NOTES ON EQUIVARIANT D-MODULES

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1. Generalities

All the varieties considered in these notes are quasi-projective algebraic varieties defined over $\mathbb{C}$.

Let X be a smooth algebraic variety. We denote by $\mathcal{O}_X$ the sheaf of regular functions on X and by $\mathcal{D}_X$ the sheaf of differential operators. The rings of global sections will be denoted by $\mathcal{O}(X)$ and $\mathcal{D}(X)$ respectively. We refer to [3] for the basic properties of $\mathcal{D}_X$-modules. All the $\mathcal{D}_X$-modules encountered in the sequel will be quasi-coherent. The category of quasi-coherent $\mathcal{D}_X$-modules is denoted by $\operatorname{Mod}\mathcal{D}_X$.

Let $F : Y \to X$ be a morphism between varieties. The comorphism of F is denoted by $F^\#$ and the inverse image of an $\mathcal{O}_X$-module M by

$$F^*M = \mathcal{O}_Y \otimes_{F^\#} M.$$

When X and Y are affine, there exists a natural map

$$F^\# : M \to F^*M, \quad v \mapsto 1_Y \otimes_{F^\#} v.$$

Suppose that $Z = X \times Y$ is the product of two varieties. Let M be an $\mathcal{O}_X$-module and N be an $\mathcal{O}_Y$-module; the $\mathcal{O}_Z$-module $M \otimes_{\mathbb{C}} N$ will be denoted by $M \boxtimes N$.

In this section we recall the definitions, and well known properties, of equivariant D-modules. Our main references are [2, 3, 5, 6, 7, 8, 11, 12, 15] where the reader will find the proof of the results stated below.

Let G be a linear algebraic group and V be a smooth affine algebraic G-variety. Let e be the unit in G and set

$$\begin{aligned}
&\mu : G \times V \to V, \quad (g,v) \mapsto g.v \ \ \text{(the action of } G \text{ on } V)\\
&\mu_G : G \times G \to G, \quad \mu_G(g,h) = gh \ \ \text{(the multiplication in } G)\\
&s : G \to G, \quad s(g) = g^{-1} \ \ \text{(the inverse in } G)\\
&\varepsilon : \{e\} \hookrightarrow G \ \ \text{(the inclusion)}\\
&p_2 : G \times V \to V, \quad p_2(g,v) = v\\
&p_{23} : G \times G \times V \to G \times V, \quad p_{23}(g,h,v) = (h,v)\\
&\varepsilon_V : V \to G \times V, \quad \varepsilon_V(v) = (e,v)
\end{aligned} \tag{1.1}$$

We then set: $\Delta = \mu_G^\#$, $S = s^\#$, $\epsilon = \varepsilon^\#$.

Let M be a rational G-module; the G-action on M is denoted by $g.a$, $g \in G$, $a \in M$. Recall that the G-module structure is equivalent to a left comodule structure $\lambda_M : M \to \mathcal{O}(G) \boxtimes M$, such that

$$g.a = a_{(1)}(g^{-1})a_{(2)}$$

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where we have used the (abbreviated) Sweedler notation $\lambda_M(a) = a_{(1)} \boxtimes a_{(2)}$, $a_{(1)} \in \mathcal{O}(G)$, $a_{(2)} \in M$. This applies in particular to $M = \mathcal{O}(V)$ with the action $(g.\varphi)(x) = \varphi(g^{-1}.x)$ for $\varphi \in \mathcal{O}(V)$, $g \in G$, $x \in V$. The corresponding coaction is denoted by λ_V. When $V = G$ and G acts via left translations, we have $\lambda_G = \Delta$.

Suppose furthermore that the G-module M has an $\mathcal{O}_V$-module structure. Then we say that M is a *G-equivariant $\mathcal{O}_X$-module* if the G and $\mathcal{O}(V)$ actions are compatible, i.e. $g.(\varphi a) = (g.\varphi)(g.a)$, $g \in G$, $\varphi \in \mathcal{O}(V)$, $a \in M$. This translates into $\lambda_M(\varphi a) = \lambda_V(\varphi)\lambda_M(a)$ for the coactions. We denote by $\mathfrak{M}(\mathcal{O}_V, G)$ the category of G-equivariant $\mathcal{O}_V$-modules.

Recall that $M \in \mathfrak{M}(\mathcal{O}_V, G)$ if and only if there exists an isomorphism of $\mathcal{O}_{G\times V}$-modules

$$\theta : p_2^* M = \mathcal{O}(G) \boxtimes M \xrightarrow{\sim} \mu^* M \tag{1.2}$$

such that

$$\varepsilon_V^*(\theta) = 1_M, \quad (\mu_G \times 1_V)^*(\theta) = (1_G \times \mu)^*(\theta) \circ p_{23}^*(\theta). \tag{1.3}$$

When the coaction λ_M is given the isomorphism θ is:

$$\theta(a \boxtimes v) = aSv_{(1)} \otimes_{\mu^\#} v_{(2)}.$$

Endow $\mathcal{O}(G)$, $\mathcal{O}(G \times V) = \mathcal{O}(G) \boxtimes \mathcal{O}(V)$ and $p_2^* M = \mathcal{O}(G) \boxtimes M$ with the action induced by left translation on G. It is then easily seen that θ is G-linear when G acts on $\mu^* M$ by

$$g.(b \otimes_{\mu^\#} m) = g.b \otimes_{\mu^\#} g.m$$

for all $g \in G$, $b \in \mathcal{O}(G) \boxtimes \mathcal{O}(V)$ and $m \in M$. For these actions, since $(\mathcal{O}(G) \boxtimes M)^G = \mathbb{C} \boxtimes M$, we obtain the isomorphism

$$\theta : \mathbb{C} \boxtimes M \xrightarrow{\sim} (\mu^* M)^G.$$

Recall that the G-action on V induces a rational G-module structure on the algebra $\mathcal{D}(V)$, given by

$$(g.D) \cdot \varphi = g.(D \cdot (g^{-1}.\varphi))$$

for all $g \in G$, $D \in \mathcal{D}(V)$, $\varphi \in \mathcal{O}(V)$. (We denote by $D \cdot \varphi$ the natural action of $\mathcal{D}(V)$ on $\mathcal{O}(V)$.) The corresponding coaction extends the coaction λ_V and we will still denote it by $\lambda_V : \mathcal{D}(V) \to \mathcal{O}(G) \boxtimes \mathcal{D}(V)$.

Let $M \in \operatorname{Mod} \mathcal{D}_V$. The module M is said to be a *weakly G-equivariant $\mathcal{D}_V$-module* if $M \in \mathfrak{M}(\mathcal{O}_V, G)$ and

$$g.(D.v) = (g.D).(g.v)$$

for all $g \in G$, $D \in \mathcal{D}(V)$ and $v \in M$. (This is equivalent to saying that $\lambda_M(D.v) = \lambda_V(D)\lambda_M(v)$.) We denote by $\mathfrak{M}(\mathcal{D}_V, G^w)$ the category of weakly G-equivariant $\mathcal{D}(V)$-modules. Then, a module $M \in \operatorname{Mod} \mathcal{D}_V$ is weakly G-equivariant if, and only if, there exists a map θ as in (1.2) & (1.3) which is $\mathcal{O}(G) \boxtimes \mathcal{D}(V)$ linear.

The differential of the G-action on V yields a Lie algebra map, τ_V, from $\mathfrak{g} = \operatorname{Lie}(G)$ to the Lie algebra Θ_V of vector fields on V. It is defined by

$$(\tau_V(\xi) \cdot \varphi)(x) = \frac{d}{dt}_{|t=0} \varphi(\exp(-t\xi).x)$$

for all $\xi \in \mathfrak{g}$, $\varphi \in \mathcal{O}(V)$, $x \in V$. Notice that $\tau_V(\xi)$ identifies with a derivation of $\mathcal{O}(V)$ and that $\tau_V(\operatorname{Ad}(g).\xi) = g.\tau_V(\xi)$ for all $g \in G$. The differential of the

G-action on $\mathcal{D}(V)$ is then given by

$$\xi.D = [\tau_V(\xi), D], \quad \text{for all } D \in \mathcal{D}(V).$$

Let $M \in \mathfrak{M}(\mathcal{O}_V, G)$. The differential of the G-action on M gives a $\mathfrak{g}$-module structure on M:

$$\xi.v = \frac{d}{dt}_{|t=0}(\exp(t\xi).v), \quad \text{for all } \xi \in \mathfrak{g} \text{ and } v \in M.$$

The module M is called a *G-equivariant $\mathcal{D}_V$-module* if $M \in \mathfrak{M}(\mathcal{D}_V, G^w)$ and

$$\tau_V(\xi).v = \xi.v, \quad \text{for all } \xi \in \mathfrak{g} \text{ and } v \in M. \tag{1.4}$$

Then, a module $M \in \operatorname{Mod} \mathcal{D}_V$ is G-equivariant if, and only if, there exists a map θ as in (1.2) & (1.3) which is $\mathcal{D}(G) \boxtimes \mathcal{D}(V)$ linear.

Remarks. (1) Let $M \in \mathfrak{M}(\mathcal{O}_V, G)$. If $v \in M^G$ we have $\lambda_M(v) = 1_G \boxtimes v$ and therefore $\theta(1_G \boxtimes v) = 1_{G\times V} \otimes_{\mu^\#} v$.

