HARMONICITY OF NORMAL ALMOST CONTACT METRIC STRUCTURES

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Abstract. We consider normal almost contact structures on a Riemannian manifold and, through their associated sections of an ad-hoc twistor bundle, study their harmonicity, as sections or as maps. We rewrite these harmonicity equations in terms of the curvature tensor and find conditions relating the harmonicity of the almost contact metric and almost complex structures of the total and base spaces of the Morimoto fibration. We apply these results to homogeneous principal circle bundles over generalised flag manifolds, in particular Aloff-Wallach spaces, to mass-produce harmonic almost contact metric structures.

1. Introduction

Though there can be a wealth of almost contact structures on an odd-dimensional manifold, few tools exist to select a better one within this multitude.

In presence of a Riemannian metric, compatible almost contact structures can be interpreted as reductions of the orthonormal frame bundle and, as such, are in one-one correspondence with sections of an ad-hoc twistor bundle, constructed as the quotient of the orthonormal frame bundle by the group $\mathrm{U}(n) \times 1 \subset \mathrm{SO}(2n+1)$, just like in the better known case of almost Hermitian structures and the group $\mathrm{U}(n) \subset \mathrm{SO}(2n)$. Fitting this homogeneous fibre bundle with a Sasaki-like metric allows one to unfold the programme of harmonic map theory: construct a functional, characterise critical points, study the associated flow, determine stable maps. The objective is to compare almost contact structures among themselves, so critical points must be understood with respectto vertical variations.

The first steps were achieved in [VW1] and an almost contact metric structure (θ, ξ, η) on a Riemannian manifold (M^{2n+1}, g) is harmonic if (we refer to Section 2 for notations and precise definitions) [VW1, Theorem 3.2]:

$$[\bar{\nabla}^{*}\bar{\nabla}\bar{J}, \bar{J}] = 0, \tag{HSE1}$$

and

$$\nabla^{*}\nabla\xi = |\nabla\xi|^{2}\xi - (1/2)\bar{J} \circ \operatorname{trace}(\bar{\nabla}\bar{J} \otimes \nabla\xi), \tag{HSE2}$$

where $\bar{J}$ (resp. $\bar{\nabla}$) is the restriction of θ to the horizontal distribution $\xi^{\perp}$ (resp. the Levi-Civita connection ∇) and $\nabla^{*}\nabla = -\operatorname{trace}\nabla^{2}$.

A supplementary condition appears when further requiring the section to be a critical point of the energy for all possible variations, i.e., a harmonic map [VW1, Theorem 3.4]:

$$\sum_{i=1}^{2n+1} \Big(g(R(E_i, X), \bar{J}(\overline{\nabla}_{E_i}\bar{J})) + 4g(R(E_i, X)\xi, \nabla_{E_i}\xi) \Big) = 0, \tag{HME}$$

for any tangent vector field X and an orthonormal frame $\{E_i\}_{i=1,\dots,2n+1}$.

Here a comparison should be drawn with its even-dimensional counterpart, as the harmonicity equation for almost Hermitian structures can be read as the limiting case of a Cartesian product with a circle and a parallel Reeb vector field. Only (HSE1) remains and harmonic almost Hermitian structures are described by the commutation of J and $\nabla^{*}\nabla J$ [Woo]. A similar remark applies for the harmonic map condition (HME).

Having the harmonic section equations (HSE1) and (HSE2), the next task is to determine classes of almost contact metric structures which are harmonic or, failing that, conditions so they become

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harmonic. The first part was done in [VW1, VW2] and [LV], and this article investigates the particular case of normal almost contact structures, that is almost contact metric structures whose cone construction is a Hermitian manifold.

After rewriting, in Theorems 3.4 and 3.5, the harmonic section equations for normal almost contact structures in terms of the curvature tensor, echoing some results in [Woo], we apply, in Section 4 these properties to the Morimoto fibration to obtain conditions linking the harmonicity of the normal almost contact structure (θ, ξ, η) on the total space of this circle bundle and that of the Hermitian structure on the base. They involve the rough Laplacian of the Reeb vector field ξ and the divergence of the tensor θ, cf. Theorems 4.1 and 4.3. As a consequence, we get that if these rough Laplacian and divergence are constant multiples of ξ, then the harmonicity of (θ, ξ, η) is equivalent to that of the complex structure on the base, see Corollary 4.7.

This relationship then leads in Section 6 to an extensive study of harmonic normal almost contact structures on homogeneous principal circle bundles over a generalised flag manifold. Specifically, let G/K be a generalised flag manifold obtained from a compact connected Lie group G. If $G/\widetilde{K} \twoheadrightarrow G/K$ is a homogeneous principal $\mathbb{S}^1$-bundle, to each Hermitian metric on G/K one can associate a (canonical) normal almost contact structure on $G/\widetilde{K}$. Applying Corollary 4.7, one concludes that the almost contact metric structure is harmonic, as a section or map, if and only if so is the Hermitian structure on G/K, see Theorems 6.15 and 6.16. The construction of this almost contact metric structure relies on the classical characterisation of invariant Hermitian metrics on G/K in terms of root systems; for the convenience of the reader, we recall in Section 5 the results that we need on the subject.

Sections 5 and 6 lead to the construction of harmonic normal almost contact structure on $\mathbb{S}^1$-bundles over a generalised flag manifold G/K, summarised by the following statement (cf. Section 5 for notations and definitions).

Theorem A. *Let G be a compact Lie group, G/K be a generalised flag manifold and denote by C the centre of K. Let $\mathsf{Q}^+ \subset \mathsf{R}$ be an invariant ordering of a root system R and let R_T^+ be the set of restrictions of the elements of Q^+ to $\mathfrak{z} = \mathrm{Lie}(C)$, so that the reductive decomposition associated to K is:*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \quad \mathfrak{m} = \oplus_{\gamma \in R_T^+} \mathfrak{m}_\gamma,$$

where $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{k} = \mathrm{Lie}(K)$ and the $\mathfrak{m}_\gamma$'s are nonequivalent irreducible K-modules. Consider the G-invariant Hermitian structure $(J, g_{G/K})$ on G/K defined by:

– the $\mathrm{Ad}(K)$-invariant complex structure $J_{\mathfrak{m}}$ on $\mathfrak{m}$ given by $J_{\mathfrak{m}} X_\lambda = Y_\lambda$, $J_{\mathfrak{m}} Y_\lambda = -X_\lambda$, for all $\lambda \in \mathsf{Q}^+$ (cf. (5.1) for the definitions of X_λ, Y_λ).

– the G-invariant metric $g_{G/K}$ given by $g_{\mathfrak{m}_\gamma} = \kappa_\gamma B$, $\kappa_\gamma > 0$, on $\mathfrak{m}_\gamma$, where B is an invariant scalar product on $\mathfrak{g}$.

Choose a sub-torus $\widetilde{C} \subset \mathfrak{z}$ of codimension one, a unit vector (for B) $X_0 \in \mathfrak{z}$ such that $\mathfrak{z}$ is the orthogonal direct sum $\mathbb{R}X_0 \oplus \mathrm{Lie}(\widetilde{C})$, and let $\widetilde{K}$ be the closed connected subgroup of K such that $\widetilde{\mathfrak{k}} = \mathrm{Lie}(\widetilde{K})$ is the orthogonal of X_0 in $\mathfrak{k}$. Then $K/\widetilde{K} \cong \mathbb{S}^1$ and $\pi : G/\widetilde{K} \twoheadrightarrow G/K$ is a G-homogeneous principal $\mathbb{S}^1$-bundle. Define:

– the G-invariant principal connection $\eta(U) = B(X_0, U)A$ for all $U \in \mathfrak{g}$;

– the endomorphism $\theta = d\pi^{-1} \circ J \circ d\pi$;

– the metric $g_{G/\widetilde{K}} = \pi^* g_{G/K} + \eta \otimes \eta$;

– the vector field $\xi_p = \frac{d}{dt}_{|t=0}(pe^{tX_0})$ for all $p \in G/\widetilde{K}$.

If J is a harmonic section then the normal almost contact structure (θ, ξ, η) on $(G/\widetilde{K}, g_{G/\widetilde{K}})$ is a harmonic section. Moreover, when the metric $g_{G/K}$ is Kählerian, i.e. $\kappa_\gamma + \kappa_{\gamma'} = \kappa_{\gamma+\gamma'}$ for $\gamma, \gamma' \in R_T^+$ such that $\gamma + \gamma' \in R_T^+$, the normal almost contact structure (θ, ξ, η) is a harmonic map.

As explained in Remark 6.17, one can choose the constants κ_γ's such that the harmonic normal almost contact structure is not Sasakian.

The article closes with a description of harmonic (map) normal almost contact metric structures on the simplest significant examples, namely the Aloff-Wallach spaces, which are principal homogeneous $\mathbb{S}^1$-bundles over the full flag variety $G/K = \mathrm{SU}(3)/T$. Thus, $G/\widetilde{K} = \mathrm{SU}(3)/T_{k,\ell}$ where $T_{k,\ell}$ is a one dimensional sub-torus of the maximal torus T depending of a couple of integers $(k, l) \neq (0, 0)$. As explained in the previous theorem, we show that the almost contact metric structure on $\mathrm{SU}(3)/T_{k,\ell}$ associated to a Kähler metric on $\mathrm{SU}(3)/T$ is a harmonic map, and we give formulas for the normal almost contact structures in terms of the integers (k, ℓ). In order to state an explicit particular result, recall that we can take T to be the subgroup of diagonal matrices in $\mathrm{SU}(3)$, so that the elements of $\mathfrak{t} = \mathrm{Lie}(T)$ can be written $i\,\mathrm{diag}[a, b, -(a+b)]$ with $a, b \in \mathbb{R}$ and $\mathrm{Lie}(T_{(k,\ell)}) = i\mathbb{R}\,\mathrm{diag}[k, \ell, -(k+\ell)]$. We choose the scalar product on $\mathfrak{su}(3)$ to be $B(U, V) = -\,\mathrm{trace}(UV)$. From Theorem A one deduces the following result (see Theorem 6.22).

Theorem B. *Write $\mathfrak{su}(3) = \mathfrak{t} \oplus \mathfrak{m}$ for the reductive decomposition associated to $\mathrm{SU}(3)/T$. Choose an invariant complex structure J on $\mathrm{SU}(3)/T$, i.e. a set $\mathsf{R}^+ = \{\alpha, \beta, \gamma = \alpha + \beta\}$ of positive roots for the root system defined by the pair $(\mathfrak{t}, \mathfrak{su}(3))$. Then $\mathfrak{m} = \mathfrak{m}_\alpha \oplus \mathfrak{m}_\beta \oplus \mathfrak{m}_\gamma$ as a T-module; define the invariant Kähler-Einstein metric g on $\mathrm{SU}(3)/T$ by $g = B$ on $\mathfrak{m}_\alpha \oplus \mathfrak{m}_\beta$ and $g = 2B$ on $\mathfrak{m}_\gamma$.*
For $k, \ell \in \mathbb{Z}$ such that $(k, \ell) \neq (0, 0)$, set $X_0 = \frac{-i}{\sqrt{6(k^2+\ell^2+k\ell)}}\,\mathrm{diag}[2\ell+k, -\ell-2k, -\ell+k]$. If $p_0 \in \mathrm{SU}(3)/T_{k,\ell}$ is the class of the identity, define θ, ξ, η by their values on $T_{p_0}(\mathrm{SU}(3)/T_{k,\ell})$ identified with $\mathfrak{m} \oplus \mathbb{R}X_0$:

$$\theta(X_\lambda) = Y_\lambda,\ \ \theta(Y_\lambda) = -X_\lambda \ \ \text{for } \lambda \in \mathsf{R}^+, \quad \theta(X_0) = 0,$$

$$\eta(X_0) = 1, \quad \eta(U) = 0 \ \ \text{for } U \in \mathfrak{m}, \qquad \xi_{p_0} = \frac{d}{dt}_{|t=0}(p_0 e^{tX_0}) = X_0.$$

Set $g_{k,\ell} = \pi^* g + \eta \otimes \eta$, hence $g_{k,\ell} = B$ on $\mathfrak{m}_\alpha \oplus \mathfrak{m}_\beta \oplus \mathbb{R}X_0$ and $g_{k,\ell} = 2B$ on $\mathfrak{m}_\gamma$.
Then $(\theta, \xi, \eta, g_{k,\ell})$ is a normal almost contact metric structure on $\mathrm{SU}(3)/T_{k,\ell}$ and is a harmonic map. One can choose J such that this structure is not Sasakian when $k \neq \ell$.

Note that we adopt the following sign for the curvature tensor: $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

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2. Normal almost contact metric structures

An *almost contact structure* on an odd-dimensional differentiable manifold M^{2n+1} is a reduction of the structure group of its tangent frame to $\mathrm{U}(n) \times 1$. More concretely, this is equivalent to the existence of a field of endomorphisms θ of the tangent space, a vector field ξ and a one-form η related by

$$\eta(\xi) = 1, \quad \theta^2 = -\mathrm{Id} + \eta \otimes \xi.$$

Then, necessarily $\theta\xi = 0$ and $\eta \circ \theta = 0$. A Riemannian metric g on M^{2n+1} is *compatible* if it satisfies

$$g(\theta X, \theta Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields X and Y tangent to M^{2n+1}, and such a metric can always be constructed from the data (θ, ξ, η). We will only consider almost contact structures compatible with the Riemannian metric. This approach is initially due to Gray [Gra] and a good treatment can be found in [Bla].

One can also lift the almost contact structure to the Cartesian line-product $\tilde{M} = M^{2n+1} \times \mathbb{R}$ and construct an almost complex structure $\tilde{J}$ by

$$\tilde{J}(X + f\partial_t) = \theta X - f\xi + \eta(X)\partial_t,$$

for $X \in \mathcal{X}(M)$ and ∂_t the canonical unit vector field tangent to $\mathbb{R}$. When $\tilde{J}$ is integrable, the almost contact structure will be called *normal*, which can be characterised by the equation

$$N_\theta + 2\mathrm{d}\eta \otimes \xi = 0,$$

where N_θ is the Nijenhuis tensor

$$N_\theta(X, Y) = \theta^2[X, Y] + [\theta X, \theta Y] - \theta[\theta X, Y] - \theta[X, \theta Y].$$

An immediate effect of this condition is that the vector field ξ, while not necessarily Killing, must have geodesic integral lines and preserve the field of endomorphisms θ, i.e. $\nabla_\xi \theta = 0$ (cf. [Bla]).

To prepare the computations of the next section, we derive an alternative characterisation of normal almost contact metric structures. Proposition 2.1 is quite probably a well-known result but, with no reference available, a proof is included for the sake of completeness.

Proposition 2.1. *Let (M^{2n+1}, g) be a Riemannian manifold equipped with an almost contact metric structure (θ, ξ, η). Then this structure is normal if and only if*

$$(\nabla_X \theta)(Y) = (\nabla_{\theta X}\theta)(\theta Y) - \eta(Y)\nabla_{\theta X}\xi,$$

for any vector fields X and Y on M^{2n+1}.

Proof. Let $\tilde{\nabla}$ denote the covariant derivative of $\tilde{M}$ and ∇ the covariant derivative of M, on the factor $\mathbb{R}$ derivation will be noted by D and ∂_t will be the unit vector field. Given two vector fields X and Y on M and functions F and G of t, we consider the vector fields $(X + F\partial_t)$ and $(Y + G\partial_t)$ tangent to $\tilde{M} = M \times \mathbb{R}$ and re-write the integrability condition of $\tilde{J}$ (cf. [BW, Prop. 7.1.3])

$$(\tilde{\nabla}_{\tilde{J}(X+F\partial_t)}\tilde{J})(\tilde{J}(Y+G\partial_t)) = (\tilde{\nabla}_{(X+F\partial_t)}\tilde{J})(Y+G\partial_t). \tag{2.1}$$

For the sake of simplicity, we will evaluate all expressions at a given point $x \in M$ and assume that, around this point, we locally extended the vector $Y \in T_xM$ such that $\nabla Y(x) = 0$.
On the one hand

$$\begin{aligned}
&\left(\tilde{\nabla}_{\tilde{J}(X,F\partial_t)}\tilde{J}\right)\left(\tilde{J}(Y+G\partial_t)\right)\\
&= -\tilde{\nabla}_{(\theta X - F\xi + \eta(X)\partial_t)}(Y+G\partial_t) - \tilde{J}(\tilde{\nabla}_{(\theta X - F\xi+\eta(X)\partial_t)}(\theta Y - G\xi + \eta(Y)\partial_t))\\
&= -\nabla_{\theta X}Y + F\nabla_\xi Y - \theta(\nabla_{\theta X}\theta Y) + G\theta(\nabla_{\theta X}\xi) + F\theta(\nabla_\xi \theta Y) - FG\theta(\nabla_\xi\xi)\\
&+ \left((\theta X)(\eta(Y)) - F\xi((\eta(Y))\right)\xi + \left(F\eta(\nabla_\xi\theta Y) - \eta((\nabla_{\theta X}\theta Y)\right)\partial_t,
\end{aligned}$$

while, on the other hand, by definition of $\tilde{J}$,

$$(\tilde{\nabla}_{(X+F\partial_t)}\tilde{J})(Y+G\partial_t) = (\nabla_X\theta)(Y) - G\nabla_X\xi + g(\nabla_X\xi, Y)\partial_t.$$

Equating these two computations by (2.1) and using different values for F and G, yields, for the M-component,

$$(\nabla_X\theta)(Y) = (\nabla_{\theta X}\theta)(\theta Y) - \eta(Y)\nabla_{\theta X}\xi \tag{2.2}$$

$$0 = \nabla_\xi Y + \theta(\nabla_\xi\theta Y) - \xi(\eta(Y))\xi \tag{2.3}$$

$$-\nabla_X\xi = \theta(\nabla_{\theta X}\xi) \tag{2.4}$$

$$\theta(\nabla_\xi\xi) = 0, \tag{2.5}$$

and for the $\mathbb{R}$-component

$$0 = g(\nabla_\xi\theta Y, \xi), \tag{2.6}$$

$$g(\nabla_X\xi, Y) = -g(\nabla_{\theta X}\theta Y, \xi). \tag{2.7}$$

Choosing the right type of vector fields, it is easily proved that Equation (2.2) implies (2.3), (2.4) and (2.5), and also Equations (2.6) and (2.7), so the normality of an almost contact metric structure is merely equivalent to (2.2). □

From the proof of Proposition 2.1, we easily obtain a series of equations which we will repeatedly use.

Corollary 2.2. *Let (M^{2n+1}, g) be a Riemannian manifold equipped with a normal almost contact metric structure (θ, ξ, η), then we have*

$$g(\nabla_X\xi, Y) + \eta(\nabla_{\theta X}\theta Y) = 0, \tag{2.8}$$

$$g(\nabla_\xi\theta Y, \xi) = 0, \tag{2.9}$$

$$\nabla_\xi Y + \theta(\nabla_\xi\theta Y) - \xi\big(\eta(Y)\big)\xi = 0, \tag{2.10}$$

$$\nabla_X\xi + \theta(\nabla_{\theta X}\xi) = 0, \tag{2.11}$$

$$\nabla_\xi\xi = 0, \tag{2.12}$$

for any vector fields X and Y on M^{2n+1}.

A consequence of Proposition 2.1 is that, on the complement to the ξ-direction, the field of endomorphisms θ behaves like an integrable complex structure. This could be seen as an almost contact version of [VW2, Lemma 2.1].

Corollary 2.3. *Let (M^{2n+1}, g) be a Riemannian manifold equipped with a normal almost contact metric structure (θ, ξ, η). Denote by $\mathcal{F}$ the contact sub-bundle, i.e. the distribution orthogonal to ξ, by $\bar{J}$ the restriction of θ to $\mathcal{F}$ and by $\bar{\nabla}$ the connection induced on $\mathcal{F}$ by ∇. Then*

$$(\bar{\nabla}_{\bar{J}X}\bar{J})(\bar{J}Y) = (\bar{\nabla}_X\bar{J})(Y),$$

for any X and Y in $\mathcal{F}$.

3. Curvature equations of harmonicity

In the more favourable cases, the two equations characterising the harmonicity of an almost contact metric structure can be expressed in terms of the curvature tensor of (M, g). This is what we establish for normal almost contact metric structures in this section, starting with a preliminary series of technical lemmas on a Riemannian manifold (M^{2n+1}, g) equipped with a normal almost contact metric structure (θ, ξ, η).

Lemma 3.1. *Let E be a horizontal vector field, that is a section of the horizontal sub-bundle $\mathcal{F}$, such that, at some point $x \in M^{2n+1}$, $(\bar\nabla E)(x) = 0$. Then, at x,*

$$[E, \bar J E] = (\bar\nabla_E \bar J)(E) + 2g(E, \nabla_{\theta E}\xi)\xi.$$

Proof. If E is a horizontal vector field then so is $\bar J E$ and, evaluating at x,

$$\begin{aligned}
[E, \bar J E] &= \bar\nabla_E(\bar J E) + g(\nabla_E(\bar J E), \xi)\xi - \bar\nabla_{\bar J E}E - g(\nabla_{\bar J E}E, \xi)\xi \\
&= (\bar\nabla_E \bar J)(E) + g(\nabla_E \theta E - \nabla_{\theta E}E, \xi)\xi \\
&= (\bar\nabla_E \bar J)(E) + g((\nabla_E \theta)(E) - \nabla_{\theta E}E, \xi)\xi.
\end{aligned}$$

By Proposition 2.1,

$$g((\nabla_E\theta)(E), \xi) = g((\nabla_{\theta E}\theta)(\theta E), \xi) = -g(\nabla_{\theta E}E, \xi).$$

Then,

$$[E, \bar J E] = (\bar\nabla_E \bar J)(E) - 2g((\nabla_{\theta E}E, \xi)\xi = (\bar\nabla_E \bar J)(E) + 2g(E, \nabla_{\theta E}\xi)\xi,$$

since ξ is vertical. □

To compute the first harmonic section equation, we need to express the commutator of $\bar J$ and its Laplacian.

Lemma 3.2. *Let E be a section of $\mathcal{F}$, such that, at some point $x \in M^{2n+1}$, $(\bar\nabla E)(x) = 0$. Then, at x,*

$$[\bar\nabla_E\bar\nabla_E \bar J, \bar J] = -2[\bar R(E, \bar J E), \bar J] - 2\bar\nabla_{(\bar\nabla_E \bar J)(E)}\bar J - [\bar\nabla_{\bar J E}\bar\nabla_{\bar J E}\bar J, \bar J],$$

where $\bar R$ is the curvature tensor of the contact sub-bundle $\mathcal{F}$ equipped with the connection $\bar\nabla$.

