of $F^!$ extends to $\mathrm{D}^b(\mathcal{D}_X)$. When F is smooth and $M \in \mathrm{Mod}\,\mathcal{D}_X$, $F^!M$ is a complex concentrated in degree $-d_{Y,X}$ and $\mathcal{H}^{-d_{Y,X}}(F^!M)$ coincides, as an $\mathcal{O}_Y$-module, with F^*M.

Assume that X is a smooth G-variety, define: $\mu : G \times X \to X$, $(g, v) \mapsto g.v$ (the action of G on X), $\mu_G : G \times G \to G$, $\mu_G(g, h) = gh$ (the multiplication in G), $\varepsilon_X : X \hookrightarrow G \times X$, $\varepsilon_X(v) = (e, v)$, $p_2 : G \times X \to X$, $p_2(g, v) = v$, $p_{23} : G \times G \times X \to G \times X$, $p_{23}(g, h, v) = (h, v)$.

Let $M \in \mathrm{Mod}\,\mathcal{D}_X$; then M is a G-equivariant $\mathcal{D}_X$-module if there exists an isomorphism of $\mathcal{D}_{G\times X}$-modules

$$\theta : p_2^*M = \mathcal{O}_G \boxtimes M \xrightarrow{\sim} \mu^*M \tag{A.1}$$

such that $\varepsilon_X^*(\theta) = 1_M$, $(\mu_G \times 1_X)^*(\theta) = (1_G \times \mu)^*(\theta) \circ p_{23}^*(\theta)$. One can then construct the category of G-equivariant $\mathcal{D}_X$-modules, which we denote by $\mathfrak{M}(\mathcal{D}_X, G)$. Notice that if $F : Y \to X$ is a G-equivariant morphism between smooth G-varieties and $M \in \mathfrak{M}(\mathcal{D}_X, G)$, the $\mathcal{D}_Y$-modules $\mathcal{H}^j(F^!M)$ have a natural structure of G-equivariant $\mathcal{D}_Y$-modules (see [17, §2]).

Remark. Suppose that X is affine. The differential of the G-action on X yields a Lie algebra map, τ_X, from $\mathfrak{g}$ to the Lie algebra of vector fields on X, defined by $(\tau_X(\xi).\varphi)(x) = \frac{d}{dt}_{|t=0}\varphi(\exp(-t\xi).x)$, $\xi \in \mathfrak{g}$, $\varphi \in \mathcal{O}(X)$, $x \in X$. Then, the $\mathcal{D}(X)$-module M is G-equivariant if, and only if, M has a rational G-module structure such that $g.(D.v) = (g.D).(g.v)$ and $\tau_X(\xi).v = \xi.v$, for all $g \in G$, $D \in \mathcal{D}(X)$, $\xi \in \mathfrak{g}$, $v \in M$, where $\xi.v = \frac{d}{dt}_{|t=0}(\exp(t\xi).v)$. (See [17, Proposition 2.6] for a proof.)

Now, let $G, \mathfrak{g}$ be as in §1 and consider the adjoint action of G on $X = \mathfrak{g}$. Let $x \in \mathbf{N}(\mathfrak{g})$ and set $\mathbf{O} = G.x$. Let M be a G-equivariant coherent $\mathcal{D}_{\mathfrak{g}}$-module such that $\operatorname{Supp} M = \overline{\mathbf{O}}$. If $x = 0$, it follows from Kashiwara's equivalence [4, VI.7.11] that M is isomorphic to $H^n_{[0]}(\mathcal{O}_{\mathfrak{g}})^{\oplus k}$ for some $k \in \mathbb{N}^*$, where $H^n_{[0]}(\mathcal{O}_{\mathfrak{g}}) \cong \mathcal{D}_{\mathfrak{g}}/\mathcal{D}_{\mathfrak{g}}\mathfrak{g}^*$. Therefore, we will suppose from now on that $x \neq 0$. Then we can find an S-triplet $\{x, y, z\}$ containing x, i.e. $[x, y] = z$, $[z, x] = 2x$, $[z, y] = -2y$ and $\mathfrak{s} = \mathbb{C}x + \mathbb{C}y + \mathbb{C}z \cong \mathfrak{sl}(2, \mathbb{C})$. Recall that $\mathfrak{g}^y = \{\xi \in \mathfrak{g} : [\xi, y] = 0\}$ and set $n = \dim \mathfrak{g}$, $m = \dim \mathfrak{g}^y$, thus $\dim \mathbf{O} = n - m$. In this situation it is well known [15, III.5.1, III.7.4] that $x + \mathfrak{g}^y$ is a transverse slice to $\mathbf{O}$ at the point x. Let $\mu_x : G \times \mathfrak{g}^y \to \mathfrak{g}$ be given by $\mu_x(g, u) = g.(x + u)$. The following lemma is classical (see, for example, [13, Proposition 2.3]).

