Semilinear elliptic equations with singular coefficients

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The problem

The purpose of this work is to use probabilistic methods to solve the Dirichlet boundary value problem for the semilinear second order elliptic PDE of the following form:

$$\begin{cases} \mathcal{A}u(x) = -f(x, u(x), \nabla u(x)), & \forall x \in D, \\ u(x)|_{\partial D} = \varphi, & \forall x \in \partial D. \end{cases} \tag{1}$$

The operator $\mathcal{A}$ is given by

$$\mathcal{A}u = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} - div(\hat{b}u)" + q(x)u, \tag{2}$$

The problem

where $a = (a_{i,j}(x))_{1\le i,j\le d} : D \to R^{d\times d}$ $(d > 2)$ is a measurable, symmetric matrix-valued function satisfying a uniform elliptic condition

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \ \forall\, \xi \in R^d \text{ and } x \in D \qquad (3)$$

and $b = (b_1, b_2, ...b_d)$, $\hat{b} = (\hat{b}_1, \hat{b}_2, ..., \hat{b}_d) : \ D \to R^d$ and $q : \ D \to R$ are merely measurable functions belonging to some L^p spaces, and $f(\cdot,\cdot,\cdot)$ is a nonlinear function. The operator $\mathcal{A}$ is rigorously determined by the following quadratic form:

$$\begin{aligned} & Q(u,v) = (-\mathcal{A}u, v) \\ = \ & \frac{1}{2}\sum_{i,j=1}^{d}\int_{R^d} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx - \sum_{i=1}^{d}\int_{R^d} b_i(x)\frac{\partial u}{\partial x_i}v(x)dx \\ & -\sum_{i=1}^{d}\int_{R^d} \hat{b}_i(x)u\frac{\partial v}{\partial x_i}dx - \int_{R^d} q(x)u(x)v(x)dx. \qquad (4) \end{aligned}$$

The problem

Let $W^{1,2}(D)$ denote the usual Sobolev space of order one:

$$W^{1,2}(D) = \{u \in L^2(D) : \nabla u \in L^2(D; R^d)\}$$

Definition

We say that $u \in W^{1,2}(D)$ is a continuous, weak solution of (1) if
(i) for any $\phi \in W_0^{1,2}(D)$,

$$\frac{1}{2}\sum_{i,j=1}^{d}\int_D a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_j}dx - \sum_{i=1}^{d}\int_D b_i(x)\frac{\partial u}{\partial x_i}\phi dx - \sum_{i=1}^{d}\int_D \hat{b}_i(x)u\frac{\partial \phi}{\partial x_i}dx$$

$$-\int_D q(x)u(x)\phi dx = \int_D f(x,u,\nabla u)\phi dx,$$

(ii) $\lim_{y\to x} u(y) = \varphi(x), \qquad \forall\, x \in \partial D$, regular.

If $f = 0$ (i.e., the linear case), and moreover $\hat{b} = 0$, the solution u to problem (1) can be solved by a Feynman-Kac formula

$$u(x) = E_x\left[\exp\left(\int_0^{\tau_D} q(X(s))ds\right)\varphi(X(\tau_D))\right] \qquad \text{for } x \in D,$$

where $X(t), t \geq 0$ is the diffusion process associated with the infinitesimal generator

$$L_1 = \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}) + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}, \tag{5}$$

τ_D is the first exit time of the diffusion process $X(t), t \geq 0$ from the domain D. Very general results are obtained, for example, in a paper by Z.Q. Chen and Z. Zhao for this case. When $\hat{b} \neq 0$, "$div(\hat{b}\cdot)$" in (2) is just a formal writing because the divergence really does not exist for the merely measurable vector field $\hat{b}$. It should be interpreted in the distributional sense.

It is exactly due to the non-differentiability of $\hat{b}$, all the previous known probabilistic methods in solving the elliptic boundary value problems could not be applied. We stress that the lower order term $div(\hat{b}\cdot)$ can not be handled by Girsanov transform or Feynman-Kac transform either. In a recent work with Z. Q. Chen (to appear in Annals of Prob.), we show that the term $\hat{b}$ in fact can be tackled by the time-reversal of Girsanov transform from the first exit time τ_D from D by the symmetric diffusion X^0 associated with $L_0 = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$.