Proof. Let E be a section of $\mathcal{F}$ and X a horizontal vector extended locally such that $\bar\nabla X = 0$ at the point $x \in M$ where we evaluate all expressions. By the Leibniz rule, we have

$$(\bar\nabla_E\bar\nabla_E\bar J)(\bar J X) = \bar\nabla_E\big((\bar\nabla_E \bar J)(\bar J X)\big) - (\bar\nabla_E\bar J)((\bar\nabla_E \bar J)(X)).$$

Using Corollary 2.3 and Lemma 3.1, the first term on the right-hand side may be expressed in terms of the curvature tensor $\bar R$ as follows

$$\begin{aligned}
&\bar\nabla_E\big((\bar\nabla_E \bar J)(\bar J X)\big) = -\bar\nabla_E\big((\bar\nabla_{\bar J E}\bar J)(X)\big) = -\bar\nabla_E\bar\nabla_{\bar J E}(\bar J X) + \bar\nabla_E(\bar J\bar\nabla_{\bar J E}X) \\
&= -\bar\nabla_E\bar\nabla_{\bar J E}(\bar J X) + \bar J(\bar\nabla_E\bar\nabla_{\bar J E}X) \\
&= -\bar\nabla_{\bar J E}\bar\nabla_E(\bar J X) + \bar J\bar\nabla_{\bar J E}\bar\nabla_E X - \bar\nabla_{[E,\bar J E]}(\bar J X) + \bar J\bar\nabla_{[E,\bar J E]}X - [\bar R(E,\bar J E), \bar J](X) \\
&= -[\bar R(E,\bar J E), \bar J](X) - \bar\nabla_{\bar J E}\bar\nabla_E(\bar J X) + \bar J\bar\nabla_{\bar J E}\bar\nabla_E X - \bar\nabla_{(\bar\nabla_E\bar J)(E)}(\bar J X) - 2g(E, \nabla_{\bar J E}\xi)\bar\nabla_\xi(\bar J X) \\
&\qquad + \bar J(\bar\nabla_{(\bar\nabla_E\bar J)(E)}X) + 2g(E, \nabla_{\bar J E}\xi)\bar J(\bar\nabla_\xi X) \\
&= -[\bar R(E,\bar J E), \bar J](X) - \bar\nabla_{\bar J E}\bar\nabla_E(\bar J X) + \bar J\bar\nabla_{\bar J E}\bar\nabla_E X - \big(\bar\nabla_{(\bar\nabla_E\bar J)(E)}\bar J\big)(X),
\end{aligned}$$

since $\bar\nabla_\xi \bar J = 0$. Now

$$\begin{aligned}
-\bar\nabla_{\bar J E}\bar\nabla_E(\bar J X) + \bar J(\bar\nabla_{\bar J E}\bar\nabla_E X) &= -\bar\nabla_{\bar J E}((\bar\nabla_E\bar J)(X)) - \bar\nabla_{\bar J E}(\bar J(\bar\nabla_E X)) + \bar J(\bar\nabla_{\bar J E}\bar\nabla_E X) \\
&= -\bar\nabla_{\bar J E}((\bar\nabla_E\bar J)(X)) = -\bar\nabla_{\bar J E}((\bar\nabla_{\bar J E}\bar J)(\bar J X)) \\
&= -(\bar\nabla_{\bar J E}\bar\nabla_{\bar J E}\bar J)(\bar J X) - (\bar\nabla_{\bar J E}\bar J)(\bar\nabla_{\bar J E}(\bar J X)) \\
&= -(\bar\nabla_{\bar J E}\bar\nabla_{\bar J E}\bar J)(\bar J X) - (\bar\nabla_{\bar J E}\bar J)((\bar\nabla_{\bar J E}\bar J)(X)),
\end{aligned}$$

because of the way we choose to extend the vector X. Therefore

$$\begin{aligned}
\bar\nabla_E\big((\bar\nabla_E \bar J)(\bar J X)\big) &= -[\bar R(E,\bar J E), \bar J](X) - \big(\bar\nabla_{(\bar\nabla_E\bar J)(E)}\bar J\big)(X) \\
&\quad - (\bar\nabla_{\bar J E}\bar\nabla_{\bar J E}\bar J)(\bar J X) - (\bar\nabla_{\bar J E}\bar J)\circ(\bar\nabla_{\bar J E}\bar J)(X).
\end{aligned} \tag{3.1}$$

By definition of the covariant derivative of $\bar\nabla_E \bar J$

$$\bar\nabla_E((\bar\nabla_E \bar J)(\bar J X)) = (\bar\nabla_E\bar\nabla_E \bar J)(\bar J X) + (\bar\nabla_E \bar J)((\bar\nabla_E \bar J)(X)),$$

hence, using Equation (3.1), we have

$$\begin{aligned}(\bar\nabla_E\bar\nabla_E \bar J)(\bar J X) &= -(\bar\nabla_E \bar J)((\bar\nabla_E \bar J)(X)) + \bar\nabla_E((\bar\nabla_E \bar J)(\bar J X))\\ &= -(\bar\nabla_E \bar J)\circ(\bar\nabla_E \bar J)(X) - [\bar R(E,\bar J E),\bar J](X) - (\bar\nabla_{(\bar\nabla_E \bar J)(E)}\bar J)(X)\\ &- (\bar\nabla_{\bar J E}\bar\nabla_{\bar J E}\bar J)(\bar J X) - (\bar\nabla_{\bar J E}\bar J)\circ(\bar\nabla_{\bar J E}\bar J)(X)\\ &= -2(\bar\nabla_E \bar J)\circ(\bar\nabla_E \bar J)(X) - [\bar R(E,\bar J E),\bar J](X) - (\bar\nabla_{(\bar\nabla_E \bar J)(E)}\bar J)(X)\\ &- (\bar\nabla_{\bar J E}\bar\nabla_{\bar J E}\bar J)(\bar J X),\end{aligned} \tag{3.2}$$

since, by Corollary 2.3,

$$(\bar\nabla_{\bar J E}\bar J)\circ(\bar\nabla_{\bar J E}\bar J)(X) = (\bar\nabla_E \bar J)\circ(\bar\nabla_E \bar J)(X).$$

To compute the second term of $[\bar\nabla_E\bar\nabla_E\bar J,\bar J](X)$ we proceed as follows:

$$\begin{aligned}(\bar\nabla_E\bar\nabla_E \bar J)(\bar J^2 X) = &-2\bar J\circ(\bar\nabla_E \bar J)\circ(\bar\nabla_E \bar J)(X) - [\bar R(E,\bar J E),\bar J](\bar J X)\\ &+ \bar J(\bar\nabla_{(\bar\nabla_E \bar J)(E)}\bar J)(X) + (\bar\nabla_{\bar J E}\bar\nabla_{\bar J E}\bar J)(X),\end{aligned}$$

then

$$\begin{aligned}-\bar J(\bar\nabla_E\bar\nabla_E \bar J)(X) = &\, 2(\bar\nabla_E \bar J)\circ(\bar\nabla_E \bar J)(X) - [\bar R(E,\bar J E),\bar J](X)\\ &- (\bar\nabla_{(\bar\nabla_E \bar J)(E)}\bar J)(X) + \bar J(\bar\nabla_{\bar J E}\bar\nabla_{\bar J E}\bar J)(X).\end{aligned}$$

Summing up this last equation with (3.2), we obtain the result. □

Lemma 3.3. *Let $\{F_i\}_{i=1,\dots,2n}$ be an orthonormal frame on the horizontal sub-bundle $\mathcal F$, such that, at the point $x\in M^{2n+1}$ where we evaluate all our expressions, $(\bar\nabla F_i)(x)=0$. Define the Lee vector field by $\bar\delta\bar J = \sum_{i=1}^{2n}(\bar\nabla_{F_i}\bar J)(F_i)$. Then*

$$\sum_{i=1}^{2n}\bar\nabla_{\bar\nabla_{\bar J F_i}(\bar J F_i)}\bar J = \bar\nabla_{\bar J\bar\delta\bar J}\bar J.$$

Proof. Combining the Leibniz rule, Corollary 2.3 and $\bar\nabla\bar J\circ\bar J = -\bar J\circ\bar\nabla\bar J$, yields

$$\sum_{i=1}^{2n}\bar\nabla_{\bar J F_i}(\bar J F_i) = \sum_{i=1}^{2n}(\bar\nabla_{\bar J F_i}\bar J)(F_i) = -\sum_{i=1}^{2n}(\bar\nabla_{F_i}\bar J)(\bar J F_i) = \bar J\bar\delta\bar J,$$

as required. □

We can now give the first harmonicity condition for a normal structure, reminiscent of [Woo, Theorem 2.8].

Theorem 3.4. *Let (M^{2n+1},g) be a Riemannian manifold equipped with a normal almost contact metric structure (θ,ξ,η). The first harmonic section equation (HSE1) for this structure can be written*

$$\sum_{i=1}^{2n}[\bar R(F_i,\bar J F_i),\bar J] = -2\bar\nabla_{\bar\delta\bar J}\bar J,$$

where $\{F_i\}_{i=1,\dots,2n}$ is an orthonormal frame on the horizontal sub-bundle $\mathcal F$.

Proof. Assume, without loss of generality, that the orthonormal frame $\{F_i\}_{i=1,\dots,2n}$ satisfies $\bar\nabla F_i(x)=0$, for all i, at the point we evaluate at. First observe that

$$-\bar\nabla^*\bar\nabla\bar J = \sum_{i=1}^{2n}(\bar\nabla_{F_i}\bar\nabla_{F_i}\bar J - \bar\nabla_{\nabla_{F_i}F_i}\bar J) + \bar\nabla_\xi\bar\nabla_\xi\bar J - \bar\nabla_{\nabla_\xi\xi}\bar J = \sum_{i=1}^{2n}\bar\nabla_{F_i}\bar\nabla_{F_i}\bar J$$

since ∇F_i is in the ξ-direction, $\bar\nabla_\xi\bar J = 0$ and $\nabla_\xi\xi = 0$. Working now with the orthonormal frame $\{\bar J F_i\}_{i=1,\dots,2n}$,

$$-\bar\nabla^*\bar\nabla\bar J = \sum_{i=1}^{2n}\bar\nabla_{\bar J F_i}\bar\nabla_{\bar J F_i}\bar J - \bar\nabla_{\nabla_{\bar J F_i}(\bar J F_i)}\bar J - \bar\nabla^2_{\xi,\xi}\bar J,$$

thus

$$-\bar\nabla^*\bar\nabla \bar J = \sum_{i=1}^{2n} \bar\nabla_{\bar JF_i}\bar\nabla_{\bar JF_i}\bar J - \bar\nabla_{\bar\nabla_{\bar JF_i}(\bar JF_i)}\bar J - g(\nabla_{\bar JF_i}(\bar JF_i),\xi)\bar\nabla_\xi \bar J = \sum_{i=1}^{2n} \bar\nabla_{\bar JF_i}\bar\nabla_{\bar JF_i}\bar J - \bar\nabla_{\bar J\bar\delta\bar J}\bar J,$$

by the previous lemma. Then,

$$-[\bar\nabla^*\bar\nabla\bar J,\bar J](X) = \sum_{i=1}^{2n}[\bar\nabla_{\bar JF_i}\bar\nabla_{\bar JF_i}\bar J,\bar J](X) - [\bar\nabla_{\bar J\bar\delta\bar J}\bar J,\bar J](X).$$

Taking traces in Lemma 3.2, we get

$$-[\bar\nabla^*\bar\nabla\bar J,\bar J](X) = -2\sum_{i=1}^{2n}[\bar R(F_i,\bar JF_i),\bar J](X) - 2(\bar\nabla_{\bar\delta\bar J}\bar J)(X) + [\bar\nabla^*\bar\nabla\bar J,\bar J](X) - [\bar\nabla_{\bar J\bar\delta\bar J}\bar J,\bar J](X).$$

Notice that

$$[\bar\nabla_{\bar J\bar\delta\bar J}\bar J,\bar J](X) = (\bar\nabla_{\bar J\bar\delta\bar J}\bar J)(\bar JX) - \bar J(\bar\nabla_{\bar J\bar\delta\bar J}\bar J)(X) = 2(\bar\nabla_{\bar\delta\bar J}\bar J)(X),$$

by Corollary 2.3. Therefore

$$[\bar\nabla^*\bar\nabla\bar J,\bar J](X) = \sum_{i=1}^{2n}[\bar R(F_i,\bar JF_i),\bar J](X) + 2(\bar\nabla_{\bar\delta\bar J}\bar J)(X),$$

and the theorem follows. □

Recall that the rough Laplacian on M is given on $V \in \mathcal{X}(M)$ by

$$\nabla^*\nabla V = -\operatorname{trace}_g \nabla^2 V = -\sum_{i=1}^{2n+1} \nabla_{E_i}(\nabla_{E_i}V) - \nabla_{\nabla_{E_i}E_i}V \tag{3.3}$$

where $\{E_i\}_{1\leqslant i\leqslant 2n+1}$ is an orthonormal frame on M. Moreover, if $\phi \in \operatorname{End} TM$, let

$$\delta\phi = \operatorname{trace}_g \nabla\phi = \sum_{i=1}^{2n+1} \nabla_{E_i}\phi(E_i) - \sum_{i=1}^{m}\phi(\nabla_{E_i}E_i). \tag{3.4}$$

Theorem 3.5. *Let (M^{2n+1},g) be a Riemannian manifold equipped with a normal almost contact metric structure (θ,ξ,η). Then the second harmonic section equation (HSE2) for the structure (θ,ξ,η) is*

$$\nabla^*\nabla\xi - |\nabla\xi|^2\xi = (1/2)\sum_{i=1}^{2n}[R(F_i,\theta F_i),\theta](\xi) + (\nabla_{\delta\theta}\theta)(\xi),$$

where $\{F_i\}_{i=1,\dots,2n}$ is an orthonormal frame on the horizontal sub-bundle $\mathcal{F}$ and $\delta\theta = \operatorname{trace}_g \nabla\theta$.

Proof. Notice that $\delta\theta = \sum_{i=1}^{2n}(\nabla_{F_i}\theta)(F_i)$ (since, by Proposition 2.1, $(\nabla_\xi\theta)(\xi) = 0$). Equation (HSE2) can be written

$$(\nabla^*\nabla\xi)^{\mathcal{F}} = -\tfrac12\sum_{i=1}^{2n}\bar J\circ(\bar\nabla_{F_i}\bar J)(\nabla_{F_i}\xi) = -\tfrac12\sum_{i=1}^{2n}\theta\circ(\nabla_{F_i}\theta)(\nabla_{F_i}\xi).$$

Working with an orthonormal frame $\{F_i\}_{i=1,\dots,2n}$ as in Theorem 3.4, then

$$\begin{aligned}\sum_{i=1}^{2n}(\nabla_{F_i}\theta)(\nabla_{F_i}\xi) &= \sum_{i=1}^{2n}\nabla_{F_i}(\theta(\nabla_{F_i}\xi)) - \theta(\nabla_{F_i}\nabla_{F_i}\xi) = \sum_{i=1}^{2n}\nabla_{F_i}\nabla_{\theta F_i}\xi + \theta(\nabla^*\nabla\xi)\\
&= \sum_{i=1}^{2n} R(F_i,\theta F_i)\xi + \nabla_{\theta F_i}\nabla_{F_i}\xi + \nabla_{[F_i,\theta F_i]}\xi + \theta(\nabla^*\nabla\xi)\\
&= \sum_{i=1}^{2n} R(F_i,\theta F_i)\xi - \nabla_{\theta F_i}(\theta\nabla_{\theta F_i}\xi) + \nabla_{[F_i,\theta F_i]}\xi + \theta(\nabla^*\nabla\xi)\\
&= \sum_{i=1}^{2n} R(F_i,\theta F_i)\xi - (\nabla_{\theta F_i}\theta)(\nabla_{\theta F_i}\xi) - \theta(\nabla_{\theta F_i}\nabla_{\theta F_i}\xi) + \nabla_{[F_i,\theta F_i]}\xi + \theta(\nabla^*\nabla\xi)\\
&= \sum_{i=1}^{2n} R(F_i,\theta F_i)\xi - (\nabla_{\theta F_i}\theta)(\nabla_{\theta F_i}\xi) - \theta(\nabla_{\nabla_{\theta F_i}(\theta F_i)}\xi) + \nabla_{[F_i,\theta F_i]}\xi + 2\theta(\nabla^*\nabla\xi).\end{aligned}$$

Notice that

$$\sum_{i=1}^{2n} \theta(\nabla_{\nabla_{\theta F_i}(\theta F_i)}\xi) = \sum_{i=1}^{2n} \nabla_{\theta(\nabla_{\theta F_i}\theta)(F_i)}\xi = -\nabla_{\delta\theta}\xi, \tag{3.5}$$

by Lemma 3.3, the geodesic integral curves of ξ and the general property (since $\theta^2 = -\mathrm{Id} + \eta\otimes\xi$):

$$(\nabla_X\theta)(\theta Y) + \theta(\nabla_X\theta)(Y) = g(\nabla_X\xi, Y)\xi + g(Y,\xi)\nabla_X\xi,$$

for any X and Y in $\mathcal{X}(M)$. Similarly, with Lemma 3.3, we have

$$\sum_{i=1}^{2n} \nabla_{[F_i,\theta F_i]}\xi = \nabla_{\bar\delta\bar J + 2\sum_{i=1}^{2n} g(F_i,\nabla_{\theta F_i}\xi)\xi}\xi = \nabla_{\bar\delta\bar J}\xi = \nabla_{\delta\theta}\xi.$$

Thus

$$\sum_{i=1}^{2n}(\nabla_{F_i}\theta)(\nabla_{F_i}\xi) = \tfrac{1}{2}\sum_{i=1}^{2n} R(F_i,\theta F_i)\xi + \theta(\nabla^*\nabla\xi) + \nabla_{\delta\theta}\xi$$

and

$$\tfrac{1}{2}\sum_{i=1}^{2n} \theta\circ(\nabla_{F_i}\theta)(\nabla_{F_i}\xi) = \tfrac{1}{4}\sum_{i=1}^{2n} \theta R(F_i,\theta F_i)\xi - \tfrac{1}{2}(\nabla^*\nabla\xi)^{\mathcal{F}} + \tfrac{1}{2}\theta\nabla_{\delta\theta}\xi.$$

Then the second harmonic equation (HSE2) is

$$(\nabla^*\nabla\xi)^{\mathcal{F}} = -\tfrac{1}{2}\sum_{i=1}^{2n} \theta R(F_i,\theta F_i)\xi - \theta\nabla_{\delta\theta}\xi.$$

Since $-\theta(\nabla_{\delta\theta}\xi) = (\nabla_{\delta\theta}\theta)(\xi)$ and $-\theta R(F_i,\theta F_i)\xi = [R(F_i,\theta F_i),\theta](\xi)$, the theorem follows. □

When the vector field ξ happens to be harmonic, as a section of the unit tangent bundle i.e. $\nabla^*\nabla\xi = |\nabla\xi|^2\xi$ (cf. [Per]), the two harmonic section equations merge into a single one.

Corollary 3.6. *Let (M^{2n+1}, g) be a Riemannian manifold equipped with a normal almost contact metric structure (θ,ξ,η). If the vector field ξ is harmonic, as unit section, then the almost contact metric structure (θ,ξ,η) is a harmonic section if and only if*

$$\sum_{i=1}^{2n}[R(F_i,\theta F_i),\theta] = -2\nabla_{\delta\theta}\theta.$$

Proof. If ξ is harmonic then the second harmonic equation simplifies to

$$0 = \sum_{i=1}^{2n}[R(F_i,\theta F_i),\theta](\xi) + 2(\nabla_{\delta\theta}\theta)(\xi).$$

The link between the $\mathcal{F}$-component of the curvature tensor of (M^{2n+1}, g) and the curvature of the connection $\bar\nabla$ was established in [Ver]:

$$\bar R(X,Y) = R^{\mathcal{F}}(X,Y) + r(\nabla_X\xi, \nabla_Y\xi),$$

where r denotes the curvature tensor of the unit sphere, i.e. $r(X,Y)Z = g(Y,Z)X - g(X,Z)Y$. If one proves that

$$\sum_{i=1}^{2n}[r(\nabla_{F_i}\xi, \nabla_{\bar J F_i}\xi), \bar J] = 0, \tag{3.6}$$

then the first harmonic section equation could be rewritten as

$$\sum_{i=1}^{2n}[R(F_i,\theta F_i),\theta] - g([R(F_i,\theta F_i),\theta],\xi)\xi = -2\nabla_{\delta\theta}\theta + 2g(\nabla_{\delta\theta}\theta,\xi)\xi, \tag{3.7}$$

which, applied to $X\in\mathcal{F}$, gives the first harmonic section equation

$$\sum_{i=1}^{2n}[R^{\mathcal{F}}(F_i,\theta F_i),\theta](X) = -2(\bar\nabla_{\delta\bar J}\bar J)(X),$$

which is the $\mathcal{F}$-part of $\sum_{i=1}^{2n}[R(F_i,\theta F_i),\theta] = -2\nabla_{\delta\theta}\theta$. When evaluated on ξ, Equation (3.7) becomes

$$\sum_{i=1}^{2n}[R(F_i,\theta F_i),\theta](\xi) - g([R(F_i,\theta F_i),\theta](\xi),\xi)\xi = -2(\nabla_{\delta\theta}\theta)(\xi) + 2g((\nabla_{\delta\theta}\theta)(\xi),\xi)\xi$$

which is the second harmonic section equation

$$\sum_{i=1}^{2n}[R(F_i,\theta F_i),\theta](\xi) = -2(\nabla_{\delta\theta}\theta)(\xi),$$

i.e. the ξ-component of $\sum_{i=1}^{2n}[R(F_i,\theta F_i),\theta] = -2\nabla_{\delta\theta}\theta$. To show that (3.6) is indeed valid, we will only need Proposition 2.1 and the definition of the tensor r. Let V be a horizontal vector field, then

$$\begin{aligned} r(\nabla_{F_i}\xi,\nabla_{\bar{J}F_i}\xi)(\bar{J}V) &= g(\nabla_{\bar{J}F_i}\xi,\bar{J}V)\nabla_{F_i}\xi - g(\nabla_{F_i}\xi,\bar{J}V)\nabla_{\bar{J}F_i}\xi \\ &= -g(\nabla_{F_i}\xi,V)\bar{J}\nabla_{\bar{J}F_i}\xi + g(\nabla_{\bar{J}F_i}\xi,V)\bar{J}\nabla_{F_i}\xi \\ &= \bar{J}\big(r((\nabla_{F_i}\xi),\nabla_{\bar{J}F_i}\xi)V\big), \end{aligned}$$

hence the conclusion. □

4. The Morimoto submersion

This section rewrites the equations of Theorems 3.4 and 3.5 for the total space of a circle bundle, in order to relate the harmonicity of its normal almost contact metric structure with the harmonicity of the Hermitian structure on the base. To ensure the normality of the structure on the total space, we use Morimoto's version of the Boothby-Wang fibration and extend [VW2]. A *Morimoto submersion* will be a holomorphic Riemannian submersion $\pi : (M^{2n+1},\theta,\xi,\eta,g_M) \longrightarrow (N^{2n},J,g_N)$, i.e. with $d\pi(\theta X) = Jd\pi(X)$, from a normal almost contact metric manifold onto a Hermitian manifold. Conditions for a circle bundle $\pi : M^{2n+1} \to N^{2n}$ to be a Morimoto fibration are given, in [Mor1], when N^{2n} is a complex manifold, in [Mor2], when M^{2n+1} carries a normal almost contact metric structure.