Lemma A.1. *There exists an affine open neighborhood U of 0 in $\mathfrak{g}^y$ such that:*

1. *$\psi = \mu_{x|Y}$ is smooth on $Y = G \times U$, $\Omega = \psi(Y) = G.(x + U)$ is a G-stable open subset of $\mathfrak{g}$;*
2. *$\Omega \cap \overline{\mathbf{O}} = \mathbf{O}$ and $\mathbf{O} \cap \{x + U\} = \{x\}$.*

Observe that, since ψ is smooth on Y, we have

$$\mathcal{H}^j(\psi^! M) = \begin{cases} 0 & \text{if } j \neq -m, \\ \psi^* M & \text{if } j = -m. \end{cases}$$

Furthermore, it follows from $\Omega \supset \mathbf{O}$ and the flatness of ψ that $\psi^* M \neq 0$.

Denote by $t_x : \mathfrak{g} \to \mathfrak{g}$ the translation by x, $t_x(v) = x + v$, and set $X = t_x(U) = x + U$. Let $\beta : U \hookrightarrow \mathfrak{g}$, $\beta_x : X \hookrightarrow \mathfrak{g}$, $\jmath_x : X \hookrightarrow x + \mathfrak{g}^y$ be the canonical inclusions (note that $\beta_x \circ t_x = t_x \circ \beta$).

Recall that $[\mathfrak{g}, x]$ is a complementary subspace to $\mathfrak{g}^y$. Let $\{v_1, \dots, v_m\}$ be a basis of $\mathfrak{g}^y$ and $\{v_{m+1}, \dots, v_n\}$ be a basis of $[\mathfrak{g}, x]$. Denote by $\{y_j = v_j^*\}_j$ the dual basis and set $f_j = t_x.y_j = y_j - y_j(x)$, $1 \leqslant j \leqslant n$. Then, $\mathcal{O}(x + \mathfrak{g}^y) = \mathcal{O}(\mathfrak{g})/J_x$ where $J_x = (f_{m+1}, \dots, f_n)\mathcal{O}(\mathfrak{g})$. Let $\mathbf{m}_x = (f_1, \dots, f_n)\mathcal{O}(\mathfrak{g})$ and $\mathbf{n}_x = (f_1, \dots, f_m)\mathcal{O}(\mathfrak{g}^y)$ be the maximal ideals asociated to $x \in \mathfrak{g}$ and $x \in (x + \mathfrak{g}^y)$ (respectively).

Theorem A.2. (1) *There exists an isomorphism of G-equivariant $\mathcal{D}_Y$-modules $\psi^! M \cong (\mathcal{O}(G) \boxtimes \beta^! t_x^* M)[n]$ (where G acts by left translation on $\mathcal{O}(G)$). Moreover,*

$$\mathcal{H}^j(\beta^! t_x^! M) = \begin{cases} 0 & \text{if } j \neq \dim \mathbf{O}, \\ \beta^* t_x^* M & \text{if } j = \dim \mathbf{O}. \end{cases}$$

(2) *Set $M_{|X} = \beta_x^* M$. Then, $\beta_x^! M = M_{|X}[-\dim \mathbf{O}]$ and $M_{|X} = \beta_x^*(M/J_x M)$ is a $\mathcal{D}_X$-module isomorphic to $H^m_{[x]}(\mathcal{O}_X)^{\oplus k}$ for some $k \geqslant 1$.*

(3) *One has $\beta^* t_x^* M \cong H^m_{[0]}(\mathcal{O}_U)^{\oplus k}$ and $\psi^! M \cong \big(\mathcal{O}_G \boxtimes H^m_{[0]}(\mathcal{O}_U)^{\oplus k}\big)[m]$.*