(2) Set

$$\mathcal{N} = \mathcal{D}_V / \mathcal{D}_V \tau_V(\mathfrak{g}). \tag{1.5}$$

Then $\mathcal{N} \in \mathfrak{M}(\mathcal{D}_V, G)$. Moreover, when G is connected every subquotient of $\mathcal{N}$ is in $\mathfrak{M}(\mathcal{D}_V, G)$ (see [9, 14]).

(3) Suppose that $M \in \mathfrak{M}(\mathcal{D}_V, G)$. Then, when G is connected:

$$\{M = \mathcal{D}_V.v \text{ with } v \in M^G\} \iff \{M \text{ is a quotient of } \mathcal{N}\}$$

The definitions of $\mathfrak{M}(\mathcal{O}_V, G)$, $\mathfrak{M}(\mathcal{D}_V, G^w)$ and $\mathfrak{M}(\mathcal{D}_V, G)$ carry over to the case when the smooth variety V is not necessarily affine, see [3, 8, 15]. For instance, $M \in \operatorname{Mod} \mathcal{D}_V$ is G-equivariant if there exists an isomorphism $\theta : \mathcal{O}_G \boxtimes M \xrightarrow{\sim} \mu^* M$ of $\mathcal{D}_{G\times V}$-modules which satisfies the conditions of (1.3).

Let $F : Y \to X$ be a G-equivariant morphism between (not necessarily affine) smooth G-varieties. Recall that if $M \in \operatorname{Mod} \mathcal{D}_X$ one defines the *inverse image* of M by setting

$$F^! M = \big(\mathcal{D}_{Y\to X} \otimes^L_{\mathcal{D}_X} M\big)[d_{Y,X}]$$

where $\mathcal{D}_{Y\to X} = F^*\mathcal{D}_X = \mathcal{O}_Y \otimes_{F^\#} \mathcal{D}_X$ and $d_{Y,X} = \dim Y - \dim X$. The inverse image is an object in the derived category $D^b(\mathcal{D}_Y)$ of $\operatorname{Mod} \mathcal{D}_Y$, and the construction of $F^!$ extends to $D^b(\mathcal{D}_X)$. Observe that $\mathcal{D}_{Y\to X}$ is a weakly G-invariant *right* $\mathcal{D}_X$-module for the G-action $g.(\varphi \otimes_{F^\#} D) = g.\varphi \otimes_{F^\#} g.D$, $\varphi \in \mathcal{O}(Y)$, $D \in \mathcal{D}(X)$.

Assume that $M \in \mathfrak{M}(\mathcal{D}_X, G)$ and let $\theta_M : p_{2,X}^* M \to \mu_X^* M$ be the associated isomorphism (with obvious notation). Set $r = \dim G$. Since $p_{2,X}$ and μ_X are smooth, $p_{2,X}^! M$ and $\mu_X^! M$ have cohomology concentrated in degree $-r$, equal to $p_{2,X}^* M$ and $\mu_X^* M$ respectively. Thus one can consider θ_M as an isomorphism of $\mathcal{D}_{G\times X}$-modules between $p_{2,X}^! M$ and $\mu_X^! M$. It follows from base change that we have an isomorphism in $D^b(\mathcal{D}_{G\times Y})$

$$\theta_{F^! M} : p_{2,Y}^! F^! M \xrightarrow{\sim} \mu_Y^! F^! M.$$

Indeed, the isomorphism $\theta_{F^! M}$ is defined by the left vertical arrow which makes the following diagram commutative:

$$\begin{array}{ccc}
\mu_Y^! F^! M & = & (1_G \times F)^! \mu_X^! M \\
\uparrow & & \uparrow {\scriptstyle (1_G\times F)^!(\theta_M)} \\
p_{2,Y}^! F^! M & = & (1_G \times F)^! p_{2,X}^! M
\end{array}$$

Therefore, since $p_{2,Y}$ and μ_Y are smooth, we obtain isomorphisms of $\mathcal{D}_{G\times Y}$-modules in cohomology:

$$\theta_{F^!M} : p_{2,Y}^* \mathcal{H}^j(F^!M) \xrightarrow{\sim} \mu_Y^* \mathcal{H}^j(F^!M).$$

This implies that $\mathcal{H}^j(F^!M)$ has a natural induced structure of G-equivariant $\mathcal{D}_Y$-module. In particular, the G-action on $\mathcal{H}^j(F^!M)$ is uniquely determined by the action of G on M. To compute this action one may proceed as follows.

Suppose that we are given a resolution $(P^\bullet, d)$ of $\mathcal{D}_{Y\to X}$ by weakly G-equivariant right $\mathcal{D}_X$-modules, see [6, Lemma 4.7] and [8, Proposition 2.1], such that each P^k is projective as a $\mathcal{D}_X$-module and d is G-equivariant. Then, since $(P^\bullet \otimes_{\mathcal{D}_X} M, d \otimes 1_M) = F^!M[-d_{Y,X}]$, the G-action on $\mathcal{H}^j(F^!M)$ is induced by the diagonal G-action on $P^\bullet \otimes_{\mathcal{D}_X} M$.

Moreover, since $\mathcal{D}_{G\times Y\to G\times X} = \mathcal{D}_G \boxtimes \mathcal{D}_{Y\to X}$, the objects $(1_G \times F)^! \mu_X^! M$ and $(1_G \times F)^! p_{2,X}^! M$ are represented, up to a shift, by

$$(\mathcal{D}_G \boxtimes P^\bullet) \otimes_{\mathcal{D}_{G\times X}} \mu_X^! M \quad \text{and} \quad (\mathcal{D}_G \boxtimes P^\bullet) \otimes_{\mathcal{D}_{G\times X}} p_{2,X}^! M.$$

Thus the isomorphism $\theta_{F^!M}$ is induced by

$$(1_G \boxtimes 1_{P^\bullet}) \otimes \theta_M : (\mathcal{D}_G \boxtimes P^\bullet) \otimes p_{2,X}^* M \longrightarrow (\mathcal{D}_G \boxtimes P^\bullet) \otimes \mu_X^* M.$$

This applies for instance to the morphisms p_2 and μ, and the isomorphism θ of (1.2) can be viewed as an isomorphism in $\mathfrak{M}(\mathcal{D}_{G\times V}, G)$ between $p_2^! M = p_2^* M[r]$ and $\mu^! M = \mu^* M[r]$.

2. Reduction to a slice

In this section we apply the results of §1 to the case where the variety V is a finite dimensional rational G-module, which we denote by E. We keep the notation of (1.1). We set $r = \dim G$, $n = \dim E$ and we fix $x \in E$. Denote by $t_x : y \mapsto y + x$ the translation by x and let $\mu_x = \mu \circ (1_G \times t_x) : G \times E \to E$, $(g, v) \mapsto g.(x + v)$.

Let $F \subseteq E$ be a linear subspace of dimension m. *We assume in this section that the following hypothesis holds:*

(†) $\qquad \mu_x : G \times F \to E$ is a smooth morphism and $E = F \oplus \mathfrak{g}.x$

Set $\mathbf{O} = G.x$, which is a quasi-affine subvariety of E. Then, by (†), $x + F$ is a transverse slice to $\mathbf{O}$ at the point x, see [13, §5.1]. Let $\mathfrak{g}^x$ be the stabilizer of x in $\mathfrak{g}$ and set $s = \dim \mathfrak{g}^x$. Then, (†) implies that

$$\dim \mathbf{O} = r - s = n - m.$$

Proposition 2.1. *There exists an affine open neighborhood U of 0 in F such that, if ψ is the restriction of μ_x to $G \times F$,*

1. *ψ is smooth on $Y := G \times U$, $\Omega = \psi(Y) = G.(x + U)$ is a G-stable open subset of E;*
2. *$\Omega \cap \overline{\mathbf{O}} = \mathbf{O}$ and $\mathbf{O} \cap \{x + U\} = \{x\}$.*

Proof. Let $T = (x + F) \times_E \mathbf{O}$ be the intersection of $x + F$ and $\mathbf{O}$; thus T is a subscheme of E. From (†) and [4, IV.17.13.8] it follows that this intersection is transverse of dimension 0 at the point x. Hence [ibid], there exists an affine open subset $0 \in U_0 \subseteq F$ such that the intersection $(x + U_0) \cap \mathbf{O}$ is transverse at x. In particular $\dim((x + U_0) \cap \mathbf{O}) = 0$, therefore $(x + U_0) \cap \mathbf{O}$ is a finite set [4, O.14.1.9], say $\{x = x_0, x_1, \ldots, x_t\}$. For each $i = 1, \ldots, t$, pick an affine open subset

$0 \in U(x_i) \subset F$ such that $x_i \notin x + U(x_i)$. Set $U_1 = U_0 \cap U(x_1) \cap \cdots \cap U(x_t)$. Then, U_1 is an affine open neighborhood of 0 and $(x + U_1) \cap \mathbf{O} = \{x\}$.

