Progresses and Problems in Theory of Nonlinear Expectations and Applications to Finance

—with Robust Central Limit Theorem and G-Brownian Motion

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Ω: a given set
$\mathcal{H}$: a linear space of real valued functions defined on Ω, s.t.

- a) $c \in \mathcal{H}$ for each constant c,
- b) $X \in \mathcal{H} \implies |X| \in \mathcal{H}$

Definition

A **Sublinear expectation** $\mathbb{E}$ is a functional $\mathbb{E} : \mathcal{H} \to \mathbb{R}$ satisfying
(i) **Monotonicity:** $\mathbb{E}[X] \geq \mathbb{E}[Y]$ if $X \geq Y$.
(ii) **Constant preserving:** $\mathbb{E}[X + c] = \mathbb{E}[X] + c$, $c \in \mathbb{R}$.
(iii) **Sub-additivity:** For each $X, Y \in \mathcal{H}$,

$$\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y].$$

(iv) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$ for $\lambda \geq 0$.
$(\Omega, \mathcal{H}, \mathbb{E})$: a **sublinear expectation space**

(i)+(ii): $\mathbb{E}$ is called **nonlinear expectation**

Definition

Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be two nonlinear expectations defined on $(\Omega, \mathcal{H})$. $\mathbb{E}_1$ is said to be **dominated** by $\mathbb{E}_2$ if

$$\mathbb{E}_1[X] - \mathbb{E}_1[Y] \leq \mathbb{E}_2[X - Y] \quad \text{for } X, Y \in \mathcal{H}.$$

$(\Omega, \mathcal{H}, \mathbb{E})$ is assumed s.t. if $X_1, \cdots, X_n \in \mathcal{H}$ then $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{Lip}(\mathbb{R}^n)$ where

$$C_{Lip}(\mathbb{R}^n) := \{\varphi : \mathbb{R}^n \mapsto \mathbb{R} : |\varphi(x) - \varphi(y)| \leq C|x - y| \quad \forall x, y \in \mathbb{R}^n\}.$$

$\mathbb{R}^n$ valued vector $X = (X_1, \cdots, X_n) \in \mathcal{H}^n$.

Representation of a Sublinear Expectation

Theorem

Let $\mathbb{E}$ a sublinear functional defined on $(\Omega, \mathcal{H})$. Then $\exists$ a family of linear functionals $\{E_\theta : \theta \in \Theta\}$ on $(\Omega, \mathcal{H})$ s. t.

$$\mathbb{E}[X] = \max_{\theta \in \Theta} E_\theta[X] \quad \text{for } X \in \mathcal{H}$$

Furthermore, if $\mathbb{E}$ is a sublinear expectation, then each E_θ is a linear expectation.

Definition

Let $X \in \mathcal{H}^d$ be given in a nonlinear (resp. sublinear) expectation spaces $(\Omega, \mathcal{H}, \mathbb{E})$. Then

$$\mathbb{F}_X[\varphi] := \mathbb{E}[\varphi(X)] : C_{Lip}(\mathbb{R}^d) \mapsto \mathbb{R}$$

forms a nonlinear (resp. sublinear) expectation spaces on $(\mathbb{R}^n, C_{Lip}(\mathbb{R}^d))$. We call it the distribution of X under $\mathbb{E}$.

Definition

In a nonlinear expectation spaces $(\Omega, \mathcal{H}, \mathbb{E})$, X_1,$X_2 \in \mathcal{H}^n$ are called **identically distributed**, denoted by $X_1 \overset{d}{=} X_2$, if

$$\mathbb{E}_1[\varphi(X_1)] = \mathbb{E}_2[\varphi(X_2)] \quad \text{for } \varphi \in C_{Lip}(\mathbb{R}^n).$$

It is clear that $X_1 \overset{d}{=} X_2$ if and only if their distributions coincide. We say that the distribution of X_1 is stronger than that of X_2 if $\mathbb{E}_1[\varphi(X_1)] \geq \mathbb{E}_2[\varphi(X_2)]$, for each $\varphi \in C_{Lip}(\mathbb{R}^n)$.

