Numerical simulation of BSDEs with drivers of quadratic growth

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Roscoff - 2010

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Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(W_t)_{t \in \mathbb{R}^+}$ be a Brownian motion in $\mathbb{R}^d$, $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$ be his augmented natural filtration, T be a nonnegative real number. We consider an SDE

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s,$$

with standard assumptions on b and σ, and a Markovian BSDE

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s.$$

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Definition

A solution to this BSDE is a pair of processes $(Y_t, Z_t)_{0\leqslant t\leqslant T}$ such that :

1. (Y, Z) is a predicable process with values in $\mathbb{R} \times \mathbb{R}^{1\times d}$,
2. $\mathbb{P} - a.s.\ t \mapsto Y_t$ is continuous and $\int_0^T |f(r, X_r, Y_r, Z_r)| + \|Z_r\|^2 dr < \infty$

Theorem (Pardoux-Peng 1990)

Let us assume that f is a Lipschitz function with respect to y and z and $\mathbb{E}\left[|g(X_T)|^2 + \int_0^T |f(r, X_r, 0, 0)|^2 dr\right] < \infty$. Then the previous equation has a unique solution (Y, Z) such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |Y_t|^2\right] < \infty, \quad \mathbb{E}\left[\int_0^T |Z_t|^2 dt\right] < \infty.$$

Time discretization

We consider a time discretization of the BSDE. We denote the time step by $h = T/n$ and $(t_k = kh)_{0 \leqslant k \leqslant n}$ stands for the discretization times. For X we take the Euler scheme :

$$\begin{aligned} X_0^n &= x \\ X_{t_{k+1}}^n &= X_{t_k}^n + hb(t_k, X_{t_k}^n) + \sigma(t_k, X_{t_k}^n)(W_{t_{k+1}} - W_{t_k}), \quad 0 \leqslant k \leqslant n. \end{aligned}$$

For (Y, Z) we use the classical dynamic programming equation

$$\begin{aligned} Y_{t_n}^n &= g(X_{t_n}^n) \\ Z_{t_k}^n &= \frac{1}{h}\mathbb{E}_{t_k}[Y_{t_{k+1}}^n (W_{t_{k+1}} - W_{t_k})], \quad 0 \leqslant k \leqslant n-1, \\ Y_{t_k}^n &= \mathbb{E}_{t_k}[Y_{t_{k+1}}^n] + h\mathbb{E}_{t_k}[f(t_k, X_{t_k}^n, Y_{t_{k+1}}^n, Z_{t_k}^n)], \quad 0 \leqslant k \leqslant n-1, \end{aligned}$$

where $\mathbb{E}_{t_k}$ stands for the conditional expectation given $\mathcal{F}_{t_k}$.

Remarks on simulation

- the dynamic programming equation is obtained by minimizing the difference
$$\mathbb{E}\left[\left(Y^n_{t_{k+1}} + h\mathbb{E}_{t_k} f(t_k, X^n_{t_k}, Y^n_{t_{k+1}}, Z) - Y - Z(W_{t_{k+1}} - W_{t_k})\right)^2\right]$$
over $\mathcal{F}_{t_k}$-measurable squared integrable random variables (Y, Z).
- After time discretization, we need to use a spatial discretization in order to compute conditional expectation.
- We suppose that g and f are Lipschitz functions with respect to x, y, z and t. If we define the error
$$e(n) = \sup_{0 \leqslant k \leqslant n} \mathbb{E}\left|Y^n_{t_k} - Y_{t_k}\right|^2 + \mathbb{E}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\left|Z^n_{t_k} - Z_t\right|^2 dt$$
then $e(n) = O(1/n)$.

References for simulation

See, for exemple :

- B. Bouchard, N. Touzi [2004],
- J. Zhang [2005],
- E. Gobet, J.P. Lemor, X. Warin [2005],
- F. Delarue, S. Menozzi [2006].

Quadratic BSDEs

What happened if f has a quadratic growth with respect to z ?

- when g is bounded : existence and uniqueness results have been proved by M. Kobylanski [2000].
- when g is unbounded : an existence result has been proved by P. Briand and Y. Hu [2006], partial uniqueness results has been proved by P. Briand and Y. Hu [2008], F. Delbaen, Y. Hu and A. R. [2010].

Such BSDEs have applications in finance : this class arises, for example, in the context of utility optimization problems with exponential utility functions (see e.g. Y. Hu, P. Imkeller and M. Müller [2005]).

