DIFFERENTIAL OPERATORS COMMUTING WITH INVARIANT FUNCTIONS

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Abstract. Let $\mathfrak{g}$ be a reductive, complex Lie algebra, with adjoint group G, let G act on the ring of differential operators $\mathcal{D}(\mathfrak{g})$ via the adjoint action and write $\tau : \mathfrak{g} \to \mathcal{D}(\mathfrak{g})$ for the differential of this action. We prove that the commutant, in $\mathcal{D}(\mathfrak{g})$, of $\mathcal{O}(\mathfrak{g})^G$ is the algebra generated by $\mathcal{O}(\mathfrak{g})$ and $\tau(\mathfrak{g})$, thereby answering a question of Barlet.

1. Introduction

Fix a reductive, complex Lie algebra $\mathfrak{g}$, with adjoint group G, let G act on the ring of differential operators $\mathcal{D}(\mathfrak{g})$ via the adjoint action and write $\tau : \mathfrak{g} \to \mathcal{D}(\mathfrak{g})$ for the differential of this action. We identify $\mathcal{O}(\mathfrak{g})$, the ring of regular functions on $\mathfrak{g}$, with $S(\mathfrak{g}^*)$ and let $\mathcal{O}(\mathfrak{g})^G$ denote the subalgebra of G-invariant functions. The aim of this note is to prove:

Theorem 1.1. *The commutant $\mathcal{C} = \mathcal{C}_{\mathcal{D}(\mathfrak{g})}(\mathcal{O}(\mathfrak{g})^G)$, in $\mathcal{D}(\mathfrak{g})$, of $\mathcal{O}(\mathfrak{g})^G$ is the algebra generated by $\mathcal{O}(\mathfrak{g})$ and $\tau(\mathfrak{g})$.*

At the level of vector fields, this result follows from [5, Theorem 2.1], in the sense that Dixmier's result implies that $\mathcal{C} \cap \operatorname{Der} \mathcal{O}(\mathfrak{g}) = \mathcal{O}(\mathfrak{g})\tau(\mathfrak{g})$. In [2], D. Barlet raised the question of whether Theorem 1.1 is true, since this would form a natural generalization of Dixmier's result. In the same paper, Barlet was able to prove the theorem in the case when $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. We would like to thank M. Raïs for bringing Barlet's question to our attention.

In the process of proving Theorem 1.1, we obtain a considerable amount of information about the structure of $\mathcal{C}$. Some particular properties are given in the next result. The unexplained definitions can be found in Section 3.

Proposition 1.2. *$\mathcal{C}$ is an Auslander-Gorenstein, CM domain and a maximal order in its quotient division ring.*

In fact, the main theorem of this paper is a result about commutative rings. To state this, let A denote the subalgebra of $\mathcal{D}(\mathfrak{g})$ generated by $\mathcal{O}(\mathfrak{g})$ and $\tau(\mathfrak{g})$ and set $E = \mathcal{O}(\mathfrak{g})\tau(\mathfrak{g}) \subset \operatorname{Der} \mathcal{O}(\mathfrak{g})$. If one filters $\mathcal{D}(\mathfrak{g})$ and its subalgebras by degree of differential operators, then it is easy to see that the associated graded rings $\operatorname{gr} A$ and $\operatorname{gr} \mathcal{C}$ are domains with the same quotient field as the symmetric algebra $\operatorname{Sym}_{\mathcal{O}(\mathfrak{g})}(E)$. Then, Theorem 1.1 and Proposition 1.2 follow easily from the following result.

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Theorem 1.3. (i) *Let* $E = \mathcal{O}(\mathfrak{g})\tau(\mathfrak{g}) \subset \operatorname{Der}\mathcal{O}(\mathfrak{g})$. *Then* $\operatorname{Sym}_{\mathcal{O}(\mathfrak{g})}(E)$ *is a factorial, complete intersection of Krull dimension* $2\dim\mathfrak{g} - \operatorname{rk}\mathfrak{g}$.

(ii) $\operatorname{gr} A = \operatorname{gr}\mathcal{C} = \operatorname{Sym}_{\mathcal{O}(\mathfrak{g})}(E)$.