Proof. (1) Notice first that, since t_x is an isomorphism, we can identify $t_x^! M$ with $t_x^* M$. From $\psi = \mu_x \circ (1_G \times \beta)$ we deduce that $\psi^! M = (1_G \times \beta)^! \mu_x^! M$. Therefore, using the G-equivariant morphisms $1_G \times \beta$ and $1_G \times t_x$, we obtain

$$(1_G \times \beta)^! (1_G \times t_x)^! (\theta) : (1_G \times \beta)^! p_2^! t_x^* M \xrightarrow{\sim} \psi^! M$$

(θ being as in (A.1)). Then, $(1_G \times \beta)^! p_2^! t_x^* M = (1_G \times \beta)^! (\mathcal{O}_G \boxtimes t_x^* M)[n] = (\mathcal{O}_G \boxtimes \beta^! t_x^* M)[n]$ yields $\psi^! M \cong (\mathcal{O}_G \boxtimes \beta^! t_x^* M)[n]$. From this isomorphism one deduces that

$\mathcal{H}^{j-n}(\psi^! M) \cong \mathcal{O}_G \boxtimes \mathcal{H}^j(\beta^! t_x^* M)$. Therefore, $\mathcal{H}^j(\beta^! t_x^* M) = 0$ unless $j = \dim \mathbf{O}$, and

$$\mathcal{H}^{\dim \mathbf{O}}(\beta^! t_x^* M) = \mathcal{H}^{\dim \mathbf{O}}\big((\mathcal{O}_U \otimes^L_{\beta^\#} t_x^* M)[\dim \mathbf{O}]\big) = \mathcal{O}_U \otimes_{\beta^\#} t_x^* M,$$

as required.

(2) Let $\tilde{\beta}_x : x + \mathfrak{g}^y \hookrightarrow \mathfrak{g}$ be the inclusion. Thus $\beta_x = \tilde{\beta}_x \circ \jmath_x$. Since $\jmath_x$ is an open immersion, $\mathcal{H}^j(\beta_x^! M) = \mathcal{O}_X \otimes_{\jmath_x^\#} \mathcal{H}^j(\tilde{\beta}_x^! M)$ for all j. Recall that $\tilde{\beta}_x^! M = \big(\mathcal{D}_{(x+\mathfrak{g}^y)\to\mathfrak{g}} \otimes^L_{\mathcal{D}_\mathfrak{g}} M\big)[m-n]$. Set $V_x = \oplus_{i=m+1}^n \mathbb{C} df_i$ and $\mathcal{C}_x^{-p} = \bigwedge^p V_x \boxtimes \mathcal{D}_\mathfrak{g}$. By [4, VI.7.4], $\tilde{\beta}_x^! M = \big(\mathcal{C}_x^\bullet \otimes_{\mathcal{D}_\mathfrak{g}} M, \partial_x\big)[m-n]$, with $\mathcal{C}_x^p \otimes_{\mathcal{D}_\mathfrak{g}} M = \bigwedge^{-p} V_x \boxtimes M$ and

$$\partial_x(df_{j_1} \wedge \cdots \wedge df_{j_p} \boxtimes v) = \textstyle\sum_{a=1}^p (-1)^{a+1} df_{j_1} \wedge \cdots \wedge \widehat{df_{j_a}} \wedge \cdots \wedge df_{j_p} \boxtimes f_{j_a} v.$$

It follows that $\mathcal{H}^{n-m}(\tilde{\beta}_x^! M) = \mathcal{H}^0\big(\mathcal{C}_x^\bullet \otimes_{\mathcal{D}_\mathfrak{g}} M, \partial_x\big) = M/J_x M = \mathcal{O}_{x+\mathfrak{g}^y} \otimes_{\tilde{\beta}_x^\#} M$. Hence, $M_{|X}$ identifies with the $\mathcal{D}_X$-module $\mathcal{H}^{\dim \mathbf{O}}(\beta_x^! M)$.