These results will lead in subsequent sections to a wide-ranging construction of harmonic invariant normal almost contact metric structures on homogeneous principal circle bundles over generalised flag manifolds.

Denote by R,∇, resp. R^N,∇^N, the curvature and the connection on M, resp. N. We recall formulas due to O'Neill [One] in our context. Let $\mathcal{H}$ and $\mathcal{V}$ be the projections onto the horizontal and vertical parts of TM and define the tensors A and T by

$$A_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F \quad T_E F = \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F.$$

However, since our vertical distribution is one-dimensional and ξ has geodesic integral curves and is self-parallel, the tensor T is identically zero.

For the Morimoto fibration, the curvature identities of [One] (with opposite sign convention for the curvatures) are, for X,Y,Z and W horizontal vector fields and X_*,Y_*,Z_* and W_* their images under $d\pi$:

$$\begin{aligned} g_M(R(X,Y)Z,W) =& g_N(R^N(X_*,Y_*)Z_*,W_*) + 2g_M(\nabla_X Y,\xi)g_M(\nabla_Z W,\xi) \\ & - g_M(\nabla_Y Z,\xi)g_M(\nabla_X W,\xi) - g_M(\nabla_Z X,\xi)g_M(\nabla_Y W,\xi) \\ g_M(R(X,Y)Z,\xi) =& -Z(g_M(\nabla_X Y,\xi)) + g_M(\nabla_Z X,\nabla_Y\xi) - g_M(\nabla_Z Y,\nabla_X\xi) \\ g_M(R(X,\xi)Y,\xi) =& -g_M(\nabla_\xi\nabla_X Y,\xi) + g_M(\nabla_\xi X,\nabla_Y\xi) - g_M(\nabla_\xi Y,\nabla_X\xi) \end{aligned} \tag{4.1}$$

The curvature expressions of the previous section enable us to link the harmonicity of the structures on the total space and the base of the Morimoto fibration. This completes a result of [VW2].

Theorem 4.1. *Let $\pi : (M^{2n+1},\theta,\xi,\eta,g_M) \longrightarrow (N^{2n},J,g_N)$ be a holomorphic Riemannian submersion, i.e. with $d\pi(\theta X) = Jd\pi(X)$, from a normal almost contact metric manifold onto a Hermitian manifold. Then the almost contact metric structure is a harmonic section if and only if J is a harmonic section and*

$$2(\nabla^*\nabla\xi)^{\mathcal{F}} + 2\theta(\nabla_{\delta\theta}\xi) + \theta\,\mathrm{grad}\,\big(g_M(\delta\theta,\xi)\big) = 0.$$

Proof. Recall from Theorem 3.4, the first harmonic section equation (HSE1):

$$\sum_{i=1}^{2n}[\bar{R}(F_i,\bar{J}F_i),\bar{J}] = -2\bar{\nabla}_{\bar{\delta}\bar{J}}\bar{J}.$$

Sublemma 4.2. *We have* $d\pi(\bar{\nabla}_{\bar{\delta}\bar{J}}\bar{J}) = \nabla^N_{\delta J}J$.

Proof. Since π is a Riemannian submersion, we have for any section X, Y in $\mathcal{F}$:

$$0 = (\nabla d\pi)(X,Y) = (\nabla^N_{d\pi(X)}\hat{Y})\circ\pi - d\pi(\nabla_X Y),$$

where $d\pi(Y) = \hat{Y}\circ\pi$. Then,

$$\begin{aligned} d\pi\left((\bar{\nabla}_{\bar{\delta}\bar{J}}\bar{J})(X)\right) &= (\nabla^N_{d\pi(\bar{\delta}\bar{J})}J\hat{X})\circ\pi - J(\nabla^N_{d\pi(\bar{\delta}\bar{J})}\hat{X})\circ\pi = (\nabla^N_{\delta J}J\hat{X})\circ\pi - J(\nabla^N_{\delta J}\hat{X})\circ\pi \\ &= (\nabla^N_{\delta J}J)(\hat{X})\circ\pi, \end{aligned}$$

using that $d\pi(\bar{\delta}\bar{J}) = (\delta J)\circ\pi$ and $d\pi(\bar{J}X) = (J\hat{X})\circ\pi$ if $d\pi(X) = \hat{X}\circ\pi$. □

Then, we concentrate on the left-hand side of the equation. Let $\{F_i\}_{i=1,\dots,2n}$ be a local orthonormal frame of the distribution $\mathcal{F}$ which projects to an orthonormal frame $\{(F_i)_*\}_{i=1,\dots,2n}$ on N. By the expression of $\bar{R}$ given in the proof of Corollary 3.6 and Equation (3.6),

$$\sum_{i=1}^{2n}[\bar{R}(F_i,\bar{J}F_i),\bar{J}] = \sum_{i=1}^{2n}[R^{\mathcal{F}}(F_i,\bar{J}F_i),\bar{J}].$$

By the curvature equations of a submersion [One] and for Z and H in $\mathcal{F}$, we have

$$\begin{aligned} &-\sum_{i=1}^{2n} g_M([R^{\mathcal{F}}(F_i,\bar{J}F_i),\bar{J}](Z),H) = \sum_{i=1}^{2n} -g_M(R^{\mathcal{F}}(F_i,\bar{J}F_i)(\bar{J}Z),H) - g_M(R^{\mathcal{F}}(F_i,\bar{J}F_i)Z,\bar{J}H) \\ &= \sum_{i=1}^{2n} -g_N(R^N((F_i)_*,J(F_i)_*)JZ_*,H_*) - \frac{1}{2}\eta([F_i,\theta F_i])\eta([\theta Z,H]) + \frac{1}{4}\eta([\theta F_i,\theta Z])\eta([F_i,H]) \\ &+ \frac{1}{4}\eta([\theta Z,F_i])\eta([\theta F_i,H]) - g_N(R^N((F_i)_*,J(F_i)_*)Z_*,JH_*) - \frac{1}{2}\eta([F_i,\theta F_i])\eta([Z,\theta H]) \\ &+ \frac{1}{4}\eta([\theta F_i,Z])\eta([F_i,\theta H]) + \frac{1}{4}\eta([Z,F_i])\eta([\theta F_i,\theta H]). \end{aligned}$$

Then, by Equations (2.8) and (2.11) of Corollary 2.2,

$$\eta([X,\theta Y]) = -\eta([\theta X,Y]) \ \text{ for any section } X,Y \text{ in } \mathcal{F},$$

and we deduce

$$\begin{aligned} -\sum_{i=1}^{2n} g_M([R^{\mathcal{F}}(F_i,\bar{J}F_i),\bar{J}](Z),H) &= \sum_{i=1}^{2n} -g_N(R^N((F_i)_*,J(F_i)_*) - g_N(R^N((F_i)_*,J(F_i)_*)Z_*,JH_*) \\ &= -\sum_{i=1}^{2n} g_N([R^N((F_i))_*,J(F_i)_*,J](Z_*),H_*). \end{aligned}$$

Combining with Sublemma 4.2, since π is a Riemannian submersion, we obtain:

$$\sum_{i=1}^{2n}[R^N((F_i)_*,J(F_i)_*),J] = -2\nabla^N_{\delta J}J.$$

which gives the first condition of [Woo].

First observe that ξ must be Killing: by the equations of a submersion [One] and for X and Y in $\mathcal{F}$, we have

$$g_M(\nabla_X\xi,Y) = -g_M(\xi,\nabla_X Y) = -g_M\left(\xi,\frac{1}{2}[X,Y]\right) = g_M(\xi,\nabla_Y X) = -g_M(\nabla_Y\xi,X).$$

By Theorem 3.5, the second harmonic section equation (HSE2) is

$$2(\nabla^*\nabla\xi)^{\mathcal{F}} = \sum_{i=1}^{2n}[R(F_i,\theta F_i),\theta](\xi) + 2(\nabla_{\delta\theta}\theta)(\xi),$$

but $[R(F_i,\theta F_i),\theta](\xi) = -\theta R(F_i,\theta F_i)\xi$, and, if $Z \in \mathcal{F}$,

$$-g_M(\theta R(F_i,\theta F_i)\xi, Z) = -g_M(R(F_i,\theta F_i)\theta Z,\xi).$$

Then by O'Neill's formulas (4.1)

$$-g_M(R(F_i,\theta F_i)\theta Z,\xi) = (\theta Z)\big(g_M(\nabla_{F_i}(\theta F_i),\xi)\big) - g_M(\nabla_{\theta Z}F_i,\nabla_{\theta F_i}\xi) + g_M(\nabla_{\theta Z}(\theta F_i),\nabla_{F_i}\xi).$$

Now, by the choice of extension, $\bar\nabla_{\theta Z}F_i = 0$ at the point we evaluate at, and

$$\begin{aligned} g_M(\nabla_{\theta Z}(\theta F_i),\nabla_{F_i}\xi) &= -g_M((\nabla_{\theta Z}\theta)(\theta^2 F_i),\nabla_{F_i}\xi) = -g_M((\nabla_Z\theta)(\theta F_i),\nabla_{F_i}\xi) \\ &= -g_M((\nabla_Z\theta)(\theta^2 F_i),\nabla_{\theta F_i}\xi) = g_M(\nabla_Z(\theta F_i) - \theta(\nabla_Z F_i),\nabla_{\theta F_i}\xi) \\ &= -g_M(\theta\nabla_Z(\theta F_i),\nabla_{F_i}\xi), \end{aligned}$$

but on the other hand

$$g_M(\nabla_{\theta Z}(\theta F_i),\nabla_{F_i}\xi) = -g_M((\nabla_Z\theta)(\theta F_i),\nabla_{F_i}\xi) = g_M(\theta\nabla_Z(\theta F_i),\nabla_{F_i}\xi).$$

So $g_M(\nabla_{\theta Z}(\theta F_i),\nabla_{F_i}\xi) = 0$ and Equation (HSE2) reduces to

$$2(\nabla^*\nabla\xi)^{\mathcal{F}} + 2\theta(\nabla_{\delta\theta}\xi) = \sum_{i,j=1}^{2n}(\theta F_j)\big(g_M(\nabla_{F_i}(\theta F_i),\xi)\big)F_j.$$

Since

$$\sum_{i=1}^{2n} g_M(\nabla_{F_i}(\theta F_i),\xi) = g_M((\nabla_{F_i}\theta)(F_i) + \theta(\nabla_{F_i}F_i),\xi) = \sum_{i=1}^{2n} g_M((\nabla_{F_i}\theta)(F_i),\xi) = g_M(\delta\theta,\xi),$$

we obtain:

$$\begin{aligned} \sum_{i,j=1}^{2n}(\theta F_j)\big(g_M(\nabla_{F_i}(\theta F_i),\xi)\big)F_j &= \sum_{j=1}^{2n}(\theta F_j)\big(g_M(\delta\theta,\xi)\big)F_j = -\sum_{j=1}^{2n}(F_j)\big(g_M(\delta\theta,\xi)\big)\theta F_j \\ &= -\theta\,\mathrm{grad}\,\big(g_M(\delta\theta,\xi)\big). \end{aligned}$$

Then Equation (HSE2) is:

$$2(\nabla^*\nabla\xi)^{\mathcal{F}} + 2\theta(\nabla_{\delta\theta}\xi) + \theta\,\mathrm{grad}\,\big(g_M(\delta\theta,\xi)\big) = 0,$$

as required. □

Theorem 4.3. *Let $\pi : (M^{2n+1},\theta,\xi,\eta,g_M) \longrightarrow (N^{2n},J,g_N)$ be a holomorphic Riemannian submersion, i.e. with $d\pi(\theta X) = Jd\pi(X)$, from a normal almost contact metric manifold onto a Hermitian manifold. If the almost contact metric structure (θ,ξ,η) is a harmonic section then it also defines a harmonic map if and only if $\xi(|\nabla\xi|^2) = 0$, i.e. the bending of ξ is constant along its curves (cf. [Wie]), and, for all X in $\mathcal{F}$:*

$$\sum_{i,j=1}^{2n} g_N(R^N((F_i)_*,X_*)(F_j)_*, J(\nabla^N_{(F_i)_*}J)(F_j)_*) = 4g_M(\nabla^*\nabla\xi,\nabla_X\xi),$$

where $\{F_i\}_{i=1,\dots,2n}$ is an orthonormal frame on the horizontal sub-bundle $\mathcal{F}$.

Proof. A harmonic almost contact metric structure will also be a harmonic map if

$$\sum_{i=1}^{2n+1}\frac{1}{4}g_M(\bar J(\bar\nabla_{E_i}\bar J),R(E_i,X)) + g_M(\nabla_{E_i}\xi,R(E_i,X)\xi) = 0, \tag{4.2}$$

for any $X \in \mathcal{X}(M)$, where $\{F_i\}_{i=1,\dots,2n}$ is an orthonormal frame of the distribution $\mathcal{F}$ and $\{E_i\}_{1\leqslant i\leqslant 2n+1}$ $= \{F_i\}_{1\leqslant i\leqslant 2n} \cup \{\xi\}$ a local orthonormal frame of TM.

First take $X \in \mathcal{F}$, extend it into a basic vector field in $\mathcal{F}$, still called X, so Equation (4.2) reads:

$$\sum_{i=1}^{2n+1}\frac{1}{4}\sum_{j=1}^{2n} g_M(\bar J(\bar\nabla_{E_i}\bar J)(F_j),R(E_i,X)(F_j)) + g_M(\nabla_{E_i}\xi,R(E_i,X)\xi) = 0, \tag{4.3}$$

and using O'Neill's formulas (4.1), the first part of (4.3) can be developed into

$$
\begin{aligned}
g_M(\bar{J}(\bar{\nabla}_{E_i}\bar{J})(F_j), R(E_i,X)(F_j)) &= g_M(\bar{J}(\bar{\nabla}_{F_i}\bar{J})(F_j), R(F_i,X)(F_j)) + g_M(\bar{J}(\bar{\nabla}_{\xi}\bar{J})(F_j), R(\xi,X)(F_j)) \\
&= g_N(R^N((F_i)_*, X_*)(F_j)_*, J(\nabla^N_{(F_i)_*}J)(F_j)_*) \\
&+ 2g_M(A_{F_i}X, A_{F_j}(\bar{J}(\bar{\nabla}_{F_i}\bar{J})(F_j))) - g_M(A_X F_j, A_{F_i}(\bar{J}(\bar{\nabla}_{F_i}\bar{J})(F_j))) \\
&- g_M(A_{F_j}F_i, A_X(\bar{J}(\bar{\nabla}_{F_i}\bar{J})(F_j))) + g_M(R(\xi,X)(F_j), \bar{J}(\bar{\nabla}_{\xi}\bar{J})(F_j)) \\
&= g_N(R^N((F_i)_*, X_*)(F_j)_*, J(\nabla^N_{(F_i)_*}J)(F_j)_*) && (4.4) \\
&+ 2g_M(\nabla_{F_i}X, \xi)g_M(\nabla_{F_j}(\bar{J}(\bar{\nabla}_{F_i}\bar{J})(F_j)), \xi) && (4.5) \\
&- g_M(\nabla_X F_j, \xi)g_M(\nabla_{F_i}(\bar{J}(\bar{\nabla}_{F_i}\bar{J})(F_j)), \xi) && (4.6) \\
&- g_M(\nabla_{F_j}F_i, \xi)g_M(\nabla_X(\bar{J}(\bar{\nabla}_{F_i}\bar{J})(F_j)), \xi) && (4.7) \\
&+ g_M(R(\xi,X)(F_j), \bar{J}(\bar{\nabla}_{\xi}\bar{J})(F_j)). && (4.8)
\end{aligned}
$$

The term (4.8) is zero because $\nabla_\xi Y$ is horizontal whenever Y is horizontal, as $\nabla_\xi \xi = 0$, and $\nabla_\xi \theta = 0$ by Corollary 2.2.
The term (4.5) vanishes because

$$
\begin{aligned}
g_M(\nabla_{F_j}(\bar{J}(\bar{\nabla}_{F_i}\bar{J})(F_j)), \xi) &= -g_M(\bar{J}(\bar{\nabla}_{F_i}\bar{J})(F_j), \nabla_{F_j}\xi) = g_M((\bar{\nabla}_{F_i}\bar{J})(F_j), \theta\nabla_{F_j}\xi) \\
&= g_M((\bar{\nabla}_{F_i}\bar{J})(F_j), \nabla_{\theta F_j}\xi) = -g_M((\bar{\nabla}_{F_i}\bar{J})(\theta F_j), \nabla_{F_j}\xi) \\
&= -g_M((\bar{\nabla}_{F_i}\bar{J})(\bar{J}F_j), \nabla_{F_j}\xi) = g_M(\bar{J}(\bar{\nabla}_{F_i}\bar{J})(F_j), \nabla_{F_j}\xi).
\end{aligned}
$$

The sum of the terms (4.6) and (4.7) is

$$
\begin{aligned}
&- \sum_{i,j=1}^{2n} g_M(\nabla_X F_j, \xi)g_M(\nabla_{F_i}(\bar{J}(\bar{\nabla}_{F_i}\bar{J})(F_j)), \xi) - \sum_{i,j=1}^{2n} g_M(\nabla_{F_j}F_i, \xi)g_M(\nabla_X(\bar{J}(\bar{\nabla}_{F_i}\bar{J})(F_j)), \xi) \\
&= -\sum_{i,j=1}^{2n} g_M(F_j, \nabla_X\xi)g_M(\bar{J}(\bar{\nabla}_{F_i}\bar{J})(F_j), \nabla_{F_i}\xi) - g_M(F_i, \nabla_{F_j}\xi)g_M(\bar{J}(\bar{\nabla}_{F_i}\bar{J})(F_j), \nabla_X\xi) \\
&= -\sum_{i=1}^{2n} g_M(\bar{J}(\bar{\nabla}_{F_i}\bar{J})(\nabla_X\xi), \nabla_{F_i}\xi) + g_M(\bar{J}(\bar{\nabla}_{F_i}\bar{J})(\nabla_{F_i}\xi), \nabla_X\xi) \\
&= 2\sum_{i=1}^{2n} g_M(\bar{J}(\bar{\nabla}_{F_i}\bar{J})(\nabla_{F_i}\xi), \nabla_X\xi) = -4g_M(\nabla^*\nabla\xi, \nabla_X\xi),
\end{aligned}
$$

by (HSE2).

The second term in (HME) when $X \in \mathcal{F}$ is, using (4.1):

$$
\begin{aligned}
&\sum_{i=1}^{2n} g_M(\nabla_{F_i}\xi, R(F_i,X)\xi) + g_M(\nabla_\xi\xi, R(\xi,X)\xi) = \sum_{i=1}^{2n} g_M(\nabla_{F_i}\xi, R(F_i,X)\xi) \\
&= \sum_{i=1}^{2n} g_M(\nabla_{\nabla_{F_i}\xi}\Big(\nabla_{F_i}X\Big), \xi) + g_M(\nabla_{F_i}X, \nabla_{\nabla_{F_i}\xi}\xi) + g_M(X, \nabla_{\nabla_{\nabla_{F_i}\xi}F_i}\xi) + g_M(\nabla_{\nabla_{F_i}\xi}X, \nabla_{F_i}\xi) \\
&= \sum_{i=1}^{2n} -g_M(\nabla_{F_i}X, \nabla_{\nabla_{F_i}\xi}\xi) + g_M(\nabla_{F_i}X, \nabla_{\nabla_{F_i}\xi}\xi) = 0.
\end{aligned}
$$

So the condition (HME) for $X \in \mathcal{F}$ is equivalent to:

$$\sum_{i,j=1}^{2n} g_N(R^N((F_i)_*, X_*)(F_j)_*, J(\nabla^N_{(F_i)_*}J)(F_j)_*) = 4g_M(\nabla^*\nabla\xi, \nabla_X\xi).$$

The first term in Equation (4.2) for the case $X = \xi$ is:

$$\sum_{i=1}^{2n+1} g_M(\bar{J}(\bar{\nabla}_{E_i}\bar{J}), R(E_i,\xi)) = \sum_{i,j=1}^{2n} g_M(\bar{J}(\bar{\nabla}_{F_i}\bar{J})(F_j), R(F_i,\xi)F_j).$$

