Representation of G-martingales as stochastic integrals with respect to G-Brownian motion

Qian LIN

Laboratoire de Mathématiques, CNRS UMR 6205, Université de Bretagne Occidentale, France.
Email: Qian.Lin@univ-brest.fr

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Xu [2009] obtained the martingale characterization of the G-Brownian motion.

The objective of the present paper is to investigate a representation of G-martingales as stochastic integrals with respect to the G-Brownian motion in the framework of sublinear expectation spaces. In this paper, we

- study stochastic integrals with respect to G-martingale;
- study representation theorem of G-martingales.
Preliminaries

We briefly recall some basic results about $G$-stochastic analysis in the following papers:


Let $\Omega$ be a given set and $\mathcal{H}$ be a linear space of real functions defined on $\Omega$ such that if $x_1, \ldots, x_n \in \mathcal{H}$ then $\varphi(x_1, \ldots, x_n) \in \mathcal{H}$, for each $\varphi \in C_{l,\text{lip}}(\mathbb{R}^m)$. Here $C_{l,\text{lip}}(\mathbb{R}^m)$ denotes the linear space of functions $\varphi$ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^n + |y|^n)|x - y|, \text{ for all } x, y \in \mathbb{R}^m,$$

for some $C > 0$ and $n \in \mathbb{N}$, both depending on $\varphi$. The space $\mathcal{H}$ is considered as a set of random variables.
Let \( \Omega = C_0(\mathbb{R}^+ \times [0, T]) \) be the space of all real valued continuous functions \((\omega_t)_{t \in \mathbb{R}^+}\) with \(\omega_0 = 0\), equipped with the distance

\[
\rho(\omega^1, \omega^2) = \sum_{i=1}^{\infty} 2^{-i} \left( \max_{t \in [0, i]} |\omega^1_t - \omega^2_t| \right) \land 1, \quad \omega^1, \omega^2 \in \Omega.
\]

For each \( T > 0 \), we consider the following space of random variables:

\[
L^0_{lip}(\mathcal{F}_T) : = \left\{ X(\omega) = \varphi(\omega_{t_1}, \ldots, \omega_{t_m}) \mid t_1, \ldots, t_m \in [0, T], \right. \\
\left. \text{for all } \varphi \in C_{lip}(\mathbb{R}^m), \ m \geq 1 \right\},
\]

\[
L^0_{lip}(\mathcal{F}) = \bigcup_{n=1}^{\infty} L^0_{lip}(\mathcal{F}_n).
\]
Sublinear expectations

**Definition**

A **Sublinear expectation** $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \mapsto \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(i) **Monotonicity**: If $X \geq Y$, then $\hat{E}[X] \geq \hat{E}[Y]$.

(ii) **Constant preserving**: $\hat{E}[c] = c$, for all $c \in \mathbb{R}$.

(iii) **Self-dominated property**: $\hat{E}[X] - \hat{E}[Y] \leq \hat{E}[X - Y]$.

(iv) **Positive homogeneity**: $\hat{E}[\lambda X] = \lambda \hat{E}[X]$, for all $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

**Remark**

*The sublinear expectation space can be regarded as a generalization of the classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the linear expectation associated with $\mathbb{P}$.*
Coherent risk measures and sublinear expectations

Let
\[ \rho(X) \doteq \hat{E}[-X], \ X \in \mathcal{H}. \]

Then \( \rho(\cdot) \) is a coherent risk measure, namely

1. **Monotonicity:** If \( X \leq Y \), then \( \rho(X) \geq \rho(Y) \).
2. **Constant preserving:** \( \rho(c) = -c \), for all \( c \in \mathbb{R} \).
3. **Self-dominated property:** \( \rho(X) - \rho(Y) \leq \rho(X - Y) \).
4. **Positive homogeneity:** \( \rho(\lambda X) = \lambda \rho(X) \), for all \( \lambda \geq 0 \).

Conversely, for every coherent risk measure \( \rho \), let

\[ \hat{E}[X] \doteq \rho(-X), \ X \in \mathcal{H}. \]

Then \( \hat{E}[\cdot] \) is a sublinear expectation.
For $p \geq 1$, $\|X\|_p = \hat{\mathbb{E}}^{\frac{1}{p}}[\|X\|^p]$, $X \in L^0_{ip}(\mathcal{F})$.