Now, the support of $M_{|X} = \mathcal{O}_X \otimes_{\beta_x^\#} M$ is clearly contained in $X \cap \operatorname{Supp} M \subset X \cap \overline{\mathbf{O}}$. But, $X \cap \overline{\mathbf{O}} \subset X \cap \Omega \cap \overline{\mathbf{O}} = X \cap \mathbf{O} = \{x\}$. Thus $M_{|X}$ is a $\mathcal{D}_X$-module whose support is contained in $\{x\}$; therefore, by Kashiwara's equivalence [4, VI.7.11], $M_{|X} \cong H^m_{[x]}(\mathcal{O}_X)^{\oplus k}$ for some k. Notice that it follows easily from $\beta_x \circ t_x = t_x \circ \beta$ (on U) that $t_x^*\big(M_{|X}\big) = \beta^* t_x^* M$. Since $\psi^* M \neq 0$, (1) implies that $k \neq 0$.

(3) is an easy consequence of (1) and (2). ☐

In the sequel, thanks to Theorem A.2(3), we will currently identify $\psi^! M$ with $\big(\mathcal{O}_G \boxtimes H^m_{[0]}(\mathcal{O}_U)^{\oplus k}\big)[m]$.

We factorize the inclusion $\imath : \{x\} \hookrightarrow \mathfrak{g}$ as the composition $\{x\} \overset{\imath_1}{\hookrightarrow} \mathbf{O} \overset{\imath_2}{\hookrightarrow} \mathfrak{g}$ and we set $M_{|\mathbf{O}} = \mathcal{H}^0(\imath_2^! M)$.

Lemma A.3. *The cohomology of $\imath_2^! M$ is concentrated in degree 0.*

Proof. Notice first that $\psi \circ \jmath_2 = \imath_2 \circ \pi$, where $\pi : G \times \{0\} \to \mathbf{O}$, $\pi(g, 0) = g.x$, and $\jmath_2 : G \times \{0\} \hookrightarrow Y$ is the natural inclusion. From $\psi^* M = \mathcal{O}_G \boxtimes H^m_{[0]}(\mathcal{O}_U)^{\oplus k}$ we get that $\psi^* M = \mathcal{H}^{-m}(\psi^! M)$ is supported on $G \times \{0\}$. Since $\jmath_2$ is a closed embedding, [4, VI.7.4] gives that $\jmath_2^! \psi^* M$ has cohomology concentrated in degree 0; equivalently, $\jmath_2^! \psi^! M$ has cohomology concentrated in degree $-m$. On the other hand,

$$\mathcal{H}^j(\pi^! \imath_2^! M) = \mathcal{H}^j\big((\mathcal{D}_{G\times\{0\}\to\mathbf{O}} \otimes^L_{\mathcal{D}_\mathbf{O}} \imath_2^! M)[m]\big) = \mathcal{H}^{j+m}\big(\mathcal{D}_{G\times\{0\}\to\mathbf{O}} \otimes^L_{\mathcal{D}_\mathbf{O}} \imath_2^! M\big).$$

From π smooth and $\mathcal{D}_{G\times\{0\}\to\mathbf{O}} = \pi^* \mathcal{D}_\mathbf{O}$ it follows that, as an $\mathcal{O}_{G\times\{0\}}$-module, $\mathcal{H}^j(\pi^! \imath_2^! M) = \mathcal{O}_{G\times\{0\}} \otimes_{\pi^\#} \mathcal{H}^{j+m}(\imath_2^! M)$. Now, since π is faithfully flat and $\pi^! \imath_2^! M = \jmath_2^! \psi^! M$, we deduce that $\imath_2^! M$ has cohomology concentrated in degree 0. ☐

Since $\imath_2 : \mathbf{O} \hookrightarrow \mathfrak{g}$ is G-equivariant, $\imath_2^! M = M_{|\mathbf{O}}$ (Lemma A.3) is in $\mathfrak{M}(\mathcal{D}_\mathbf{O}, G)$. Therefore, see [7, Proposition 4.11.1] and [8, §4], $M_{|\mathbf{O}}$ is the (integrable) connection on $\mathbf{O}$ associated to the representation of $A(\mathbf{O})$ on the geometric fibre $M_{|\mathbf{O}}(x) = (M_{|\mathbf{O}})_x / \mathbf{m}_x (M_{|\mathbf{O}})_x$. In particular, $M_{|\mathbf{O}}$ is flat as an $\mathcal{O}_\mathbf{O}$-module and it follows that

$$\imath^! M = \imath_1^! \imath_2^! M = \imath_1^! M_{|\mathbf{O}} = \big(\mathcal{D}_{\{x\}\to\mathbf{O}} \otimes^L_{\mathcal{D}_\mathbf{O}} M_{|\mathbf{O}}\big)[m-n]$$

has cohomology concentrated in degree $\dim \mathbf{O}$ with $\mathcal{H}^{\dim \mathbf{O}}(\imath^! M) = M_{|\mathbf{O}}(x)$.