Since $\mathbf{O}$ is open in its closure, we can find a G-stable open subset $V \subset E$ such that $V \cap \overline{\mathbf{O}} = \mathbf{O}$. Define an open neighborhood of 0 in F by:

$$U_2 = p_2\psi^{-1}(V) = \{u \in F : \exists\, g \in G, g.(x+u) \in V\}$$

Let $v \in U_2$; since ψ is G-equivariant, we have $\psi(G \times \{v\}) \subset V$. Hence, $G \times U_2 \subset \psi^{-1}(V)$. Observe that, if $0 \in U \subset U_2$ is any open subset, $\psi(G \times U)$ is open and $\psi(G \times U) \cap \overline{\mathbf{O}} = \mathbf{O}$.

Now, let U_1 be as in the first paragraph and choose an affine open subset $0 \in U \subseteq U_1 \cap U_2$. Then U satisfies the required properties. □

We will keep the notation of Proposition 2.1 for the rest of this section. In particular, ψ will be the smooth morphism

$$\psi : Y = G \times U \longrightarrow E, \quad \psi(g,u) = g.(x+u).$$

Set $X = x + U$ and notice that t_x induces an isomorphism (of varieties) from U onto X, with inverse t_{-x}. Define:

$$\beta : U \hookrightarrow E,\ \beta_x : X \hookrightarrow E,\ \imath : \{x\} \hookrightarrow E \ \text{ (the natural inclusions)} \tag{2.1}$$

Observe that $\beta_x = t_x \circ \beta \circ t_{-x}$ and $\psi = \mu_x \circ (1_G \times \beta)$.

Let $M \in \mathfrak{M}(\mathcal{D}_E, G)$. We assume that M is a coherent $\mathcal{D}_E$-module, i.e. M is a finitely generated G-equivariant $\mathcal{D}(E)$-module. Recall from §1 that we have an isomorphism of G-equivariant $\mathcal{D}_{G\times E}$-modules,

$$\theta : p_2^! M = p_2^* M[r] \xrightarrow{\sim} \mu^* M[r] = \mu^! M.$$

By using the translation t_x we can construct the $\mathcal{D}_E$-module $t_x^* M$, which identifies with $t_x^! M$. Observe that $t_x \in \mathrm{Aut}(E)$ induces an automorphism of $\mathcal{O}(E)$, $(t_x.\varphi)(y) = \varphi(y-x)$ for $\varphi \in \mathcal{O}(E)$, $y \in E$, and therefore yields an automorphism of $\mathcal{D}(E)$, $(t_x.D)(\varphi) = t_x.D(t_{-x}.\varphi)$ for $D \in \mathcal{D}(E)$. Then, the $\mathcal{D}(E)$-module $t_x^* M$ can be identified with the vector space M endowed with the action $D \centerdot u = (t_x.D)u$. As in §1 we have maps $t_x^\# : M \to t_x^* M$ and $t_{-x}^\# : M \to t_{-x}^* M$. We set:

$$t_{-x}.v = t_x^\#(v), \quad t_x.v = t_{-x}^\#(v).$$

We will adopt a similar notation for the inverse images by t_x, or t_{-x}, for modules over X, or U.

Lemma 2.2. *There exists an isomorphism of G-equivariant $\mathcal{D}_{G\times E}$-modules:*

$$\theta_x : p_2^! t_x^* M \xrightarrow{\sim} \mu_x^! M$$

Proof. Since p_2 and μ are smooth, $p_2^! t_x^! M = p_2^* t_x^* M[r]$ and $\mu_x^! M = \mu^* M[r]$. Notice that $\mu_x^! M = (1_G \times t_x)^! \mu^! M = (1_G \times t_x)^* \mu^! M$ and $p_2^! t_x^* M = (\mathcal{O}(G) \boxtimes t_x^* M)[r] = (1_G \times t_x)^* p_2^! M$. Now, the G-equivariant isomorphism $1_G \times t_x : G \times E \to G \times E$ yields the isomorphism, in $\mathfrak{M}(\mathcal{D}_{G\times E}, G)$,

$$\theta_x = (1_G \times t_x)^*(\theta) : (1_G \times t_x)^* p_2^! M = p_2^! t_x^* M \xrightarrow{\sim} (1_G \times t_x)^* \mu^! M = \mu_x^! M$$

as desired. □

Remark. The isomorphism θ_x yields the isomorphism

$$\theta_x : \mathcal{H}^{-r}(p_2^! t_x^* M) = \mathcal{O}(G) \boxtimes t_x^* M \xrightarrow{\sim} \mu_x^* M = \mathcal{H}^{-r}(\mu_x^! M),$$

which is given by $\theta_x(a \boxtimes t_{-x}.v) = aSv_{(1)} \otimes_{\mu_x^\#} v_{(2)}$ for $a \in \mathcal{O}(G)$ and $v \in M$.

Since ψ is smooth and G-equivariant, $\psi^* M$ is coherent [3, VI.4.8] and $\psi^! M = \psi^* M[r+m-n] \in \mathfrak{M}(\mathcal{D}_{G\times U}, G)$. Thus,

$$\mathcal{H}^j(\psi^! M) = \begin{cases} 0 & \text{if } j \neq -s = n-r-m, \\ \psi^* M & \text{if } j = -s = n-r-m. \end{cases} \tag{2.2}$$

The G-action on $\psi^* M = \mathcal{O}(G\times U) \otimes_{\psi^\#} M$ is given by $g.(b \otimes_{\psi^\#} v) = g.b \otimes_{\psi^\#} g.v$, $b \in \mathcal{O}(G\times U)$, $v \in M$.

Lemma 2.3. *There exists an isomorphism of G-equivariant $\mathcal{D}_Y$-modules*

$$\psi^! M \cong (\mathcal{O}(G) \boxtimes \beta^! t_x^* M)[r]$$

and $\beta^! t_x^! M = (\beta^* t_x^* M)[m-n]$. Hence

$$\mathcal{H}^j(\beta^! t_x^! M) = \begin{cases} 0 & \textit{if } j \neq \dim \mathbf{O}, \\ \mathcal{O}_U \otimes_{\beta^\#} t_x^* M & \textit{if } j = \dim \mathbf{O}. \end{cases}$$

Proof. Notice first that, since t_x is an isomorphism, we can identify $t_x^! M$ with $t_x^* M$. From $\psi = \mu_x \circ (1_G \times \beta)$ we deduce that $\psi^! M = (1_G \times \beta)^! \mu_x^! M$. Therefore, using the G-equivariant morphism $1_G \times \beta$, we obtain

$$(1_G \times \beta)^!(\theta_x) : (1_G \times \beta)^! p_2^! t_x^* M \xrightarrow{\sim} \psi^! M.$$

Then, $(1_G \times \beta)^! p_2^! t_x^* M = (1_G \times \beta)^! (\mathcal{O}(G) \boxtimes t_x^* M)[r] = (\mathcal{O}(G) \boxtimes \beta^! t_x^* M)[r]$ yields $\psi^! M \cong (\mathcal{O}(G) \boxtimes \beta^! t_x^* M)[r]$. From this isomorphism one deduces that $\mathcal{H}^{j-r}(\psi^! M) \cong \mathcal{O}_G \boxtimes \mathcal{H}^j(\beta^! t_x^* M)$. Then, by (2.2), $\mathcal{H}^j(\beta^! t_x^* M) = 0$ unless $j = n-m$, and

$$\mathcal{H}^{n-m}(\beta^! t_x^* M) = \mathcal{H}^{n-m}\big((\mathcal{O}_U \otimes^L_{\beta^\#} t_x^* M)[m-n]\big) = \mathcal{O}_U \otimes_{\beta^\#} t_x^* M.$$

This completes the proof of the lemma. □

In order to simplify the notation we set

$$t_x^* M_{|U} = \mathcal{H}^{\dim \mathbf{O}}(\beta^! t_x^! M) = \mathcal{O}_U \otimes_{\beta^\#} t_x^* M.$$

Recall that the natural inclusion $\beta_x : X \hookrightarrow E$ is equal to $t_x \circ \beta \circ t_{-x}$. Therefore, $\beta_x^! M = t_{-x}^! \beta^! t_x^! M$ has non-zero cohomology only in degree $\dim \mathbf{O} = n-m$, where $\mathcal{H}^{\dim \mathbf{O}}(\beta_x^! M) = \mathcal{O}_X \otimes_{\beta^\#} M$. We set

$$M_{|X} = \mathcal{H}^{\dim \mathbf{O}}(\beta_x^! M) = \mathcal{O}_X \otimes_{\beta_x^\#} M.$$

Thus we have:

$$\beta_x^! M = M_{|X}[m-n] \tag{2.3}$$

Notice that it follows easily from $\beta_x \circ t_x = t_x \circ \beta$ (on U) that

$$t_x^*\big(M_{|X}\big) = t_x^* M_{|U}. \tag{2.4}$$

Let S be a complementary subspace to F, $E = F \bigoplus S$. Let $\{v_1, \dots, v_m\}$ be a basis of F and $\{v_{m+1}, \dots, v_n\}$ be a basis of S. Denote by $\{y_j = v_j^*\}_j$ the dual basis and set $f_j = t_x.y_j = y_j - y_j(x)$, $1 \leqslant j \leqslant n$. Then,

$$\mathcal{O}(F) = S(E^*)/J, \quad \mathcal{O}(x+F) = S(E^*)/J_x$$

where $J = (y_{m+1}, \dots, y_n)S(E^*)$ and $J_x = t_x.J = (f_{m+1}, \dots, f_n)S(E^*)$.