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2. The symmetric algebra of the module generated by $\tau(\mathfrak{g})$

In this section we prove Theorem 1.3 from the introduction. We begin with some preliminary notation and results.

As before, we fix a complex, reductive Lie algebra $\mathfrak{g}$ of dimension n and rank ℓ. Write G for the adjoint group of $\mathfrak{g}$. Define the categorical quotient of $\mathfrak{g}$ by $\mathcal{O}(\mathfrak{g}/\!\!/G) = \operatorname{Spec}\mathcal{O}(\mathfrak{g})^G$ and let $u : \mathfrak{g} \to \mathfrak{g}/\!\!/G$ denote the quotient morphism. We will write $\mathcal{O}$ for $\mathcal{O}(\mathfrak{g})$. Define

$$\mathcal{X}_i \;=\; \{y \in \mathfrak{g} \;:\; \operatorname{rk} d_y u \le i\},$$

where $d_y u : T_y\mathfrak{g} \to T_{u(y)}\mathfrak{g}/\!\!/G$ denotes the differential of u. Observe that each $\mathcal{X}_i$ is a closed G-subvariety of $\mathfrak{g}$. Recall that $y \in \mathfrak{g}$ is called *regular* if its centralizer in $\mathfrak{g}$ is of dimension ℓ. Then [10, Theorem 10.1], $\operatorname{rk} d_y u = \ell$ if and only if y is regular.

We would like to thank D. Panyushev for the proof of the following proposition, which is considerably easier than our original proof.

Proposition 2.1. *One has:* $\operatorname{codim}\mathcal{X}_i \ge \ell - i + 2$, *for* $0 \le i \le \ell - 1$.

Proof. Notice that u induces a surjective morphism $\varpi : \mathcal{X}_i \to \mathcal{X}_i/\!\!/G$ and that, for all $x \in \mathcal{X}_i$, the differential $d_x\varpi$ is the restriction of $d_x u$ to $T_x\mathcal{X}_i$. Set $r = \max\{\operatorname{rk} d_x\varpi : x \in \mathcal{X}_i\}$. Then, by [7, Proposition III.10.6] and the definition of $\mathcal{X}_i$, we obtain that $\dim \mathcal{X}_i/\!\!/G \le r \le i$.

Since $\mathcal{X}_i$ is stable under the $\mathbb{C}^*$-action $y \mapsto \lambda y$, $\lambda \in \mathbb{C}^*$, the point 0 belongs to each irreducible component of $\mathcal{X}_i$. Hence, $\dim\mathcal{X}_i \le \dim\mathcal{X}_i/\!\!/G + \dim\varpi^{-1}(\varpi(0))$ (see [11, AI.3.3] or [7, Ex. II.3.22]). But $\varpi^{-1}(\varpi(0)) = \mathcal{X}_i \cap \mathbf{N}$, where $\mathbf{N}$ denotes the nilpotent cone of $\mathfrak{g}$, and, since $i \le \ell - 1$, $\mathcal{X}_i \cap \mathbf{N}$ is contained in the subvariety of non-regular nilpotent elements. Therefore $\dim\varpi^{-1}(\varpi(0)) \le n - \ell - 2$ and it follows that $\dim\mathcal{X}_i \le i + n - \ell - 2$, as required. □

Remark . Proposition 2.1 generalizes the well-known fact that $\mathcal{X}_{\ell-1}$ has codimension at least three (see, for example, [18, Theorem 4.12]). It is natural to conjecture that Proposition 2.1 can be improved to the statement that $\operatorname{codim}\mathcal{X}_i \ge 3(\ell - i)$ for $0 \le i \le \ell - 1$. D. Panyushev informs us that he has been able to prove this by a case by case analysis.