BMO tool

Definition

For a brownian martingale $\Phi_t = \int_0^t \phi_s dW_s$, $t \in [0, T]$, we say that Φ is a BMO martingale if

$$\|\Phi\|_{BMO} = \sup_{\tau \in [0,T]} \mathbb{E}\left[\int_\tau^T \phi_s^2 ds \middle| \mathcal{F}_\tau\right]^{1/2} < +\infty,$$

where the supremum is taken over all stopping times in $[0, T]$.

BMO tool

the very important feature of BMO martingales is the following lemma :

Lemma

Let Φ be a BMO martingale. Then we have :

1. *The stochastic exponential*

$$\mathcal{E}(\Phi)_t = \mathcal{E}_t = \exp\left(\int_0^t \phi_s dW_s - \frac{1}{2}\int_0^t |\phi_s|^2 ds\right), \quad 0 \leqslant t \leqslant T,$$

   is a uniformly integrable martingale.

2. *Thanks to the reverse Hölder inequality, there exists $p > 1$ such that $\mathcal{E}_T \in L^p$. The maximal p with this property can be expressed in terms of the BMO norm of Φ.*

Theorem (Briand, Confortola (2008), Ankirchner and al. (2007))

We suppose that

$$\begin{aligned}
|f(t,x,y,z)| &\leqslant M_f(1+|y|+|z|^2),\\
|f(t,x,y,z)-f(t,x',y',z')| &\leqslant K_{f,x}\,|x-x'|+K_{f,y}\,|y-y'|\\
&\quad +(K_{f,z}+L_{f,z}(|z|+|z'|))\,|z-z'|\,,\\
|g(x)| &\leqslant M_g.
\end{aligned}$$

The SDE-BSDE system has a unique solution (X,Y,Z) *such that* $\mathbb{E}[\sup_{t\in[0,T]}|X|^2]<+\infty$*,* Y *is a bounded measurable process and* $\mathbb{E}[\int_0^T |Z_s|^2\,ds]<+\infty$*. The martingale* $Z*W$ *belongs to the space of BMO martingales and* $\|Z*W\|_{BMO}$ *only depends on* T*,* M_g *and* M_f*. Moreover, there exists* $r>1$ *such that* $\mathcal{E}(Z*W)\in L^r$*.*

Proposition (Briand, Confortola (2008), Ankirchner and al. (2007))

If we denote (Y^i, Z^i) *the solution of a BSDE with a terminal condition* g_i *and a driver* f_i*, then we have*

$$\mathbb{E}[\sup_{t\in[0,T]} \left|Y_t^1 - Y_t^2\right|^2] + \mathbb{E}[\int_0^T \left|Z_s^1 - Z_s^2\right|^2 ds]$$

$$\leqslant \mathbb{E}\left[\left|g_1(X_T) - g_2(X_T)\right|^{2q} + \left(\int_0^T \left|(f_1 - f_2)(s, X_s, Y_s^2, Z_s^2)\right| ds\right)^{2q}\right]^{1/q}.$$

where $1/r + 1/q = 1$.

Goal

The aim of our work is to give a time discretization scheme for quadratic BSDEs, and to obtain a "good" convergence rate for this scheme.

The exponential transformation

When the generator has the specific form

$$f(t,x,y,z) = l(t,x,y) + a(t,z) + \frac{\gamma}{2}|z|^2,$$

with a and l Lipschitz functions and a homogeneous with respect to z, it is possible to use an exponential transform (also known as the Cole-Hopf transformation) : $(e^{\gamma Y}, \gamma e^{\gamma Y} Z)$ is the solution of a BSDE with a driver of linear growth. See P. Imkeller, G. dos Reis and J. Zhang [2010].

g Lipschitz

Proposition

If g is a Lipschitz function with a Lipschitz constant K_g and σ does not depend on x, then, $\forall t \in [0, T]$,

$$|Z_t| \leqslant C(1 + K_g).$$

In this situation the driver becomes a Lipschitz function with respect to z, and so we are allowed to use the classical discrete time approximation.

If g is α-Hölder, we have an explicit uniform Lipschitz approximation g_N of g with $K_{g_N} = N$. Then we do an approximation of (Y, Z) by the solution (Y^N, Z^N) to the BSDE

$$Y_t^N = g_N(X_T) + \int_t^T f(s, X_s, Y_s^N, Z_s^N)ds - \int_t^T Z_s^N dW_s.$$

- Thanks to BMO tools we have an error estimate for this approximation : $CN^{\frac{-\alpha}{1-\alpha}}$.
- We also need to have the error estimate for the time approximation of our BSDE with linear growth : $Ce^{CN^2}n^{-1}$.