Let $\jmath : U \hookrightarrow F$ and $\jmath_x : X \hookrightarrow x+F$ be the natural inclusions.

Lemma 2.4. *We have:*

(1) $M_{|X} = \mathcal{O}_X \otimes_{\jmath_x^\#} (M/J_x M)$;

(2) $t_x^* M_{|U} = \mathcal{O}_U \otimes_{\jmath^\#} (t_x^* M / J t_x^* M) = \mathcal{O}_U \otimes_{\jmath^\#} t_x^*(M/J_x M)$.

Proof. 1. Let $\tilde{\beta}_x : x + F \hookrightarrow E$ be the inclusion. Thus $\beta_x = \tilde{\beta}_x \circ \jmath_x$. Since $\jmath_x$ is an open immersion, $\mathcal{H}^j(\beta_x^! M) = \mathcal{O}_X \otimes_{\jmath_x^\#} \mathcal{H}^j(\tilde{\beta}_x^! M)$ for all j. Recall that $\tilde{\beta}_x^! M = \big(\mathcal{D}_{x+F\to E} \otimes^L_{\mathcal{D}_E} M\big)[m-n]$. Set $V_x = \oplus_{i=m+1}^n \mathbb{C} df_i$ and $\mathcal{C}_x^{-p} = \bigwedge^p V_x \boxtimes \mathcal{D}_E$. By [3, VI.7.4], $\tilde{\beta}_x^! M = \big(\mathcal{C}_x^\bullet \otimes_{\mathcal{D}_E} M, \partial_x\big)[m-n]$, with $\mathcal{C}_x^p \otimes_{\mathcal{D}_E} M = \bigwedge^{-p} V_x \boxtimes M$ and

$$\partial_x(df_{j_1} \wedge \cdots \wedge df_{j_p} \boxtimes v) = \textstyle\sum_{a=1}^p (-1)^{a+1} df_{j_1} \wedge \cdots \wedge \widehat{df_{j_a}} \wedge \cdots \wedge df_{j_p} \boxtimes f_{j_a} v.$$

It follows that $\mathcal{H}^{n-m}(\tilde{\beta}_x^! M) = \mathcal{H}^0\big(\mathcal{C}_x^\bullet \otimes_{\mathcal{D}_E} M, \partial_x\big) = M/J_x M = \mathcal{O}_{x+F} \otimes_{\tilde{\beta}_x^\#} M$.

2. follows from 1. and (2.4). □

Let $\operatorname{Supp} M$ be the support of the $\mathcal{D}_E$-module M. Since M is coherent and G-equivariant, $\operatorname{Supp} M$ is a closed G-stable subvariety of E and we have $\operatorname{Supp} M = \bigcup_{v \in M} \operatorname{Supp} \mathcal{O}_E.v$. *From now on we assume that*

$$\operatorname{Supp} M \subseteq \overline{\mathbf{O}}.$$

Recall that the local cohomology group $H^m_{[0]}(\mathcal{O}_U)$ is a $\mathcal{D}_U$-module isomorphic to $\mathcal{D}_U \big/ (\mathcal{D}_U y_1 + \cdots + \mathcal{D}_U y_m)$. It is easily seen that $t_{-x}^* H^m_{[0]}(\mathcal{O}_U) = H^m_{[x]}(\mathcal{O}_X) \cong \mathcal{D}_X \big/ (\mathcal{D}_X f_1 + \cdots + \mathcal{D}_X f_m)$.

Proposition 2.5. *Let M be as above. Then, there exists $k \in \mathbb{N}$ such that*

(1) $M_{|X} \cong H^m_{[x]}(\mathcal{O}_X)^{\oplus k}$, $t_x^* M_{|U} \cong H^m_{[0]}(\mathcal{O}_U)^{\oplus k}$;
(2) $\psi^! M \cong \big(\mathcal{O}_G \boxtimes H^m_{[0]}(\mathcal{O}_U)^{\oplus k}\big)[s]$;
(3) *if* $\operatorname{Supp} M = \overline{\mathbf{O}}$, *then* $k \geqslant 1$.

Proof. 1. Clearly, the support of $M_{|X} = \mathcal{O}_X \otimes_{\beta_x^\#} M$ is contained in $X \cap \operatorname{Supp} M \subseteq X \cap \overline{\mathbf{O}}$. But, $X \cap \overline{\mathbf{O}} \subseteq X \cap \Omega \cap \overline{\mathbf{O}} = X \cap \mathbf{O} = \{x\}$. Thus $M_{|X}$ is a $\mathcal{D}_X$-module whose support is contained in $\{x\}$; therefore, by Kashiwara's equivalence [3, VI.7.11], $M_{|X} = H^m_{[x]}(\mathcal{O}_X)^{\oplus k}$ for some k. The proof is similar for $t_x^* M_{|U}$.

2. follows from 1. and Lemma 2.3.

3. The hypothesis implies that there exists $v \in M$ such that $\mathbf{O} \cap \operatorname{Supp} \mathcal{O}_E.v \neq \emptyset$. Then, since ψ is flat and $\Omega \supset \mathbf{O}$, we have $1 \otimes_{\beta^\#} v \in \psi^* M \setminus \{0\}$. Thus $\psi^* M \neq 0$, i.e. $k \geqslant 1$. □

Denote by $\mathbf{m}_x = (f_1, \dots, f_n)\mathcal{O}(E)$ and $\mathbf{n}_x = (f_1, \dots, f_m)\mathcal{O}(F)$ the maximal ideals asociated to $x \in E$ and $x \in (x+F)$ (respectively). Set $\mathbb{C}_x = \mathcal{O}(E)/\mathbf{m}_x \mathcal{O}(E)$.

Theorem 2.6. *Let $\imath : \{x\} \hookrightarrow E$ be the inclusion. Set*

$$\omega^{-1} = df_1 \wedge \cdots \wedge df_m \ \text{ and } \ T = \{u \in M/J_x M : \mathbf{n}_x u = 0\}.$$

Then,

(1) $T \cong \big\{v \in H^m_{[x]}(\mathcal{O}_X)^{\oplus k} : \mathbf{n}_x v = 0\big\}$ *is a $\mathbb{C}$-vector space of dimension k;*
(2) $\imath^! M$ *has cohomology concentrated in degree* $\dim \mathbf{O}$ *and*

$$\mathcal{H}^{\dim \mathbf{O}}(\imath^! M) = \operatorname{Tor}_m^{\mathcal{O}(E)}(\mathbb{C}_x, M) = \omega^{-1} \otimes_{\mathbb{C}_x} T.$$

Proof. Recall [3, VI.4.2] that $\mathcal{H}^j(\imath^! M) = \operatorname{Tor}_{-j+n}^{\mathcal{O}(E)}(\mathbb{C}_x, M)$. Let $\gamma : \{x\} \hookrightarrow X$ be the inclusion. Then, $\imath = \beta_x \circ \gamma$ and $\imath^! M = \gamma^! \beta_x^! M$. Recall from (2.3) that $\beta_x^! M = M_{|X}[m-n]$. By Lemma 2.4 and Proposition 2.5, we know that $M_{|X} =$

$\mathcal{O}_X \otimes_{\jmath_x^\#} (M/J_x M) \cong H^m_{[x]}(\mathcal{O}_X)^{\oplus k}$ is supported on $\{x\}$. Thus, $T_{|X} = \{u \in M_{|X} : \mathbf{n}_x u = 0\}$ is a $\mathbb{C}$-vector space of dimension k. Furthermore, by [3, VI.7.4],

$$\mathcal{H}^j(\gamma^! M_{|X}) = \begin{cases} 0 & \text{if } j \neq 0, \\ \omega^{-1}_{\{x\}/X} \otimes_{\mathbb{C}_x} T_{|X} & \text{if } j = 0. \end{cases}$$

Since $\omega^{-1}_{\{x\}/X} = df_1 \wedge \cdots \wedge df_m$, we obtain that $\imath^! M$ has cohomology concentrated in degree $n-m$ with $\mathcal{H}^{n-m}(\imath^! M) = \omega^{-1} \otimes_{\mathbb{C}_x} T_{|X}$. To finish the proof it suffices to apply the following standard result to the module $N = M/J_x M$: Let N be any $\mathcal{O}_{x+F}$-module; then, if $N' = \mathcal{O}_X \otimes_{\jmath_x^\#} N$, one has

$$\{u \in N : \mathbf{n}_x u = 0\} \xrightarrow{\sim} \{u' \in N' : \mathbf{n}_x u' = 0\}$$

through the natural map $\jmath_x^\# : N \to N'$, $\jmath_x^\#(u) = 1_X \otimes_{\jmath_x^\#} u$. □

One can factorize the inclusion $\imath : \{x\} \hookrightarrow E$ as follows: $\imath : \{x\} \overset{\imath_1}{\hookrightarrow} \mathbf{O} \overset{\imath_2}{\hookrightarrow} E$. We now compute $\imath_2^! M$, the "restriction to $\mathbf{O}$".