Let $Y=\sum_{i,j=1}^{2n}\bar{J}(\bar{\nabla}_{F_i}\bar{J})(F_j)$ then $Y=\sum_{i,j=1}^{2n}\theta(\nabla_{F_i}\theta)(F_j)$. Using O'Neill's formula or the fact that ξ is a Killing vector field, so $\nabla^2_{X,Y}\xi=-R(\xi,X)Y$, we have

$$\begin{aligned}
&\sum_{i,j=1}^{2n} g_M(\bar{J}(\bar{\nabla}_{F_i}\bar{J})(F_j),R(F_i,\xi)F_j)=\sum_{i,j=1}^{2n} g_M(\theta(\nabla_{F_i}\theta)(F_j),\nabla_{F_i}\nabla_{F_j}\xi-\nabla_{\nabla_{F_i}F_j}\xi)\\
&=\sum_{i,j=1}^{2n} g_M(\theta(\nabla_{F_i}\theta)(\theta F_j),\nabla_{F_i}\nabla_{\theta F_j}\xi-\nabla_{\nabla_{F_i}\theta F_j}\xi)\\
&=\sum_{i,j=1}^{2n} g_M(-\theta^2(\nabla_{F_i}\theta)(F_j),(\nabla_{F_i}\theta)(\nabla_{F_j}\xi)+\theta(\nabla_{F_i}\nabla_{F_j}\xi)-\nabla_{(\nabla_{F_i}\theta)(F_j)+\theta(\nabla_{F_i}F_j)}\xi)\\
&=\sum_{i,j=1}^{2n} g_M(-\theta^2(\nabla_{F_i}\theta)(F_j),(\nabla_{F_i}\theta)(\nabla_{F_j}\xi))+g_M(-\theta^2(\nabla_{F_i}\theta)(F_j),\theta(\nabla_{F_i}\nabla_{F_j}\xi))\\
&+g_M(\theta^2(\nabla_{F_i}\theta)(F_j),\nabla_{(\nabla_{F_i}\theta)(F_j)}\xi))
\end{aligned}$$

and the last term is zero because ξ is Killing. It follows that

$$2\sum_{i,j=1}^{2n} g_M(\theta(\nabla_{F_i}\theta)(F_j),\nabla_{F_i}\nabla_{F_j}\xi)$$

is equal to:

$$\begin{aligned}
&\sum_{i,j=1}^{2n} g_M(-\theta^2(\nabla_{F_i}\theta)(F_j),(\nabla_{F_i}\theta)(\nabla_{F_j}\xi))=\sum_{i,j=1}^{2n} g_M(-\theta[\theta(\nabla_{F_i}\theta)(F_j)],(\nabla_{F_i}\theta)(\nabla_{F_j}\xi))\\
&=\sum_{i,j=1}^{2n} g_M(-\theta[-(\nabla_{F_i}\theta)(\theta F_j)+g_M(\nabla_{F_i}\xi,F_j)\xi],(\nabla_{F_i}\theta)(\nabla_{F_j}\xi))=\sum_{i,j=1}^{2n} g_M(\theta(\nabla_{F_i}\theta)(\theta F_j),(\nabla_{F_i}\theta)(\nabla_{F_j}\xi))\\
&=\sum_{i,j=1}^{2n} g_M(-(\nabla_{F_i}\theta)(\theta^2 F_j)+g_M(\nabla_{F_i}\xi,\theta F_j)\xi,(\nabla_{F_i}\theta)(\nabla_{F_j}\xi))\\
&=\sum_{i,j=1}^{2n} g_M((\nabla_{F_i}\theta)(F_j),(\nabla_{F_i}\theta)(\nabla_{F_j}\xi))+\sum_{i,j=1}^{2n} g_M(\nabla_{F_i}\xi,\theta F_j)g_M(\xi,(\nabla_{F_i}\theta)(\nabla_{F_j}\xi)).
\end{aligned}$$

The first term on the right-hand side of the equation is:

$$\begin{aligned}
\sum_{i,j=1}^{2n} g_M((\nabla_{F_i}\theta)(F_j),(\nabla_{F_i}\theta)(\nabla_{F_j}\xi))&=\sum_{i,j,k=1}^{2n} g_M((\nabla_{F_i}\theta)(F_j),(\nabla_{F_i}\theta)(g_M(\nabla_{F_j}\xi,F_k)F_k))\\
&=\sum_{i,j,k=1}^{2n} g_M((\nabla_{F_i}\theta)(F_j),(\nabla_{F_i}\theta)(F_k))g_M(\nabla_{F_j}\xi,F_k)=0.
\end{aligned}$$

The second term on the right-hand side of the equation is:

$$\begin{aligned}
&\sum_{i,j=1}^{2n} g_M(\nabla_{F_i}\xi,\theta F_j)g_M(\xi,(\nabla_{F_i}\theta)(\nabla_{F_j}\xi))=\sum_{i,j=1}^{2n} g_M(\nabla_{F_i}\xi,\theta F_j)g_M(\theta(\nabla_{F_i}\xi),\nabla_{F_j}\xi)\\
&=-\sum_{i,j=1}^{2n} g_M(\nabla_{F_i}\xi,\theta F_j)g_M(\nabla_{F_i}\xi,\nabla_{\theta F_j}\xi)=-\sum_{i,j=1}^{2n} g_M(\nabla_{F_i}\xi,\nabla_{\nabla_{F_i}\xi}\xi)=0.
\end{aligned}$$

Then the first term of (HME) for $X=\xi$ is equal to zero.

As to the second term of the harmonic map equation when $X = \xi$, we have, using the third and last of the Formulas (4.1)

$$\sum_{i=1}^{2n+1} g_M(\nabla_{E_i}\xi, R(E_i,\xi)\xi) = -\sum_{i=1}^{2n} g_M(R(F_i,\xi)\nabla_{F_i}\xi,\xi)$$
$$= \sum_{i=1}^{2n} g_M(\nabla_\xi(\nabla_{F_i}\nabla_{F_i}\xi),\xi) - g_M(\nabla_\xi F_i, \nabla_{\nabla_{F_i}\xi}\xi) + g_M(\nabla_\xi\nabla_{F_i}\xi, \nabla_{F_i}\xi)$$
$$= \sum_{i=1}^{2n} \xi\Big(g_M(\nabla_{F_i}\nabla_{F_i}\xi,\xi)\Big) + \frac{1}{2}\xi\Big(g_M(\nabla_{F_i}\xi,\nabla_{F_i}\xi)\Big) = \xi\Big(g_M(\sum_{i=1}^{2n}\nabla_{F_i}\nabla_{F_i}\xi + \frac{1}{2}|\nabla\xi|^2\xi,\xi)\Big)$$
$$= -\frac{1}{2}\xi\Big(g_M(\nabla^*\nabla\xi,\xi)\Big) = -\frac{1}{2}\xi\Big(|\nabla\xi|^2\Big).$$

So the condition (HME) for $X = \xi$ is equivalent to $\xi\Big(|\nabla\xi|^2\Big) = 0$. □

Corollary 4.4. *Let $\pi : (M^{2n+1}, \theta, \xi, \eta, g_M) \longrightarrow (N^{2n}, J, g_N)$ be a holomorphic Riemannian submersion from a normal almost contact metric manifold onto a Hermitian manifold.*

If ξ is a harmonic unit vector field then the almost contact metric structure is a harmonic section if and only if J is a harmonic section and $2\nabla_{\delta\theta}\xi + \mathrm{grad}\big(g_M(\delta\theta,\xi)\big) = 0$. Moreover (θ,ξ,η) defines a harmonic map if and only if J is a harmonic map and $\xi(|\nabla\xi|^2) = 0$.

If ξ is parallel, then the almost contact metric structure is a harmonic section if and only if J is a harmonic section. Moreover (θ,ξ,η) defines a harmonic map if and only if J is a harmonic map.

One can endow a principal $\mathbb{S}^1$-bundle $\pi : M \twoheadrightarrow N$ over a complex manifold N, equipped with a connection η, of a normal almost contact structure (see [Mor1]). Moreover, from a Hermitian metric on N one deduces a normal almost contact metric structure on the total space M, cf. [Hat, Mor1, Ogi]. We may apply Theorem 4.1 and Corollary 4.4 in this situation, as we now explain.

Let $\exp(\mathbb{R}A) = \mathbb{S}^1$ be a one-dimensional compact torus where $\mathfrak{s} = \mathrm{Lie}(\mathbb{S}^1) = \mathbb{R}A$. Let $\pi : M \twoheadrightarrow N$ be a principal $\mathbb{S}^1$-bundle over a (connected) differentiable manifold N. Fix a principal connection $\eta : N \to \mathfrak{s}$; if we identify $\mathfrak{s}$ with $\mathbb{R}$ via $A \mapsto 1$, one can view η as an $\mathbb{S}^1$-invariant $\mathbb{R}$-valued form on N. Since $\mathbb{S}^1$ is abelian, the curvature of η is given by $d\eta(X,Y) = -\eta([X,Y])$ for $X, Y \in \mathcal{X}(M)$. The connection η yields a splitting $TM = HM \oplus VM$ where $VM = \mathrm{Ker}\, d\pi$ is the vertical bundle and $HM = \mathrm{Ker}\,\eta$ is the horizontal bundle. Let $p \in M$, the inverse of the isomorphism $d\pi(p) : H_pM \xrightarrow{\sim} T_{\pi(p)}N$ is denoted by $\psi(p)$. The vector field induced by the (right) action of $\mathbb{S}^1$ on M is denoted by ξ, that is to say $\xi_p = \frac{d}{dt}_{|t=0} pe^{tX_0}$ for all $p \in M$. From [Hat, §2] [Mor1, Theorem 6] and [Ogi, §3] we get the following result.

Theorem 4.5. *Let J be an almost complex structure on N and denote by (g_N, J) an almost Hermitian metric. Set*

$$g_M = \pi^* g_N + \eta \otimes \eta,$$

and define $\theta \in \mathrm{End}\, TM$ by

$$\theta_p(X_p) = \psi(p)(J d\pi(p).X_p) \ \text{for all}\ X \in \mathcal{X}(M)\ \text{and}\ p \in M. \tag{4.9}$$

Then, $\sigma = (\theta, \xi, \eta, g_M)$ is an almost contact metric structure on M such that $\pi \circ \theta = J \circ \pi$. The Reeb vector field ξ is Killing for g_M.
Furthermore, the almost contact structure (θ,ξ,η) is normal if and only if J is integrable and $d\eta = \pi^*\Sigma$ for some $\Sigma \in \Omega^2(N,\mathbb{R})$ such that Σ is J-invariant, i.e. $\Sigma(JX,JY) = \Sigma(X,Y)$ for all $X, Y \in \mathcal{X}(N)$.

The Kähler form Ω^{g_N} associated to an almost Hermitian metric (g_N, J) on N is defined by

$$\Omega^{g_N}(X,Y) = g_N(JX,Y).$$

Let $\sigma = (\theta, \xi, \eta, g_M)$ be a normal almost contact metric structure on M as above (hence J is integrable). One says that σ is Sasakian if $d\eta = \pi^*\Omega^{g_N}$, cf. [Ogi, Definitions 2.2 & 4.2], and that σ is c-Sasakian if there exists $c > 0$ such that $(\theta, \xi, \eta, cg_M)$ is Sasakian. The following corollary is a combination of results proved in [Ogi, §4].

Corollary 4.6. *Let $\pi : M \twoheadrightarrow N$ be a principal $\mathbb{S}^1$-bundle over an almost Hermitian manifold (N, J, g_N). Assume that the almost contact metric structure (θ, ξ, η, g_M) defined in Theorem 4.5 is normal. Then, the following assertions are equivalent.*

(i) *(θ, ξ, η, g_M) is a c-Sasakian structure;*
(ii) *(g_N, J) is a Kähler metric and there exists $c > 0$ such that $\Sigma = c\,\Omega^{g_N}$, where $d\eta = \pi^*\Sigma$ as in Theorem 4.5.*

Proof. It suffices to give the proof for $c = 1$.
(i) $\Rightarrow$ (ii): If (θ, ξ, η, g_M) is Sasakian, we know that $d\eta = \pi^*\Omega^{g_N} = \pi^*\Sigma$, from which it follows $\Omega^{g_N} = \Sigma$ since $\Omega^{g_N}(X, Y) = \pi^*\Omega^{g_N}(X^\psi, Y^\psi) = \pi^*\Sigma(X^\psi, Y^\psi) = \Sigma(X, Y)$, where we denote by X^ψ the horizontal lift of X. Moreover, (g_N, J) is Kähler by [Ogi, §4, Corollary].
(ii) $\Rightarrow$ (i): Since $\pi^*\Omega^{g_N} = \pi^*\Sigma = d\eta$, (θ, ξ, η, g_M) is almost Sasakian, see [Ogi, Definitions 4.2], and as (θ, ξ, η, g_M) is normal, it is Sasakian. □

Recall from the introduction that J being a harmonic section on the manifold (N, g_N, J) is equivalent to $[(\nabla^N)^*\nabla^N J, J] = 0$ and J is a harmonic map if, moreover, the following equation

$$\sum_{i=1}^{2n} g_N(\nabla^N_{F_i} J, R^N(F_i, X)J) = 0 \text{ for all } X \in \mathcal{X}(N)$$

is satisfied. Here, $(F_i)_{1\leqslant i\leqslant 2n}$ is an orthonormal frame on N. Observe that if (g_N, J) is a Kähler metric, i.e. $\nabla^N J = 0$, one has $[(\nabla^N)^*\nabla^N J, J] = 0$ and $\nabla^N_{F_i} J = 0$, thus J is a harmonic map.

Corollary 4.7. *Under the previous notation, assume that $\delta\theta$ and $\nabla^*\nabla\xi$ are constant multiples of ξ. Then:*

(1) *σ is a harmonic section if and only if J is a harmonic section;*
(2) *σ is a harmonic map if and only if J is a harmonic map;*
(3) *if (g_N, J) is a Kähler metric, σ is a harmonic map.*

Proof. By (2.12) we have $\nabla_\xi\xi = 0$ (actually ξ is Killing vector field, cf. Theorem 4.5). Therefore $\nabla_{\delta\theta}\xi = 0$ and the function $g_M(\delta\theta, \xi)$ is constant, thus $2\theta\nabla_{\delta\theta}\xi + \theta\,\mathrm{grad}(g_M(\delta\theta, \xi)) = 0$ by Theorem 4.1 and (1) follows from Corollary 4.4.
It is known that if $\nabla^*\nabla\xi = f\xi$ for some function f, then $f = |\nabla\xi|^2$ (see, for example, [Str, Lemma 7.1, & Proposition 8.1]). Thus $|\nabla\xi|^2$ is constant, $\xi\big(|\nabla\xi|^2\big) = 0$ and Corollary 4.4 yields (2). Finally, (3) follows since J is a harmonic map in the Kählerian case. □

In section 6 we will apply Corollary 4.7 to homogeneous principal $\mathbb{S}^1$-bundles over generalised flag manifolds associated to a compact semi-simple Lie group. In the next section we fix the notation used to described almost Hermitian structures on generalised flag manifolds.

5. Generalised flag manifolds

Let R be a reduced root system. If we fix a basis B of R, we denote by R^+ the set of positive roots. By [Bou, Chapitre 8, § 2 & § 4 Théorème 1] one knows that there exists a split semi-simple real Lie algebra $\mathfrak{g}_\mathbb{R}$ and a Cartan subalgebra $\mathfrak{t}_\mathbb{R}$ of $\mathfrak{g}_\mathbb{R}$ such that the root system of $(\mathfrak{t}_\mathbb{R}, \mathfrak{g}_\mathbb{R})$ is equal to R. We then have, for any choice of positive roots,

$$\mathfrak{g}_\mathbb{R} = \mathfrak{t}_\mathbb{R} \oplus_{\alpha\in\mathsf{R}^+} \left(\mathfrak{g}^\alpha_\mathbb{R} \oplus \mathfrak{g}^{-\alpha}_\mathbb{R}\right)$$

where

$$\mathfrak{g}^\lambda_\mathbb{R} = \{v \in \mathfrak{g}_\mathbb{R} : [h, v] = \lambda(h)v \text{ for all } h \in \mathfrak{t}_\mathbb{R}\}$$

if $\lambda \in \mathsf{R}$. Set $\mathfrak{g}_\mathbb{C} = \mathfrak{g}_\mathbb{R} \otimes \mathbb{C}$, $\mathfrak{t}_\mathbb{C} = \mathfrak{t}_\mathbb{R} \otimes \mathbb{C}$, $\mathfrak{g}^\lambda_\mathbb{C} = \mathfrak{g}^\lambda_\mathbb{R} \otimes \mathbb{C}$, etc., then, $\mathfrak{g}_\mathbb{C}$ is a complex semi-simple Lie algebra We denote by $G_\mathbb{C}$ a connected complex semi-simple Lie group with Lie algebra $\mathfrak{g}_\mathbb{C} \neq 0$.

Denote by $\langle\ |\ \rangle$ a positive real scalar multiple of the Killing form on $\mathfrak{g}_\mathbb{C}$. The restriction of $\langle\ |\ \rangle$ to $\mathfrak{t}_\mathbb{R}$ defines a Euclidean scalar product on $\mathfrak{t}_\mathbb{R}$, see [Bou, Chapitre 8, § 2, Remarque 2, p. 80]. Thus, for each $\phi \in \mathfrak{t}^*_\mathbb{R}$ there exits a unique $h_\phi \in \mathfrak{t}_\mathbb{R}$ such that $\langle h_\phi \mid h\rangle = \phi(h)$ for all $h \in \mathfrak{t}_\mathbb{R}$. We have $\mathfrak{t}_\mathbb{R} = \oplus_{\beta\in\mathsf{B}}\mathbb{R}h_\beta$ if B is a basis of R. For each $\lambda \in \mathsf{R}$ one can choose a root vector $E_\lambda \in \mathfrak{g}^\lambda_\mathbb{R}$ such that $\mathfrak{g}^\lambda_\mathbb{R} = \mathbb{R}E_\lambda$, $[E_\lambda, E_{-\lambda}] = h_\lambda \in \mathfrak{t}_\mathbb{R}$; hence,

$$[h_\lambda, E_\mu] = \mu(h_\lambda)E_\mu = \langle h_\mu \mid h_\lambda\rangle E_\mu$$

for all $\mu \in \mathsf{R}$. One has $\langle E_\lambda \mid E_{-\lambda}\rangle = 1$ and $\langle E_\lambda \mid E_\mu\rangle = 0$ if $\mu \neq -\lambda$.

One defines a compact Cartan subalgebra by

$$\mathfrak{t} = i\mathfrak{t}_{\mathbb{R}} = \bigoplus_{\beta \in \mathsf{B}} \mathbb{R} i h_\beta.$$

If $\mathfrak{l} \subset \mathfrak{t}$ is a subspace we set $\mathfrak{l}_{\mathbb{R}} = \{x \in \mathfrak{t}_{\mathbb{R}} : ix \in \mathfrak{l}\}$, thus $\mathfrak{l} = i\mathfrak{l}_{\mathbb{R}}$. Set, for $\lambda \in \mathsf{R}$:

$$X_\lambda = \frac{1}{\sqrt{2}}(E_\lambda - E_{-\lambda}), \quad Y_\lambda = \frac{i}{\sqrt{2}}(E_\lambda + E_{-\lambda}). \tag{5.1}$$

Thus, $[X_\lambda, Y_\lambda] = ih_\lambda$ and for all $H \in \mathfrak{t}_{\mathbb{R}}$:

$$[iH, X_\lambda] = \lambda(H) Y_\lambda, \quad [iH, Y_\lambda] = -\lambda(H) X_\lambda.$$

Notice that

$$\langle X_\lambda \mid Y_\lambda \rangle = 0, \;\; \langle X_\lambda \mid X_\lambda \rangle = \langle Y_\lambda \mid Y_\lambda \rangle = -1, \tag{5.2}$$

$$\langle Y_\lambda \mid Y_\mu \rangle = \langle X_\lambda \mid X_\mu \rangle = \langle X_\lambda \mid Y_\mu \rangle = 0 \;\text{ for all } \lambda \neq \mu \text{ in } \mathsf{R}^+. \tag{5.3}$$

Set $\mathfrak{m}_\lambda = \mathfrak{m}_{-\lambda} = \mathbb{R}X_\lambda \oplus \mathbb{R}Y_\lambda$ and $\mathfrak{n} = \bigoplus_{\lambda \in \mathsf{R}^+} \mathfrak{m}_\lambda$. The $\mathfrak{t}$-modules $\mathfrak{m}_\lambda$ are pairwise non-isomorphic irreducible $\mathfrak{t}$-modules. One has $[\mathfrak{t}, \mathfrak{m}_\lambda] \subset \mathfrak{m}_\lambda$, $[\mathfrak{m}_\lambda, \mathfrak{m}_\lambda] = \mathbb{R}ih_\lambda$ and, if $\lambda \neq \pm\mu$:

$$[\mathfrak{m}_\lambda, \mathfrak{m}_\mu] \subset \mathfrak{m}_{\lambda+\mu} \oplus \mathfrak{m}_{\lambda-\mu} \tag{5.4}$$

with the convention that $\mathfrak{m}_\nu = 0$ when $\nu \notin \mathsf{R}$, see [WG, Lemma 9.2]. Under this notation, the Lie subalgebra

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$$

is a compact real form of $\mathfrak{g}_{\mathbb{C}}$. We denote by G and T the closed connected Lie subgroups of $G_{\mathbb{C}}$ with Lie algebras $\mathfrak{g}$ and $\mathfrak{t}$. Hence, G is compact and T is a maximal torus of G.

One obtains a Euclidean scalar product on $\mathfrak{g}$ by setting, cf. [Bou, Chapitre 9, § 1, p. 16]:

$$B(U, V) = -\langle U \mid V \rangle \;\text{ for all } U, V \in \mathfrak{g}.$$

If $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are subspaces of $\mathfrak{g}$ such that $\mathfrak{c} = \mathfrak{a} \oplus \mathfrak{b}$ with $\mathfrak{a}$ orthogonal to $\mathfrak{b}$ with respect to B (i.e. $\langle \mid \rangle$) we will write $\mathfrak{c} = \mathfrak{a} \ominus \mathfrak{b}$. In this case, we denote by $x_{\mathfrak{a}}$ the orthogonal projection of $x \in \mathfrak{c}$ onto $\mathfrak{a}$.

By (5.2), $B(X_\lambda, X_\lambda) = B(Y_\lambda, Y_\lambda) = 1$, $B(X_\lambda, Y_\lambda) = 0$. Therefore $\{X_\lambda, Y_\lambda\}$ is an orthonormal basis (for B) of $\mathfrak{m}_\lambda$ and, by (5.3), $\big((X_\lambda, Y_\lambda)\,;\lambda \in \mathsf{R}^+\big)$ is an orthonormal basis of $\mathfrak{n}$. One verifies that $\langle h_\alpha \mid E_\lambda \rangle = 0$ for all $\alpha \in \mathsf{R}$, hence $B(h_\alpha, X_\lambda) = B(h_\alpha, Y_\lambda) = 0$ and $B(\mathfrak{t}, \mathfrak{n}) = B(\mathfrak{m}_\lambda, \mathfrak{m}_\mu) = 0$, if $\lambda \neq \pm\mu$.

A *generalised flag manifold* is a homogeneous space G/K where K is the centraliser of a torus $C \subset G$; we may assume that $C \subset T$ is the connected component of the centre of K. When $C = T$, we obtain the full flag manifold G/T. The group K is compact and connected; we set $\mathfrak{k} = \mathrm{Lie}(K)$, which is a reductive Lie algebra. We mostly follow [Arv, WG] for notation and results on generalised flag manifolds and we will assume that C is non-trivial, i.e. $K \neq G$.

Let $\mathfrak{z} = \mathrm{Lie}(C)$ be the centre of $\mathfrak{k}$. Since $\mathfrak{t} \subset \mathfrak{k} \subset \mathfrak{g}$ one can find a root subsystem $\mathsf{P} \subset \mathsf{R} \subset \mathfrak{t}_{\mathbb{R}}^*$ such that, setting $\mathfrak{n}^{\mathsf{P}} = \sum_{\beta \in \mathsf{P}} \mathfrak{m}_\beta$ and $\mathfrak{t}^{\mathsf{P}} = \sum_{\lambda \in \mathsf{P}} \mathbb{R}\, ih_\lambda$, one has $\mathfrak{t} = \mathfrak{z} \oplus \mathfrak{t}^{\mathsf{P}} \subset \mathfrak{k} = \mathfrak{t} \oplus \mathfrak{n}^{\mathsf{P}}$. Set:

$$\mathsf{Q} = \mathsf{R} \smallsetminus \mathsf{P}, \qquad \mathfrak{m} = \sum_{\alpha \in \mathsf{Q}} \mathfrak{m}_\alpha.$$

Then $B(\mathfrak{k}, \mathfrak{m}) = 0$ and $\mathfrak{g} = \mathfrak{k} \ominus \mathfrak{m}$ is the reductive decomposition of $\mathfrak{g}$ associated to K, i.e. to the homogeneous space G/K.