Let $\mathcal{H} = L^p_G(\mathcal{F})$ (resp. $\mathcal{H}_t = L^p_G(\mathcal{F}_t)$) be the completion of $L^0_{ip}(\mathcal{F})$ (resp. $L^0_{ip}(\mathcal{F}_t)$) under the norm $\| \cdot \|_p$.

$(L^p_G(\mathcal{F}), \| \cdot \|_p)$ is a Banach space.

$L^p_G(\mathcal{F}_t) \subset L^p_G(\mathcal{F}_T) \subset L^p_G(\mathcal{F})$, for all $0 \leq t \leq T < \infty$.

**Remark**

Bounded and measurable random variables in general are not in $L^p_G(\mathcal{F})$ (e.g. $I_A$). Thus, the powerful techniques of stopping times in classical situations cannot be applied to $G$-stochastic analysis. This is a main difficulty faced in the calculus.
Independence

**Definition**

In a sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\), a random vector \(Y = (Y_1, \cdots, Y_n), Y_i \in \mathcal{H}\), is said to be **independent** of another random vector \(X = (X_1, \cdots, X_m), X_i \in \mathcal{H}\), if for each test function \(\varphi \in C_{l,lip}(\mathbb{R}^{m+n})\) we have

\[
\hat{E}[\varphi(X, Y)] = \hat{E}[\hat{E}[\varphi(x, Y)]_{x=x}].
\]

**Remark**

*Independence means the distribution of \(Y\) does not change the realization of \(X (X = x)\).*
Remark

$Y$ is independent of $X$ does not imply that $X$ is independent of $Y$.

Example

$\hat{E}[X] = \hat{E}[-X] = 0, \hat{E}[X^+] > 0, \hat{E}[Y^2] > -\hat{E}[-Y^2] > 0.$

- If $X$ is independent of $Y$, then $\hat{E}[XY^2] = 0$.
- But if $Y$ is independent of $X$, then $\hat{E}[XY^2] > 0$. 
G-normal distribution

Definition

**G-normal distribution:**

\[ \xi \sim \mathcal{N}(0, [\sigma_1^2, \sigma_2^2]), \text{ if for all } \varphi \in C_{l,\text{lip}}(\mathbb{R}), \]

\[ u(t, x) := \hat{E}[\varphi(x + \sqrt{t}\xi)], \quad (t, x) \in [0, \infty) \times \mathbb{R} \]

is the solution of the following PDE:

\[ \partial_t u = G(\partial_{xx}^2 u), \quad u|_{t=0} = \varphi, \]

where \( G(\alpha) = \frac{1}{2} \sup_{\sigma_1 \leq \sigma \leq \sigma_2} \alpha \sigma^2, \quad 0 \leq \sigma_1 \leq \sigma_2. \)
Remark

In the case where $\sigma_1 = \sigma_2 > 0$, then $\mathcal{N}(0, [\sigma_1^2, \sigma_2^2])$ is just the classical normal distribution $\mathcal{N}(0, \sigma_2^2)$.

Remark

If $X \sim \mathcal{N}(0, [\sigma_1^2, \sigma_2^2])$ and $\varphi$ is convex, then

$$\hat{E}[\varphi(X)] = \frac{1}{\sqrt{2\pi\sigma_2^2}} \int_{-\infty}^{\infty} \varphi(y) \exp\left(-\frac{(x - y)^2}{2\sigma_2^2 t}\right) dy.$$  

Remark

Let $X \sim \mathcal{N}(0, [\sigma_1^2, \sigma_2^2])$. If $\varphi$ is concave and $\sigma_1^2 > 0$, then

$$\hat{E}[\varphi(X)] = \frac{1}{\sqrt{2\pi\sigma_1^2}} \int_{-\infty}^{\infty} \varphi(y) \exp\left(-\frac{(x - y)^2}{2\sigma_1^2 t}\right) dy.$$
G-Brownian motion

For simplicity, we assume $0 \leq \sigma_1 = \sigma \leq 1, \sigma_2 = 1$ in the following.