Proposition 2.7. *$\imath_2^! M$ has cohomology concentrated in degree* 0 *and, as an $\mathcal{O}_{\mathbf{O}}$-module, $\mathcal{H}^0(\imath_2^! M) = \mathrm{Tor}_m^{\mathcal{O}_E}(\mathcal{O}_{\mathbf{O}}, M)$.*

Proof. Notice first the following commutative diagram of G-equivariant morphisms

$$\begin{array}{ccc} Y = G \times U & \xrightarrow{\psi} & E \\ {\scriptstyle \jmath_2}\uparrow & & \uparrow{\scriptstyle \imath_2} \\ G \times \{0\} & \xrightarrow{\pi} & \mathbf{O} \end{array}$$

where $\pi(g, 0) = g.x$.

Recall that, by Proposition 2.5, $\psi^! M \cong \psi^* M[s]$ with $\psi^* M = \mathcal{O}_G \boxtimes H^m_{[0]}(\mathcal{O}_U)^{\oplus k}$. Thus $\psi^* M = \mathcal{H}^{-s}(\psi^! M)$ is supported on $G \times \{0\}$. Since $\jmath_2 : G \times \{0\} \hookrightarrow Y$ is a closed embedding, [3, VI.7.4] gives that $\jmath_2^! \psi^* M$ has cohomology concentrated in degree 0; equivalently, $\jmath_2^! \psi^! M$ has cohomology concentrated in degree $-s$.

On the other hand,

$$\mathcal{H}^j(\pi^! \imath_2^! M) = \mathcal{H}^j\big((\mathcal{D}_{G\times\{0\}\to\mathbf{O}} \otimes^L_{\mathcal{D}_{\mathbf{O}}} \imath_2^! M)[s]\big) = \mathcal{H}^{j+s}\big(\mathcal{D}_{G\times\{0\}\to\mathbf{O}} \otimes^L_{\mathcal{D}_{\mathbf{O}}} \imath_2^! M\big).$$

Since π is smooth and $\mathcal{D}_{G\times\{0\}\to\mathbf{O}} = \pi^* \mathcal{D}_{\mathbf{O}}$, it follows that, as an $\mathcal{O}_{G\times\{0\}}$-module,

$$\mathcal{H}^j(\pi^! \imath_2^! M) = \mathcal{O}_{G\times\{0\}} \otimes_{\pi^\#} \mathcal{H}^{j+s}(\imath_2^! M).$$

Now, since π is faithfully flat and $\pi^! \imath_2^! M = \jmath_2^! \psi^! M$, the previous paragraph implies that $\imath_2^! M$ has cohomology concentrated in degree 0. By definition, $\imath_2^! M = \big(\mathcal{D}_{\mathbf{O}} \otimes^L_{\mathcal{D}_E} M\big)[-m]$. Hence, see [3, VI.4.2], $\mathcal{H}^0(\imath_2^! M) = \mathrm{Tor}_m^{\mathcal{O}_E}(\mathcal{O}_{\mathbf{O}}, M)$. □

We set:

$$M_{|\mathbf{O}} = \mathcal{H}^0(\imath_2^! M) \tag{2.5}$$

Denote by G^x the stabilizer of x in G and let G^x_0 be its connected component. Define the component group of $\mathbf{O}$ by

$$A(\mathbf{O}) = G^x / G^x_0.$$

Let $\pi : G \twoheadrightarrow \mathbf{O}$, $g \mapsto g.x$, be the natural morphism and L be a rational representation of G^x. Define an $\mathcal{O}_{\mathbf{O}}$-module $\mathcal{L}$ by setting, for any open subset $W \subseteq \mathbf{O}$,

$$\Gamma(W, \mathcal{L}) = \big\{ f : \pi^{-1}(W) \to L : f(gh) = h^{-1}.f(g) \text{ for all } g \in \pi^{-1}(W),\ h \in G^x \big\}.$$

Then $\mathcal{L} \in \mathfrak{M}(\mathcal{O}_{\mathbf{O}}, G)$ [7, Theorem 4.8.1]. When L is a representation of the (finite) group $A(\mathbf{O})$, i.e. when G_0^x acts trivially on L, $\mathcal{L} \in \mathfrak{M}(\mathcal{D}_{\mathbf{O}}, G)$ and, conversely, any G-equivariant $\mathcal{D}_{\mathbf{O}}$-module is of this form, see [7, Proposition 4.11.1] and [8, §4]. An object of $\mathfrak{M}(\mathcal{D}_{\mathbf{O}}, G)$ will be called a *connection* on $\mathbf{O}$. The representation of $A(\mathbf{O})$ associated to a connection $\mathcal{L}$ is the "geometric fibre at the point x", $\mathcal{L}(x) = \mathbb{C}_x \otimes_{\mathcal{O}_{\mathbf{O}}} \mathcal{L} = \mathcal{L}_x/\mathbf{m}_x\mathcal{L}_x$, where the $A(\mathbf{O})$-action is coming from the natural action of G^x.

Proposition 2.8. *The $\mathcal{D}_{\mathbf{O}}$-module $M_{|\mathbf{O}}$ is a connection and its geometric fibre at the point x is*

$$M_{|\mathbf{O}}(x) = \omega^{-1} \otimes_{\mathbb{C}} T.$$

Proof. Since $\imath_2 : \mathbf{O} \hookrightarrow E$ is G-equivariant, the results of §1 ensure that $\imath_2^! M = M_{|\mathbf{O}}$ (see Proposition 2.7) is in $\mathfrak{M}(\mathcal{D}_{\mathbf{O}}, G)$. Therefore, by the remarks above, $M_{|\mathbf{O}}$ is a connection. In particular, $M_{|\mathbf{O}}$ is flat as an $\mathcal{O}_{\mathbf{O}}$-module and it follows that

$$\imath^! M = \imath_1^! \imath_2^! M = \imath_1^! M_{|\mathbf{O}} = \big(\mathcal{D}_{\{x\}\to\mathbf{O}} \otimes^L_{\mathcal{D}_{\mathbf{O}}} M_{|\mathbf{O}}\big)[m-n]$$

has cohomology concentrated in degree $n-m$, where $\mathcal{H}^{n-m}(\imath^! M) = \mathbb{C}_x \otimes_{\mathcal{O}_{\mathbf{O}}} M_{|\mathbf{O}} = M_{|\mathbf{O}}(x)$. The proposition then follows from Theorem 2.6. □

Remark. We will see (in a particular case) in the next section how to compute the $A(\mathbf{O})$-action on $\omega^{-1} \otimes_{\mathbb{C}} T$.

Recall that $t_x^* M_{|U} = \mathcal{O}_U \otimes_{\beta^\#} t_x^* M$, $M_{|X} = \mathcal{O}_X \otimes_{\beta_x^\#} M$. Thus we have maps $\beta^\# : t_x^* M \to t_x^* M_{|U}$, $\beta^\#(t) = 1_U \otimes_{\beta^\#} t$, and $\beta_x^\# : M \to M_{|X}$, $\beta_x^\#(v) = 1_X \otimes_{\beta_x^\#} v$. We set:

$$\rho(v) = \beta^\#(t_{-x}.v) = 1_U \otimes_{\beta^\#} t_{-x}.v \tag{2.6}$$

Recall also from Lemma 2.4 that $t_x^* M_{|U} = \mathcal{O}_U \otimes_{\jmath^\#} (t_x^* M / J t_x^* M)$ and $M_{|X} = \mathcal{O}_X \otimes_{\jmath_x} (M/J_x M)$. Let $\varpi : t_x^* M \twoheadrightarrow t_x^* M / J t_x^* M$ and $\varpi_x M \twoheadrightarrow M/J_x M$ be the canonical projections. It is easy to see that, since $t_x^* M / J t_x^* M = t_x^*(M/JM)$, one has $t_{-x}.(\varpi_x(v)) = \varpi(t_{-x}.v)$. One obtains from the definitions that, for all $v \in M$,

$$1_X \otimes_{\jmath_x^\#} \varpi_x(v) = 1_X \otimes_{\beta_x^\#} v, \quad \rho(v) = 1_U \otimes_{\beta^\#} t_{-x}.v = 1_U \otimes_{\jmath^\#} \varpi(t_{-x}.v).$$

It is also easily seen that the map $t_x^\# : M_{|X} \to t_x^*(M_X) = t_x^* M_{|U}$ is a bijection given by

$$t_x^\#(a \otimes_{\jmath_x^\#} \varpi_x(v)) = t_{-x}.a \otimes_{\jmath^\#} \varpi(t_{-x}.v) = (t_{-x}.a)\rho(v)$$

for all $a \in \mathcal{O}(X)$ and $v \in M$.