Lemma

Let $(\Omega, \mathcal{H}, \mathbb{E})$ *be a sublinear expectation space. Let* $X \in \mathcal{H}^d$ *be given such that,* $\exists \delta > 0$, *such that* $\varphi(X)|X|^\delta \in \mathcal{H}$ *for each* $\varphi \in C_{Lip}(\mathbb{R}^d)$. *Then for each sequence* $\{\varphi_n\}_{n=1}^\infty \subset C_{Lip}(\mathbb{R}^d)$ *satisfying* $\varphi_n \downarrow 0$, *we have* $\mathbb{E}[\varphi_n(X)] \downarrow 0$.

Lemma

We assume the same condition as the above lemma. Then there exists a family of probability measures $\{F_\theta\}_{\theta\in\Theta}$ *defined on* $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ *such that*

$$\mathbb{F}_X[\varphi] = \sup_{\theta\in\Theta} \int_{\mathbb{R}^d} \varphi(x) F_\theta(dx), \quad \varphi \in C_{l,Lip}(\mathbb{R}^d).$$

The following notion of independence plays a key role in the nonlinear expectation theory.

Definition

In a nonlinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, a random vector $Y \in \mathcal{H}^n$ is said to be **independent** from another random vector $X \in \mathcal{H}^m$ under $\mathbb{E}[\cdot]$ if for each test function $\varphi \in C_{Lip}(\mathbb{R}^{m+n})$ we have

$$\mathbb{E}[\varphi(X, Y)] = \mathbb{E}[\mathbb{E}[\varphi(x, Y)]_{x=X}].$$

The independence property of two random vectors X, Y involves only the "joint distribution" of (X, Y). The following result tells us how to construct random vectors with given "marginal distributions" and with a specific direction of independence.

Definition.

Let $(\Omega_i, \mathcal{H}_i, \mathbb{E}_i)$, $i = 1, 2$ be two sublinear (resp. nonlinear) expectation spaces. We denote

$$\mathcal{H}_1 \otimes \mathcal{H}_2 := \{Z(\omega_1, \omega_2) = \varphi(X(\omega_1), Y(\omega_2)) : (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2,$$
$$(X, Y) \in \mathcal{H}_1^m \times \mathcal{H}_2^n,\ \varphi \in C_{Lip}(\mathbb{R}^{m+n})\},$$

and, for each random variable of the above form $Z(\omega_1, \omega_2) = \varphi(X(\omega_1), Y(\omega_2))$,

$$(\mathbb{E}_1 \otimes \mathbb{E}_2)[Z] := \mathbb{E}_1[\bar{\varphi}(X)], \text{ where } \bar{\varphi}(x) := \mathbb{E}_2[\varphi(x, Y)],\ x \in \mathbb{R}^m.$$

□

Definition (continue) .

$(\Omega_1 \times \Omega_2, \mathcal{H}_1 \otimes \mathcal{H}_2, \mathbb{E}_1 \otimes \mathbb{E}_2)$ forms a sublinear (resp. nonlinear) expectation space. We call it the **product space** of sublinear (resp. nonlinear) expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$. In this way, we can define the product space

$$(\prod_{i=1}^{n} \Omega_i, \bigotimes_{i=1}^{n} \mathcal{H}_i, \bigotimes_{i=1}^{n} \mathbb{E}_i)$$

of given sublinear (resp. nonlinear) expectation spaces $(\Omega_i, \mathcal{H}_i, \mathbb{E}_i)$, $i = 1, 2, \cdots, n$. In particular, when $(\Omega_i, \mathcal{H}_i, \mathbb{E}_i) = (\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ we have the product space of the form $(\Omega_1^n, \mathcal{H}_1^{\otimes n}, \mathbb{E}_1^{\otimes n})$. □

Let $X, \bar{X}$ be two n-dimensional random vectors on a sublinear (resp. nonlinear) expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. $\bar{X}$ is called an independent copy of X if $\bar{X} \overset{d}{=} X$ and $\bar{X}$ is independent from X.