Let $G^\phi = \{g \in G : g.a = a \text{ for all } a \in \mathfrak{s}\}$ be the centralizer of the Lie subalgebra $\mathfrak{s}$ and denote by G_0^ϕ its identity component. Recall that G^ϕ is reductive and that the map $G^\phi \hookrightarrow G^x$ induces the identification $A(\mathbf{O}) = G^\phi/G^\phi_0$ [2, Proposition 2.4]. Observe that the subspaces

$$\bigoplus_{i=1}^m \mathbb{C}y_i,\quad \bigoplus_{i=m+1}^n \mathbb{C}y_i,\quad \bigoplus_{i=1}^m \mathbb{C}f_i,\quad \bigoplus_{i=m+1}^n \mathbb{C}f_i$$

are G^ϕ-stable. Set

$$\omega^{-1} = df_1 \wedge \cdots \wedge df_m \ \text{ and } \ T = \{\bar{u} \in M/J_xM : \mathbf{n}_x.\bar{u} = 0\}.$$

It is easily shown that $\omega^{-1} = dy_1 \wedge \cdots \wedge dy_m$ is G^ϕ-invariant. Furthermore, G^ϕ acts naturally on M/J_xM and T. Therefore, $\mathbb{C}\,\omega^{-1} \otimes_{\mathbb{C}} T$ carries a representation of G^ϕ isomorphic to T. A consequence of the next result is that G^ϕ_0 acts trivially on T (it is not difficult to prove this fact directly).

Theorem A.4. *The $\mathcal{D}_{\mathbf{O}}$-module $M_{|\mathbf{O}}$ is the connection on $\mathbf{O}$ defined by representation $\mathcal{H}^{\dim \mathbf{O}}(\imath^! M) = \omega^{-1} \otimes_{\mathbb{C}} T$ of the group $A(\mathbf{O})$. Furthermore, $T \cong \big\{v \in H^m_{[x]}(\mathcal{O}_X)^{\oplus k} : \mathbf{n}_x v = 0\big\}$ is a $\mathbb{C}$-vector space of dimension k.*

Proof. Write $\imath = \tilde{\beta}_x \circ \gamma$, where $\gamma : \{x\} \hookrightarrow x + \mathfrak{g}^y$ and $\tilde{\beta}_x : x + \mathfrak{g}^y \hookrightarrow \mathfrak{g}$. Since the maps γ and $\tilde{\beta}_x$ are G^ϕ-equivariant, we have a natural G^ϕ-action on $M_{|\mathbf{O}}(x) = \mathcal{H}^{\dim \mathbf{O}}(\imath^! M)$ which yields the representation of the group $A(\mathbf{O})$ that we want to compute.

Set $V_x = \oplus_{i=m+1}^n \mathbb{C}df_i = \oplus_{i=m+1}^n \mathbb{C}dy_i$; then V_x is a G^ϕ-stable subspace of $\mathfrak{g}^*$. We have seen in the proof of Theorem A.2 that (setting $\partial_{II} = \partial_x$),

$$\tilde{\beta}_x^! M = \big(\mathcal{C}^\bullet_{II} = \mathcal{C}^\bullet_x \otimes_{\mathcal{D}_\mathfrak{g}} M, \partial_{II}\big)[m-n].$$

Let G^ϕ act diagonally on $\mathcal{C}^p_{II} = \mathcal{C}^{p+m-n}_x \otimes_{\mathcal{D}_\mathfrak{g}} M = \bigwedge^{-p+\dim \mathbf{O}} V_x \boxtimes M$. Then, ∂_{II} is G^ϕ-equivariant and the G^ϕ-action on the cohomology group $\mathcal{H}^j(\tilde{\beta}_x^! M) \in \mathfrak{M}(\mathcal{D}_{x+\mathfrak{g}^y}, G^\phi)$ is induced by the diagonal action of G^ϕ on $\mathcal{C}^\bullet_{II}$. Notice, in particular, that $\mathcal{H}^{\dim \mathbf{O}}(\tilde{\beta}_x^! M) = \mathrm{Tor}_0^{\mathcal{O}_\mathfrak{g}}(\mathcal{O}_{x+\mathfrak{g}^y}, M) = M/J_xM$ is endowed with the natural action of G^ϕ. By Theorem A.2(1), $\mathcal{H}^j(\beta_x^! M) = \mathcal{O}_X \otimes_{\mathcal{O}_{x+\mathfrak{g}^y}} \mathcal{H}^j(\tilde{\beta}_x^! M) = 0$ when $j \neq \dim \mathbf{O}$. Thus,