**Definition**

A process $B$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called **G-Brownian motion** if for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n < \infty$, $B_{t_1}, \ldots, B_{t_n} \in \mathcal{H}$, the following properties are satisfied:

(i) $B_0 = 0$;

(ii) For each $t, s \geq 0$, $B_{t+s} - B_t \sim \mathcal{N}(0, [\sigma^2 s, s])$;

(iii) For each $t, s \geq 0$, $B_{t+s} - B_t$ is independent of $(B_{t_1}, \ldots, B_{t_n})$, for each $n \in \mathbb{N}$ and $t_n \leq t$. 


**Theorem**

Let $\hat{E}$ be a G-expectation. Then there exists a weekly compact family of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\hat{E}[X] = \max_{P \in \mathcal{P}} E_P[X], \text{ for all } X \in \mathcal{H},$$

where $E_P[\cdot]$ is the linear expectation with respect to $P \in \mathcal{P}$.

**Definition**

- Choquet capacity: $c(A) = \sup_{P \in \mathcal{P}} P(A), A \in \mathcal{B}(\Omega)$.
- A set $A$ is called polar if $c(A) = 0$ and a property holds quasi-surely (q.s.) if it holds outside a polar set.
As in the classical stochastic analysis, the definition of a modification of a process plays an important role.

**Definition**

Let $I$ be a set of indexes, and $\{X_t\}_{t \in I}$ and $\{Y_t\}_{t \in I}$ two processes indexed by $I$. We say that $Y$ is a modification of $X$ if for all $t \in I$, $X_t = Y_t$ q.s.
Finally, we recall the definition of a G-martingale introduced by Peng [2006].

**Definition**

A process $M = \{M_t, t \geq 0\}$ is called a G-martingale (respectively, G-supermartingale, and G-submartingale) if for each $t \in [0, \infty)$, $M_t \in L^1_G(F_t)$ and for each $s \in [0, t]$, we have

$$\hat{E}[M_t | \mathcal{H}_s] = M_s,$$

(respectively $\leq M_s$, and $\geq M_s$) q.s.

**Definition**

A process $M = \{M_t, t \geq 0\}$ is called a symmetric G-martingale, if $M$ and $-M$ are G-martingales.

**Remark**

$B_t$ is symmetric G-martingale, but $B_t^2 - t$ is not symmetric G-martingale.
Our objective: representation theorem for G-martingales

Recall: classical representation theorem for martingales

**Theorem**

Let $M$ be a square integrable continuous martingale. $M_t^2 - \int_0^t f_s^2 ds$ is a martingale, for some adapted process $f$ such that $\int_0^T f_s^2 ds < \infty$, a.s.,. Then there exists a Brownian motion $B$ such that

$$M_t = \int_0^t f_s dB_s.$$
Peng [2006] introduced stochastic integrals with respect to G-Brownian motion.

Xu [2009] introduced stochastic integrals with respect to symmetric G-martingales \( M \), with \( \{ M_t^2 - t \}_{t \in [0,T]} \) being a G-martingale.

In order to obtain representation of G-martingale, it is necessary to extend the notion of G-stochastic integrals.
Let $p \geq 1$ and $T > 0$. Let $\{A_t, t \in [0, T]\}$ be a continuous and increasing process such that for all $t \in [0, T]$, $A_t \in \mathcal{H}_t$, $A_0 = 0$ and $\hat{E}[A_T] < \infty$. We first consider the following space of step processes:

$$M_{G}^{p,0}(0, T) = \left\{ \eta: \eta_t = \sum_{j=0}^{n-1} \xi_{t_j} I_{[t_j, t_{j+1})}, \ 0 = t_0 < t_1 < \cdots < t_n = T, \right. \left. \xi_{t_j} \in L_{G}^p(\mathcal{F}_{t_j}), j = 0, \cdots, n-1, \text{for all } n \geq 1 \right\},$$

and we define the following norm in $M_{G}^{p,0}(0, T)$:

$$\| \eta \|_p = \left( \hat{E} \left[ \int_0^T |\eta_t|^p dA_t \right] \right)^{\frac{1}{p}} = \left( \hat{E} \left[ \sum_{j=0}^{n-1} |\xi_{t_j}|^p (A_{t_{j+1}} - A_{t_j}) \right] \right)^{\frac{1}{p}}.$$
We denote by $M_{G,A}^p(0,T)$ the completion of $M_{G}^{p,0}(0,T)$ under the norm $\| \cdot \|_p$. If $A_t = t$, then we denote by $M_{G}^p(0,T)$ the completion of $M_{G}^{p,0}(0,T)$ under the norm $\| \cdot \|_p$.