Fix a G-invariant, non-degenerate, symmetric bilinear form κ on $\mathfrak{g}$ and let $\tilde\kappa : \mathfrak{g}^* \to \mathfrak{g}$ be the induced isomorphism. Thus, $\tilde\kappa$ induces an isomorphism between differential one-forms on $\mathfrak{g}$ and vector fields on $\mathfrak{g}$. If $f \in \mathcal{O}^G$, then we define a G-invariant vector field $\operatorname{grad}(f) \in \mathcal{O} \otimes_{\mathbb{C}} \mathfrak{g}$ to be the image of df under $\widetilde{\kappa}$. Equivalently, if we fix an orthonormal basis $\{e_i\}$ of $\mathfrak{g}$ and write $x_i = e_i^* \in \mathfrak{g}^*$, then

$$\operatorname{grad}(f) = \textstyle\sum_{j=1}^n \frac{\partial f}{\partial x_j} \otimes e_j = \sum_{j=1}^n \frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_j}. \tag{2.1}$$

By Chevalley's Theorem, $\mathcal{O}^G$ is a polynomial ring, say $\mathcal{O}^G = \mathbb{C}[u_1, \ldots, u_\ell]$ for homogeneous, algebraically independent polynomials $\{u_i\}_i$. Set $\nabla_i = \operatorname{grad}(u_i)$, for

$1 \le i \le \ell$. If $\tau : \mathfrak{g} \to \operatorname{Der} \mathcal{O}$ is the differential of the adjoint action of G on $\mathfrak{g}$, then write $E = \mathcal{O}\tau(\mathfrak{g})$. We will also write τ for the induced map:

$$\tau : \mathcal{O} \otimes_{\mathbb{C}} \mathfrak{g} \longrightarrow E \;\subseteq\; \operatorname{Der} \mathcal{O}.$$

Notice that if $\theta \in \mathcal{O} \otimes_{\mathbb{C}} \mathfrak{g}$, the vector field $\tau(\theta)$ is given by $\tau(\theta)_y = [y, \theta_y]$ for all $y \in \mathfrak{g}$. It follows easily that if θ is G-invariant, then $\tau(\theta) = 0$. In particular, one has $\tau(\nabla_i) = 0$ for all i. In fact rather more is true:

Lemma 2.2. *There is a short exact sequence*

$$0 \longrightarrow \bigoplus_{i=1}^{\ell} \mathcal{O}\nabla_i \longrightarrow \mathcal{O} \otimes_{\mathbb{C}} \mathfrak{g} \xrightarrow{\tau} E \longrightarrow 0. \tag{2.2}$$

Proof. This is [16, Theorem 2.5.4]. Using the identification of $\mathfrak{g}$ with $\mathfrak{g}^*$ under $\tilde{\kappa}$, it also follows from [14, Theorem 1.9]. $\square$

Corollary 2.3. *If* $\operatorname{Sym}_{\mathcal{O}}(E)$ *denotes the symmetric algebra of the* $\mathcal{O}$*-module* E*, then,* $\operatorname{Sym}_{\mathcal{O}}(E) \;\cong\; \operatorname{Sym}_{\mathcal{O}}(\mathcal{O} \otimes_{\mathbb{C}} \mathfrak{g})/(\nabla_1, \dots, \nabla_\ell)$.

Proof. This follows from the universal property of symmetric algebras. $\square$

Set $\operatorname{Sym}(E) = \operatorname{Sym}_{\mathcal{O}}(E)$. The main aim of this section is to understand the structure of $\operatorname{Sym}(E)$, for which we use the results from [1] and [8].

Let $I_t(\mathbf{u})$ be the ideal generated by the $t \times t$ minors of the matrix $\mathbf{u} = \left[\frac{\partial u_i}{\partial x_j}\right]$ and consider the following condition for $s \ge 0$:

$$\operatorname{ht} I_t(\mathbf{u}) \;\ge\; \ell - t + 1 + s, \quad \text{for } 1 \le t \le \ell. \tag{$\mathcal{F}_s$}$$

Observe that, if we regard the short exact sequence (2.2) as a sequence

$$0 \longrightarrow \mathcal{O}^\ell \xrightarrow{\beta} \mathcal{O}^n \longrightarrow E \longrightarrow 0,$$

then (2.1) implies that $I_t(\mathbf{u})$ is the ideal generated by the $t \times t$ minors of the map β. Thus, the ideals $I_{n-t}(\mathbf{u})$ are nothing more than the Fitting ideals of E (see, for example, [17, 1.1]). In particular, they are independent of the presentation of E and our condition $(\mathcal{F}_s)$ coincides with that of [8].