Proposition.

Let X_i be an n_i-dimensional random vector on sublinear (resp. nonlinear) expectation space $(\Omega_i, \mathcal{H}_i, \mathbb{E}_i)$ for $i = 1, \cdots, n$, respectively. We denote

$$Y_i(\omega_1, \cdots, \omega_n) := X_i(\omega_i), \quad i = 1, \cdots, n.$$

Then Y_i, $i = 1, \cdots, n$, are random vectors on $(\prod_{i=1}^n \Omega_i, \bigotimes_{i=1}^n \mathcal{H}_i, \bigotimes_{i=1}^n \mathbb{E}_i)$. Moreover we have $Y_i \overset{d}{=} X_i$ and Y_{i+1} is independent from $(Y_1, \cdots, Y_i)$, for each i.

Furthermore, if $(\Omega_i, \mathcal{H}_i, \mathbb{E}_i) = (\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $X_i \overset{d}{=} X_1$, for all i, then we also have $Y_i \overset{d}{=} Y_1$. In this case Y_i is said to be an **independent copy** of Y_1 for $i = 2, \cdots, n$. □

Remark.

In the above construction the integer n can be also infinite. In this case each random variable $X \in \bigotimes_{i=1}^{\infty} \mathcal{H}_i$ belongs to $(\prod_{i=1}^{k} \Omega_i, \bigotimes_{i=1}^{k} \mathcal{H}_i, \bigotimes_{i=1}^{k} \mathbb{E}_i)$ for some positive integer $k < \infty$ and

$$\bigotimes_{i=1}^{\infty} \mathbb{E}_i[X] := \bigotimes_{i=1}^{k} \mathbb{E}_i[X].$$

□

Remark.

The situation "Y is independent from X" often appears when Y occurs after X, thus a robust expectation should take the information of X into account. □

Example

We consider a situation where two random variables X and Y in $\mathcal{H}$ are identically distributed and their common distribution is

$$\mathbb{F}_X[\varphi] = \mathbb{F}_Y[\varphi] = \sup_{\theta\in\Theta} \int_{\mathbb{R}} \varphi(y) F(\theta, dy) \quad \text{for } \varphi \in C_{Lip}(\mathbb{R}),$$

where for each $\theta \in \Theta$, $\{F(\theta, A)\}_{A\in\mathcal{B}(\mathbb{R})}$ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. In this case, "$Y$ is independent from X" means that the joint distribution of X and Y is

$$\mathbb{F}_{X,Y}[\psi] = \sup_{\theta_1\in\Theta} \int_{\mathbb{R}} \left[\sup_{\theta_2\in\Theta} \int_{\mathbb{R}} \psi(x, y) F(\theta_2, dy) \right] F(\theta_1, dx) \quad \text{for } \psi \in C_{Lip}(\mathbb{R}^2).$$

Law of Large Numbers and Central Limit Theorem

We present the law of large numbers (LLN) and central limit theorem (CLT) under sublinear expectations.

Theorem

(**Law of large numbers**) *Let* $\{Y_i\}_{i=1}^{\infty}$ *be a sequence of* $\mathbb{R}^d$*-valued random variables on a sublinear expectation space* $(\Omega, \mathcal{H}, \mathbb{E})$*. We assume that* $Y_{i+1} \overset{d}{=} Y_i$ *and* Y_{i+1} *is indep. from* $\{Y_1, \cdots, Y_i\}$ *for each* $i = 1, 2, \cdots$. *Then* $\bar{S}_n = \frac{1}{n}\sum_{i=1}^n Y_i$ *converges in law to a maximal distribution:*

$$\lim_{n\to\infty} \mathbb{E}[\varphi(\bar{S}_n)] = \mathbb{E}[\varphi(\eta)] = \max_{\underline{\mu}\le v\le\overline{\mu}} \varphi(v),$$

$$\text{with } \overline{\mu} = \mathbb{E}[X_1], \ \ \underline{\mu} = -\mathbb{E}[-X_1].$$

"worst case risk measure"—**maximal distribution**.