$\mathcal{N} = \left\{ M | M \text{ is a continuous symmetric G-martingale such that } M^2 - A \text{ is a G-supermartingale} \right\}$.

**Definition**

For any $M \in \mathcal{N}$ and $\eta \in M_{G}^{2,0}(0,T)$ of the form

$$\eta_t = \sum_{j=0}^{n-1} \xi_{t_j} I_{[t_j,t_{j+1})}(t),$$

we define

$$I(\eta) = \int_0^T \eta_t dM_t = \sum_{j=0}^{n-1} \xi_{t_j} (M_{t_{j+1}} - M_{t_j}).$$
Proposition

For all $M \in \mathcal{N}$, the mapping $I : M_{G,0}^2(0, T) \rightarrow L_G^2(\mathcal{F}_T)$ is a linear continuous mapping and, thus, can be continuously extended to $I : M_{G,A}^2(0, T) \rightarrow L_G^2(\mathcal{F}_T)$. Moreover, for all $\eta \in M_{G,A}^2(0, T)$, the process $\left\{ \int_0^t \eta_s dM_s \right\}_{t \in [0,T]}$ is a symmetric $G$-martingale and

$$\hat{\mathbb{E}} \left[ \left\| \int_0^T \eta_t dM_t \right\|^2 \right] \leq \hat{\mathbb{E}} \left[ \int_0^T |\eta_t|^2 dA_t \right].$$  (1)
For $0 \leq s \leq t \leq T$ and $\eta \in M^2_{G,A}(0, T)$, we denote

$$\int_s^t \eta_u dM_u = \int_0^T I_{[s,t]}(u) \eta_u dM_u.$$ 

It is now straightforward to see that we have the following properties of the stochastic integral of G-martingales.

**Proposition**

Let $0 \leq s < r \leq t \leq T$. For all $M \in \mathcal{N}$ and $\theta, \eta \in M^2_{G,A}(0, T)$, we have

(i) $\int_s^t \eta_u dM_u = \int_s^r \eta_u dM_u + \int_r^t \eta_u dM_u$;

(ii) $\int_s^t (\eta_u + \alpha \theta_u) dM_u = \int_s^t \eta_u dM_u + \alpha \int_s^t \theta_u dM_u$, for all $\alpha$ bounded random variable in $L^p_G(\mathcal{F}_s)$;

(iii) $\hat{E}[X + \int_r^T \eta_u dM_u | \mathcal{H}_s] = \hat{E}[X | \mathcal{H}_s]$, for all $X \in L^p_G(\mathcal{F})$. 
For proving the continuity of the stochastic integral regarded as a process, we need the following Doob inequality for symmetric G-martingale.

**Theorem**

*If X is a right-continuous symmetric G-martingale running over an interval \([0, T]\) of \(\mathbb{R}\), then for every \(p > 1\) such that \(X_T \in L^p_G(\mathcal{F})\),

\[
\hat{\mathbb{E}}\left[ \sup_{0 \leq t \leq T} |X_t|^p \right] \leq \left( \frac{p}{p - 1} \right)^p \hat{\mathbb{E}}[|X_T|^p].
\]
Theorem

For all $M \in \mathcal{N}$ and $\eta \in M_{G,A}^2(0, T)$, there exists a continuous modification of stochastic integral

$$\int_0^t \eta_s dM_s, \quad 0 \leq t \leq T.$$
Now we give the Burkholder-Davis-Gundy inequality for the stochastic integral with respect to G-martingales.

**Theorem**

For every $q > 0$, there exist a positive constant $C_q$ such that, for all $M \in \mathcal{N}$ and all $\eta \in M_{G,A}^2(0,T)$,

$$\hat{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t \eta_s dM_s \right|^{2q} \right] \leq C_q \hat{E} \left[ \left( \int_0^T \eta_s^2 dA_s \right)^q \right].$$
Assumptions:

- \( \hat{\mathbb{E}}[A_T^2] < \infty \)
- For all \( \{\pi^n\}_{n \geq 1} \) sequence of partitions \( \pi^n = \{0 = t^n_0 < t^n_1 \cdots < t^n_n = T\} \) of \([0, T]\) such that \( |\pi^n| \to 0, \text{ as } n \to \infty, \hat{\mathbb{E}}[\sum_{i=0}^{n-1} (A_{t^n_{i+1}} - A_{t^n_i})^2] \to 0, n \to \infty. \)