Proposition 2.4. (i) The condition $(\mathcal{F}_2)$ is satisfied by E.

(ii) $\operatorname{Sym}(E)$ is a factorial domain of Krull dimension $2n - \ell$. In particular, $\operatorname{Sym}(E)$ is a complete intersection and is Gorenstein.

(iii) If P is a prime ideal of $\mathcal{O}$ with $\operatorname{ht} P \ge 2$, then $\operatorname{ht} P \operatorname{Sym}(E) \ge 2$.

Proof. Write $\widetilde{\mathcal{X}}_{i-1}$ for the zero set of $I_i(\mathbf{u})$; thus

$$\widetilde{\mathcal{X}}_{i-1} = \left\{ x \in \mathfrak{g} : \operatorname{rk}\left(\nabla_1(x), \dots, \nabla_\ell(x)\right) \le i - 1 \right\}.$$

Since the ∇_j are the images of the du_j under the isomorphism $\tilde{\kappa}$, clearly $\widetilde{\mathcal{X}}_{i-1} = \left\{ x \in \mathfrak{g} : \operatorname{rk}\left(d_x u_1, \dots, d_x u_\ell\right) \le i - 1 \right\}$. Since $u_1, \dots, u_\ell$ define the quotient map $u : \mathfrak{g} \to \mathfrak{g}/\!\!/G$, this implies that $\widetilde{\mathcal{X}}_i = \mathcal{X}_i$. Hence, part (i) is a reformulation of Proposition 2.1.

By Lemma 2.2, E has projective dimension at most 1. Thus, part (ii) follows from part (i), combined with [1, Propositions 3 and 6]. By [8, Remarks, pp. 664-5], the condition of part (iii) is equivalent to the condition $(\mathcal{F}_2)$. $\square$

We end this section by giving the geometric significance of Proposition 2.4. This should be compared with [9, § 2] which proves weaker results for much more general G-varieties.

The map τ induces a homomorphism of algebras

$$\tilde{\tau} : \mathcal{O}(\mathfrak{g} \times \mathfrak{g}^*) = \mathrm{Sym}_{\mathcal{O}}(\mathcal{O} \otimes \mathfrak{g}) \longrightarrow \mathcal{O}(T^*\mathfrak{g}) = \mathrm{Sym}_{\mathcal{O}}(\mathrm{Der}\,\mathcal{O}).$$

Clearly, the image of $\tilde{\tau}$ is the subring $\mathcal{O}[\tau(\mathfrak{g})]$ of $\mathrm{Sym}_{\mathcal{O}}(\mathrm{Der}\,\mathcal{O})$ generated by $\mathcal{O}$ and $\tau(\mathfrak{g})$. After identification of $\mathfrak{g}^*$ with $\mathfrak{g}$ through $\tilde{\kappa}$, the associated morphism to $\tilde{\tau}$ is:

$$\nu : T^*\mathfrak{g} \cong \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \times \mathfrak{g}, \qquad \nu(x, y) = (x, [y, x])$$

Let $\tilde{T}\mathfrak{g}$ denote the closure of the image of ν; thus, $\tilde{T}\mathfrak{g}$ is an irreducible affine subvariety of $\mathfrak{g} \times \mathfrak{g}$ with coordinate ring $\mathcal{O}(\tilde{T}\mathfrak{g}) \cong \mathcal{O}[\tau(\mathfrak{g})]$.

Corollary 2.5. (i) $\mathrm{Sym}(E) = \mathcal{O}(\tilde{T}\mathfrak{g})$.

(ii) The variety $\tilde{T}\mathfrak{g}$ is a factorial complete intersection in $\mathfrak{g} \times \mathfrak{g}$.

Proof. By universality, $\tilde{\tau}$ induces a surjective morphism $\pi : \mathrm{Sym}(E) \twoheadrightarrow \mathcal{O}[\tau(\mathfrak{g})]$. If we prove that $\dim \tilde{T}\mathfrak{g} \geq 2n - \ell$, then the corollary will follow from Proposition 2.4(ii).