Recall from Lemma 2.2, and the remark thereafter, that θ_x yields an isomorphism $\mathcal{O}(G) \boxtimes \mu_x^* M \xrightarrow{\sim} \mu_x^* M$, $\theta_x(a \boxtimes t_{-x}.v) = aSv_{(1)} \otimes_{\mu_x^\#} v_{(2)}$. It follows that $\bar\theta_x = (1_G \times \beta)^*(\theta_x)$ is the isomorphism:

$$\bar\theta_x : \mathcal{O}(G) \boxtimes t_x^* M_{|U} \xrightarrow{\sim} \psi^* M, \quad \bar\theta_x(a \boxtimes \rho(v)) = aSv_{(1)} \otimes_{\psi^\#} v_{(2)}.$$

Lemma 2.9. *Let $\alpha : \{e\} \times U \hookrightarrow Y$ be the inclusion and $\varphi : \{e\} \times U \xrightarrow{\sim} X$ be the restriction of ψ. Then, $\mathcal{H}^{\dim \mathbf{O}}(\alpha^! \psi^! M) = \alpha^* \psi^* M = \varphi^* M_{|X}$ and $\bar\theta_x$ induces an isomorphism*

$$\alpha^*(\bar\theta_x) : \mathbb{C} \boxtimes t_x^* M_{|U} \xrightarrow{\sim} \varphi^* M_{|X}, \quad 1 \boxtimes (t_{-x}.a)\rho(v) \mapsto \varphi^\#(a \otimes_{\jmath_x^\#} \varpi_x(v)),$$

$a \in \mathcal{O}(X)$, $v \in M$.

Proof. Recall that

$$\alpha^! \psi^! M = \big(\mathcal{D}_{\{e\}\times U \to \mathcal{Y}} \otimes^L_{\mathcal{D}_\mathcal{Y}} \psi^! M\big)[-r] = \big(\mathcal{D}_{\{e\}\times U \to \mathcal{Y}} \otimes^L_{\mathcal{D}_\mathcal{Y}} \psi^* M\big)[m-n].$$

It follows that $\mathcal{H}^{n-m}(\alpha^!\psi^! M) = \alpha^*\psi^* M$. On the other hand, $\psi \circ \alpha = \beta_x \circ \varphi$ yields $\alpha^*\psi^* M = \varphi^*\beta_x^* M = \varphi^* M_{|X}$.

Let $v \in M$, then we have:

$$\begin{aligned}
\alpha^*(\bar\theta_x)(1 \boxtimes \rho(v)) &= 1_{\{e\}\times U} \otimes_{\alpha^\#} Sv_{(1)} \otimes_{\psi^\#} v_{(2)} \\
&= (Sv_{(1)})(e) \otimes_{(\psi\circ\alpha)^\#} v_{(2)} \\
&= 1 \otimes_{(\psi\circ\alpha)^\#} v \\
&= \varphi^\#(1_X \otimes_{\beta_x^\#} v) \\
&= \varphi^\#(1_X \otimes_{\jmath_x^\#} \varpi_x(v)),
\end{aligned}$$

as required. □

From now on, in order to simplify the notation, we will identify the $\mathcal{D}_Y$-modules $\psi^* M$ and $\mathcal{O}(G) \boxtimes t_x^* M_{|U}$ through the isomorphism $\bar\theta_x$. By Lemma 2.9, the $\mathcal{D}_{\{e\}\times U}$-module $\varphi^* M_{|X}$ then identifies with $\mathbb{C} \boxtimes t_x^* M_{|U}$ via $\varphi^\#$; this imply that we will identify the elements $1\boxtimes\rho(v)$ and $\varphi^\#(1_X \otimes_{\jmath_x} \varpi_x(v))$.

Let $o : \{0\} \hookrightarrow U$ be the inclusion and recall that $\gamma : \{x\} \hookrightarrow X$. We have a commutative diagram:

$$\begin{array}{ccc}
Y & \xrightarrow{\ \psi\ } & E \\
{\scriptstyle\alpha}\big\uparrow & & \big\uparrow{\scriptstyle\beta_x} \\
\{e\}\times U & \xrightarrow{\ \varphi\ } & X \\
{\scriptstyle\varepsilon\times o}\big\uparrow & & \big\uparrow{\scriptstyle\gamma} \\
\{e\}\times\{0\} & \xrightarrow[\ \bar\varphi\ ]{} & \{x\}
\end{array}$$

It follows that $\bar\varphi$ gives an isomrphism:

$$\imath^! M = \gamma^! \beta_x^! M \xrightarrow{\sim} (\varepsilon \times o)^! \alpha^! \psi^! M$$

Since $\mathbb{C} \boxtimes t_x^* M_{|U}$ is supported on $\{e\}\times\{0\}$ (see Proposition 2.5), $(\varepsilon\times o)^!\alpha^!\psi^! M$ has cohomology concentrated in degree $\dim \mathbf{O}$ and $\bar\varphi^\#$ yields the isomorphism

$$\bar\varphi^\# : \mathcal{H}^{\dim \mathbf{O}}(\imath^! M) \xrightarrow{\sim} \mathbb{C} \boxtimes \mathcal{H}^0(o^! t_x^* M_{|U}).$$

Let $\mathbf{n}_0 = (y_1, \dots, y_m)\mathcal{O}(F)$ be the defining ideal of the point $0 \in F$. Since $f_i = y_i - y_i(x)$, we have

$$\omega^{-1}_{\{0\}/U} = dy_1 \wedge \cdots \wedge dy_m = \omega^{-1} = df_1 \wedge \cdots \wedge df_m.$$

Theorem 2.10. *Set $T_0 = \{u \in t_x^* M / J t_x^* M : \mathbf{n}_0 u = 0\}$. Then,*

1. *$T_0 \cong \big\{u \in H^m_{[0]}(\mathcal{O}_U)^{\oplus k} : \mathbf{n}_0 u = 0\big\}$ is a vector space of dimension k and $\mathcal{H}^0(o^! t_x^* M_{|U}) = \omega^{-1} \otimes_\mathbb{C} T_0$;*
2. *the isomorphism $\bar\varphi^\# : \mathcal{H}^{\dim \mathbf{O}}(\imath^! M) \xrightarrow{\sim} \mathbb{C} \boxtimes \mathcal{H}^0(o^! t_x^* M_{|U})$ coincides with*

$$\bar\varphi^\# : \omega^{-1} \otimes_\mathbb{C} T \longrightarrow \mathbb{C} \boxtimes (\omega^{-1} \otimes_\mathbb{C} T_0), \quad \omega^{-1} \otimes \varpi_x(v) \mapsto 1 \boxtimes (\omega^{-1} \otimes \varpi(t_{-x}.v)).$$

Proof. Since $\mathcal{O}_U \otimes_{J^\#} \big(t_x^* M / J t_x^* M\big) \cong H^m_{[0]}(\mathcal{O}_U)^{\oplus k}$ (see Lemma 2.4 and Proposition 2.5), the proof of 1. is the same as the proof of Theorem 2.6. Observe in particular that

$$\big\{u \in t_x^* M_{|U} : \mathbf{n}_0 u = 0\big\} \equiv \big\{u \in t_x^* M / J t_x^* M : \mathbf{n}_0 u = 0\big\}.$$

The assertion 2. then follows from 1. and the identification of $\varphi^\#(1_X \otimes_{J^\#} \varpi_x(v))$ with $1 \boxtimes \rho(v) = 1 \boxtimes (1_U \otimes_{J^\#} \varpi(t_{-x}.v))$ □

Remark. We notice for further use the following consequence of Theorem 2.10. Let $v \in M^G$. Then, $\lambda_M(v) = 1_G \boxtimes v$ and therefore

$$\psi^\#(v) = 1_Y \otimes_{\psi^\#} v = \bar{\theta}_x(1_G \boxtimes \rho(v)).$$

Thus we may identify $\psi^\#(v)$ with $1_G \boxtimes \rho(v)$. Assume moreover that $\mathbf{n}_0 \rho(v) = 0$, then $\rho(v) = 1_U \otimes_{J^\#} \varpi(t_{-x}.v) \in \mathcal{H}^0(o^! t_x^* M_{|U})$ can be identified with $\varpi(t_{-x}.v) \in T_0$.

3. The case of the adjoint representation

In this section we consider the case where G is the adjoint group of a semisimple Lie algebra $\mathfrak{g}$ of dimension n. We are going to apply the results of §2 to the case of the adjoint action of G on $E = \mathfrak{g}$. Moreover, we will assume that the element $x \in \mathfrak{g}$ is nilpotent, hence $\mathbf{O} = G.x$ is a nilpotent orbit. We fix a coherent equivariant $\mathcal{D}_\mathfrak{g}$-module $M \in \mathfrak{M}(\mathcal{D}_\mathfrak{g}, G)$ such that $\operatorname{Supp} M = \overline{\mathbf{O}}$.

Suppose that $x = 0$. Then, by Kashiwara's equivalence [3, VI.7.11] one has $M \cong H^n_{\{0\}}(\mathcal{O}_\mathfrak{g})^{\oplus k}$ for some $k \geqslant 1$. In this case $A(\mathbf{O}) = \{e\}$ and the connection $M_{|\mathbf{O}}$ is the vector space $\mathbb{C}^k$.

Therefore, we now will suppose that $x \neq 0$. Then, we can find an S-triplet $\{x, y, z\}$ containing x, i.e. $[x, y] = z$, $[z, x] = 2x$, $[z, y] = -2y$ and $\mathfrak{s} = \mathbb{C}x + \mathbb{C}y + \mathbb{C}z \cong \mathfrak{sl}(2, \mathbb{C})$. We take $F = \mathfrak{g}^y = \{\xi \in \mathfrak{g} : [\xi, y] = 0\}$. Thus,

$$n = r = \dim \mathfrak{g}, \quad m = s = \dim \mathfrak{g}^y.$$

In this situation it is well known [13, III.5.1, III.7.4] that $x + F = x + \mathfrak{g}^y$ is a transverse slice to $\mathbf{O}$ at the point x. Thus the condition (†) of §2 is satisfied. We adopt the notation of the previous section; in particular, we have a smooth morphism $\psi : Y = G \times U \to \mathfrak{g}$ as in Proposition 2.1. We can summarize the results about the equivariant $\mathcal{D}_Y$-module $\psi^* M$ in the following theorem, see Proposition 2.5, Theorem 2.6 and Proposition 2.8. Recall that we have the natural embedding $\imath : \{x\} \overset{\imath_1}{\hookrightarrow} \mathbf{O} \overset{\imath_2}{\hookrightarrow} \mathfrak{g}$.