Introduction

The solution to equation (1) (when $f = 0$) is given by

$$\begin{aligned} & u(x) \\ = \;& E_x^0\Big[\varphi(X^0(\tau_D))\exp\Big\{\int_0^{\tau_D}(a^{-1}b)(X^0(s))\cdot dM^0(s) \\ & + \Big(\int_0^{\tau_D}(a^{-1}\hat{b})(X^0(s))\cdot dM^0(s)\Big)\circ r_{\tau_D} + \int_0^{\tau_D} q(X^0(s))ds \\ & \quad -\frac{1}{2}\int_0^{\tau_D}(b-\hat{b})a^{-1}(b-\hat{b})^*(X^0(s))ds\Big\}\Big], \end{aligned} \tag{6}$$

where $M^0(s)$ is the martingale part of the diffusion X^0, r_t denotes the reverse operator.

Nonlinear elliptic PDEs (i.e., $f \neq 0$ in (1)) are generally very hard to solve. One can not expect explicit expressions for the solutions. However, in recent years backward stochastic differential equations (BSDEs) have been used effectively to solve certain nonlinear PDEs. This was initiated by S. Peng. The general approach is to represent the solution of the nonlinear equation (1) as the solution of certain BSDEs associated with the diffusion process generated by the linear operator $\mathcal{A}$. But so far, only the cases where $\hat{b} = 0$ and b being bounded were considered. The main difficulty for treating the general operator $\mathcal{A}$ in (2) with $\hat{b} \neq 0$, $q \neq 0$ is that there are no associated diffusion processes anymore. The mentioned methods eased to work. Our approach is to transform the problem (1) to a similar problem for which the operator $\mathcal{A}$ does not have the "bad" term $\hat{b}$.

Preliminaries

Introduce two diffusion processes which will be used later.
Let $(\Omega, \mathcal{F}, \mathcal{F}_t, X(t), P_x, x \in R^d)$ be the Feller diffusion process whose infinitesimal generator is given by

$$L_1 = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}) + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}, \tag{7}$$

where $\mathcal{F}_t$ is the completed, minimal admissible filtration generated by $X(s), s \geq 0$. The associated non-symmetric, semi-Dirichlet form with L_1 is defined by

$$\begin{aligned} & Q_1(u,v) = (-L_1 u, v) \\ = \; & \frac{1}{2}\sum_{i,j=1}^{d}\int_{R^d} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx - \sum_{i=1}^{d}\int_{R^d} b_i(x)\frac{\partial u}{\partial x_i}v(x)dx. \end{aligned} \tag{8}$$

Preliminaries

The process $X(t), t \geq 0$ is not a semimartingale in general. However, it is known (e.g. [FOT] and [LZ]) that the following Fukushima's decomposition holds:

$$X(t) = x + M(t) + N(t) \qquad P_x - a.s., \tag{9}$$

where $M(t)$ is a continuous square integrable martingale with

$$< M^i, M^j >_t = \int_0^t a_{i,j}(X(s))ds, \tag{10}$$

and $N(t)$ is a continuous process of zero quadratic variation. Later we also write $X_x(t)$, $M_x(t)$ to emphasize the dependence on the initial value x.

Let $\mathcal{M}$ denote the space of square integrable martingales w.r.t. the filtration $\mathcal{F}_t, t \geq 0$. The following result is a martingale representation theorem, which paves the way to study the backward stochastic differential equations associated with the martingale part M.

Theorem (1)

For any $L \in \mathcal{M}$, there exist predictable processes $H_i(t), i = 1, ..., d$ such that

$$L_t = \sum_{i=1}^{d} \int_0^t H_i(s) dM^i(s). \tag{11}$$

Preliminaries

We will denote by $(\Omega, \mathcal{F}^0, \mathcal{F}_t^0, X^0(t), P_x^0, x \in R^d)$ the diffusion process generated by

$$L_0 = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x) \frac{\partial}{\partial x_j}). \tag{12}$$

The corresponding Fukushima's decomposition is written as

$$X^0(t) = x + M^0(t) + N^0(t), t \geq 0$$

For $v \in W^{1,2}(R^d)$, the Fukushima's decomposition for the Dirichlet process $v(X^0(t))$ reads as

$$v(X^0(t)) = v(X^0(0)) + M^v(t) + N^v(t), \tag{13}$$

where $M^v(t) = \int_0^t \nabla v(X^0(s)) \cdot dM^0(s)$, $N^v(t)$ is a continuous process of zero energy (the zero energy part).