Definition

(**maximal distribution**) A random variable η in $(\Omega, \mathcal{H}, \mathbb{E})$ is called **maximal distributed,** denoted by $\eta \overset{d}{=} M([\underline{\mu}, \overline{\mu}])$, if

$$\mathbb{E}[\varphi(\eta)] = \max_{\underline{\mu} \leq y \leq \overline{\mu}} \varphi(y) \text{ with } \underline{\mu} = \mathbb{E}[\eta], \;\; \overline{\mu} = -\mathbb{E}[-\eta].$$

Remark.

In general a maximal distributed η satisfies

$$a\eta + b\bar{\eta} \overset{d}{=} (a+b)\eta \quad \text{for } a, b \geq 0,$$

where $\bar{\eta}$ is an independent copy of η. □

Central limit theorem with zero-mean.

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of $\mathbb{R}^d$-valued random variables on a sublinear $(\Omega, \mathcal{H}, \mathbb{E})$. We assume that $X_{i+1} \overset{d}{=} X_i$ and X_{i+1} is independent from $\{X_1, \cdots, X_i\}$ for each $i = 1, 2, \cdots$. We further assume that

$$\mathbb{E}[X_1] = \mathbb{E}[-X_1] = 0.$$

Then we have convergence in law:

$$\lim_{n\to\infty} \mathbb{E}[\varphi(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i)] = \mathbb{E}[\varphi(X)], \quad X \overset{d}{=} N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$$

where

$$\overline{\sigma}^2 = \mathbb{E}[X_1^2], \quad \underline{\sigma}^2 = -\mathbb{E}[-X_1^2].$$

□

Definition

(G-**normal distribution**) A d-dimensional random variable X on a sublinear $(\Omega, \mathcal{H}, \mathbb{E})$ is called (centralized) G-**normal distributed** if and

$$aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2} X \quad \text{for } a, b \geq 0,$$

where $\bar{X}$ is an independent copy of X. We denote $X \overset{d}{=} N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$, with $\overline{\sigma}^2 = \mathbb{E}[X^2]$ and $\underline{\sigma}^2 = -\mathbb{E}[-X]$ ($\mathbb{E}[X] = -\mathbb{E}[-X] = 0$).

If $X \overset{d}{=} N(0, [\underline{\sigma}^2, \overline{\sigma}^2]$, then

- For each **convex** function φ, we have

$$\mathbb{E}[\varphi(X)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\overline{\sigma}^2 y) \exp(-\frac{y^2}{2}) dy.$$

- For each **concave** function φ, we have

$$\mathbb{E}[\varphi(X)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\underline{\sigma}^2 y) \exp(-\frac{y^2}{2}) dy.$$

Remark.

When $d = 1$, the sequence $\{\bar{S}_n\}_{n=1}^{\infty}$ converges in law to $N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$, where $\overline{\sigma}^2 = \mathbb{E}[X_1^2]$ and $\underline{\sigma}^2 = -\mathbb{E}[-X_1^2]$. In particular, if $\overline{\sigma}^2 = \underline{\sigma}^2$, then it becomes a classical central limit theorem. □

Central Limit Theorem with law of large numbers.