**Proposition**

Let \( M \in \mathcal{N} \). Then the quadratic variation of \( M \) exists and

\[
\langle M \rangle_t = M_t^2 - 2 \int_0^t M_s dM_s, \text{ for all } t \geq 0.
\]

**Remark**

The quadratic variation of \( M \) is increasing and continuous.
Now we can give another kind of the Burkholder-Davis-Gundy inequalities for the stochastic integral with respect to G-martingales.

**Theorem**

For every $p > 0$, there exist two positive constants $c_p$ and $C_p$ such that, for all $M \in \mathcal{N}$ and all $\eta \in M_{G,A}^2(0, T)$,

$$c_p \hat{\mathbb{E}}\left[\left(\int_0^T \eta_s^2 d\langle M \rangle_s\right)^p\right] \leq \hat{\mathbb{E}}\left[\sup_{t \in [0, T]} \left| \int_0^t \eta_s dM_s \right|^{2p}\right] \leq C_p \hat{\mathbb{E}}\left[\left(\int_0^T \eta_s^2 d\langle M \rangle_s\right)^p\right].$$
Proposition

For a fixed $T \geq 0$, $M$ is a symmetric $G$-martingale such that $M^2 - A$ and $-M^2 + \sigma_0^2 A$ be $G$-martingales. If $f \in M^1_{G,A}(0, T)$, then

$$X_t := \int_0^t f_s d\langle M \rangle_s - 2 \int_0^t G(f_s) dA_s, \quad t \in [0, T]$$

is a decreasing $G$-martingale.

Recall $G(\alpha) = \frac{1}{2}(\alpha^+ - \sigma^2 \alpha^-)$, $\alpha \in \mathbb{R}$.

Corollary

$$\int_0^t f_s d\langle B \rangle_s - 2 \int_0^t G(f_s) ds, \quad t \in [0, T],$$ is a $G$-martingale.
With respect to a linear expectation, if $X$ is a continuous martingale with finite variation, then $X$ is a constant.

But it is not true in G-stochastic analysis.

Example

$\langle B \rangle_t - t$ is a continuous G-martingale with finite variation. But $\langle B \rangle_t - t$ is not a constant. It is a decreasing stochastic process.
Representation theorem of G-martingales

Special case of the martingale representation is the Lévy characterization theorem of Brownian motion.

- Recall: Lévy characterization theorem of Brownian motion.
- With respect to a linear expectation we have

**Lemma**

A process $M$ is a Brownian motion if

1. $M$ is continuous and $M_0 = 0$;
2. $M$ is a local martingale;
3. $M_t^2 - t$ is a local martingale.

**Lemma**

A process $M \in \mathcal{M}_G^2(0,T)$ is a G-Brownian motion with a parameter $0 < \sigma \leq 1$ if

1. $M$ is continuous and $M_0 = 0$;
2. $M$ is a symmetric G-martingale;
3. For any $t \geq 0$, $M_t^2 - t$ is a G-martingale;
4. For any $t \geq 0$, $\hat{\mathbb{E}}[-M_t^2] = -\sigma^2 t$.

**Remark**

In our framework, we do not need the assumption $M \in \mathcal{M}_G^2(0,T)$. 
Main Results — Representation of G-martingales

The following representation of G-martingales as stochastic integrals with respect to G-Brownian motion is the main result in this section.

**Theorem**

Let $0 < \sigma \leq 1$ and $f \in M^2_G(0, T)$ be such that $\mathbb{H}[\int_0^T |f_s|^4 ds] < \infty$. Moreover, if there exists a constant $C$ (small enough) such that $0 < C \leq |f|$ and the following hold

1. $M$ is a symmetric G-martingale and $M_0 = 0$;
2. $M_t^2 - \int_0^t f_s^2 ds$ and $-M_t^2 + \sigma^2 \int_0^t f_s^2 ds$ are G-martingales, for $t \in [0, T]$,

then there exists a G-Brownian motion $B$ such that $M_t = \int_0^t f_s dB_s$, for all $t \in [0, T]$. 


Thanks for your attention!