Let $\rho : \tilde{T}\mathfrak{g} \twoheadrightarrow \mathfrak{g}$ denote the projection onto the first factor. By [11, AI.3.3] there exists a dense open subset $U \subseteq \tilde{T}\mathfrak{g}$ such that $\dim \tilde{T}\mathfrak{g} = \dim \mathfrak{g} + \dim \rho^{-1}(\rho(u))$ for all $u \in U$. Since $\tilde{T}\mathfrak{g}$ is irreducible, we can pick $u = (x, y) \in \rho^{-1}(\mathfrak{g}') \cap U$, where $\mathfrak{g}'$ denotes the set of generic elements in $\mathfrak{g}$. Now,

$$\rho^{-1}(\rho(u)) \supseteq \rho^{-1}(\rho(u)) \cap \mathrm{Im}\,\nu = \{(x, [\mathfrak{g}, x])\}.$$

Since x is generic, $\dim \rho^{-1}(\rho(u)) \geq \dim[\mathfrak{g}, x] = n - \ell$ and the result follows. □

3. The commutant of $\mathcal{O}(\mathfrak{g})^G$

As usual, we identify $\mathcal{D}(\mathfrak{g})$, as a vector space, with $\mathcal{O} \otimes_{\mathbb{C}} S(\mathfrak{g})$, where $\mathcal{O} = \mathcal{O}(\mathfrak{g})$ and the symmetric algebra $S(\mathfrak{g})$ is identified with the constant coefficient differential operators on $\mathfrak{g}$. We will always filter $\mathcal{D}(\mathfrak{g})$ by degree of differential operators and so, as algebras, $\mathrm{gr}\,\mathcal{D}(\mathfrak{g}) = \mathrm{Sym}_{\mathcal{O}}(\mathrm{Der}\,\mathcal{O}) \cong \mathcal{O} \otimes_{\mathbb{C}} S(\mathfrak{g})$. Write A for the subring of $\mathcal{D}(\mathfrak{g})$ generated by $\mathcal{O}$ and $\tau(\mathfrak{g})$ and let $\mathcal{C}$ denote the commutant of $\mathcal{O}^G$, as in the introduction. Obviously, A is contained in $\mathcal{C}$.

Lemma 3.1. *Let $x \in \mathfrak{g}$ be a regular point and set $R = \mathcal{O}_{\mathfrak{g},x}$ for the local ring of $\mathfrak{g}$ at x. Then, there exists a basis of derivations $\{\partial_i : 1 \leq i \leq n\}$ of $\mathrm{Der}\,R$ such that $\partial_i(u_j) = \delta_{ij}$ for all $1 \leq i, j \leq \ell$ and $R\tau(\mathfrak{g}) = \bigoplus_{i=\ell+1}^{n} R\partial_i$.*

Proof. Let $\mathbf{m}$ denote the maximal ideal of R. By [10, Theorem 0.1], the $\{d_x u_i : 1 \leq i \leq \ell\}$ are linearly independent. The $\{d_x u_i\}$ may also be regarded as elements of $\mathbf{m}/\mathbf{m}^2$, under the usual identification of $T_x^*\mathfrak{g}$ with $\mathbf{m}/\mathbf{m}^2$. Thus, for some scalars λ_i, the set $\{u_1 - \lambda_1, \ldots, u_\ell - \lambda_\ell\}$ is part of a system of parameters, say $\{z_1 = u_1 - \lambda_1, \ldots, z_\ell = u_\ell - \lambda_\ell, z_{\ell+1}, \ldots, z_n\}$ for $\mathbf{m}$. Let $\partial_i \in \mathrm{Der}\,R$ be defined by $\partial_i(z_j) = \delta_{ij}$.