Let $\varpi : \mathfrak{t}_{\mathbb{R}}^* \to \mathfrak{z}_{\mathbb{R}}^*$ be the restriction map. The elements of $R_T = \varpi(\mathsf{R}) = \varpi(\mathsf{Q})$ are called T-roots and parameterise the irreducible summands of the K-module $\mathfrak{m}$ as follows, cf. [Arv, Theorem 7.3]. If $\gamma \in R_T \subset \mathfrak{z}_{\mathbb{R}}^*$, set:

$$\varpi^{-1}(\gamma) = \{\alpha \in \mathsf{Q} \,:\, \alpha_{|\mathfrak{z}} = \gamma\} \subset \mathsf{Q}. \tag{5.5}$$

The $\mathfrak{k}_{\mathbb{C}}$-module $\mathfrak{m}_{\mathbb{C}} = \mathfrak{m} \otimes \mathbb{C}$ decomposes as the sum of the pairwise non-isomorphic irreducible modules $\mathfrak{m}_\gamma^{\mathbb{C}} = \sum_{\alpha \in \varpi^{-1}(\gamma)} \mathbb{C}E_\alpha$, i.e. $\mathfrak{m}_{\mathbb{C}} = \bigoplus_{\gamma \in R_T} \mathfrak{m}_\gamma^{\mathbb{C}}$. Choose a set of positive roots R^+ and define for $\gamma \in R_T$:

$$\mathfrak{m}_\gamma = \mathfrak{m}_{-\gamma} = (\mathfrak{m}_\gamma^{\mathbb{C}} \oplus \mathfrak{m}_{-\gamma}^{\mathbb{C}}) \cap \mathfrak{g} = \bigoplus_{\alpha \in \varpi^{-1}(\gamma) \cap \mathsf{R}^+} \mathfrak{m}_\alpha.$$

Then, the $\mathfrak{m}_\gamma$ are nonequivalent irreducible K-modules. In order to write $\sum_\gamma \mathfrak{m}_\gamma$ as a direct sum, for each class in $R_T/\{\pm 1\}$ we choose a representative element $\gamma \in R_T$ and we denote by R_T^+ the set of these representative elements. Observe that

$$\mathsf{Q} \cap \mathsf{R}^+ = \bigsqcup_{\gamma \in R_T^+} (\varpi^{-1}(\gamma) \cap \mathsf{R}^+), \quad \mathfrak{m} = \bigoplus_{\gamma \in R_T^+} \mathfrak{m}_\gamma = \bigoplus_{\alpha \in \mathsf{Q} \cap \mathsf{R}^+} \mathfrak{m}_\alpha.$$

Notice that $\{(X_\alpha, Y_\alpha) : \alpha \in \varpi^{-1}(\gamma) \cap \mathsf{R}^+\}$ is an orthonormal basis (for B) of $\mathfrak{m}_\gamma$ and that $B(\mathfrak{m}_\gamma, \mathfrak{m}_{\gamma'}) = 0$ if $\gamma \neq \gamma'$.

We say that $J \in \mathrm{End}(\mathfrak{m})$ is an invariant complex structure on $\mathfrak{m}$ if $J^2 = -\mathrm{id}$ and $\mathrm{Ad}(k)J = J$ for all $k \in K$. We extend J to $\mathfrak{g}$ by $J = 0$ on $\mathfrak{k}$. Recall that if $z \in \mathfrak{g}$ one defines the Kostant-Kirillov-Souriau (KKS for short) form ω_z by

$$\omega_z(A, A') = B(z, [A, A']) \text{ for all } A, A' \in \mathfrak{g}.$$

Notice that if $v \in \mathfrak{t}_\mathbb{R}$ one has $\omega_{iv}(X_\lambda, Y_\lambda) = \lambda(v)$ since $B(iv, ih_\lambda) = \langle v \mid h_\lambda \rangle = \lambda(v)$. The proof of the next classical proposition is left to the reader.

Proposition 5.1. *Let J be an invariant complex structure on $\mathfrak{m}$ and $z \in \mathfrak{z}$.*
(1) *For all $\gamma \in R_T$, there exists $\epsilon(\gamma) = \pm 1$ such that if $\lambda \in \varpi^{-1}(\gamma)$ one has $JX_\lambda = \epsilon(\gamma) Y_\lambda$ and $JY_\lambda = -\epsilon(\gamma) X_\lambda$.*
(2) *The form ω_z is J-invariant and K-invariant, i.e.,*

$$\omega_z(JA, JA') = \omega_z(A, A') = \omega_z(\mathrm{Ad}(k)A, \mathrm{Ad}(k)A') \text{ for all } A, A' \in \mathfrak{g} \text{ and } k \in K.$$

Moreover, $\omega_z(A, A') = B(z, [A, A']_\mathfrak{t}) = B(z, [A, A']_\mathfrak{z})$ and

$$\omega_z(A, A') = \omega_z(A_\mathfrak{m}, A'_\mathfrak{m}) = \omega_z(JA_\mathfrak{m}, JA'_\mathfrak{m}).$$

(3) *Suppose that $z' \in \mathfrak{z}$ and $\omega_z = \omega_{z'}$ on $\mathfrak{m} \times \mathfrak{m}$, then $z = z'$.*

The next two theorems summarise results proved in [Arv, Propositions 7.5, 7.6, 7.7] and [WG, Theorem 4.7, Proposition 9.3, Theorem 9.4]. Let $e \in G$ be the identity and $p_0 = eK \in G/K$; recall that $T_{p_0}(G/K)$ identifies with $\mathfrak{m}$.

Theorem 5.2. (1) *The set of G-invariant almost complex structures J on G/K is in bijection with the set of $\mathrm{Ad}(K)$-invariant complex structure $J_\mathfrak{m}$ on $\mathfrak{m}$ or, equivalently, with the set of signs $\{\epsilon(\gamma) = \pm 1 : \gamma \in R_T^+\}$. The structure $J_\mathfrak{m}$ is defined by $J_\mathfrak{m} X_\lambda = \epsilon(\gamma) Y_\lambda$, $J_\mathfrak{m} Y_\lambda = -\epsilon(\gamma) X_\lambda$ for all $\lambda \in \varpi^{-1}(\gamma) \cap \mathsf{R}^+$.*
(2) *Any G-invariant metric $g_{G/K}$ on G/K is compatible with any invariant almost complex structure. It is determined by an $\mathrm{Ad}(K)$-invariant scalar product $g_\mathfrak{m}$ on $\mathfrak{m}$ defined on each $\mathfrak{m}_\gamma$, $\gamma \in R_T^+$, by $g_{\mathfrak{m}_\gamma} = \kappa_\gamma B$ for some $\kappa_\gamma > 0$, hence, $g_\mathfrak{m}(X_\lambda, X_\lambda) = g_\mathfrak{m}(Y_\lambda, Y_\lambda) = \kappa_\gamma$ for all $\lambda \in \varpi^{-1}(\gamma) \cap \mathsf{R}^+$.*

Set $\mathfrak{z}_{\mathrm{reg}} = \mathfrak{z}_\mathbb{R} \smallsetminus \big(\bigcup_{\gamma \in R_T} \mathrm{Ker}(\gamma)\big)$ and define a Weyl chamber to be a connected component of $\mathfrak{z}_{\mathrm{reg}}$. The invariant orderings Q^+, as defined in [Arv, Chapter 7, §7], are in bijection with the Weyl chambers in $\mathfrak{z}_\mathbb{R}$. If C is a Weyl chamber, pick $z \in \mathsf{C}$; then $\mathsf{Q}^+ = \{\alpha \in \mathsf{Q} : \alpha(z) > 0\}$ gives an invariant ordering. From Q^+ one deduces a set of positive roots $\mathsf{R}^+ \supset \mathsf{Q}^+ = \mathsf{R}^+ \cap \mathsf{Q}$. When an invariant ordering is fixed, we will choose R^+ in this way. Then, one may take $R_T^+ = \varpi(\mathsf{Q}^+)$ and if $\gamma \in R_T^+$ one has, cf. (5.5) :

$$\varpi^{-1}(\gamma) = \varpi^{-1}(\gamma) \cap \mathsf{R}^+ = \{\lambda \in \mathsf{Q}^+ : \varpi(\lambda) = \gamma\}.$$

The $\mathrm{Ad}(K)$-invariant complex structure $J_\mathfrak{m}$ associated to the invariant ordering is given by choosing the signs $\epsilon(\gamma) = +1$ for $\gamma \in R_T^+$, hence:

$$J_\mathfrak{m} X_\lambda = Y_\lambda, \quad J_\mathfrak{m} Y_\lambda = -X_\lambda, \quad \text{for all } \lambda \in \mathsf{Q}^+. \tag{5.6}$$

Theorem 5.3. *Retain the notation of Theorem 5.2.*
(1) *There is a bijection between the following sets:*
- (i) *G-invariant complex structures on G/K;*
- (ii) *$\mathrm{Ad}(K)$-invariant complex structures $J_\mathfrak{m}$ on $\mathfrak{m}$ as in (5.6) defined by a choice of an invariant ordering Q^+, i.e. a Weyl chamber C.*

(2) *Let J be an invariant complex structure on G/K and $g_{G/K}$ be a G-invariant metric on G/K. The following are equivalent:*
- (i) *$(g_{G/K}, J)$ is Kähler;*
- (ii) *for all $\gamma, \gamma' \in R_T^+ = \varpi(\mathsf{Q}^+)$ such that $\gamma + \gamma' \in R_T^+$, one has $\kappa_\gamma + \kappa_{\gamma'} = \kappa_{\gamma+\gamma'}$;*

(iii) *the Kähler form* $\Omega_{\mathfrak{m}}(U,V) = g_{\mathfrak{m}}(J_{\mathfrak{m}}U,V)$ *is equal to the KKS form* $\omega_{ih|\mathfrak{m}\times\mathfrak{m}}$ *for some* h *in the Weyl chamber defined by* Q^+.

(3) *The metric* $g_{G/K}$ *on* G/K *is Kähler-Einstein if and only if* $\Omega_{\mathfrak{m}} = c\omega_{ih_\rho}$ *where* $\rho = \frac{1}{2}\sum_{\alpha\in\mathsf{Q}^+}\alpha$ *and* c *is a positive real constant.*

The list of generalised flag manifolds for classical groups can be found in [Arv, Chapter 7, §4, p. 100]. Examples illustrating the previous definitions and results are given in [Arv, Chapter 7, §§ 5 & 8, pp. 102-103 & 109-111] for the full flag manifold $\mathrm{SU}(3)/T$ and the generalised flag manifold $\mathrm{SO}(8)/(\mathrm{U}(2)\times\mathrm{U}(2))$. (See also the example for $\mathrm{SU}(3)/T$ given after Proposition 6.21.) Harmonic maps between generalised flag manifolds are studied in [Gue].

6. Homogeneous principal bundles over a generalised flag manifold

Let $M = G/K$, $K = Z_G(C)$, be a generalised flag manifold as in Section 5. Choose a sub-torus $\widetilde{C} \subset C$ of codimension one. Set $\widetilde{\mathfrak{z}} = \mathrm{Lie}(\widetilde{C})$ and pick a unit vector $X_0 \in \mathfrak{z}$, i.e. $B(X_0,X_0) = 1$, such that $B(X_0,\widetilde{\mathfrak{z}}) = 0$, hence $\mathfrak{z} = \mathbb{R}X_0 \ominus \widetilde{\mathfrak{z}}$. Then, if $\widetilde{\mathfrak{k}} = \widetilde{\mathfrak{z}} \oplus [\mathfrak{k},\mathfrak{k}]$ we get $\mathfrak{k} = \mathbb{R}X_0 \ominus \widetilde{\mathfrak{k}}$ and $\widetilde{\mathfrak{k}}$ is the orthogonal of X_0 in $\mathfrak{k}$. One has $Y_0 = -iX_0 \in \mathfrak{z}_{\mathbb{R}}$ and $\widetilde{\mathfrak{z}} = \mathfrak{z} \cap \widetilde{\mathfrak{k}}$. The algebra $\widetilde{\mathfrak{k}}$ is reductive and $\widetilde{\mathfrak{t}} = \widetilde{\mathfrak{z}} \ominus \mathfrak{t}^{\mathsf{P}}$ is a Cartan subalgebra of $\widetilde{\mathfrak{k}}$. Let $\widetilde{K}$ be the closed connected subgroup of G such that $\mathrm{Lie}(\widetilde{K}) = \widetilde{\mathfrak{k}}$, then $K/\widetilde{K} \cong C/\widetilde{C} \cong \mathbb{S}^1$ and we obtain a G-homogeneous principal $\mathbb{S}^1$-bundle:

$$\pi : M = G/\widetilde{K} \twoheadrightarrow N = G/K.$$

From $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} = \widetilde{\mathfrak{k}} \oplus \mathbb{R}X_0 \oplus \mathfrak{m}$ one deduces that the $\widetilde{K}$-invariant decomposition associated to the reductive homogeneous space $G/\widetilde{K}$ is:

$$\mathfrak{g} = \widetilde{\mathfrak{k}} \ominus \widetilde{\mathfrak{m}} \;\text{ with } \widetilde{\mathfrak{m}} = \mathbb{R}X_0 \ominus \mathfrak{m} \text{ (hence } [\widetilde{\mathfrak{k}},\widetilde{\mathfrak{m}}] \subset \widetilde{\mathfrak{m}}).$$

Set $\tilde{p}_0 = e\widetilde{K}$. One can identify $\widetilde{\mathfrak{m}}$ with $T_{\tilde{p}_0}(G/\widetilde{K})$ and the differential $d\pi(\tilde{p}_0) : T_{\tilde{p}_0}(G/\widetilde{K}) \twoheadrightarrow T_{p_0}(G/K)$ with the orthogonal projection $\widetilde{\mathfrak{m}} \twoheadrightarrow \mathfrak{m}$. Recall that there is a bijection between G-invariant tensors on $G/\widetilde{K}$ and $\mathrm{Ad}(\widetilde{K})$-invariant tensors on $\widetilde{\mathfrak{m}}$, cf. [CS]. We will emphasise this bijection in a few cases which enable us to construct invariant structures on $G/\widetilde{K}$.

Let $g \in G$, we denote by $\mathsf{L}_g : G/\widetilde{K} \to G/\widetilde{K}$, or $\mathsf{L}_g : G/K \to G/K$, the left translation by g, i.e. $x \mapsto gx$. If $\mathcal{O} \subset G/\widetilde{K}$ is open and $Z \in \mathcal{X}(\mathcal{O})$, the vector field $g_*Z \in \mathcal{X}(g\mathcal{O})$ is defined by

$$g_*Z_{gq} = d\mathsf{L}_g(q).Z_q \quad \text{for all } q \in \mathcal{O}.$$

A vector field $V \in \mathcal{X}(G/\widetilde{K})$ is said to be left-invariant if $g_*V = V$ for all $g \in G$, that is $V_{gq} = d\mathsf{L}_g(q).V_q$ for all $g \in G$ and $q \in G/\widetilde{K}$. We denote by $\mathcal{X}(G/\widetilde{K})^G$ the vector space of left-invariant vector fields. Suppose that $V \in \widetilde{\mathfrak{m}}$ centralises $\widetilde{\mathfrak{k}}$. The one-parameter subgroup $\exp(\mathbb{R}V)$ acts by right translation on $G/\widetilde{K}$, and one defines the vector field $V^\#$ on $G/\widetilde{K}$ by:

$$V^\#_q = \frac{d}{dt}\Big|_{t=0}\left(qe^{tV}\right) \quad \text{for } q \in G/\widetilde{K}.$$

It is easy to see that $V^\#$ is left-invariant. The next (well-known) lemma recalls that we can obtain $\mathcal{X}(G/\widetilde{K})^G$ through that construction.

Lemma 6.1. *There exists a linear bijection* f *between* $\mathcal{X}(G/\widetilde{K})^G$ *and* $\widetilde{\mathfrak{m}}^{\widetilde{K}} = \{V \in \widetilde{\mathfrak{m}} : \mathrm{Ad}(h)V = V \text{ for all } h \in \widetilde{K}\} = \{V \in \widetilde{\mathfrak{m}} : [\widetilde{\mathfrak{k}},V] = 0\}$. *One has* $f(X) = X_{\tilde{p}_0}$ *and* $f^{-1}(V) = V^\#$.

Using the previous lemma one can prove the following proposition. Since we will not use this result, the proof is left to the interested reader.

Proposition 6.2. *The space* $\mathcal{X}(G/\widetilde{K})^G$ *is equal to* $\mathbb{R}X_0^\#$ *unless* $\widetilde{\mathfrak{t}} = i\,\mathrm{Ker}(\alpha)$ *for an* $\alpha \in \mathsf{Q}$ *such that, setting* $\delta = \varpi(\alpha)$, *the* $\widetilde{\mathfrak{k}}_{\mathbb{C}}$*-module* $\mathfrak{m}^{\mathbb{C}}_\delta = \mathbb{C}E_\alpha$ *is of dimension one. In this case one has* $\mathcal{X}(G/\widetilde{K})^G = \mathbb{R}X_0^\# \oplus \mathbb{R}X_\alpha^\# \oplus \mathbb{R}Y_\alpha^\#$ *and* $ih_\alpha = \pm\sqrt{\langle h_\alpha \mid h_\alpha\rangle}X_0 \in \mathfrak{z}$.

Definition 6.3. Set $\xi = X_0^\#$. We say that $\widetilde{\mathfrak{k}}$ is *generic* if $\widetilde{\mathfrak{m}}^{\widetilde{\mathfrak{k}}} = \mathbb{R}X_0$, i.e. $\mathcal{X}(G/\widetilde{K})^G = \mathbb{R}\xi$.

Notice that, by the previous proposition, $\widetilde{\mathfrak{k}}$ is generic except in a finite number of cases.

We now study some invariant vector fields which are important in the construction of harmonic normal almost contact metric structures on $G/\widetilde{K}$, cf. §4.5.

Endow $G/\widetilde{K}$ with a G-invariant metric $g_{G/\widetilde{K}}$, i.e.:

$$(g_{G/\widetilde{K}})_{kp}(d\mathrm{L}_k(p).U_p, d\mathrm{L}_k(p).V_p) = (g_{G/\widetilde{K}})_k(U_p, V_p)$$

for all $U, V \in \mathcal{X}(G/\widetilde{K})$, $p \in G/\widetilde{K}$ and $k \in G$. Since $G/\widetilde{K}$ is a reductive homogeneous space, $g_{G/\widetilde{K}}$ is uniquely determined by an $\mathrm{Ad}(\widetilde{K})$-invariant Euclidean product $g_{\widetilde{\mathfrak{m}}}$ on $\widetilde{\mathfrak{m}}$. Denote by ∇ the Levi-Civita connection associated to $g_{G/\widetilde{K}}$. Then ∇ is G-invariant, that is to say:

$$g_* \nabla_X Y = \nabla_{g_* X} g_* Y \quad \text{for all } g \in G \text{ and } X, Y \in \mathcal{X}(G/\widetilde{K}).$$

There is a bijection between the space of G-invariant connections on $G/\widetilde{K}$ and the space of bilinear maps $\Lambda : \widetilde{\mathfrak{m}} \times \widetilde{\mathfrak{m}} \to \widetilde{\mathfrak{m}}$ which are $\widetilde{K}$-invariant, i.e. $\Lambda(\mathrm{Ad}(h)U, \mathrm{Ad}(h)V) = \Lambda(U, V)$ for $U, V \in \widetilde{\mathfrak{m}}$ and $h \in \widetilde{K}$ (see [Nom] and [KN] for the general theory). We quickly recall, for further use, the construction of a connection ∇ starting from Λ as given in [Nom, §7]. Firstly, if $\widetilde{\pi} : G \twoheadrightarrow G/\widetilde{K}$ is the natural projection, there exists $\mathcal{U}$ contained in a neighbourhood of $e \in G$ such that $\widetilde{\pi} : \mathcal{U} \to \widetilde{\mathcal{U}} = \widetilde{\pi}(\mathcal{U})$ is a diffeomorphism. Then, for $X \in \widetilde{\mathfrak{m}}$ one can define $X^* \in \mathcal{X}(\widetilde{\mathcal{U}})$ by setting $X^*_{k\tilde{p}_0} = d\mathrm{L}_k(\tilde{p}_0).X$ for all $k \in \mathcal{U}$. The connection ∇ is completely determined on $\widetilde{\mathcal{U}}$ by $\nabla_{X^*} Y^* = \Lambda(X, Y)^*$ for $X, Y \in \widetilde{\mathfrak{m}}$. This local construction then furnishes the invariant connection ∇ on the manifold $G/\widetilde{K}$. In particular, if $\mathcal{O}$ is an open set of $G/\widetilde{K}$, one has on the translated open set $g\mathcal{O}$:

$$g_* \nabla_X Y = \nabla_{g_* X} g_* Y \quad \text{for all } X, Y \in \mathcal{X}(\mathcal{O}). \tag{6.1}$$

By [Nom, Theorem 13.1] and [KN, Theorem 3.3 & p. 202] the map Λ corresponding to the Levi-Civita connection ∇ associated to the chosen invariant metric $g_{G/\widetilde{K}}$ is given by the following formula for $X, Y \in \widetilde{\mathfrak{m}}$:

$$\Lambda(X, Y) = \frac{1}{2}[X, Y]_{\widetilde{\mathfrak{m}}} + U(X, Y) \tag{6.2}$$

where $U(X, Y) \in \widetilde{\mathfrak{m}}$ is determined by

$$2g_{\widetilde{\mathfrak{m}}}(U(X, Y), Z) = g_{\widetilde{\mathfrak{m}}}(X, [Z, Y]_{\widetilde{\mathfrak{m}}}) + g_{\widetilde{\mathfrak{m}}}(Y, [Z, X]_{\widetilde{\mathfrak{m}}}) \quad \text{for } Z \in \widetilde{\mathfrak{m}}. \tag{6.3}$$

Let $\vartheta \in \operatorname{End} T(G/\widetilde{K})$; one says that ϑ is G-invariant if:

$$d\mathrm{L}_g(q) \circ \vartheta_q = \vartheta_{gq} \circ d\mathrm{L}_g(q) \quad \text{for all } g \in G \text{ and } q \in G/\widetilde{K}.$$

Recall, cf. (3.4), that $\delta\vartheta \in \mathcal{X}(G/\widetilde{K})$ is defined by

$$\delta\vartheta = \sum_{i=1}^{m} \nabla_{E_i}\vartheta(E_i) - \sum_{i=1}^{m} \vartheta(\nabla_{E_i} E_i),$$

where $(E_i)_{1\leqslant i\leqslant m}$ is an orthonormal frame on a open subset $\mathcal{O}$ of $G/\widetilde{K}$. Recall also, see (3.3), that for $V \in \mathcal{X}(G/\widetilde{K})$ we have defined $\nabla^*\nabla V \in \mathcal{X}(G/\widetilde{K})$ by

$$\nabla^*\nabla V = -\sum_{i=1}^{m} \nabla_{E_i}(\nabla_{E_i} V) - \nabla_{\nabla_{E_i} E_i} V.$$

We will use the following remark.