Theorem 3.1. (1) *The $\mathcal{D}_{G \times U}$-module $\psi^* M$ is isomorphic to $\mathcal{O}_G \boxtimes H^m_{[0]}(\mathcal{O}_U)^{\oplus k}$ for some $k \geqslant 1$.*

(2) *The $\mathcal{D}_\mathbf{O}$-module $M_{|\mathbf{O}} = \mathcal{H}^0(\imath_2^! M)$ is the connection defined by the k-dimensional representation $\mathcal{H}^{\dim \mathbf{O}}(\imath^! M) = \omega^{-1} \otimes_\mathbb{C} T$ of the finite group $A(\mathbf{O})$, where $\omega^{-1} = dy_1 \wedge \cdots \wedge dy_m$ and $T = \{\varpi_x(v) \in M / J_x M : \mathbf{n}_x \varpi_x(v) = 0\}$.*

We now want to be more explicit on the action of $A(\mathbf{O})$ on $M_{|\mathbf{O}}(x) = \omega^{-1} \otimes T$. We first recall the following "Levi decomposition" of the stabilizer G^x.

Lemma 3.2. ([1, Proposition 2.4]) *Let $G^\phi = \{g \in G : g.a = a \text{ for all } a \in \mathfrak{s}\}$ be the centralizer of the Lie subalgebra $\mathfrak{s}$ and denote by G_0^ϕ its identity component. Then, G^ϕ is reductive and there exists a semidirect product decomposition $G^x = U^x.G^\phi$,* *where U^x is a normal unipotent subgroup. Furthermore, the map $G^\phi \hookrightarrow G^x$ induces the identification $A(\mathbf{O}) = G^\phi/G^\phi_0$.*

Recall that we have chosen a decomposition $\mathfrak{g} = \mathfrak{g}^y \oplus S$. Since $F = \mathfrak{g}^y$ is G^ϕ-stable and G^ϕ is reductive, we may choose S to be G^ϕ-stable (e.g. $S = [\mathfrak{g}, x] = T_x(G.x)$). Hence, with the notation of §2, the subspaces

$$\bigoplus_{i=1}^m \mathbb{C}y_i, \quad \bigoplus_{i=m+1}^n \mathbb{C}y_i, \quad \bigoplus_{i=1}^m \mathbb{C}f_i, \quad \bigoplus_{i=m+1}^n \mathbb{C}f_i$$

are G^ϕ-stable. Observe that $\mathbb{C}\,\omega^{-1} = \mathbb{C}dy_1 \wedge \cdots \wedge dy_m$ carries a representation of G^ϕ. Furthermore, G^ϕ acts naturally on M/J_xM and T. We will need the following well known result.

Lemma 3.3. *Let $g \in G^\phi$ and $\xi \in \mathfrak{g}^\phi = \mathrm{Lie}(G^\phi)$. Then,*

1. $\det \mathrm{Ad}_{\mathfrak{g}^y}(g) = \det \mathrm{Ad}_{\mathfrak{g}/\mathfrak{g}^y}(g) = 1$*;*
2. $g.\omega^{-1} = \omega^{-1}$*;*
3. $\mathrm{tr}\,\mathrm{ad}_{\mathfrak{g}^y}(\xi) = \mathrm{tr}\,\mathrm{ad}_{\mathfrak{g}/\mathfrak{g}^y}(\xi) = 0$.

Proof. Since $\mathfrak{g}$ is semisimple, $\det \mathrm{Ad}_{\mathfrak{g}}(g) = 1$ for all $g \in G$. Recall that the Killing form B induces a symplectic form B_y on $\mathfrak{g}/\mathfrak{g}^y$ by the formula $B_y(\bar{\xi}, \bar{\eta}) = B(y, [\xi, \eta])$. It is easily seen that B_y is G^y-invariant. Hence, if $g \in G^\phi$, $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{g}^y}(g)$ belongs to the symplectic group $\mathrm{Sp}(\mathfrak{g}/\mathfrak{g}^y, B_y)$. This implies $\det \mathrm{Ad}_{\mathfrak{g}/\mathfrak{g}^y}(g) = 1$. Now, the G^ϕ-stable decomposition $\mathfrak{g} = \mathfrak{g}^y \oplus S$ with $S \cong \mathfrak{g}/\mathfrak{g}^y$ (as a G^ϕ-module) yields $\det \mathrm{Ad}_{\mathfrak{g}^y}(g) = 1$. This proves 1., and 2. follows from $g.\omega^{-1} = \det \mathrm{Ad}_{\mathfrak{g}^y}(g^{-1})\,\omega^{-1}$. The proof of 3. is similar. □

Theorem 3.4. *The representation of $A(\mathbf{O})$ on the fibre $M_{|\mathbf{O}}(x) = \omega^{-1} \otimes_{\mathbb{C}} T$ is induced by the natural G^ϕ-action on T.*

Proof. Write $\imath = \tilde{\beta}_x \circ \tilde{\gamma}$, where $\tilde{\gamma} : \{x\} \hookrightarrow x+F$ and $\tilde{\beta}_x : x+F \hookrightarrow \mathfrak{g}$. The maps $\tilde{\gamma}$ and $\tilde{\beta}_x$ being G^ϕ-equivariant, we have a natural G^ϕ-action on $M_{|\mathbf{O}}(x) = \mathcal{H}^{\dim \mathbf{O}}(\imath^! M)$ which yields the representation of the group $A(\mathbf{O})$ that we want to compute.

Set $V_x = \oplus_{i=m+1}^n \mathbb{C}df_i = \oplus_{i=m+1}^n \mathbb{C}dy_i$; then V_x is a G^ϕ-stable subspace of $\mathfrak{g}^*$. We have seen in the proof of Lemma 2.4 that (setting $\partial_{II} = \partial_x$),

$$\tilde{\beta}_x^! M = \big(\mathcal{C}_{II}^\bullet = \mathcal{C}_x^\bullet \otimes_{\mathcal{D}_\mathfrak{g}} M, \partial_{II}\big)[m-n].$$

Let G^ϕ act diagonally on $\mathcal{C}_{II}^p = \mathcal{C}_x^{p+m-n} \otimes_{\mathcal{D}_\mathfrak{g}} M = \bigwedge^{-p+\dim \mathbf{O}} V_x \boxtimes M$. Then, ∂_{II} is G^ϕ-equivariant and the G^ϕ-action on the cohomology group $\mathcal{H}^j(\tilde{\beta}_x^! M) \in \mathfrak{M}(\mathcal{D}_{x+F}, G^\phi)$ is induced by the diagonal action of G^ϕ on $\mathcal{C}_{II}^\bullet$ (see §1). Notice, in particular, that

$$\mathcal{H}^{\dim \mathbf{O}}(\tilde{\beta}_x^! M) = \mathrm{Tor}_0^{\mathcal{O}_\mathfrak{g}}(\mathcal{O}_{x+F}, M) = M/J_xM$$

is endowed with the natural action of G^ϕ. By (2.3), $\mathcal{H}^j(\beta_x^! M) = \mathcal{O}_X \otimes_{\mathcal{O}_{x+F}} \mathcal{H}^j(\tilde{\beta}_x^! M) = 0$ when $j \neq \dim \mathbf{O}$. Thus,

$$\mathrm{Supp}\,\mathcal{H}^j(\tilde{\beta}_x^! M) \subseteq (x+F) \setminus X \subset (x+F) \setminus \{x\} \quad \text{if } j \neq \dim \mathbf{O}.$$

Now,

$$\imath^! M = \tilde{\gamma}^! \tilde{\beta}_x^! M = \big(\mathcal{D}_{\{x\} \to x+F} \otimes^L_{\mathcal{D}_{x+F}} \tilde{\beta}_x^! M\big)[-m]$$

can be computed as follows. Notice that $\mathbf{n}_x/\mathbf{n}_x^2 = \oplus_{j=1}^m \mathbb{C}df_j$ and consider the complex $\big(\mathcal{C}_I^\bullet, \partial_I\big)$ where $\mathcal{C}_I^p = \bigwedge^{-p}(\mathbf{n}_x/\mathbf{n}_x^2) \boxtimes \mathcal{D}_{x+F}$ and

$$\partial_I(df_{j_1} \wedge \cdots \wedge df_{j_p} \boxtimes D) = \textstyle\sum_{s=1}^p (-1)^{s+1} df_{j_1} \wedge \cdots \wedge \widehat{df_{j_s}} \wedge \cdots \wedge df_{j_p} \boxtimes f_{j_s} D.$$