BSDEs with deterministic terminal times

Let $f(s, y, z, \omega) : [0, T] \times R \times R^d \times \Omega \to R$ be a given progressively measurable function. For simplicity, we omit the random parameter ω. Assume that f is continuous in y, z and satisfies

(A.1) $(y_1 - y_2)(f(s, y_1, z) - f(s, y_2, z)) \leq -d_1(s)|y_1 - y_2|^2$,

(A.2) $|f(s, y, z_1) - f(s, y, z_2)| \leq d_2|z_1 - z_2|$,

(A.3) $|f(s, y, z)| \leq |f(s, 0, z)| + K(1 + |y|)$,

where $d_1(\cdot)$ is a progressively measurable stochastic process and d_2, K are constants. Let $\xi \in L^2(\Omega, \mathcal{F}_T, P)$. Let λ be the constant defined in (3).

BSDEs with deterministic terminal times

Theorem (2)

Assume $E\left[e^{-\int_0^T 2d_1(s)ds}|\xi|^2\right] < \infty$ *and*

$$E\left[\int_0^T e^{-\int_0^s 2d_1(u)du}|f(s,0,0)|^2 ds\right] < \infty.$$

Then, there exists a unique ($\mathcal{F}_t$-adapted) solution (Y, Z) *to the following BSDE:*

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s))ds - \int_t^T < Z(s), dM(s) >, \quad (14)$$

where $Z(s) = (Z_1(s), ..., Z_d(s))$*.*

BSDEs with random terminal times

Let $f(t,y,z)$ satisfy (A.1)-(A.3) in Section 2.1. In this subsection, set $d(s) = -2d_1(s) + \delta d_2^2$. The following result provides existence and uniqueness for BDEs with random terminal time. Let τ be a stopping time. Suppose ξ is $\mathcal{F}_\tau$-measurable.

Theorem (3)

Assume $E[e^{\int_0^\tau d(s)ds}|\xi|^2] < \infty$ *and*

$$E\left[\int_0^\tau e^{\int_0^s d(u)du}|f(s,0,0)|^2 ds\right] < \infty, \tag{15}$$

for some $\delta > \frac{1}{\lambda}$. *Then, there exists a unique solution* (Y,Z) *to the BSDE:*

$$Y(t) = \xi + \int_{\tau\wedge t}^\tau f(s,Y(s),Z(s))ds - \int_{\tau\wedge t}^\tau < Z(s), dM(s) > . \tag{16}$$

Moreover, the solution (Y,Z) *satisfies*

BSDEs with random terminal times

$$E\left[\int_0^{\tau} e^{\int_0^s d(u)du} Y^2(s)ds\right] < \infty, \quad E\left[\int_0^{\tau} e^{\int_0^s d(u)du} |Z(s)|^2 ds\right] < \infty, \tag{17}$$

and

$$E\left[\sup_{0\le s\le \tau} \{e^{\int_0^s d(u)du} Y^2(s)\}\right] < \infty. \tag{18}$$

A particular case

Let $f(x,y,z) : R^d \times R \times R^d \to R$ be a Borel measurable function. Assume that f is continuous in y, z and satisfies

(B.1) $(y_1 - y_2)(f(x,y_1,z) - f(x,y_2,z)) \leq -c_1(x)|y_1 - y_2|^2$

(B.2) $|f(x,y,z_1) - f(x,y,z_2)| \leq c_2|z_1 - z_2|$.

(B.3) $|f(x,y,z)| \leq |f(x,0,z)| + K(1 + |y|)$.