Let $\{(X_i, Y_i)\}_{i=1}^{\infty}$ be a sequence of $2d$-valued r.v. in a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Assume that $(X_{i+1}, Y_{i+1}) \overset{d}{=} (X_i, Y_i)$ and (X_{i+1}, Y_{i+1}) is indep. of $\{(X_1, Y_1), \cdots, (X_i, Y_i)\}$. We further assume that $\mathbb{E}[X_1] = \mathbb{E}[-X_1] = 0$. Then

$$\lim_{n\to\infty} \mathbb{E}[\varphi(\sum_{i=1}^{n}(\frac{X_i}{\sqrt{n}} + \frac{Y_i}{n}))] = \mathbb{E}[\varphi(X + \eta)],$$

for all functions $\varphi \in C_{Lip}(\mathbb{R}^d)$, where the pair (X, η) is G-distributed. □

Definition

The pair (X, η) is called G-**distributed**. It satisfies

$$(aX + b\bar{X}, a^2\eta + b^2\bar{\eta}) \overset{d}{=} (\sqrt{a^2 + b^2}X, (a^2 + b^2)\eta), \quad \text{for } a, b \geq 0,$$

where $(\bar{X}, \bar{\eta})$ is an independent copy of (X, η). Thus X is G-normal and η is maximal distributed.

$$G(p, A) := \mathbb{E}[\langle p, \eta \rangle + \frac{1}{2} \langle AX, X \rangle], \quad p \in \mathbb{R}^d, \quad A \in \mathbb{S}(d).$$

$\mathbb{S}(d)$ is the collection of all $d \times d$ symmetric matrices.

Definition

Remark.

$G(p,X)$ satisfies

$$\begin{cases} G(p+\bar{p},A+\bar{A}) & \leq G(p,A)+G(\bar{p},\bar{A}), \\ G(\lambda p,\lambda A) & =\lambda G(p,A), \quad \forall \lambda \geq 0, \\ G(p,A) & \geq G(p,\bar{A}), \quad \text{if } A\geq \bar{A}. \end{cases}$$

□

The above pair (X, η) is characterized via

$$u(t,x,y) = \mathbb{E}[\varphi(x + \sqrt{t}X, y + t\eta)], \quad x, y \in \mathbb{R}^d, \ t \geq 0,$$

by the following parabolic partial differential equation (PDE) defined on $[0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$:

$$\partial_t u - G(D_y u, D_x^2 u) = 0, \quad u|_{t=0} = \varphi.$$

If $d = 1$, then $X \overset{d}{=} N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$, $\eta \overset{d}{=} M([\underline{\mu}, \overline{\mu}^2])$

Corollary

If both (X, η) *and* $(\bar{X}, \bar{\eta})$ *are* G*-distributed with the same* G*, i.e.,*

$$G(p, A) := \mathbb{E}[\frac{1}{2}\langle AX, X\rangle + \langle p, \eta\rangle] = \mathbb{E}[\frac{1}{2}\langle A\bar{X}, \bar{X}\rangle + \langle p, \bar{\eta}\rangle] \quad \forall (p, A),$$

then $(X, \eta) \overset{d}{=} (\bar{X}, \bar{\eta})$*. In particular,* $X \overset{d}{=} -X$.

Problem

How to estimate $[\underline{\mu}, \overline{\mu}]$ and $[\underline{\sigma}^2, \overline{\sigma}^2]$ with a given data $\{x_i\}_{i=0}^N$? More generaly, How to determine $G(p, A)$? How to use the above LLN and CLT to do statistics?

- [Zengjing Chen 2009]:

$$\hat{c}(\liminf \frac{1}{n}\sum_{i=1}^{n} Y_i < \underline{\mu}) = 0,$$
$$\hat{c}(\limsup \frac{1}{n}\sum_{i=1}^{n} Y_i > \overline{\mu}) = 0.$$

- [Peng2009] Data based sublinear mean

$$\tilde{\mathbb{E}}[\psi(Y)] := \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \psi(y_i)$$

- [Peng2009] Data based sublinear mean

$$\tilde{\mathbb{E}}[\psi(Y)] := \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\psi(y_i)$$

- In risk control: try to use $N(0,[\underline{\sigma}^2,\overline{\sigma}^2])$-model.