If $D \in \mathrm{Der}\,R$, then $\widetilde{D} = D - \sum_{i=1}^{\ell} D(u_i)\partial_i$ satisfies $\widetilde{D}(u_j) = 0$, for $1 \leq j \leq \ell$. Thus, $\widetilde{D}(\mathcal{O}^G) = 0$ and so, by [5, Theorem 2.1] (or directly), $\widetilde{D} \in R\tau(\mathfrak{g})$. Hence, $\mathrm{Der}\,R = R\tau(\mathfrak{g}) \oplus \left(\oplus_{i=1}^{\ell} R\partial_i\right)$. Since

$$R\tau(\mathfrak{g}) \subseteq \left\{D \in \mathrm{Der}\,R : D(u_j) = 0 \text{ for } 1 \leq j \leq \ell\right\} = \bigoplus_{i=\ell+1}^{n} R\partial_i,$$

the result follows. □

Theorem 3.2. *Let A and $\mathcal{C}$ be given the filtrations induced from that on $\mathcal{D}(\mathfrak{g})$. Then* $\operatorname{gr} A = \operatorname{gr} \mathcal{C} \cong \operatorname{Sym}(E)$.

Proof. Since $\operatorname{gr} \mathcal{C} \subset \operatorname{gr} \mathcal{D}(\mathfrak{g}) \cong \mathcal{O}(T^*\mathfrak{g})$, certainly $\operatorname{gr} A \subseteq \operatorname{gr} \mathcal{C}$ are domains. Also, as $\tau(\mathfrak{g})$ consists of derivations, we may regard $\tau(\mathfrak{g}) \subseteq \operatorname{Der} \mathcal{O} \subseteq \operatorname{gr} \mathcal{D}(\mathfrak{g})$. Hence the ring $\mathcal{O}[\tau(\mathfrak{g})]$ is contained in $\operatorname{gr} A$ and, by Corollary 2.5(i), the natural map $\pi : \operatorname{Sym}(E) \to \mathcal{O}[\tau(\mathfrak{g})]$ is an isomorphism.

Let $x \in \mathfrak{g}$ be a regular point and let $\mathcal{S} = \{f \in \mathcal{O} : f(x) \neq 0\}$. Given a ring C containing $\mathcal{O}$, we write C_x for the localization $C_{\mathcal{S}}$ (given that it exists). Then, we claim that

$$(\operatorname{gr} A)_x = (\operatorname{gr} \mathcal{C})_x \cong \operatorname{Sym}(E)_x, \tag{3.1}$$

where the isomorphism is induced by π^{-1}.

By mimicking the proof of Richardson's Lemma [11, II.3.4], one can show that this suffices to prove the theorem. In more detail, assume that (3.1) is true. Since $\operatorname{gr} \mathcal{C}$ and $\operatorname{Sym}(E)$ are domains, (3.1) certainly implies that

$$\operatorname{Sym}(E) \overset{\pi}{\hookrightarrow} \operatorname{gr} A \subseteq \operatorname{gr} \mathcal{C}$$

and that $\operatorname{gr} \mathcal{C}$ and $\operatorname{Sym}(E)$ have the same field of fractions. Moreover, $\{x \in \mathfrak{g} : (\operatorname{gr} \mathcal{C})_x \neq \operatorname{Sym}(E)_x\}$ is contained in the set of non-regular elements of $\mathfrak{g}$. By [10, Theorem 0.1], this is precisely the subspace $\mathcal{X}_{\ell-1}$ and, by Proposition 2.1 or [18, Theorem 4.12], $\operatorname{codim} \mathcal{X}_{\ell-1} \geq 3$. Thus, for any $b \in \operatorname{gr} \mathcal{C}$, there exists an ideal I of $\mathcal{O}$ of height at least 3 such that $bI \subseteq \operatorname{Sym}(E)$. By Proposition 2.4(iii), $\operatorname{ht}_{\operatorname{Sym}(E)} I \operatorname{Sym}(E) \geq 2$. Hence, $b \in \operatorname{Sym}(E)_{\mathbf{p}}$ for every height one prime $\mathbf{p}$ of $\operatorname{Sym}(E)$. Since $\operatorname{Sym}(E)$ is Cohen-Macaulay, it satisfies the (S_2)-condition [12, p. 125], and therefore $b \in \operatorname{Sym}(E)$.