Let D be a bounded regular domain. Define

$$\tau_x = \inf\{t \geq 0 : X_x(t) \notin D\} \tag{19}$$

Given $g \in C_b(R^d)$. Consider for each $x \in D$ the following BSDE:

$$\begin{aligned} Y_x(t) &= g(X_x(\tau_x)) + \int_{t\wedge\tau_x}^{\tau_x} f(X_x(s), Y_x(s), Z_x(s))ds \\ &\quad - \int_{t\wedge\tau_x}^{\tau_x} < Z_x(s), dM_x(s) >, \end{aligned} \tag{20}$$

where $M_x(s)$ is the martingale part of $X_x(s)$. As a consequence of Theorem 3, we have

Theorem (4)

Suppose

$$E_x[exp(\int_0^{\tau_x} \left(-2c_1(X(s)) + \delta c_2^2\right) ds)] < \infty,$$

for some $\delta > \frac{1}{\lambda}$ *and*

$$E_x[\int_0^{\tau_x} |f(X(s), 0, 0)|^2 ds] < \infty.$$

The BSDE (20) admits a unique solution $(Y_x(t), Z_x(t))$*. Furthermore,*

$$\sup_{x \in \bar{D}} |Y_x(0)| < \infty. \tag{21}$$

Linear PDEs

Consider the second order differential operator:

$$L_2 = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}) + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i} + q(x). \tag{22}$$

Let D be a bounded domain with regular boundary (w. r. t. the Laplace operator. Δ) and $F(x)$ a measurable function satisfying

$$|F(x)| \leq C + C|q(x)|. \tag{23}$$

Take $\varphi \in C(\partial D)$ and consider the Dirichlet boundary value problem:

$$\begin{cases} L_2 u = F & \text{in } D, \\ u = \phi & \text{on } \partial D. \end{cases} \tag{24}$$

Linear PDEs

Theorem (5)

Assume (23) and that there exists $x_0 \in D$ *such that* $E_{x_0}[exp(\int_0^{\tau_D} q(X_s)ds)] < \infty$. *Then there is a unique, continuous weak solution* u *to the Dirichlet boundary value problem (24) which is given by*

$$u(x) = E_x[\varphi(X_{\tau_D}) + \int_0^{\tau_D} e^{\int_0^t q(X(s))ds} F(X(t))dt]. \qquad (25)$$

Semilinear PDEs

Let $g(x, y, z) : R^d \times R \times R^d \to R$ be a Borel measurable function that satisfies

(C.1) $(y_1 - y_2)(g(x, y_1, z) - g(x, y_2, z)) \leq -k_1(x)|y_1 - y_2|^2$,

(C.2) $|g(x, y, z_1) - g(x, y, z_2)| \leq k_2|z_1 - z_2|$,

(C.3) $|g(x, y, z)| \leq C + C|q(x)|$.

Consider the semilinear Dirichlet boundary value problem:

$$\begin{cases} L_2 u = -g(x, u(x), \nabla u(x)) & \text{in } D, \\ u = \phi & \text{on } \partial D. \end{cases} \tag{26}$$

Semilinear PDEs

Theorem (6)

Assume

$$E_x[exp(\int_0^{\tau_x} \left(q(X(s)) - 2k_1(X(s)) + \delta k_2^2\right) ds)] < \infty,$$

for some $\delta > \frac{1}{\lambda}$ *and*

$$E_x[\int_0^{\tau_x} |q(X(s))|^2 ds] < \infty.$$

The Dirichlet boundary value problem (26) has a unique continuous weak solution.

Idea of the proof. Step 1. Set $f(x,y,z) = q(x)y + g(x,y,z)$. According to Theorem 4, for every $x \in D$ the following BSDE:

$$\begin{aligned} Y_x(t) &= \phi(X_x(\tau_x)) + \int_{t\wedge\tau_x}^{\tau_x} f(X_x(s), Y_x(s), Z_x(s))ds \\ &\quad - \int_{t\wedge\tau_x}^{\tau_x} < Z_x(s), dM_x(s) >, \end{aligned} \tag{27}$$

admits a unique solution $(Y_x(t), Z_x(t)), t \geq 0$. Put $u_0(x) = Y_x(0)$ and $v_0(x) = Z_x(0)$. By the strong Markov property of X and the uniqueness of the BSDE (27), it is seen that

$$Y_x(t) = u_0(X_x(t)), \quad Z_x(t) = v_0(X_x(t)), \quad 0 \leq t \leq \tau_x. \tag{28}$$

Step 2. Consider the following problem:

$$\begin{cases} L_1 u = -f(x, u_0(x), v_0(x)) & \text{in } D, \\ u = \varphi & \text{on } \partial D, \end{cases} \tag{29}$$

where L_1 is defined as in Section 2.