Remark 6.4. *If $(E_i)_{1\leqslant i\leqslant m}$ is an orthonormal frame on an open subset $\mathcal{O}$ of $G/\widetilde{K}$ and $g \in G$, then $(g_* E_i)_{1\leqslant i\leqslant m}$ is an orthonormal frame on the open subset $g\mathcal{O}$.*

The remark follows from the invariance of $g_{G/\widetilde{K}}$ since, if $q \in \mathcal{O}$ and $k \in G$, we obtain:

$$(g_{G/\widetilde{K}})_{kq}(k_* E_{i|kq}, k_* E_{j\,|kq}) = (g_{G/\widetilde{K}})_{kq}(d\mathrm{L}_k(q).E_{i|q}, d\mathrm{L}_k(q).E_{j\,|q}) = (g_{G/\widetilde{K}})_k(E_{i|q}, E_{j\,|q}) = \delta_{ij}.$$

Lemma 6.5. (1) *If $\vartheta \in \operatorname{End} T(G/\widetilde{K})$ is G-invariant, the vector field $\delta\vartheta$ is G-invariant.*
(2) *If $V \in \mathcal{X}(G/\widetilde{K})^G$, the vector field $\nabla^*\nabla V$ is G-invariant.*
(3) *Let ϑ be G-invariant and $V \in \mathcal{X}(G/\widetilde{K})^G$. If $\widetilde{\mathfrak{k}}$ is generic, $\delta\vartheta \in \mathbb{R}\xi$ and $\nabla^*\nabla V \in \mathbb{R}\xi$.*

Proof. (1) We need to show $\delta\vartheta = g_*\delta\vartheta$ for all $g \in G$. Let $\mathcal{O}$ and $(E_i)_i$ be as above. We begin with the following observation. Let $X \in \mathcal{X}(\mathcal{O})$ and $g \in G$, then the vector field $\vartheta(X) : q \mapsto \vartheta_q(X_q)$ satisfies

$$g_*\vartheta(X) = \vartheta(g_*X) \text{ on the open set } g\mathcal{O}. \tag{6.4}$$

Indeed, using the invariance of ϑ we have on $g\mathcal{O}$:

$$g_*\vartheta(X)_{gq} = d\mathrm{L}_g(q).\vartheta_q(X)_q = \vartheta_{gq}(d\mathrm{L}_g(q).X_q) = \vartheta_{vq}(g_*X_{|gq}) = \vartheta(g_*X)_{gq}.$$

Therefore $g_*\vartheta(X) = \vartheta(g_*X)$ on $g\mathcal{O}$. Then, by definition of $\delta\vartheta$, Remark 6.4, (6.1) and (6.4), one gets

$$\delta\vartheta = \sum_i \nabla_{g_*E_i}\vartheta(g_*E_i) - \vartheta(\nabla_{g_*E_i}g_*E_i) = \sum_i g_*\nabla_{E_i}\vartheta(E_i) - g_*\vartheta(\nabla_{E_i}E_i) = g_*\delta\vartheta.$$

(2) By Remark 6.4 the vector field $\nabla^*\nabla V$ on $g\mathcal{O}$ is equal to $-\sum_{i=1}^m \nabla^2_{g_*E_i,g_*E_i}V$. Thus, on $g\mathcal{O}$ we have:

$$\begin{aligned}\nabla^*\nabla V &= -\sum_i \nabla^2_{g_*E_i,g_*E_i}g_*V = -\sum_i \nabla_{g_*E_i}(\nabla_{g_*E_i}g_*V) - \nabla_{\nabla_{g_*E_i}g_*E_i}g_*V \quad \text{(by invariance of } V)\\ &= g_*\Big\{-\sum_i \nabla_{E_i}(\nabla_{E_i}V) - \nabla_{\nabla_{E_i}E_i}V\Big\} = g_*\nabla^*\nabla V \quad \text{(by invariance of } \nabla).\end{aligned}$$

(3) follows from Definition 6.3 combined with (1) and (2). □

Recall that $\pi : G/\widetilde{K} \twoheadrightarrow G/K$ is a principal $\mathbb{S}^1$-bundle. Fix a G-invariant metric $g_{G/K}$ on G/K given by scalars $\kappa_\gamma > 0$, $\gamma \in R_T^+$, and let J be a G-invariant almost complex structure on G/K associated to an invariant complex structure $J_{\mathfrak{m}}$ on $\mathfrak{m}$, see Theorem 5.2.

Set $\mathfrak{s} = \mathrm{Lie}(\mathbb{S}^1) = \mathfrak{k}/\widetilde{\mathfrak{k}} = \mathbb{R}A$ with $A = [X_0 + \widetilde{\mathfrak{k}}]$. We begin by recalling a few (well-known) facts about the construction of G-invariant principal connections η on the principal bundle $G/\widetilde{K}$.
Such a connection is an element of $\Omega^1(G/\widetilde{K}, \mathfrak{s})$ such that

$$\eta_x(A^\#_x) = A, \quad \eta_{xk}(d\mathrm{R}_k(x).X) = \mathrm{Ad}(k)^{-1}\eta_x(X) = \eta_x(X) \ \text{ (since } \mathbb{S}^1 \text{ is abelian)},$$

for all $x \in G/\widetilde{K}$, $X \in T_x(G/\widetilde{K})$ and $k \in \mathbb{S}^1$. Here, R_k is the right translation by $k \in \mathbb{S}^1$ and $A^\#$ is the vector field defined by $A^\#_{g\tilde{p}_0} = \frac{d}{dt}_{|t=0}(ge^{tX_0}\tilde{p}_0)$, hence $A^\# = \xi$ (see Definition 6.3). The connection η is said to be invariant if $\eta \in \Omega(G/\widetilde{K}, \mathfrak{s})^G$, i.e.,

$$\eta_{gx}(d\mathrm{L}_g(x).X) = \eta_x(X) \ \text{ for all } x \in G/\widetilde{K} \text{ and } g \in G.$$

Using the fact that $\mathbb{S}^1$ is abelian, the invariant connections are described by the following proposition, see, for example, [CS, pages 18 & 39, Theorem 1.4.5]. Recall that, here, the exterior derivative of η is given by $d\eta(X,Y) = -\eta([X,Y])$ for $X, Y \in \mathcal{X}(G/\widetilde{K})$.

Proposition 6.6. (1) *There exists a bijection between the set of G-invariant principal connections and the set of linear maps $\nu : \mathfrak{g} \to \mathfrak{s}$ such that:*

(a) *$\nu_{|\mathfrak{k}}$ coincides with the canonical projection from $\mathfrak{k}$ to $\mathfrak{s} = \mathfrak{k}/\widetilde{\mathfrak{k}}$;*
(b) *$\nu(\mathrm{Ad}(t)v) = \nu(v)$ for all $t \in K$ and $v \in \mathfrak{g}$.*

(2) *the connection form of η equals $d\eta \in \Omega^2(G/\widetilde{K}, \mathfrak{s})^G$ and is determined by its value at $\tilde{p}_0$:*

$$d\eta_{g\tilde{p}_0}(U,V) = d\eta_{\tilde{p}_0}(d\mathrm{L}_{g^{-1}}(g\tilde{p}_0).U, d\mathrm{L}_{g^{-1}}(g\tilde{p}_0).V)$$

for all $g \in G$ and $U, V \in T_{g\tilde{p}_0}(G/\widetilde{K})$.

In the previous bijection, if $\nu : \mathfrak{g} \to \mathfrak{s}$ one defines $\eta_{\tilde{p}_0}$ by $\eta_{\tilde{p}_0}(\bar{u}) = \nu(u)$ for all $\bar{u} \in T_{\tilde{p}_0}(G/\widetilde{K}) \equiv \mathfrak{g}/\widetilde{\mathfrak{k}}$ (by condition (a) this is well defined).

Lemma 6.7. *One defines a G-invariant principal connection η on the principal bundle $\pi : G/\widetilde{K} \twoheadrightarrow G/K$ by setting*

$$\eta_{\tilde{p}_0}(U) = B(X_0, U)A \ \text{ for all } U \in \mathfrak{g}.$$

Define $\Sigma \in \Omega^2(G/K, \mathfrak{s})^G$ by $\Sigma_{p_0}(U,V) = \omega_{-X_0}(U,V)A$. Then $d\eta = \pi^*\Sigma$ and Σ is J-invariant.

Proof. We first verify the conditions (a) and (b) of Proposition 6.6. The orthogonal projection of $U \in \mathfrak{k}$ onto $\mathbb{R}X_0$ is equal to $B(X_0, U)X_0$, therefore its image in $\mathfrak{s} = \mathfrak{t}/\widetilde{\mathfrak{t}}$ is $\eta_{\tilde{p}_0}(U) = B(X_0, U)A$, this gives (a). Let $t \in K$ and $v \in \mathfrak{g}$. Then we have:

$$\eta_{\tilde{p}_0}(\mathrm{Ad}(t)v) = B(X_0, \mathrm{Ad}(t)v)A = B(\mathrm{Ad}(t)^{-1}X_0, v)A = B(X_0, v)A = \eta_{\tilde{p}_0}(v).$$

This proves (b). Thus, Proposition 6.6 applies and η is a G-invariant connection.
Recall that $\omega_{X_0}(U, V) = B(X_0, [U, V])$ and that ω_{X_0} is $\mathrm{Ad}(K)$-invariant, see Proposition 5.1. Thus Σ_{p_0} identifies with an $\mathrm{Ad}(K)$-invariant element of $\Omega^2(\mathfrak{g}/\mathfrak{k}, \mathfrak{s})$. It follows that we can define a G-invariant form $\Sigma \in \Omega^2(G/K, \mathfrak{s})^G$ by its value at p_0. Then, since π, $d\eta$ and Σ are G-invariant it suffices to prove the second claim at the point $\tilde{p}_0$. Recall that

$$d\pi(\tilde{p}_0) : T_{\tilde{p}_0}(G/\widetilde{K}) \equiv \widetilde{\mathfrak{m}} \to T_{p_0}(G/K) \equiv \mathfrak{m}$$

coincides with the projection from $\widetilde{\mathfrak{m}}$ onto $\mathfrak{m}$ given by $\widetilde{\mathfrak{m}} = \mathbb{R}X_0 \oplus \mathfrak{m}$. If $X, Y \in T_{\tilde{p}_0}(G/\widetilde{K}) \equiv \widetilde{\mathfrak{m}}$, one has

$$d\eta_{\tilde{p}_0}(X, Y) = -\eta_{\tilde{p}_0}([X, Y]) = -B(X_0, [X, Y])A = -B(X_0, [X_{\mathfrak{m}}, Y_{\mathfrak{m}}])A,$$

cf. Proposition 5.1. Therefore,

$$d\eta_{\tilde{p}_0}(X, Y) = \Sigma_{p_0}(d\pi(\tilde{p}_0).X, d\pi(\tilde{p}_0).Y) = (\pi^*\Sigma)_{\tilde{p}_0}(X, Y).$$

The J-invariance of Σ follows from the $J_{\mathfrak{m}}$-invariance of ω_{-X_0}, cf. Proposition 5.1. □

Notation 6.8. In order to recover the notation used in [Hat, Mor1, Mor2, Ogi] we identify the one dimensional Lie algebra $\widetilde{\mathfrak{t}} = \mathbb{R}A$ with $\mathbb{R}$ by sending A to 1. Under this notation, $\eta \in \Omega^1(G/\widetilde{K}, \mathbb{R})^G$, $\Sigma \in \Omega^2(G/K, \mathbb{R})^G$, and:

$$\eta_{\tilde{p}_0}(X) = B(X_0, X), \quad \Sigma_{p_0}(U, V) = -\omega_{X_0}(U, V) = -B(X_0, [U, V]),$$

for all $X \in \mathfrak{g}$, $U, V \in \mathfrak{m} \equiv T_{p_0}(G/K)$. Since $\eta_{\tilde{p}_0}(\xi_{\tilde{p}_0}) = B(X_0, X_0) = 1$ we obtain $\eta(\xi) = 1$ by G-invariance of η.

The connection η furnishes the decomposition $T(G/\widetilde{K}) = H(G/\widetilde{K}) \oplus V(G/\widetilde{K})$ where $d\pi : H(G/\widetilde{K}) = \mathrm{Ker}\,\eta \xrightarrow{\sim} T(G/K)$ and $V(G/\widetilde{K}) = \mathrm{Ker}\, d\pi$ is a sub-bundle of rank one.
Recall that we set $\psi(p) = d\pi(p)^{-1} : T_{\pi(p)}(G/K) \xrightarrow{\sim} H_p(G/\widetilde{K})$. One has $V_{g\tilde{p}_0}(G/\widetilde{K}) = d\mathrm{L}_g(\tilde{p}_0).V_{\tilde{p}_0}(G/\widetilde{K})$ by invariance of π, and the invariance of η implies that $H_{g\tilde{p}_0}(G/\widetilde{K}) = d\mathrm{L}_g(\tilde{p}_0).H_{\tilde{p}_0}(G/\widetilde{K})$. Clearly, ξ is a $\mathcal{C}^\infty$-generator of $V(G/\widetilde{K})$.

Recall from [Hat] the construction of the endomorphism θ, see (4.9); for all $p \in G/\widetilde{K}$ and $X \in \mathcal{X}(G/\widetilde{K})$, we set:

$$\theta_p(X_p) = \psi(p)(J\, d\pi(p).X_p). \tag{6.5}$$

Proposition 6.9. *Let θ be as in (6.5). The endomorphism θ is G-invariant and the triple (θ, ξ, η) is a G-invariant almost contact structure on $G/\widetilde{K}$ such that $\pi \circ \theta = J \circ \pi$.*

Proof. Using [Hat, §2, Theorem 1] one obtains that (θ, ξ, η) is an almost contact structure on $G/\widetilde{K}$ such that $\pi \circ \theta = J \circ \pi$. By construction η and ξ are G-invariant. Therefore it suffices to prove that θ is G-invariant. Notice first that ψ is G-invariant. Indeed, for all $p \in G/\widetilde{K}$, $\psi(gp)$ is the inverse of $d\pi(gp) = d\mathrm{L}_g(\pi(p)) \circ d\pi(p) \circ d\mathrm{L}_{g^{-1}}(gp)$; it follows that $\psi(gp) = d\mathrm{L}_g(p) \circ \psi(p) \circ d\mathrm{L}_{g^{-1}}(g\pi(p))$ as required. Then, using the G-invariance of J, π and ψ, one easily gets $\theta_{gp} \circ d\mathrm{L}_g(p) = d\mathrm{L}_g(p) \circ \theta_p$. Thus θ is G-invariant. □

Recall that $g_{G/K}$ is compatible with the almost complex structure J, i.e. $(G/K, J, g_{G/K})$ is almost Hermitian, cf. Theorem 5.2. Define a metric $g_{G/\widetilde{K}}$ on $G/\widetilde{K}$ by

$$g_{G/\widetilde{K}} = \pi^* g_{G/K} + \eta \otimes \eta, \tag{6.6}$$

that is to say: $(g_{G/\widetilde{K}})_p(U, V) = g_{G/K}(d\pi(p).U, d\pi(p).V) + \eta(U)\eta(V)$ for all $U, V \in T_p(G/\widetilde{K})$.

Lemma 6.10. *The metric $g_{G/\widetilde{K}}$ is G-invariant. The $\widetilde{K}$-invariant scalar product $g_{\widetilde{\mathfrak{m}}}$ on $\widetilde{\mathfrak{m}} = \mathbb{R}X_0 \oplus \mathfrak{m}$ defining $g_{G/\widetilde{K}}$ is given by $g_{\widetilde{\mathfrak{m}}} = B$ on $\mathbb{R}X_0$ and $g_{\widetilde{\mathfrak{m}}} = g_{\mathfrak{m}}$ on $\mathfrak{m}$.*

Proof. The invariance of $g_{G/\widetilde{K}}$ follows from the invariance of $d\pi$, $g_{G/K}$ and η. Through the usual identifications, at the point $\tilde{p}_0$ one has

$$\begin{aligned} g_{\tilde{\mathfrak{m}}}(U,V) &= (g_{G/\widetilde{K}})_{\tilde{p}_0}(B(X_0,U)X_0 + U_{\mathfrak{m}}, B(X_0,V)X_0 + V_{\mathfrak{m}}) = B(X_0,U)B(X_0,V) + (g_{G/K})_{p_0}(U_{\mathfrak{m}}, V_{\mathfrak{m}}) \\ &= \eta_{\tilde{p}_0}(U)\eta_{\tilde{p}_0}(V) + (\pi^* g_{G/K})_{\tilde{p}_0}(U,V). \end{aligned}$$

This proves the second claim. □

We now combine the previous results to get the next theorem, already obtained in [Cor, Theorem 4.1] by different methods. We have defined above an almost contact metric structure (θ,ξ,η) on $G/\widetilde{K}$ and a metric $g_{G/\widetilde{K}}$. Recall that if J is a complex structure on G/K, it is determined by the choice of an invariant ordering Q^+, i.e. by a Weyl chamber $\mathsf{C} \subset \mathfrak{z}_{\mathbb{R}}$, cf. Theorem 5.3 (1). Moreover, $(g_{G/K}, J)$ is Kähler if and only if the form $\Omega_{\mathfrak{m}}$ is equal to ω_{ih_κ} for some $h_\kappa \in \mathsf{C}$, see Theorem 5.3 (2).

Theorem 6.11. *Let J be a G-invariant almost complex structure and $g_{G/K}$ be a G-invariant almost Hermitian metric on G/K.*
(1) Endow the principal bundle $\pi : M = G/\widetilde{K} \twoheadrightarrow N = G/K$ with the almost contact structure (θ,ξ,η) and the metric $g_{G/\widetilde{K}}$ defined by (6.6). Then $(\theta,\xi,\eta,g_{G/\widetilde{K}})$ is G-invariant and defines an almost contact metric structure on $G/\widetilde{K}$.
(2) Assume that J is a complex structure on G/K. Then $(\theta,\xi,\eta,g_{G/\widetilde{K}})$ is a normal almost contact metric structure on $G/\widetilde{K}$.
(3) The structure $(\theta,\xi,\eta,g_{G/\widetilde{K}})$ is c-Sasakian if and only if $(g_{G/K}, J)$ is Kähler and $ch_\kappa = iX_0 = -Y_0 \in \mathsf{C}$ for some $c > 0$. In this case, $-Y_0 \in \mathsf{C}$ and $\gamma(Y_0) = -c\kappa_\gamma < 0$ for all $\gamma \in R_T^+ = \varpi(\mathsf{Q}^+)$.

Proof. (1) is consequence of Proposition 6.9, Lemma 6.10 and Theorem 4.5.
(2) Since J is a complex structure, $g_{G/K}$ is Hermitian. Recall that $d\eta = \pi^*\Sigma$ by Lemma 6.7 and that Σ is J-invariant, cf. Lemma 6.7. Then, Theorem 4.5 yields the result.
(3) Corollary 4.6 says that $(\theta,\xi,\eta,g_{G/\widetilde{K}})$ is c-Sasakian if and only if $(g_{G/K}, J)$ is Kähler and $c\Omega^{g_{G/K}} = \Sigma$; the later is equivalent (by invariance and Lemma 6.7) to $c\omega_{ih_\kappa} = \omega_{-X_0}$. Therefore, by Proposition 5.1 (3), $c\Omega^{g_{G/K}} = \Sigma$ if and only if $ch_\kappa = iX_0$. Then, $iX_0 = -Y_0 \in \mathsf{C}$ and if $\alpha \in \varpi^{-1}(\gamma)$ we get:

$$\begin{aligned} c\kappa_\alpha &= c\omega_{ih_\kappa}(X_\alpha, Y_\alpha) = c\langle h_\kappa \mid h_\alpha\rangle = c\alpha(h_\kappa) \\ &= \omega_{-iY_0}(X_\alpha, Y_\alpha) = -\langle Y_0 \mid h_\alpha\rangle = -\alpha(Y_0) = -\gamma(Y_0). \end{aligned}$$

This proves the last assertion. □

Let $(g_{G/K}, J)$ be a G-invariant almost Hermitian metric on G/K as in the previous section and let $\sigma = (\theta,\xi,\eta,g_{G/\widetilde{K}})$ be the almost contact metric structure obtained in Theorem 6.11 (2).

Remark 6.12. Suppose that $(g_{G/K}, J)$ is Hermitian. Using Lemma 6.5 (3) and Corollary 4.7 we can deduce the following result: assume that $\tilde{\mathfrak{k}}$ is generic, then σ is a harmonic section/map if and only if J is a harmonic section/map.

The statement of the previous remark is, in fact, true for any choice of $\tilde{\mathfrak{k}}$. In order to prove this result we need to compute formulas involving the G-invariant Levi-Civita connection ∇ associated to $g_{G/\widetilde{K}}$.