$$\operatorname{Supp} \mathcal{H}^j(\tilde{\beta}_x^! M) \subset (x + \mathfrak{g}^y) \setminus X \subset (x + \mathfrak{g}^y) \setminus \{x\} \ \text{ if } j \neq \dim \mathbf{O}.$$

Now, $\imath^! M = \gamma^! \tilde{\beta}_x^! M = \big(\mathcal{D}_{\{x\} \to x+\mathfrak{g}^y} \otimes^L_{\mathcal{D}_{x+\mathfrak{g}^y}} \tilde{\beta}_x^! M\big)[-m]$ can be computed as follows. Notice that $\mathbf{n}_x/\mathbf{n}_x^2 = \oplus_{j=1}^m \mathbb{C}df_j$ and consider the complex $\big(\mathcal{C}^\bullet_I, \partial_I\big)$ where $\mathcal{C}^p_I = \bigwedge^{-p}(\mathbf{n}_x/\mathbf{n}_x^2) \boxtimes \mathcal{D}_{x+\mathfrak{g}^y}$ and

$$\partial_I(df_{j_1} \wedge \cdots \wedge df_{j_p} \boxtimes D) = \textstyle\sum_{s=1}^p (-1)^{s+1} df_{j_1} \wedge \cdots \wedge \widehat{df_{j_s}} \wedge \cdots \wedge df_{j_p} \boxtimes f_{j_s} D.$$

Observe that G^ϕ acts diagonally on $\mathcal{C}^p_I$ and that ∂_I is G^ϕ-equivariant. Let $\big(\mathcal{C}^\bullet_{\mathrm{tot}}, \partial_{\mathrm{tot}}\big)$ be the total complex associated to the double complex $\mathcal{C}^\bullet = \mathcal{C}^\bullet_I \otimes_{\mathcal{D}_{x+\mathfrak{g}^y}} \mathcal{C}^\bullet_{II}$. Then,

$$\imath^! M = \gamma^! \tilde{\beta}_x^! M = \big(\mathcal{C}^\bullet_{\mathrm{tot}}, \partial_{\mathrm{tot}}\big)[-m]$$

and therefore $\mathcal{H}^j(\imath^! M) = \mathcal{H}^{j-m}(\mathcal{C}^\bullet_{\mathrm{tot}})$. This group is computed by the spectral sequence $E_2^{pq} = \mathcal{H}^p_I(\mathcal{H}^q_{II}(\mathcal{C}^\bullet)) \Longrightarrow \mathcal{H}^{p+q}(\mathcal{C}^\bullet_{\mathrm{tot}})$. But, $E_2^{pq} = \mathrm{Tor}_{-p}^{\mathcal{O}_{x+\mathfrak{g}^y}}(\mathbb{C}_x, \mathcal{H}^q(\mathcal{C}_{II}))$ as $\mathcal{O}_{x+\mathfrak{g}^y}$-module, where $\mathbb{C}_x = \mathcal{O}(\mathfrak{g})/\mathbf{m}_x\mathcal{O}(\mathfrak{g})$, and we have noticed that the support of $\mathcal{H}^q(\mathcal{C}_{II}) = \mathcal{H}^q(\tilde{\beta}_x^! M)$ is contained in $(x + \mathfrak{g}^y) \setminus \{x\}$ when $q \neq \dim \mathbf{O}$. Therefore $E_2^{pq} = 0$ for all $q \neq \dim \mathbf{O}$ and $E_2^{p\,\dim \mathbf{O}} = \mathrm{Tor}_{-p}^{\mathcal{O}_{x+\mathfrak{g}^y}}(\mathbb{C}_x, \mathcal{H}^{\dim \mathbf{O}}(\tilde{\beta}_x^! M))$. Hence, the spectral sequence E_2^{pq} collapses to $E_2^{p\dim\mathbf{O}} = \mathcal{H}^{p+\dim\mathbf{O}}(\mathcal{C}^\bullet_{\mathrm{tot}})$. In particular, we obtain