By Theorem 4.1, problem (29) has a unique continuous weak solution $u(x)$.
Step 3. We show that $u(x) = u_0(x)$ and hence u is the solution.

Semilinear PDEs with singular coefficients

Consider the semilinear second order elliptic PDEs of the following form:

$$\begin{cases} \mathcal{A}u(x) = -f(x, u(x)), \ \forall x \in D, \\ u(x)|_{\partial D} = \varphi, \ \ \forall x \in \partial D, \end{cases} \tag{30}$$

where the operator $\mathcal{A}$ is given by

$$\mathcal{A} = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}) + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i} - div(\hat{b}\cdot)\text{''} + q(x)$$

Consider the following conditions:

(D.1) $(y_1 - y_2)(f(x, y_1) - f(x, y_2)) \leq -J_1(x)|y_1 - y_2|^2$

(D.2) $|f(x, y, z)| \leq C.$

The following theorem is the main result:

Theorem (7)

Suppose that $(D.1), (D.2)$ *hold and*

$$E_x^0\Bigg[\exp\bigg\{\int_0^{\tau_D}(a^{-1}b)(X^0(s))\cdot dM^0(s)+\int_0^{\tau_D}q(X^0(s))ds \\ +\left(\int_0^{\tau_D}(a^{-1}\hat{b})(X^0(s))\cdot dM^0(s)\right)\circ r_{\tau_D} \\ -\frac{1}{2}\int_0^{\tau_D}(b-\hat{b})a^{-1}(b-\hat{b})^*(X^0(s))ds-2\int_0^{\tau_D}J_1(X^0(s))ds\bigg\}\Bigg] \\ <\infty \tag{31}$$

for some $x \in D$, where X^0 is the diffusion generated by L_0 as in Section 2. Then there exists a unique, continuous weak solution to equation (30).

Key steps of the proof

Ideas of the proof. Step 1. Set

$$
\begin{aligned}
Z_t = \exp\Bigg\{ & \int_0^t (a^{-1}b)(X^0(s)) \cdot dM^0(s) + \int_0^t q(X^0(s))ds \\
& + \left(\int_0^t (a^{-1}\hat{b})(X^0(s)) \cdot dM^0(s) \right) \circ r_t \\
& - \frac{1}{2} \int_0^t (b-\hat{b})a^{-1}(b-\hat{b})^*(X^0(s))ds - 2\int_0^t J_1(X^0(s))ds \Bigg\}
\end{aligned} \tag{32}
$$

Put

$$
\hat{M}(t) = \int_0^t (a^{-1}\hat{b})(X^0(s)) \cdot dM^0(s) \qquad \text{for } t \geq 0.
$$

Let $R > 0$ so that $D \subset B_R := B(0, R)$. It is known from [CZ], there exits a bounded function $v \in W_0^{1,p}(B_R) \subset W_0^{1,2}(B_R)$ such that

$$(\hat{M}(t)) \circ r_t = -\hat{M}(t) + N^v(t),$$

where N^v is the zero energy part of the Fukushima decomposition for the Dirichlet process $v(X^0(t))$. Furthermore, v satisfies the following equation in the distributional sense:

$$div(a\nabla v) = -2div(\hat{b}) \qquad \text{in } B_R. \tag{33}$$

Note that by Sobolev embedding theorem, $v \in C(R^d)$ if we extend $v = 0$ on D^c.