We start by introducing more notation. Recall that $X_0 = iY_0$ with $Y_0 \in \mathfrak{z}_{\mathbb{R}}$ and $\mathfrak{m}_\gamma = \oplus_{\alpha\in\varpi^{-1}(\gamma)\cap \mathsf{R}^+}\mathfrak{m}_\alpha$ for $\gamma \in R_T^+$. The metric $g_{G/\widetilde{K}}$ is determined by the scalar product $g_{\tilde{\mathfrak{m}}}$ on $\tilde{\mathfrak{m}}$ as in Lemma 6.10 and the scalar product $g_{\mathfrak{m}}$ on each $\mathfrak{m}_\gamma$ is equal to $\kappa_\gamma B$ for some $\kappa_\gamma > 0$, cf. Theorem 5.2. For $\gamma \in R_T^+$ and $\alpha \in \varpi^{-1}(\gamma)$ we set:

$$X'_\alpha = \frac{1}{\sqrt{\kappa_\gamma}}X_\alpha, \quad Y'_\alpha = \frac{1}{\sqrt{\kappa_\gamma}}Y_\alpha, \quad c_\gamma = \gamma(Y_0), \quad a_\gamma = \frac{c_\gamma}{\kappa_\gamma} = \frac{\gamma(Y_0)}{\kappa_\gamma}.$$

Then the elements (X'_α, Y'_α), $\alpha \in \varpi^{-1}(\gamma)$, provide an orthonormal basis of $\mathfrak{m}_\alpha$ and

$$\{X_0\,;(X'_\alpha, Y'_\alpha)\,:\,\alpha \in \varpi^{-1}(\gamma) \cap \mathsf{R}^+, \gamma \in R_T^+\}$$

is an orthonormal basis (for $g_{\tilde{\mathfrak{m}}}$) of $\tilde{\mathfrak{m}}$. Easy computations yield, for all $\lambda \in \mathsf{R}^+ \cap \varpi^{-1}(\gamma)$:

$$B(ih_\lambda, X_0) = \langle h_\lambda \mid Y_0 \rangle = c_\gamma, \quad (ih_\lambda)_{\tilde{\mathfrak{m}}} = c_\gamma X_0,$$
$$[X_0, X'_\lambda] = c_\gamma Y'_\lambda, \quad [X_0, Y'_\lambda] = -c_\gamma X'_\lambda$$
$$[X'_\lambda, Y'_\lambda] = \frac{1}{\kappa_\gamma} ih_\lambda, \quad [X'_\lambda, Y'_\lambda]_{\tilde{\mathfrak{m}}} = a_\gamma X_0, \quad g_{\tilde{\mathfrak{m}}}(X_0, [X'_\lambda, Y'_\lambda]_{\tilde{\mathfrak{m}}}) = a_\gamma.$$

Recall that we can find $\mathcal{U}$ containing $e \in G$ such that $\tilde{\pi} : \mathcal{U} \xrightarrow{\sim} \widetilde{\mathcal{U}} = \tilde{\pi}(\mathcal{U})$ is a diffeomorphism onto the open neighbourhood $\widetilde{\mathcal{U}}$ of $\tilde{p}_0 \in G/\widetilde{K}$. If $X \in \tilde{\mathfrak{m}} \equiv T_{\tilde{p}_0}(G/\widetilde{K})$ we defined a vector field $X^* \in \mathcal{X}(\widetilde{\mathcal{U}})$ by $X^*_{k\tilde{p}_0} = d\mathsf{L}_k(\tilde{p}_0).X$ for $k \in \mathcal{U}$. We know that the connection ∇ is determined by the $\mathrm{Ad}(\widetilde{K})$-invariant bilinear map $\Lambda : \tilde{\mathfrak{m}} \times \tilde{\mathfrak{m}} \to \tilde{\mathfrak{m}}$ such that $\Lambda(X, Y) = \frac{1}{2}[X, Y]_{\tilde{\mathfrak{m}}} + U(X, Y)$, cf. (6.2) and (6.3). From the definition of ξ we see that $X_0^* = \xi_{|\widetilde{\mathcal{U}}}$ and, since $g_{G/\widetilde{K}}$ is G-invariant, the set

$$\{X_0^*\,; X'^{\,*}_\alpha, Y'^{\,*}_\alpha\}_{\{\alpha \in \varpi^{-1}(\gamma) \cap \mathsf{R}^+, \gamma \in R_T^+\}}$$

is an orthonormal frame on $\widetilde{\mathcal{U}}$, cf. [Nom, p. 51]. To simplify the notation we set

$$F_\alpha = X'^{\,*}_\alpha, \quad \Phi_\alpha = Y'^{\,*}_\alpha, \quad \text{for } \alpha \in \varpi^{-1}(\gamma) \cap \mathsf{R}^+.$$

Lemma 6.13. *Let $\gamma \in R_T^+$ and $\alpha \in \varpi^{-1}(\gamma) \cap \mathsf{R}^+$. On the open subset $\widetilde{\mathcal{U}}$, one has:*

$$\begin{aligned}
&(0)\ \ \nabla_\xi \xi = 0,\\
&(1)\ \ \nabla_\xi F_\alpha = \big(c_\gamma - \frac{a_\gamma}{2}\big)\Phi_\alpha, \quad (2)\ \ \nabla_{F_\alpha}\xi = -\frac{a_\gamma}{2}\Phi_\alpha,\\
&(3)\ \ \nabla_\xi \Phi_\alpha = \big(-c_\gamma + \frac{a_\gamma}{2}\big)F_\alpha, \quad (4)\ \ \nabla_{\Phi_\alpha}\xi = \frac{a_\gamma}{2}F_\alpha,\\
&(5)\ \ \nabla_{F_\alpha}\Phi_\alpha = \frac{a_\gamma}{2}\xi, \quad (6)\ \ \nabla_{\Phi_\alpha}F_\alpha = -\frac{a_\gamma}{2}\xi,\\
&(7)\ \ \nabla_{F_\alpha}F_\alpha = 0, \quad (8)\ \ \nabla_{\Phi_\alpha}\Phi_\alpha = 0.
\end{aligned}$$

Proof. Since $U(X, Y) = U(Y, X)$, we have $\Lambda(Y, X) = \Lambda(X, Y) - [X, Y]_{\tilde{\mathfrak{m}}}$ and it suffices to check the formulas $(0), (1), (3), (5), (7), (8)$. Notice that to determine $U(X, Y)$ we need to compute the value of

$$f_{X,Y}(Z) = 2g_{\tilde{\mathfrak{m}}}(U(X, Y), Z) = g_{\tilde{\mathfrak{m}}}(X, [Z, Y]_{\tilde{\mathfrak{m}}}) + g_{\tilde{\mathfrak{m}}}(Y, [Z, X]_{\tilde{\mathfrak{m}}})$$

for $Z = X_0, X'_\lambda, Y'_\lambda$, $\lambda \in \mathsf{R}^+$. The computation in the different cases is straightforward; we only give it in case (1). Recall that $g_{\tilde{\mathfrak{m}}}(X_0, \tilde{\mathfrak{m}}) = g_{\tilde{\mathfrak{m}}}(\mathfrak{m}_\lambda, \mathfrak{m}_\mu) = 0$ (for $\lambda \neq \mu$) and $[\mathfrak{m}_\alpha, \mathfrak{m}_\lambda] \subset \mathfrak{m}_{\alpha+\lambda} + \mathfrak{m}_{\alpha-\lambda}$ if $\alpha \neq \pm\lambda$, cf. (5.4); this implies that if $V \in [\mathfrak{m}_\alpha, \mathfrak{m}_\lambda]$, we have

$$g_{\tilde{\mathfrak{m}}}(X_0, V_{\tilde{\mathfrak{m}}}) = g_{\tilde{\mathfrak{m}}}(X'_\alpha, V_{\tilde{\mathfrak{m}}}) = g_{\tilde{\mathfrak{m}}}(Y'_\alpha, V_{\tilde{\mathfrak{m}}}) = 0.$$

(1) One easily sees that $f_{X_0, X'_\alpha}(X_0) = f_{X_0, X'_\alpha}(X'_\alpha) = 0$. Let us give the details for the cases $Z = Y'_\alpha$ and $Z = X'_\lambda$, $\lambda \neq \alpha$. One has

$$\begin{aligned}
f_{X_0, X'_\alpha}(Y'_\alpha) &= g_{\tilde{\mathfrak{m}}}(X_0, [Y'_\alpha, X'_\alpha]_{\tilde{\mathfrak{m}}}) + g_{\tilde{\mathfrak{m}}}(X'_\alpha, [Y'_\alpha, X_0]_{\tilde{\mathfrak{m}}})\\
&= -a_\gamma + g_{\tilde{\mathfrak{m}}}(X'_\alpha, c_\gamma X'_\alpha) = -a_\gamma + c_\gamma.
\end{aligned}$$

If $Z = X'_\lambda$, $\lambda \neq \alpha$, we have $[X'_\lambda, X'_\alpha] \subset \mathfrak{m}_{\lambda+\alpha} + \mathfrak{m}_{\alpha-\lambda}$ and $[X'_\lambda, X_0] \in \mathfrak{m}_\lambda$. This implies $f_{X_0, X'_\alpha}(X'_\lambda) = 0$. A similar calculation shows that $f_{X_0, X'_\alpha}(Y'_\lambda) = 0$. It follows that $U(X_0, X'_\alpha) = \frac{1}{2}(-a_\gamma + c_\gamma)Y'_\alpha$. Since $\frac{1}{2}[X_0, X'_\alpha]_{\tilde{\mathfrak{m}}} = \frac{1}{2}c_\gamma Y'_\alpha$ we obtain $\Lambda(X_0, X'_\alpha) = (c_\gamma - \frac{1}{2}a_\gamma)Y'_\alpha$. Consequently, $\nabla_\xi F_\alpha = (c_\gamma - \frac{1}{2}a_\gamma)\Phi_\alpha$. □

Recall that the almost complex structure J on G/K is given by $J_{\mathfrak{m}} X_\lambda = \epsilon(\gamma) Y_\lambda$ and $J_{\mathfrak{m}}(Y_\lambda) = -\epsilon(\gamma) X_\lambda$, $\epsilon(\gamma) = \pm 1$, for all $\lambda \in \varpi^{-1}(\gamma) \cap \mathsf{R}^+$, see Theorem 5.2.

Lemma 6.14. *Let $\alpha \in \mathsf{Q} \cap \mathsf{R}^+$ and $\gamma = \varpi(\alpha)$. Then, $\theta F_\alpha = \epsilon(\gamma)\Phi_\alpha$ and $\theta\Phi_\alpha = -\epsilon(\gamma) F_\alpha$ on $\widetilde{\mathcal{U}}$.*

Proof. We make the calculation for F_α, the same method will give $\theta\Phi_\alpha$. Recall first that the projection $d\pi(\tilde{p}_0) : T_{\tilde{p}_0}(G/\widetilde{K}) \twoheadrightarrow T_{p_0}(G/K)$ can be identified with the orthogonal projection from $\tilde{\mathfrak{m}}$ onto $\mathfrak{m}$. Its inverse, $\psi(\tilde{p}_0)$, then identifies with the inclusion $\mathfrak{m} \hookrightarrow \tilde{\mathfrak{m}}$. In other terms, the orthogonal decomposition (with respect to $g_{G/\widetilde{K}}$) $T_{\tilde{p}_0}(G/\widetilde{K}) = H_{\tilde{p}_0}(G/\widetilde{K}) \oplus V_{\tilde{p}_0}(G/\widetilde{K})$ identifies with the orthogonal decomposition $\tilde{\mathfrak{m}} = \mathfrak{m} \oplus \mathbb{R}X_0$.
The vector fields F_α and Φ_α are determined by $(F_\alpha)_{|\tilde{p}_0} = X'_\alpha$ and $(\Phi_\alpha)_{|\tilde{p}_0} = Y'_\alpha$ in $\tilde{\mathfrak{m}} \equiv T_{\tilde{p}_0}(G/\widetilde{K})$. By the previous remarks, $d\pi(\tilde{p}_0).(F_\alpha)_{|\tilde{p}_0} = X'_\alpha$ and $\psi(\tilde{p}_0).Y'_\alpha = Y'_\alpha = (\Phi_\alpha)_{|\tilde{p}_0}$. Let $k \in \mathcal{U}$. We get

$\theta_{k\tilde{p}_0}(F_\alpha)_{|k\tilde{p}_0} = \theta_{k\tilde{p}_0} d\mathsf{L}_k(\tilde{p}_0).(F_\alpha)_{|\tilde{p}_0} = d\mathsf{L}_k(\tilde{p}_0).\theta_{\tilde{p}_0}(F_\alpha)_{|\tilde{p}_0}$ by invariance of θ and the definition of F_α. By definition of θ we have $\theta_{\tilde{p}_0}(F_\alpha)_{|\tilde{p}_0} = \psi(\tilde{p}_0)(J_{p_0} d\pi(\tilde{p}_0).(F_\alpha)_{|\tilde{p}_0})$. Thus, through the previous identifications we obtain:

$$\psi(\tilde{p}_0)(J_{p_0} d\pi(\tilde{p}_0).(F_\alpha)_{|\tilde{p}_0}) = \psi(\tilde{p}_0)(J_{p_0} X'_\alpha) = \epsilon(\gamma)\psi(\tilde{p}_0)(Y'_\alpha) = \epsilon(\gamma) Y'_\alpha = \epsilon(\gamma)(\Phi_\alpha)_{|\tilde{p}_0}.$$

Hence, $\theta_{k\tilde{p}_0}(F_\alpha)_{|k\tilde{p}_0} = \epsilon(\gamma) d\mathsf{L}_k(\tilde{p}_0).(\Phi_\alpha)_{|\tilde{p}_0} = \epsilon(\gamma)(\Phi_\alpha)_{|k\tilde{p}_0}$ and we get $\theta F_\alpha = \epsilon(\gamma)\Phi_\alpha$ on $\widetilde{\mathcal{U}}$. □

Observe that if two G-invariant vector fields coincide on $\widetilde{\mathcal{U}}$, they are equal on $G/\widetilde{K}$. This remark and Lemma 6.13 (0) imply that $\nabla_\xi \xi = 0$ (actually, ξ is a Killing vector field). Furthermore, to compute the invariant vector fields $\delta\theta$ and $\nabla^*\nabla\,\xi$ (see Lemma 6.5 and Proposition 6.9) it suffices to do it on $\widetilde{\mathcal{U}}$.

Theorem 6.15. *Let $(\theta, \xi, \eta, g_{G/\widetilde{K}})$ be the almost contact metric structure on $G/\widetilde{K}$ defined in Theorem 6.11 (2). For $\gamma \in R_T^+$, set $n_\gamma = |\varpi^{-1}(\gamma) \cap \mathsf{R}^+|$. Under the previous notation we have:*

$$\nabla_\xi \xi = 0, \quad \delta\theta = \Big(\sum_{\gamma \in R_T^+} \epsilon(\gamma)\, n_\gamma \frac{\gamma(Y_0)}{\kappa_\gamma}\Big)\xi, \quad \nabla^*\nabla\xi = \frac{1}{2}\Big(\sum_{\gamma \in R_T^+} n_\gamma \frac{\gamma(Y_0)^2}{\kappa_\gamma^2}\Big)\xi.$$

Assume that J is a complex structure, then the normal almost contact structure $\sigma = (\theta, \xi, \eta, g_{G/\widetilde{K}})$ is a harmonic section/map if and only if J is a harmonic section/map.

Proof. Since $\theta\xi = \nabla_\xi \xi = 0$, the invariant vector fields $\delta\theta$ and $\nabla^*\nabla\xi$ have the following expressions on $\widetilde{\mathcal{U}}$:

$$\delta\theta = \Big(\sum_{\alpha \in \mathsf{R}^+ \cap \mathsf{Q}} \nabla_{F_\alpha}\theta F_\alpha + \nabla_{\Phi_\alpha}\theta\Phi_\alpha\Big) + \nabla_\xi\theta\xi = \sum_{\gamma \in R_T^+} \sum_{\alpha \in \varpi^{-1}(\gamma)\cap \mathsf{R}^+} \Big(\nabla_{F_\alpha}\theta F_\alpha + \nabla_{\Phi_\alpha}\theta\Phi_\alpha\Big),$$

$$\nabla^*\nabla\xi = -\sum_{\gamma \in R_T^+} \sum_{\alpha \in \varpi^{-1}(\gamma)\cap \mathsf{R}^+} \Big\{\Big(\nabla_{F_\alpha}(\nabla_{F_\alpha}\xi) - \nabla_{\nabla_{F_\alpha}F_\alpha}\xi\Big) + \Big(\nabla_{\Phi_\alpha}(\nabla_{\Phi_\alpha}\xi) - \nabla_{\nabla_{\Phi_\alpha}\Phi_\alpha}\xi\Big)\Big\}.$$

Let $\alpha \in \varpi^{-1}(\gamma) \cap \mathsf{R}^+$. By Lemma 6.14 and Lemma 6.13 we get

$$\nabla_{F_\alpha}\theta F_\alpha + \nabla_{\Phi_\alpha}\theta\Phi_\alpha = \epsilon(\gamma)\Big(\nabla_{F_\alpha}\Phi_\alpha - \nabla_{\Phi_\alpha}F_\alpha\Big) = \epsilon(\gamma)\Big(\frac{1}{2}a_\gamma\xi + \frac{1}{2}a_\gamma\xi\Big) = \epsilon(\gamma)a_\gamma\xi.$$

Since $a_\gamma = \gamma(Y_0)/\kappa_\gamma$ we have proved the desired formula for $\delta\theta$.

The computation of $\nabla^*\nabla\xi$ is similar. Indeed,

$$\begin{aligned}&\Big(\nabla_{F_\alpha}(\nabla_{F_\alpha}\xi) - \nabla_{\nabla_{F_\alpha}F_\alpha}\xi\Big) + \Big(\nabla_{\Phi_\alpha}(\nabla_{\Phi_\alpha}\xi) - \nabla_{\nabla_{\Phi_\alpha}\Phi_\alpha}\xi\Big)\\ &= \Big(\nabla_{F_\alpha}(-\frac{a_\gamma}{2}\Phi_\alpha) + \nabla_{\Phi_\alpha}(\frac{a_\gamma}{2}F_\alpha)\Big) = \Big(-\frac{a_\gamma^2}{4}\xi - \frac{a_\gamma^2}{4}\Big)\xi = -\frac{1}{2}a_\gamma^2\xi.\end{aligned}$$

The formula for $\nabla^*\nabla\,\xi$ follows. The last claim is a consequence of Corollary 4.7. □

Assume that J is a complex structure on G/K. Let $\sigma = (\theta, \xi, \eta, g_{G/\widetilde{K}})$ be the normal almost contact metric structure on $G/\widetilde{K}$ given by Theorem 6.11. Recall that conditions for $(g_{G/K}, J)$ to be Kähler are given in Theorem 5.3, either in terms of the scalars κ_γ or of the KKS form defining the Kähler form $\Omega^{g_{G/K}}$.

Theorem 6.16. *Assume that $(g_{G/K}, J)$ is Kähler on $N = G/K$ and endow $M = G/\widetilde{K}$ with the normal almost contact metric structure σ. Then, this structure is a harmonic map.*

Proof. We know that J is a harmonic map when $(g_{G/K}, J)$ is Kähler. Therefore the result is consequence of Theorem 6.15 and Corollary 4.7. □

Remarks 6.17. (0) When $\tilde{\mathfrak{k}}$ is generic, the proof of Theorem 6.16 is shorter since, thanks to Lemma 6.5, we do not need Lemma 6.13 to apply Corollary 4.7.

Let C be the Weyl chamber defining the complex structure J.

(1) It is not difficult to construct a harmonic normal almost contact metric structure $(\theta, \xi, \eta, g_{G/\widetilde{K}})$ on $G/\widetilde{K}$, as in Theorem 6.16, which is not c-Sasakian. By Theorem 6.11 it suffices to choose the Kähler metric $g_{G/K}$ such that, for all $c > 0$ there exists $\gamma \in R_T^+$ such that $c\kappa_\gamma \neq \gamma(iX_0)$, that is to say choose $h_\kappa \in \mathsf{C}$ such that $-Y_0 \notin \mathbb{R}_+ h_\kappa$. (This is always the case if one starts with a choice of C such that $iX_0 = -Y_0 \notin \mathsf{C}$.)

(2) Recall that $g_{G/K}$ is Kähler-Einstein when the form $\Omega_{\mathfrak{m}}$ is equal (up to a scalar multiple $c > 0$) to $c\omega_{ih_\rho}$

with $\rho = \frac{1}{2}\sum_{\alpha\in\mathsf{Q}^+}\alpha$ (see Theorem 5.3). In this case $k_\gamma = c\,\alpha(h_\rho)$ for all $\alpha \in \varpi^{-1}(\gamma)$. The structure $(\theta, \xi, \eta, g_{G/\widetilde{K}})$ is then Sasakian if and only if $iX_0 = -Y_0 = c\,h_\rho$. We can always produce a principal $\mathbb{S}^1$-bundle $\pi : G/\widetilde{K} \twoheadrightarrow G/K$ and a harmonic normal almost contact metric structure $(\theta, \xi, \eta, g_{G/\widetilde{K}})$ on $G/\widetilde{K}$ which is not Sasakian. Indeed, choose $\widetilde{\mathfrak{z}}$ such that $\mathfrak{z} = \mathbb{R}X_0 \oplus \widetilde{\mathfrak{z}}$ with $-Y_0 \notin \mathbb{R}^*_+ h_\sigma$ and define $\widetilde{\mathfrak{k}}$ to be the orthogonal (with respect to B) of $\mathbb{R}X_0$ in $\mathfrak{k}$.
(3) In Theorem 6.16 we assume that $g_{G/K}$ is a Kähler metric in order to have the harmonicity of the section/map J. A natural question is: what is the weakest hypothesis on $g_{G/K}$ to ensure that J is a harmonic section/map? One can observe that on a generalised flag manifold G/K equipped with a complex structure J, then

$$(g_{G/K}, J) \text{ Kähler} \iff (g_{G/K}, J) \text{ quasi-Kähler (i.e. } (1,2)\text{-symplectic)} \iff (g_{G/K}, J) \text{ nearly-Kähler}.$$

This follows from [WG, Theorem 9.15 & Theorem 9.17]. Furthermore, every G-invariant almost Hermitian metric on G/K is semi-Kähler by [WG, Theorem 8.9 (ii)].

We now study the harmonicity of the Reeb vector field ξ. Let (M, g_M) be a compact Riemannian manifold and let ∇ be the associated Levi-Civita connection. Denote by UM the unit tangent bundle defined for $x \in M$ by $U_xM = \{v \in T_xM \,:\, (g_M)_x(v,v) = 1\}$. By definition, a unit vector field V on M is a section of UM (thus it defines a vector field on M). We endow the tangent bundle TM with a g-natural metric as defined in [DP, 5.1] (recall that the Sasaki metric is an example of such a metric [DP, (5.6)]). The unit tangent bundle is equipped with the restriction of this metric. One says that a unit vector field V is a harmonic vector field, resp. a harmonic map, if V is a harmonic as a section of UM, resp. as a map from M to UM. Denote by $R(X,Y) = \nabla^2_{X,Y} - \nabla^2_{Y,X}$ the curvature tensor and set

$$S(V) = \operatorname{trace}_{g_M}(R(\nabla_. V, V).) = \sum_i R(\nabla_{E_i}V, V)E_i,$$

where $(E_i)_i$ is an orthonormal frame for g_M. Combining [ACP, Theorem 2], [Str, Lemma 7.1 & Proposition 8.1], [ACP, Corollary 2 & Proposition 3] and [HY, page 85], we have the following characterisation of the harmonicity of V.