$$\begin{aligned}\mathcal{H}^{\dim\mathbf{O}}(\imath^! M) = \mathcal{H}^{\dim\mathbf{O}-m}(\mathcal{C}^\bullet_{\mathrm{tot}}) &= \mathrm{Tor}_m^{\mathcal{O}_{x+\mathfrak{g}^y}}(\mathbb{C}_x, \mathcal{H}^{\dim\mathbf{O}}(\tilde{\beta}_x^! M)) \\ &= \mathrm{Tor}_m^{\mathcal{O}_{x+\mathfrak{g}^y}}(\mathbb{C}_x, M/J_xM) = \omega^{-1}\otimes_{\mathbb{C}} T.\end{aligned}$$

Furthermore, the group G^ϕ acts diagonally on the complexes $\mathcal{C}^\bullet$, $\mathcal{C}_{\mathrm{tot}}$ and it follows from the previous computation that the action of $A(\mathbf{O})$ on $\mathcal{H}^{\dim\mathbf{O}}(\imath^! M)$ is coming from the induced action of G^ϕ on $E_2^{-m\dim\mathbf{O}} = \omega^{-1}\otimes_{\mathbb{C}} T$.

To finish the proof it suffices, since $M_{|X} = \mathcal{O}_X \otimes_{\jmath_x^\#} M/J_xM \cong H^m_{[x]}(\mathcal{O}_X)^{\oplus k}$ by Theorem A.2, to apply the following standard result to the module $N = M/J_xM$: Let N be any $\mathcal{O}_{x+\mathfrak{g}^y}$-module; then, if $N' = \mathcal{O}_X \otimes_{\jmath_x^\#} N$, one has

$$\{u \in N : \mathbf{n}_x.u = 0\} \xrightarrow{\sim} \{u' \in N' : \mathbf{n}_x.u' = 0\}$$

through the natural map $\jmath_x^\# : N \to N'$. □

Recall that we identify $\psi^* M$ with $\mathcal{O}_G \boxtimes H^m_{[0]}(\mathcal{O}_U)^{\oplus k} \in \mathfrak{M}(\mathcal{D}_Y, G)$, where G acts by left translation on $\mathcal{O}_G$; thus $(\psi^* M)^G = H^m_{[0]}(\mathcal{O}_U)^{\oplus k}$. By Theorem A.2 we can identify $M_{|X}$ with $H^m_{[x]}(\mathcal{O}_X)^{\oplus k}$ and, by Theorem A.4, the $A(\mathbf{O})$-module $M_{|\mathbf{O}}(x)$ identifies with T.

Let $v \in M^G$ and denote by $\bar{v}$ the class of v in M/J_xM. Then, $\psi^\#(v) \in (\psi^* M)^G$ and we write $\psi^\#(v) = 1_G \boxtimes \rho(v)$ with $\rho(v) \in H^m_{[0]}(\mathcal{O}_U)^{\oplus k}$. Denote by $\varphi : \{e\} \times U \xrightarrow{\sim} X$ the restriction of ψ. Following the previous identifications, it is not difficult to see that φ yields an isomorphism $\mathbb{C} \boxtimes H^m_{[0]}(\mathcal{O}_U)^{\oplus k} \xrightarrow{\sim} M_{|X} = \mathcal{O}_X \otimes_{\jmath_x^\#} M/J_xM$ such that $1\boxtimes\rho(v) \leftrightarrow \jmath_x^\#(\bar{v})$. For simplicity we will still denote the element $\jmath_x^\#(\bar{v})$ by $\bar{v}$. Set $\mathbf{n}_0 = (y_1, \dots, y_m)\mathcal{O}(\mathfrak{g}^y)$ and $T_0 = \{f \in H^m_{[0]}(\mathcal{O}_U)^{\oplus k} : \mathbf{n}_0.f = 0\}$. Under the previous identifications, we have $\mathbb{C} \boxtimes T_0 \xrightarrow{\sim} T$.