Finally, if we take $N = (\frac{C}{\varepsilon} \log n)^{1/2}$ with ε small, then the global error bound becomes

$$C_\varepsilon (\log n)^{\frac{-\alpha}{2(1-\alpha)}}.$$

Truncated BSDE

An other idea is to do an approximation of (Y, Z) by the solution (Y^N, Z^N) to the truncated BSDE

$$Y_t^N = g(X_T) + \int_t^T f(s, X_s, Y_s^N, h_N(Z_s^N))ds - \int_t^T Z_s^N dW_s,$$

where $h_N : \mathbb{R}^{1\times d} \to \mathbb{R}^{1\times d}$ is a smooth modification of the projection on the open Euclidean ball of radius N about 0. An error estimate is obtain by P. Imkeller and G. dos Reis [2009], but the same drawback appears.

A time-dependent estimate of Z

Theorem (Delbaen, Hu, Bao (2010), R. (2010))

We suppose that b is differentiable with respect to x and σ is differentiable with respect to t. There exists $\lambda \in \mathbb{R}^+$ such that $\forall \eta \in \mathbb{R}^d$

$$\left| {}^t\eta\sigma(s)[{}^t\sigma(s){}^t\nabla b(s,x) - {}^t\sigma'(s)]\eta \right| \leqslant \lambda \left| {}^t\eta\sigma(s) \right|^2. \tag{3.1}$$

Moreover, suppose that g is lower (or upper) semi-continuous. Then, $\forall t \in [0, T[$,

$$|Z_t| \leqslant C_z + C'_z(T-t)^{-1/2}.$$

Sketch of the proof (1/2)

We suppose that

- f does not depends on x and y,
- g is C^1 with respect to x and f is C^1 with respect to z.

Then Y and Z are differentiable with respect to x the initial condition of X, and

$$\begin{aligned}\nabla Y_t &= \nabla g(X_T)\nabla X_T + \int_t^T \nabla_z f \nabla Z_s ds - \int_t^T \nabla Z_s dW_s \\ &= \nabla g(X_T)\nabla X_T - \int_t^T \nabla Z_s d\tilde{W}_s.\end{aligned}$$

That is to say ∇Y is a $\mathbb{Q}$-martingale.

Sketch of the proof (2/2)

Thanks to the Malliavin calculus we have :
$Z_t = \nabla Y_t (\nabla X_t)^{-1}\sigma(t)$. By applying the Itô formula to the process $\left|e^{\lambda t}\nabla Y_t(\nabla X_t)^{-1}\sigma(t)\right|^2$, we show that $\left|e^{\lambda t} Z_t\right|^2$ is a $\mathbb{Q}$-submartingale. Finally

$$e^{2\lambda t}\left|Z_t\right|^2(T-t) \leqslant \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{2\lambda s}\left|Z_s\right|^2 ds \middle| \mathcal{F}_t\right] \leqslant e^{2\lambda T}\left\|Z\right\|_{BMO(\mathbb{Q})}.$$

Remark

This type of estimation is well known in the case of drivers with linear growth as a consequence of the Bismut-Elworthy formula. In our case, σ does not depends on x but we do not need to suppose that σ is invertible.

How can we use this time-dependent estimate of Z ?

In the Lipschitz case, to obtain a bound for the error

$$\sup_{0\leqslant k\leqslant n} \mathbb{E}\left[\left|Y^n_{t_k} - Y_{t_k}\right|^2\right]$$

we show such an estimate :

$$\mathbb{E}\left[\left|Y^n_{t_k} - Y_{t_k}\right|^2\right] \leqslant (1 + Ch + K^2_{f,z}h)\mathbb{E}\left[\left|Y^n_{t_{k+1}} - Y_{t_{k+1}}\right|^2\right] + h$$

and then we use the Gronwall's lemma.

How can we use this time-dependent estimate of Z ?

In our case we have

$$\mathbb{E}\left[\left|Y^n_{t_k}-Y_{t_k}\right|^2\right]\leqslant (1+C(t_k-t_{k+1})+K\frac{t_{k+1}-t_k}{T-t_{k+1}})\mathbb{E}\left[\left|Y^n_{t_{k+1}}-Y_{t_{k+1}}\right|^2\right]+h.$$

So, the idea is to find a new time net such that $\frac{t_{k+1}-t_k}{T-t_{k+1}}$ is a constant : We define the n first discretization times by

$$t_k = T\left(1-\left(\frac{\varepsilon}{T}\right)^{k/(n-1)}\right).$$

ε is a parameter : $t_{n-1}=T-\varepsilon$. We will set $\varepsilon := T/n^a$ with a a parameter.