Notes and Comments

The contents of this chapter are mainly from Peng (2008) [66] (see also Peng (2007) [62]).

The notion of G-normal distribution was firstly introduced by Peng (2006) [61] for 1-dimensional case, and Peng (2008) [65] for multi-dimensional case. In the classical situation, a distribution satisfying equation (–ch2e1) is said to be stable (see Lévy (1925) [43] and (1965) [44]). In this sense, our G-normal distribution can be considered as the most typical stable distribution under the framework of sublinear expectations.

Marinacci (1999) [47] used different notions of distributions and independence via capacity and the corresponding Choquet expectation to obtain a law of large numbers and a central limit theorem for non-additive probabilities (see also Maccheroni and Marinacci (2005) [48]). But since a sublinear expectation can not be characterized by the corresponding capacity, our results can not be derived from theirs. In fact, our results show that the limit in CLT, under uncertainty, is a G-normal distribution in which the distribution uncertainty is not just the parameter of the classical normal distributions (see Exercise –exxee1).

The notion of viscosity solutions plays a basic role in the definition and properties of G-normal distribution and maximal distribution. This notion was initially introduced by Crandall and Lions (1983) [15]. This is a fundamentally important notion in the theory of nonlinear parabolic and elliptic PDEs. Readers are referred to Crandall, Ishii and Lions (1992) [16] for rich references of the beautiful and powerful theory of viscosity solutions. For books on the theory of viscosity solutions and the related HJB equations, see Barles (1994) [6], Fleming and Soner (1992) [26] as well as Yong and Zhou (1999) [?].

We note that, for the case when the uniform elliptic condition holds, the viscosity solution (–e320) becomes a classical $C^{1+\frac{\alpha}{2},2+\alpha}$-solution (see Krylov (1987) [42] and the recent works in Cabre and Caffarelli (1997) [**?**] and Wang (1992) [73]). In 1-dimensional situation, when $\underline{\sigma}^2 > 0$, the G-equation becomes the following Barenblatt equation:

$$\partial_t u + \gamma|\partial_t u| = \triangle u, \ |\gamma| < 1.$$

This equation was first introduced by Barenblatt (1979) [5] (see also Avellaneda, Levy and Paras (1995) [4]).

Definition

A d-dimensional process $(B_t)_{t\geq 0}=(B_t^1,\cdots,B_t^d)_{t\geq 0}$ defined on a sublinear expectation space $(\Omega,\mathcal{H},\mathbb{E})$ is called a G**–Brownian motion** if $|B_t^i|^k\in\mathcal{H}$, for $i=1,\cdots,d$ and $k=1,2,3$, and

(i) $B_0(\omega)=0$;

(ii) For each $t,s\geq 0$, $B_{t+s}-B_t\overset{d}{=}B_s$ and $B_{t+s}-B_t$ is independent from $(B_{t_1},B_{t_2},\cdots,B_{t_n})$, for each $n\in\mathbb{N}$ and $0\leq t_1\leq\cdots\leq t_n\leq t$.

(iii) $\lim_{t\downarrow 0}\mathbb{E}[|B_t|^3]t^{-1}=0$.

(iv) $\mathbb{E}[B_t]=\mathbb{E}[-B_t]=0$.

The following theorem gives a characterization of G-Brownian motion's distribution.