Corollary A.5. *Assume that $M = \mathcal{D}_{\mathfrak{g}}.v$ with $v \in M^G$. Then $\psi^* M$ is generated by $1_G \boxtimes \rho(v)$. Moreover, if $k = 1$ and $\rho(v) \in T_0$, then $M_{|\mathbf{O}}(x) = \mathbb{C}\bar{v}$ is the trivial representation of $A(\mathbf{O})$ and the connection $M_{|\mathbf{O}}$ is the standard $\mathcal{D}_{\mathbf{O}}$-module $\mathcal{O}_{\mathbf{O}}$.*

Proof. Since ψ is smooth, it is easy to see [13, Lemma 3.2] that $\psi^* M = \mathcal{D}_Y \psi^\#(v)$. Now, assume that $k = 1$ and $\mathbf{n}_0.\rho(v) = 0$. Since $\dim T_0 = 1$ and $\rho(v) \neq 0$, we obtain that $T_0 = \mathbb{C}\rho(v)$ and, by the previous identifications, $T = \mathbb{C}\bar{v}$. Since $v \in M^G$, the group G^ϕ acts trivially on $M_{|\mathbf{O}}(x) = T$. The corollary then follows from Theorem A.4. □

Acknowledgement. We wish to thank M. Raïs for bringing the reference [14] to our attention, and M. Van den Bergh for a useful correspondence.

References

[1] D. Barbasch and D. Vogan, The local structure of characters, *J. Funct. Anal.*, **37** (1980), 27-55.
[2] ———, Unipotent representations of complex semisimple groups, *Annals of Mathematics*, **121** (1985), 41-110.
[3] J. Bernstein and V. Lunts, Localization for derived categories of $(\mathfrak{g}, K)$-modules, *J. Amer. Math. Soc.*, **8** (1995), 819-856.
[4] A. Borel et al., *Algebraic D-modules*, Academic Press, Boston, 1987.
[5] Harish-Chandra, Differential operators on a semisimple Lie algebra, *Amer. J. Math.*, **79** (1957), 87-120.

[6] R. Hotta and M. Kashiwara, The invariant holonomic system on a semisimple Lie algebra, *Invent. Math.*, **75** (1984), 327-358.

[7] M. Kashiwara, Representation theory and D-modules on flag varieties, *in* "Orbites unipotentes et représentations, III", Astérisque, 173-174 (1989), 55-109.

[8] M. Kashiwara and W. Schmid, Quasi-Equivariant $\mathcal{D}$-modules, Equivariant Derived Category, and Representaions of Reductive Lie Groups, *in* "Lie theory and geometry", 457-488, Progr. Math., 123, Birkhuser, Boston, 1994.

[9] B. Kostant, Lie group representations on polynomial rings, *Amer. J. Math.*, **85** (1963), 327- 404.

[10] T. Levasseur, Notes on equivariant D-modules, (1997).

[11] T. Levasseur and J. T. Stafford, The kernel of an homomorphism of Harish-Chandra, *Ann. Sci. École Norm. Sup.*, **29** (1996), 385-397.

[12] ———, Semi-simplicity of invariant holonomic systems on a reductive Lie algebra, *Amer. J. Math.*, **119** (1997), 1095-1117.

[13] ———, Invariant differential operators on the tangent space of some symmetric spaces, preprint, (1997).

[14] W. Lichtenstein, Multipole potentials for $SU(n)$ and $SO(n)$, *J. Differential Geometry*, **17** (1982), 655-660.

[15] P. Slodowy, *Simple singularities and Simple Algebraic Groups*, Lecture Notes in Mathematics 815, Springer-Verlag, Berlin/New York, 1980.

[16] M. Van den Bergh, Some rings of differential operators for Sl_2-invariants are simple, *J. Pure and Applied Algebra*, **107** (1996), 309-335.

[17] ———, Some generalities on G-equivariant quasi-coherent $\mathcal{O}_X$ and $\mathcal{D}_X$-modules, Preprint Université Louis Pasteur, Strasbourg, (1994).

[18] N. Wallach, Invariant differential operators on a reductive Lie algebra and Weyl group representations, *J. Amer. Math. Soc.*, **6** (1993), 779-816.

Département de Mathématiques, Université de Poitiers, 86022 Poitiers, France.
E-mail address: `levasseu@mathlabo.univ-poitiers.fr`