How can we use this time-dependent estimate of Z ?

Lemma

$$\prod_{i=0}^{n-2}\left(1+C(t_{i+1}-t_i)+K\frac{t_{i+1}-t_i}{T-t_{i+1}}\right)\leqslant Cn^{aK}.$$

Our algorithm (1/2)

Due to technical reason, we have to approximate our BSDE by an other one. Let $(Y_t^{N,\varepsilon}, Z_t^{N,\varepsilon})$ the solution of the BSDE

$$Y_t^{N,\varepsilon} = g_N(X_T) + \int_t^T f^\varepsilon(s, X_s, Y_s^{N,\varepsilon}, Z_s^{N,\varepsilon})ds - \int_t^T Z_s^{N,\varepsilon} dW_s. \tag{3.2}$$

with

$$f^\varepsilon(s,x,y,z) := \mathbb{1}_{s<T-\varepsilon} f(s,x,y,z) + \mathbb{1}_{s\geqslant T-\varepsilon} f(s,x,y,0),$$

and g_N a N-Lipschitz approximation of g.

Our algorithm (2/2)

We denote $\rho_s : \mathbb{R}^{1\times d} \to \mathbb{R}^{1\times d}$ the projection on the ball

$$B\left(0, C_z + \frac{C'_z}{(T-s)^{1/2}}\right).$$

Finally we denote $(Y^{N,\varepsilon,n}, Z^{N,\varepsilon,n})$ our time approximation of $(Y^{N,\varepsilon}, Z^{N,\varepsilon})$. This couple is obtained by a slight modification of the classical dynamic programming equation :

$$\begin{aligned}
Y_{t_n}^{N,\varepsilon,n} &= g_N(X_{t_n}^n) \\
Z_{t_k}^{N,\varepsilon,n} &= \rho_{t_{k+1}}\left(\frac{1}{h_k}\mathbb{E}_{t_k}[Y_{t_{k+1}}^{N,\varepsilon,n}(W_{t_{k+1}} - W_{t_k})]\right), \quad 0 \leqslant k \leqslant n-1, \\
Y_{t_k}^{N,\varepsilon,n} &= \mathbb{E}_{t_k}[Y_{t_{k+1}}^{N,\varepsilon,n}] + h_k\mathbb{E}_{t_k}[f(t_k, X_{t_k}^n, Y_{t_{k+1}}^{N,\varepsilon,n}, Z_{t_k}^{N,\varepsilon,n})], \quad 0 \leqslant k \leqslant n -
\end{aligned}$$

A first speed of convergence

Theorem

Let us recall that $\varepsilon = T/n^a$. *We set* $N = n^b$. *We assume that* g *is* α*-Hölder. Then we can set* a *and* b *such that for all* $\eta > 0$, *there exists a constant* $C_\eta > 0$ *that verifies*

$$\sup_{0 \leqslant k \leqslant n} \mathbb{E}\left[\left|Y_{t_k}^{N,\varepsilon,n} - Y_{t_k}\right|^2\right] + \sum_{k=0}^{n-1} \mathbb{E}\left[\int_{t_k}^{t_{k+1}} \left|Z_{t_k}^{N,\varepsilon,n} - Z_t\right|^2 dt\right] \leqslant \frac{C_\eta}{n^p},$$

where

$$p = \frac{2\alpha}{(2-\alpha)(2+K(1+\eta)) - 2 + 2\alpha}.$$

K *is an explicit constant. It depends on constants that appear in assumptions on* g *and* f.

A better speed of convergence

Theorem

If, moreover, b is bounded, then we can take K as small as we want :

$$\sup_{0\leqslant k\leqslant n} \mathbb{E}\left[\left|Y_{t_k}^{N,\varepsilon,n} - Y_{t_k}\right|^2\right] + \sum_{k=0}^{n-1} \mathbb{E}\left[\int_{t_k}^{t_{k+1}} \left|Z_{t_k}^{N,\varepsilon,n} - Z_t\right|^2 dt\right] \leqslant \frac{C_\eta}{n^{\alpha-\eta}},$$

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