Theorem 6.18. (1) *The unit vector field V is a harmonic vector field if and only if $\nabla^*\nabla V = fV$ for some $f \in \mathcal{C}^\infty(M, \mathbb{R})$ and in this case one has $f = |\nabla V|^2$.*
(2) *Endow UM with the Sasaki metric. The unit vector field V is a harmonic map if and only if V is a harmonic vector field and $S(V) = 0$.*
(3) *Assume that the unit vector field V is Killing. Then V is a harmonic map for the Sasaki metric if and only if V is a harmonic map for any g-natural metric.*

Suppose that $M = G/\widetilde{K}$ is a homogeneous space ($\widetilde{K}$ being a closed connected subgroup of G) equipped with a G-invariant metric $g_{G/\widetilde{K}}$.

Lemma 6.19. *Assume that V is a G-invariant vector field on $G/\widetilde{K}$. Then, $S(V)$ is G-invariant.*

Proof. Let $g \in G$. The invariance of ∇ implies $g_*\nabla^2_{X,Y}Z = \nabla^2_{g_*X, g_*Y}g_*Z$, thus $g_*R(X,Y)Z$ is equal to $g_*R(g_*X, g_*Y)g_*Z$. Let $(E_i)_i$ be an orthonormal frame on an open subset $\mathcal{O}$ of $G/\widetilde{K}$. Then, on $\mathcal{O}$, we get by invariance of V:

$$S(V) = R(\nabla_{g_*E_i}V, V)g_*E_i = R(\nabla_{g_*E_i}g_*V, g_*V)g_*E_i = g_*.R(\nabla_{E_i}V, V)E_i = g_*S(V).$$

This shows that $S(V)$ is an invariant vector field. □

Now, assume that $\pi : M = G/\widetilde{K} \twoheadrightarrow N = G/K$ is a principal $\mathbb{S}^1$-bundle over a generalised flag manifold G/K as in the previous sections. Endow $G/\widetilde{K}$ with the almost contact metric structure $(\theta, \xi, \eta, g_{G/\widetilde{K}})$ defined in Theorem 6.11 (1). By construction, the Reeb vector field ξ is a G-invariant unit vector field on $G/\widetilde{K}$. On a contact metric manifold the harmonicity of the Reeb vector field ξ, and is consequences, has been studied in details in [ACP, §7].

Theorem 6.20. *The map $\xi : G/\widetilde{K} \to U(G/\widetilde{K})$ is a harmonic map for any g-natural metric on $T(G/\widetilde{K})$.*

Proof. We have shown in Theorem 6.15 that $\nabla^*\nabla\xi \in \mathbb{R}\xi$, therefore, by Theorem 6.18 (1), ξ is a harmonic unit vector field. Since ξ is Killing, by Theorem 6.18 (2) & (3), it suffices to prove that $S(\xi) = 0$. Furthermore, by invariance of $S(\xi)$ it is enough to show that $S(\xi)$ vanishes on the open subset $\widetilde{\mathcal{U}}$ used in

Lemma 6.13 and we can apply the formulas of that lemma. We calculate $S(\xi)$ in the orthonormal frame $\{\xi,(F_\alpha,\Phi_\alpha)_\alpha\}$. Since $\nabla_\xi\xi=0$ the vector field $S(\xi)$ is sum of terms of the form

$$R(\nabla_{F_\alpha}\xi,\xi)F_\alpha+R(\nabla_{\Phi_\alpha}\xi,\xi)\Phi_\alpha.$$

From the definition we obtain

$$\begin{aligned}R(\nabla_{F_\alpha}\xi,\xi)F_\alpha&=\nabla^2_{\nabla_{F_\alpha}\xi,\xi}F_\alpha-\nabla^2_{\xi,\nabla_{F_\alpha}\xi}F_\alpha\\&=\nabla_{\nabla_{F_\alpha}\xi}(\nabla_\xi F_\alpha)-\nabla_{\nabla_{\nabla_{F_\alpha}\xi}\xi}F_\alpha-\big\{\nabla_\xi(\nabla_{\nabla_{F_\alpha}\xi}F_\alpha)-\nabla_{\nabla_\xi(\nabla_{F_\alpha}\xi)}F_\alpha\big\}.\end{aligned}$$

Recall (with the notation of Lemma 6.13) that

$$\nabla_\xi F_\alpha=(c_\gamma-\frac{a_\gamma}{2})\Phi_\alpha,\nabla_{F_\alpha}\xi=-\frac{a_\gamma}{2}\Phi_\alpha,\nabla_\xi\Phi_\alpha=\frac{a_\gamma}{2}F_\alpha\text{ and }\nabla_\xi\Phi_\alpha=-(c_\gamma-\frac{a_\gamma}{2})F_\alpha.$$

Therefore, $\nabla_{F_\alpha}F_\alpha=0$ implies that:

$$\begin{aligned}R(\nabla_{F_\alpha}\xi,\xi)F_\alpha&=\frac{a_\gamma}{2}\nabla_{\nabla_{\Phi_\alpha}\xi}F_\alpha+\frac{a_\gamma}{2}\big\{\nabla_\xi(\nabla_{\Phi_\alpha}F_\alpha)-\nabla_{\nabla_\xi\Phi_\alpha}F_\alpha\big\}\\&=\frac{a_\gamma}{2}\big\{\nabla_\xi(-\frac{a_\gamma}{2}\xi)-\nabla_{-(c_\gamma-\frac{a_\gamma}{2})F_\alpha}F_\alpha\big\}=0.\end{aligned}$$

A similar calculation gives $R(\nabla_{\Phi_\alpha}\xi,\xi)\Phi_\alpha=0$. Thus $S(\xi)=0$ and ξ is a harmonic map. □

Example. Let $G=\mathrm{SU}(m+1)$, $K=S(\mathrm{U}(1)\times\mathrm{U}(m))$, $\widetilde{K}=\mathrm{SU}(m)\subset K$. Then the generalised flag manifold $N=G/K=\mathrm{SU}(m+1)/S(\mathrm{U}(1)\times\mathrm{U}(m))$ is an irreducible Hermitian symmetric space. It follows that any K-invariant scalar product on $\mathfrak{m}$ is proportional to B and that there exist two almost complex structures on G/K, which are complex structures. We fix one of them, denoted by J. Any G-invariant Kähler metric is Kähler-Einstein and up to a positive scalar there exists a "unique" G-invariant Kähler-Einstein metric on G/K, that we denote by $g_{G/K}$. The manifold $(G/K,g_{G/K},J)$ then identifies with $\mathbb{C}P^m$ equipped with its standard structure. Hence, since $M=G/\widetilde{K}=\mathrm{SU}(m+1)/\mathrm{SU}(m)$ is isomorphic to the sphere $\mathbb{S}^{2m+1}$, the fibration $G/\widetilde{K}\twoheadrightarrow G/K$ is the Hopf fibration $\pi:\mathbb{S}^{2m+1}\twoheadrightarrow\mathbb{C}P^m$ as described in [Bes, 9.81]. If we equip $G/\widetilde{K}$ with the structure $(\theta,\xi,\eta,g_{G/\widetilde{K}})$, as above, we have shown that $\xi:G/\widetilde{K}\to U(G/\widetilde{K})$ is a harmonic map for any g-natural metric on $T(G/\widetilde{K})$.

Finally, we illustrate the previous results by an example in the case where $N=G/T$ is the full flag manifold, i.e. $K=T$ is a maximal torus. In this setting the notation simplifies since we have:

$$\mathfrak{g}=\mathfrak{t}\oplus\mathfrak{m},\quad\mathfrak{m}=\mathfrak{n}=\oplus_{\alpha\in\mathsf{R}^+}\mathfrak{m}_\alpha,\quad\mathfrak{z}=\mathfrak{t},\quad\mathsf{P}=\varnothing,\quad\mathsf{Q}=\mathsf{R},\quad\varpi=\mathrm{id}_{\mathfrak{t}_\mathbb{R}},\quad R_T=\mathsf{R}.$$

Therefore the irreducible K-modules $\mathfrak{m}_\gamma$ are the irreducible T-modules $\mathfrak{m}_\alpha=\mathbb{R}X_\alpha\oplus\mathbb{R}Y_\alpha$. The choice of an invariant ordering is the choice of a set of positive roots R^+, i.e. the choice of a Weyl chamber C in $\mathfrak{t}_\mathbb{R}$. The G-invariant complex structure J on G/T associated to R^+ is defined by $J_\mathfrak{m}X_\alpha=Y_\alpha$, $J_\mathfrak{m}Y_\alpha=-X_\alpha$ for $\alpha\in\mathsf{R}^+$. Let $g_{G/T}$ be a G-invariant metric on G/T, then the Euclidean product $g_\mathfrak{m}$ is given by $g_\mathfrak{m}=\kappa_\alpha B$ on $\mathfrak{m}_\alpha$, $\alpha\in\mathsf{R}^+$. The metric $g_{G/T}$ is Kähler if and only if $\kappa_\alpha+\kappa_\beta=\kappa_{\alpha+\beta}$ if $\alpha+\beta\in\mathsf{R}^+$; equivalently, the Kähler form $\Omega_\mathfrak{m}$ is equal to the KKS form ω_{ih_κ} with $h_\kappa\in\mathsf{C}$. Choose a sub-torus $\widetilde{T}$ of T such that $T/\widetilde{T}\cong\mathbb{S}^1$, let $X_0=iY_0\in\mathfrak{t}$ (with $Y_0\in\mathfrak{t}_\mathbb{R}$) be such that $B(X_0,X_0)=1$ and $\mathfrak{t}=\mathbb{R}X_0\ominus\widetilde{\mathfrak{t}}$. Then $\pi:G/\widetilde{T}\twoheadrightarrow G/T$ is a principal $\mathbb{S}^1$-bundle. Recall that a normal almost contact metric structure $(\theta,\xi,\eta,g_{G/\widetilde{T}})$ on $G/\widetilde{T}$ is constructed from: a Hermitian metric $(g_{G/T},J)$, the connection η such that $\eta_{\tilde{p}_0}(U)=B(X_0,U)$, the metric $g_{G/\widetilde{T}}=\pi^*g_{G/T}+\eta\otimes\eta$ and the endomorphism θ lifting J to the horizontal bundle.

We give an example of this construction in the case:

$$G=\mathrm{SU}(3)\subset G_\mathbb{C}=\mathrm{SL}(3,\mathbb{C}),\quad\mathfrak{g}=\mathfrak{su}(3)\subset\mathfrak{g}_\mathbb{C}=\mathfrak{sl}(3,\mathbb{C}).$$

The split real form of $\mathfrak{g}_\mathbb{C}$ is $\mathfrak{g}_\mathbb{R}=\mathfrak{sl}(3,\mathbb{R})$ and we choose the $G_\mathbb{C}$-invariant bilinear form $\langle A\mid B\rangle=\mathrm{trace}(AB)$ on $\mathfrak{g}_\mathbb{C}$. The split Cartan subalgebra $\mathfrak{t}_\mathbb{R}$ consists of the traceless diagonal real matrices, which will be written $\mathrm{diag}[x_1,x_2,x_3]$ with $x_1+x_2+x_3=0$. The compact Cartan subalgebra is

$$\mathfrak{t}=i\mathfrak{t}_\mathbb{R}=\{i\,\mathrm{diag}[a,b,c]\ :\ \mathrm{diag}[a,b,c]\in\mathfrak{t}_\mathbb{R}\}.$$

For $i=1,2,3$, let $\varepsilon_i\in\mathfrak{t}^*_\mathbb{C}$ be the linear form which maps $\mathrm{diag}[x_1,x_2,x_3]$ to x_i. Set $\alpha_j=\varepsilon_j-\varepsilon_{j+1}$ for $j=1,2$ and $\alpha_3=\alpha_1+\alpha_2=\varepsilon_1-\varepsilon_3$. Then one has $\mathsf{R}=\{\pm\alpha_1,\pm\alpha_2,\pm\alpha_3\}=\{\varepsilon_i-\varepsilon_j,\ 1\leqslant i\neq j\leqslant 3\}$.

For $\lambda = \varepsilon_k - \varepsilon_\ell$ the element E_λ is the unit matrix $E_{k\ell}$. Hence,

$$X_\lambda = \frac{1}{\sqrt{2}}(E_{k\ell} - E_{\ell k}), \quad Y_\lambda = \frac{i}{\sqrt{2}}(E_{k\ell} + E_{\ell k}).$$

The algebra $\mathfrak{g}$ decomposes as the orthogonal direct sum $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$. One has $\mathfrak{m} = \mathfrak{m}_{\pm\alpha_1} \oplus \mathfrak{m}_{\pm\alpha_2} \oplus \mathfrak{m}_{\pm\alpha_3}$, where $\mathfrak{m}_{\pm\alpha_i} = \mathbb{R}X_{\pm\alpha_i} \oplus \mathbb{R}Y_{\pm\alpha_i}$. Any T-invariant scalar product $g_\mathfrak{m}$ on $\mathfrak{m}$ is defined by $(g_\mathfrak{m})_{|\mathfrak{m}_{\alpha_j}} = \kappa_{\alpha_j} B$ with $\kappa_{\alpha_j} > 0$. Finally, recall that a T-invariant complex structure $J_\mathfrak{m}$ on $\mathfrak{m}$ is determined by $J_\mathfrak{m} X_{\alpha_j} = \epsilon(\alpha_j) Y_{\alpha_j}$, $J_\mathfrak{m} Y_{\alpha_j} = -\epsilon(\alpha_j) X_{\alpha_j}$ with $\epsilon(\alpha_j) = \pm 1$ for $j = 1, 2, 3$.

We fix two integers $k, \ell \in \mathbb{Z}$ such that $(k, \ell) \neq (0, 0)$ and set $\Gamma = \sqrt{k^2 + \ell^2 + k\ell}$. Define an element of $\mathfrak{t}$ by $X_{-1} = \frac{i}{\Gamma\sqrt{2}} \mathrm{diag}[k, \ell, -(k + \ell)]$ and set

$$\widetilde{\mathfrak{t}} = \mathfrak{t}_{k,\ell} = \mathbb{R}X_{-1}, \quad \widetilde{T} = T_{k,\ell} = \exp(\widetilde{\mathfrak{t}}) \subset T.$$

Therefore, $\widetilde{T}$ is the one-dimensional sub-torus of T such that

$$\widetilde{T} = T_{k,\ell} = \{\mathrm{diag}[e^{itk}, e^{it\ell}, e^{-it(k+\ell)}] \,:\, t \in \mathbb{R}\}.$$

Let $X_0 \in \mathfrak{t}$ such that $B(X_{-1}, X_0) = 0$, $B(X_0, X_0) = 1$:

$$X_0 = \frac{-i}{\Gamma\sqrt{6}} \mathrm{diag}[2\ell + k, -(\ell + 2k), -\ell + k] = iY_0, \quad Y_0 = \frac{-1}{\Gamma\sqrt{6}} \mathrm{diag}[2\ell + k, -(\ell + 2k), -\ell + k].$$

Notice that (X_{-1}, X_0) is an orthonormal basis, for B, of $\mathfrak{t} = \mathfrak{t}_{k,\ell} \oplus \mathbb{R}X_0 = \mathbb{R}X_{-1} \oplus \mathbb{R}X_0$. The group $\mathbb{S}^1 = T/\widetilde{T}$ is a one-dimensional torus such that $\mathrm{Lie}(\mathbb{S}^1) = \mathfrak{s} = \mathfrak{t}/\widetilde{\mathfrak{t}} = \mathbb{R}A$, with $A = [X_0 + \widetilde{\mathfrak{t}}]$. One can therefore define a principal $\mathbb{S}^1$-bundle

$$\pi : M = G/T_{k,\ell} \twoheadrightarrow N = G/T.$$

We say that this homogeneous bundle is an *Aloff-Wallach space* [AW]. Recall that we set $\widetilde{\mathfrak{m}} = \mathbb{R}X_0 \oplus \mathfrak{m} = \mathbb{R}X_0 \oplus \mathfrak{m}_{\alpha_1} \oplus \mathfrak{m}_{\alpha_2} \oplus \mathfrak{m}_{\alpha_3}$, and that $\widetilde{\mathfrak{m}}$ is a $\widetilde{T}$-module under the adjoint action. One easily sees, by computing $\lambda(X_{-1})$ for $\lambda \in \mathsf{R}$, that:

$$i\widetilde{\mathfrak{t}} = \mathrm{Ker}(\alpha_1) \iff k = \ell, \;\; i\widetilde{\mathfrak{t}} = \mathrm{Ker}(\alpha_2) \iff 2\ell + k = 0, \;\; i\widetilde{\mathfrak{t}} = \mathrm{Ker}(\alpha_3) \iff 2k + \ell = 0.$$

Then, $\widetilde{\mathfrak{t}}$ is generic if and only if $k \neq \ell$, $2\ell + k \neq 0$, $2k + \ell \neq 0$. Proposition 6.2 becomes in this example:

Proposition 6.21. *The space of G-invariant vector fields $\mathcal{X}(G/T_{k,\ell})^G$ is equal to:*

(1) $\mathbb{R}\xi$ *when* $k \neq \ell$, $2\ell + k \neq 0$, $2k + \ell \neq 0$;
(2) $\mathbb{R}\xi \oplus \mathbb{R}X^\#_{\alpha_1} \oplus Y^\#_{\alpha_1}$ *if* $k = \ell$;
(3) $\mathbb{R}\xi \oplus \mathbb{R}X^\#_{\alpha_2} \oplus Y^\#_{\alpha_2}$ *if* $2\ell + k = 0$;
(4) $\mathbb{R}\xi \oplus \mathbb{R}X^\#_{\alpha_3} \oplus Y^\#_{\alpha_3}$ *if* $2k + \ell = 0$.

To give an example, we now choose the complex structure J on G/T associated to the basis $\mathsf{B} = \{\alpha = \alpha_3, \, \beta = -\alpha_2\}$ of R, hence $\mathsf{R}^+ = \{\alpha = \alpha_3, \, \beta = -\alpha_2, \, \gamma = \alpha + \beta = \alpha_1\}$ and the corresponding Weyl chamber is $\mathsf{C} = \{\mathrm{diag}[a, b, c] \in \mathfrak{t}_\mathbb{R} : a > c > b\}$. Then, we have

$$(X_\alpha = X_{\alpha_3}, \, Y_\alpha = Y_{\alpha_3}), \; (X_\beta = -X_{\alpha_2}, \, Y_\beta = Y_{\alpha_2}), \; (X_\gamma = X_{\alpha_1}, \, Y_\gamma = Y_{\alpha_1}),$$

and $J_\mathfrak{m} X_\lambda = Y_\lambda$, $J_\mathfrak{m} Y_\lambda = -X_\lambda$ for $\lambda = \alpha, \beta, \gamma$. Moreover, under the choice of Y_0 made above:

$$\alpha(Y_0) = -\frac{\sqrt{3}}{\Gamma\sqrt{2}}\ell, \quad \beta(Y_0) = -\frac{\sqrt{3}}{\Gamma\sqrt{2}}k, \quad \gamma(Y_0) = -\frac{\sqrt{3}}{\Gamma\sqrt{2}}(\ell + k).$$

Thus, $-Y_0 \in \mathsf{C}$ if and only if $k > 0$ and $\ell > 0$. Notice that $\mathfrak{m}_\alpha = \mathfrak{m}_{\alpha_3}$, $\mathfrak{m}_\beta = \mathfrak{m}_{\alpha_2}$ and $\mathfrak{m}_\gamma = \mathfrak{m}_{\alpha_1}$. Let $(g_{G/T}, J)$ be a Hermitian metric on G/T determined by a T-invariant Euclidean product $g_\mathfrak{m}$ as above, i.e. by a triple $(\kappa_\alpha, \kappa_\beta, \kappa_\gamma)$ of positive scalars. Then, $(g_{G/T}, J)$ is Kähler if and only if $\kappa_\alpha + \kappa_\beta = \kappa_\gamma$; equivalently, for all $U, V \in \mathfrak{m}$, $g_\mathfrak{m}(U, V) = \omega_{ih_\kappa}(U, J_\mathfrak{m} V)$with $h_\kappa \in \mathsf{C}$ and in this case $\kappa_\alpha = \alpha(h_\kappa), \kappa_\beta = \beta(h_\kappa)$ (thus $\kappa_\gamma = \gamma(h_\kappa)$).

Recall that from $J, X_0, g_{G/T}$ we can construct an almost contact metric structure $\sigma = (\theta, \xi, \eta, g_{G/\widetilde{T}})$ on $G/\widetilde{T} = G/T_{k,\ell}$. Theorem 6.16 yields:

Theorem 6.22. *Let J be as above and choose a Kähler metric $(g_{G/T}, J)$ on G/T. Then, the normal almost contact metric structure structure σ on $G/T_{k,\ell}$ is a harmonic map.*

By Theorem 6.11, σ is c-Sasakian if and only if there exists $c > 0$ such that $-Y_0 = ch_\kappa$, i.e. $c\kappa_\alpha = \frac{\sqrt{3}}{\Gamma\sqrt{2}}\ell$ and $c\kappa_\beta = \frac{\sqrt{3}}{\Gamma\sqrt{2}}k$. Therefore, it suffices to pick $h_\kappa \in \mathsf{C}$ such that $h_\kappa \notin -\mathbb{R}_+^* Y_0$ to get that σ is not c-Sasakian. In particular, if k or ℓ is < 0, the structure σ is not c-Sasakian. Moreover, the metric $g_{G/T}$ is Kähler-Einstein if and only if h_κ is a multiple of $h_\rho = h_\gamma = \mathrm{diag}[1, -1, 0]$. Therefore, for this choice of $g_{G/T}$, the structure σ is c-Sasakian when $-Y_0 = c\,\mathrm{diag}[1, -1, 0]$, which is equivalent to $k = \ell$ and $\frac{\sqrt{3}}{\Gamma\sqrt{2}}\ell = c > 0$. Thus, if $k \neq \ell$ or $k = \ell < 0$ the structure σ is not Sasakian with $g_{G/T}$ Kähler-Einstein.

For completeness, we give the expression of the vector fields $\delta\theta$, $\nabla^*\nabla\xi$ obtained in Theorem 6.15:

$$\delta\theta = -\frac{\sqrt{3}}{\Gamma\sqrt{2}}\Big(\frac{\ell}{\kappa_\alpha} + \frac{k}{\kappa_\beta} + \frac{(\ell+k)}{\kappa_\gamma}\Big)\xi, \quad \nabla^*\nabla\xi = \frac{3}{4\Gamma^2}\Big(\frac{\ell^2}{\kappa_\alpha^2} + \frac{k^2}{\kappa_\beta^2} + \frac{(\ell+k)^2}{\kappa_\gamma^2}\Big)\xi.$$

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