Inf-convolution of Choquet integrals and
applications in optimal risk transfer

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We are concerned with optimal risk sharing between two economic agents using convex risk measures as criteria. We focus on risk measures defined on Lp spaces, that we can represent as Choquet integrals with respect to a capacity. Their key property is comonotonicity, which means these mappings are additive for comonotone random variables.
When the risk measures are the same with different risk aversion coefficients [1], or when the risk measures are law invariant [4], the problem is well treated in the literature. We concentrate here on risk measures which are not necessarily law invariant and which can act on non essentially bounded random variables.

We will first present an extention of the representation theorem as Choquet integrals of comonotonic monetary risk measures defined on bounded random variables [3], to convex and comonotonic functionals defined on Lp, the key ingredient being a recent extension by S. Biagini and M. Frittelli of the Namioka-Klee theorem to convex mappings [2].

Then we will show how we can always make coincide at a given point a Choquet integral with a distortion risk measure, provided the capacity that we consider has the same null sets as a given reference probability measure. This enables us to prove a Kusuoka-type representation theorem [5] for functionals which are again, not necessarily law invariant.

Finally, we will show that the inf-convolution of Choquet integrals with respect to two different general sub-modular capacities is again a Choquet integral with respect to some capacity, and we compute it in an explicit manner: it takes the form of a sum of layers on the total risk. This means that the inf-convolution of convex and comonotone risk measures is given by a generalization of the so-called Excess-of-Loss contract in reinsurance, with more threshold values. The domain of attainable losses is divided in layers, the bounds of the layers corresponding to quantile values of the total risk, and each layer is alternatively at the charge of one of the two agents. This result extends easily to the multiple agents setting, and the combinatorics of the optimal risk transfer is summarized in the distortion functions associated to the capacity of each agent and to the aggregate risk.

We will give examples of random variables with the same law, but generating different optimal risk transfers.
The Average Value-at-Risk (AVaR) being a particular law invariant Choquet integral, we give some examples of explicit solutions to inf-convolutions involving the AvaR risk measure.

References

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