Thus, it remains to prove (3.1). Let $R = \mathcal{O}_x = \mathcal{O}_{\mathfrak{g},x}$ and keep the notation of Lemma 3.1. It is immediate from that lemma that $D \in \mathcal{D}(\mathfrak{g})_x$ satisfies $[D, u_j] = 0$ if and only if $D \in R\langle \partial_j : j \neq i\rangle$. Consequently, $\mathcal{C}_x = A_x = R\langle \partial_{\ell+1}, \dots, \partial_n\rangle$.

Let $\overline{\partial}_k$ denote the image of ∂_k in $\operatorname{gr} \mathcal{D}(\mathfrak{g})$. Obviously, Lemma 3.1 also implies that $R\tau(\mathfrak{g}) = \bigoplus_{k=\ell+1}^{n} R\overline{\partial}_k$, where $R\tau(\mathfrak{g})$ is now regarded as a subspace of $\operatorname{Der} R \subset \operatorname{gr} \mathcal{D}(\mathfrak{g})_x \cong R \otimes_{\mathbb{C}} S(\mathfrak{g})$. Thus,

$$\operatorname{gr} \mathcal{C}_x = \operatorname{gr} A_x = \operatorname{gr} R\langle \partial_{\ell+1}, \dots, \partial_n\rangle = R[\overline{\partial}_{\ell+1}, \dots, \overline{\partial}_n] = R[\tau(\mathfrak{g})].$$

Since $\pi : \operatorname{Sym}(E)_x \to R[\tau(\mathfrak{g})]$ is an isomorphism, this completes the proof of (3.1) and hence of the theorem. □

The Gelfand-Kirillov dimension of a module M will be denoted $\operatorname{GKdim} M$. If a Noetherian ring A has finite injective dimension, then A is called *Auslander-Gorenstein* if A satisfies the following condition: For any integers $0 \leq i < j$ and finitely generated (right) A-module M, one has $\operatorname{Ext}^i_A(N, A) = 0$ for all (left) A-submodules N of $\operatorname{Ext}^j_A(M, A)$. Set $\mathrm{j}(M) = \min\{j : \operatorname{Ext}^j_A(M, A) \neq 0\}$. The algebra A is *CM* if $\mathrm{j}(M) + \operatorname{GKdim} M = \operatorname{GKdim} A$ holds for all finitely generated, non-zero A-modules M.

Corollary 3.3. (i) The commutant $\mathcal{C}$ of $\mathcal{O}^G$ in $\mathcal{D}(\mathfrak{g})$ is the ring generated by $\mathcal{O}$ and $\tau(\mathfrak{g})$. Moreover, $\mathcal{C}$ is an Auslander-Gorenstein, CM, Noetherian domain and a maximal order.

(ii) As a (left or right) $\mathcal{O}$-module, $\mathcal{C} \cap \mathcal{D}(\mathfrak{g})_m$ is generated by the elements

$$\{\tau(\xi_1)\tau(\xi_2)\cdots\tau(\xi_k) : \xi_i \in \mathfrak{g} \text{ and } k \leq m\}.$$

(iii) The centre of $\mathcal{C}$ is $\mathcal{O}(\mathfrak{g})^G$.

Proof. (i) By Theorem 3.2, $\mathcal{C}$ is generated by $\mathcal{O}$ and $\tau(\mathfrak{g})$. By that theorem and Proposition 2.4, $\operatorname{gr}\mathcal{C}$ satisfies the other conditions given in part (i). Let $M = \bigcup_{n\in\mathbb{N}} M_n$ be a filtered right $\mathcal{C}$-module such that $\operatorname{gr} M$ is a finitely generated $\operatorname{gr}\mathcal{C}$-module. By [3, Theorem 3.9], $\mathcal{C}$ is Auslander-Gorenstein and $\mathrm{j}_{\mathcal{C}}(M) = \mathrm{j}_{\operatorname{gr}\mathcal{C}}(\operatorname{gr} M)$. However, [13, Corollary 1.4] implies that $\operatorname{GKdim}\operatorname{gr} M = \operatorname{GKdim} M$ and hence that $\mathcal{C}$ is CM. Finally, [15] implies that $\mathcal{C}$ is a maximal order.