Key steps of the proof

Thus,

$$
\begin{aligned}
& \left(\int_0^t (a^{-1}\hat{b})(X^0(s)) \cdot dM^0(s)\right) \circ r_t \\
= & -\int_0^t (a^{-1}\hat{b})(X^0(s)) \cdot dM^0(s) + N^v(t) \\
= & -\int_0^t (a^{-1}\hat{b})(X^0(s)) \cdot dM^0(s) + v(X^0(t)) - v(X^0(0)) - M^v(t) \\
= & -\int_0^t (a^{-1}\hat{b})(X^0(s)) \cdot dM^0(s) + v(X^0(t)) \\
& -v(X^0(0)) - \int_0^t \nabla v(X^0(s)) dM^0(s).
\end{aligned}
$$

Key steps of the proof

Hence

$$
\begin{aligned}
Z_t \;=\; & \frac{e^{v(X^0(t))}}{e^{v(X^0(0))}} \exp\left(\int_0^t a^{-1}(b - \hat{b} - a\nabla v)(X^0(s)) \cdot dM^0(s) \right. \\
& + \int_0^t \left(\frac{1}{2}(b - \hat{b} - a\nabla v)a^{-1}(b - \hat{b} - a\nabla v)^* \right) (X^0(s))ds \\
& - 2\int_0^t J_1(X^0(s))ds + \int_0^t q(X^0(s))ds \\
& \left. + \int_0^t \left(\frac{1}{2}(\nabla v)a(\nabla v)^* - < b - \hat{b}, \nabla v > \right) (X^0(s))ds \right) \quad (34)
\end{aligned}
$$

Key steps of the proof

Step 2. Set $h(x) = e^{v(x)}$ and

$$\hat{f}(x, y) = h(x) f(x, h^{-1}(x) y)$$

Introduce

$$\hat{\mathcal{A}} = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial}{\partial x_j}) + \sum_{i=1}^{d} [b_i(x) - \hat{b}_i(x) - (a \nabla v)_i(x)] \frac{\partial}{\partial x_i}$$

$$- < b - \hat{b}, \nabla v > (x) + \frac{1}{2} (\nabla v) a (\nabla v)^*(x) + q(x).$$

and consider the following nonlinear elliptic partial differential equation:

$$\begin{cases} \hat{\mathcal{A}} \hat{u}(x) = -\hat{f}(x, \hat{u}(x)), \ \forall x \in D, \\ \hat{u}(x)|_{\partial D} = h(x) v(x), \qquad \forall x \in \partial D. \end{cases} \tag{35}$$

Key steps of the proof

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \hat{X}(t), \hat{P}_x, x \in R^d)$ be the diffusion process whose infinitesimal generator is given by

$$\hat{L} = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}) + \sum_{i=1}^{d}[b_i(x) - \hat{b}_i(x) - (a\nabla v)_i(x)]\frac{\partial}{\partial x_i}$$

It is known from [LZ] that $\hat{P}_x$ is absolutely continuous with respect to P_x^0 and

$$\frac{d\hat{P}_x}{dP_x^0}|_{\mathcal{F}_t} = \hat{Z}_t,$$

where

$$\begin{aligned}\hat{Z}_t \quad = \quad & \exp\left(\int_0^t (a^{-1}(b - \hat{b} - a\nabla v)(X^0(s)) \cdot dM^0(s)\right.\\ & \left. - \int_0^t \left(\frac{1}{2}(b - \hat{b} - a\nabla v)a^{-1}(b - \hat{b} - a\nabla v)^*\right)(X^0(s))ds\right)\end{aligned}$$

Key steps of the proof

In view of (34), condition (31) implies that

$$\hat{E}_x\left[\exp\left(-2\int_0^{\tau_D} J_1(X^0(s))ds+\int_0^{\tau_D} q(X^0(s))ds\right.\right.$$
$$\left.\left.+\int_0^{\tau_D}\left(\frac{1}{2}(\nabla v)a(\nabla v)^* - < b-\hat{b},\nabla v >\right)(X^0(s))ds\right]<\infty,$$

where $\hat{E}_x$ means that the expectation is taken under $\hat{P}_x$. From Theorem 6, it follows that equation (35) admits a unique weak solution $\hat{u}$.

Key steps of the proof

Step 3. Set $u(x) = h^{-1}(x)\hat{u}(x)$. We verify that u is a weak solution to equation (30) using the equation

$$div(a\nabla v) = -2div(\hat{b})$$

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