Observe that G^ϕ acts diagonally on $\mathcal{C}_I^p$ and that ∂_I is G^ϕ-equivariant. Let $\big(\mathcal{C}_{\rm tot}^\bullet, \partial_{\rm tot}\big)$ be the total complex associated to the double complex $\mathcal{C}^\bullet = \mathcal{C}_I^\bullet \otimes_{\mathcal{D}_{x+F}} \mathcal{C}_{II}^\bullet$. Then,

$$\imath^! M = \tilde{\gamma}^! \tilde{\beta}_x^! M = \big(\mathcal{C}_{\rm tot}^\bullet, \partial_{\rm tot}\big)[-m]$$

and therefore $\mathcal{H}^j(\imath^! M) = \mathcal{H}^{j-m}(\mathcal{C}_{\rm tot}^\bullet)$. This group is computed by the spectral sequence:

$$E_2^{pq} = \mathcal{H}_I^p(\mathcal{H}_{II}^q(\mathcal{C}^\bullet)) \Longrightarrow \mathcal{H}^{p+q}(\mathcal{C}_{\rm tot}^\bullet)$$

But, $E_2^{pq} = \mathrm{Tor}_{-p}^{\mathcal{O}_{x+F}}(\mathbb{C}_x, \mathcal{H}^q(\mathcal{C}_{II}))$ as $\mathcal{O}_{x+F}$-module, and we have noticed that the support of $\mathcal{H}^q(\mathcal{C}_{II}) = \mathcal{H}^q(\tilde{\beta}_x^! M)$ is contained in $(x+F) \setminus \{x\}$ when $q \neq \dim \mathbf{O}$. Therefore $E_2^{pq} = 0$ for all $q \neq \dim \mathbf{O}$ and $E_2^{p \dim \mathbf{O}} = \mathrm{Tor}_{-p}^{\mathcal{O}_{x+F}}(\mathbb{C}_x, \mathcal{H}^{\dim \mathbf{O}}(\tilde{\beta}_x^! M))$. Hence, the spectral sequence E_2^{pq} collapses to $E_2^{p \dim \mathbf{O}} = \mathcal{H}^{p+\dim \mathbf{O}}(\mathcal{C}_{\rm tot}^\bullet)$. In particular, we obtain

$$\begin{aligned}\mathcal{H}^{\dim \mathbf{O}}(\imath^! M) &= \mathcal{H}^{\dim \mathbf{O} - m}(\mathcal{C}_{\rm tot}^\bullet) = \mathrm{Tor}_m^{\mathcal{O}_{x+F}}(\mathbb{C}_x, \mathcal{H}^{\dim \mathbf{O}}(\tilde{\beta}_x^! M)) \\ &= \mathrm{Tor}_m^{\mathcal{O}_{x+F}}(\mathbb{C}_x, M/J_x M) = \omega^{-1} \otimes_{\mathbb{C}} T\end{aligned}$$

(as expected). Furthermore, the group G^ϕ acts diagonally on the complexes $\mathcal{C}^\bullet$, $\mathcal{C}_{\rm tot}$ and it follows from the previous computation that the action of $A(\mathbf{O})$ on $\mathcal{H}^{\dim \mathbf{O}}(\imath^! M)$ is coming from the induced action of G^ϕ on $E_2^{-m \dim \mathbf{O}} = \omega^{-1} \otimes_{\mathbb{C}} T$. Then, by Lemma 3.3,

$$g.(\omega^{-1} \otimes \varpi_x(v)) = g.\omega^{-1} \otimes g.\varpi_x(v) = \omega^{-1} \otimes \varpi_x(g.v)$$

for all $\varpi_x(v) \in T$. Hence the result. □

Remark. A consequence of Theorem 3.4 is that the identity component G_0^ϕ acts trivially on $\omega^{-1} \otimes_{\mathbb{C}} T$. It is not difficult to prove this fact directly. Denote by $\tau_\phi : \mathfrak{g}^\phi \to \mathrm{Der}\, \mathcal{O}(x+F)$ the differential of the (adjoint) action of G^ϕ on $x+F$ (thus $\tau_\phi = \tau_{x+F}$ in the notation of §1). Let $\xi \in \mathfrak{g}^\phi$. Since $\mathcal{O}(x+F) = \mathcal{O}(\mathfrak{g})/J_x \mathcal{O}(\mathfrak{g}) \equiv \mathbb{C}[y_1, \dots, y_m]$, we may write $\tau_\phi(\xi) = \sum_{j=1}^m \xi_j \partial_j$, where $\partial_j = \frac{\partial}{\partial y_j}$, $1 \leqslant j \leqslant m$. Lemma 3.3 and a straightforward computation yield

$$\tau_\phi(\xi) = \textstyle\sum_{j=1}^m \partial_j \xi_j + \mathrm{tr}\,\mathrm{ad}_{\mathfrak{g}^y}(\xi) = \sum_{j=1}^m \partial_j \xi_j.$$

Notice that $\xi_j(x) = y_j([x,\xi]) = 0$, hence $\xi_j \in \mathbf{n}_x$. Recall that $M/J_x M = \mathcal{H}^{\dim \mathbf{O}}(\tilde{\beta}_x^! M) \in \mathfrak{M}(\mathcal{D}_{x+F}, G^\phi)$. Then, for all $\varpi_x(v) \in T \subset M/J_x M$,

$$\begin{aligned}\xi.(\omega^{-1} \otimes \varpi_x(v)) &= \frac{d}{dt}_{|t=0}(e^{t\xi}.\omega^{-1} \otimes e^{t\xi}.\varpi_x(v)) \\ &= \omega^{-1} \otimes \frac{d}{dt}_{|t=0}(e^{t\xi}.\varpi_x(v)) \quad \text{(by Lemma 3.3)} \\ &= \omega^{-1} \otimes \tau_\phi(\xi)\varpi_x(v) \quad (\text{since } M/J_x M \in \mathfrak{M}(\mathcal{D}_{x+F}, G^\phi)) \\ &= \textstyle\sum_{j=1}^m \omega^{-1} \otimes \partial_j \xi_j \varpi_x(v) \\ &= 0.\end{aligned}$$

Thus $\mathfrak{g}^\phi = \mathrm{Lie}(G_0^\phi)$ acts trivially on $\omega^{-1} \otimes_{\mathbb{C}} T$ and the result follows.

We end these notes by the following particular case of Theorem 3.4. Recall that $\mathbf{n}_0 = (y_1, \dots, y_m)\mathcal{O}(F)$ and $\rho(v) = 1_U \otimes_{\jmath^\#} \varpi(t_{-x}.v)$.

Corollary 3.5. *Let $M = \mathcal{D}_{\mathfrak{g}} v \in \mathfrak{M}(\mathcal{D}_{\mathfrak{g}}, G)$ with $v \in M^G$. Then, $t_x^* M_{|U} = \mathcal{D}_U \rho(v)$ and $\psi^* M = \mathcal{D}_Y(1_G \boxtimes \rho(v))$. Furthermore, if $k = 1$ and $\mathbf{n}_0 \rho(v) = 0$, then we have*

(i) $T = \mathbb{C}\varpi_x(v)$;

(ii) *the representation of* $A(\mathbf{O})$ *on* $M_{|\mathbf{O}}(x)$ *is the trivial representation and the connection* $M_{|\mathbf{O}}$ *is isomorphic to the standard* $\mathcal{D}_{\mathbf{O}}$*-module* $\mathcal{O}_{\mathbf{O}}$.

Proof. Since ψ is smooth, it is easy to see [10, Lemma 3.2] that $\psi^* M = \mathcal{D}_Y \psi^{\#}(v)$. As explained in §2 (cf. Lemma 2.9, Theorem 2.10 and Remark at the end of §2) we may identify the $\mathcal{D}_Y$-module $\psi^* M$ with $\mathcal{O}(G) \boxtimes t_x^* M_{|U}$, and, since $v \in M^G$, $\psi^{\#}(v)$ identifies with $1_G \boxtimes \rho(v)$. Thus,

$$\psi^* M = \mathcal{O}_G \boxtimes t_x^* M_{|U} = \mathcal{D}_{G\times U}(1_G \boxtimes \rho(v)) = \mathcal{O}_G \boxtimes \mathcal{D}_U \rho(v),$$

proving the first assertions of the corollary.

Now, assume that $k = 1$ and $\mathbf{n}_0\rho(v) = 0$. Then, $t_x^* M_{|U} \cong H^m_{[0]}(\mathcal{O}_U)$ and $\rho(v)$ identifies with $\varpi(t_{-x}.v)$ inside $T_0 = \{u \in t_x^* M_{|U} : \mathbf{n}_0 u = 0\}$ (loc. cit.). Since $\dim T_0 = 1$ and $\rho(v) \neq 0$, we obtain $T_0 = \mathbb{C}\rho(v)$. It follows then from Theorem 2.10 that $\omega^{-1} \otimes_{\mathbb{C}} T = \mathbb{C}(\omega^{-1} \otimes \varpi_x(v))$. By Theorem 3.4, since $v \in M^G$, the group G^ϕ acts trivially on $M_{|\mathbf{O}}(x) = \omega^{-1} \otimes_{\mathbb{C}} T$. The isomorphism $M_{|\mathbf{O}} \cong \mathcal{O}_{\mathbf{O}}$ then follows from Proposition 2.8. □

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