(ii) This follows from the fact that, in $\operatorname{gr}\mathcal{C} = \operatorname{Sym}(E)$, a homogeneous element $\overline{c}$ of degree m can be written $\overline{c} = \sum f_{i_1,\dots,i_m}\xi_{i_1}\cdots\xi_{i_m}$, for some $f_{i_1,\dots,i_m} \in \mathcal{O}$ and $\xi_{i_j} \in \tau(\mathfrak{g})$.

(iii) Let Z denote the centre of $\mathcal{C}$. Clearly both $\tau(\mathfrak{g})$ and $\mathcal{O}$ commute with $\mathcal{O}^G$ and so $\mathcal{O}^G \subseteq Z$. Conversely, Z is contained in the commutant, in $\mathcal{D}(\mathfrak{g})$, of $\mathcal{O}$. Hence, $Z \subseteq \mathcal{O}$. Since $\mathcal{O}^G$ is the commutant, in $\mathcal{O}$, of $\tau(\mathfrak{g})$, the result follows. □

Corollary 3.4. *Both* $\mathcal{C}$ *and* $\operatorname{Sym}(E)$ *are free (left or right) modules over* $\mathcal{O}(\mathfrak{g})^G$.

Proof. Set $\mathcal{O} = \mathcal{O}(\mathfrak{g})$ and $S = \operatorname{Sym}(E) = \bigoplus_{m=0}^{\infty}\operatorname{Sym}_m(E)$. We first prove the result for $\mathcal{C}$, assuming that $\operatorname{Sym}_m(E)$ is a free $\mathcal{O}^G$-module for all $m \in \mathbb{N}$. Note that the isomorphism $\operatorname{gr}\mathcal{C} \cong S$ of Theorem 3.2 is a graded isomorphism of $\mathcal{O}$-algebras, for the natural graded structure of the two objects. In other words $\mathcal{C}_m/\mathcal{C}_{m-1} \cong \operatorname{Sym}_m(E)$, for all m, where $\mathcal{C}_m = \mathcal{C} \cap \mathcal{D}(\mathfrak{g})_m$. Hence, each $\mathcal{C}_m/\mathcal{C}_{m-1}$ is a free $\mathcal{O}^G$-module; it follows routinely that $\mathcal{C}$ is also free over $\mathcal{O}^G$.

We now prove the result for $\operatorname{Sym}_m(E)$. Note, first, that S is a quotient of the polynomial ring

$$T = \operatorname{Sym}_{\mathcal{O}}(\mathcal{O}\otimes\mathfrak{g}) \cong \mathcal{O}[y_1,\dots,y_n] \cong \mathbb{C}[x_1,\dots,x_n,y_1\dots,y_n],$$

which we now grade by giving each generator x_i and y_j degree one. Since the u_i are homogeneous in $\mathcal{O}$, the $\nabla_i = \operatorname{grad}(u_i)$ are homogeneous in T and so, by Corollary 2.3, $\operatorname{Sym}_m(E)$ is a graded $\mathcal{O}^G$-module.

Set $P = \sum_{i=1}^{\ell} u_i S$. By [6, Proposition 2.16] and its proof (which depends upon a case by case analysis), S/P is a domain of dimension $2n - 2\ell = \dim S - \ell$. Hence, the u_j form a regular sequence in S, and therefore in each module $\operatorname{Sym}_m(E)$. Thus, by [4, § 8, Proposition 8 and § 9, Corollaire 2], $\operatorname{Sym}_m(E)$ is a graded free $\mathcal{O}^G$-module. □

Corollary 3.3 and Corollary 3.4 should be compared with [9] which (as a very special case) shows that the commutant of $\mathcal{D}(\mathfrak{g})^G$ is simply $\mathbb{C}\langle\tau(\mathfrak{g})\rangle$ ($\cong U(\mathfrak{g})$ when $\mathfrak{g}$ is semisimple). Moreover, both rings are free modules over the centre of $\mathcal{D}(\mathfrak{g})^G$ (which is also the centre of $\mathbb{C}\langle\tau(\mathfrak{g})\rangle$).

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