Theorem

Let $(B_t)_{t\geq 0} = (B_t^1, \cdots, B_t^d)_{t\geq 0}$ be a d-dimensional process defined on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Then $B_s/\sqrt{s} \overset{d}{=} B_1$ is G-normally distributed with

$$G(A) := \mathbb{E}[\frac{1}{2}\langle AB_1, B_1\rangle], \quad A \in \mathbb{S}(d).$$

$\Omega =: C_0^d(\mathbb{R}^+)$ the space of all $\mathbb{R}^d$–valued continuous paths $(\omega_t)_{t\in\mathbb{R}^+}$, with $\omega_0 = 0$, equipped with

$$\rho(\omega^1,\omega^2) := \sum_{i=1}^{\infty} 2^{-i}[(\max_{t\in[0,i]} |\omega_t^1 - \omega_t^2|) \wedge 1].$$

We set $\Omega_T := \{\omega_{\cdot\wedge T} : \omega \in \Omega\}$. We will consider the canonical process $B_t(\omega) = \omega_t$, $t \in [0,\infty)$, for $\omega \in \Omega$.

For each fixed $T \in [0, \infty)$, we set

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1 \wedge T}, \cdots, B_{t_n \wedge T}) : n \in \mathbb{N}, t_1, \cdots, t_n \in [0, \infty), \varphi \in C_{Lip}(\mathbb{R}$$

It is clear that $L_{ip}(\Omega_t) \subseteq L_{ip}(\Omega_T)$, for $t \leq T$. We also set

$$L_{ip}(\Omega) := \bigcup_{n=1}^{\infty} L_{ip}(\Omega_n).$$

Let $(\xi_i)_{i=1}^{\infty}$ $\xi_i \in \widetilde{\mathcal{H}}$ is a sublinear expectation space $(\widetilde{\Omega}, \widetilde{\mathcal{H}}, \widetilde{\mathbb{E}})$ such that ξ_i is G-normal distributed for a given function G :, and ξ_{i+1} is independent from $(\xi_1, \cdots, \xi_i)$ for each $i = 1, 2, \cdots$.

We now introduce a sublinear expectation $\hat{\mathbb{E}}$ defined on $L_{ip}(\Omega)$ via the following procedure: for each $X \in L_{ip}(\Omega)$ with

$$X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})$$

for some $\varphi \in C_{Lip}(\mathbb{R}^{d\times n})$ and $0 = t_0 < t_1 < \cdots < t_n < \infty$, we set

$$\hat{\mathbb{E}}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})]$$

$$:= \widetilde{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0}\xi_1, \cdots, \sqrt{t_n - t_{n-1}}\xi_n)].$$

The related conditional expectation of $X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})$ under Ω_{t_j} is defined by

$$\hat{\mathbb{E}}[X|\Omega_{t_j}] = \hat{\mathbb{E}}[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})|\Omega_{t_j}]$$
$$:= \widetilde{\mathbb{E}}[\varphi(x_1, \cdots, x_j, \sqrt{t_{j+1} - t_j}\xi_{j+1}, \cdots, \sqrt{t_n - t_{n-1}}\xi_n)] \Big|_{\substack{x_1 = B_{t_1} \\ \vdots \\ x_j = B_{t_j} - B_{t_{j-1}}}}$$

It is easy to check that $\hat{\mathbb{E}}[\cdot]$ consistently defines a sublinear expectation on $L_{ip}(\Omega)$ and $(B_t)_{t\geq 0}$ is a G-Brownian motion. Since $L_{ip}(\Omega_T) \subseteq L_{ip}(\Omega)$, $\hat{\mathbb{E}}[\cdot]$ is also a sublinear expectation on $L_{ip}(\Omega_T)$.

Definition

The sublinear expectation $\hat{\mathbb{E}}[\cdot]$: $L_{ip}(\Omega) \to \mathbb{R}$ defined through the above procedure is called a G–**expectation**. The corresponding canonical process $(B_t)_{t\geq 0}$ on the sublinear expectation space $(\Omega, L_{ip}(\Omega), \hat{\mathbb{E}})$ is called a G–Brownian motion.

In the rest of this book, when we talk about G–Brownian motion, we mean that the canonical process $(B_t)_{t\geq 0}$ is under G-expectation.

Proposition.

We list the properties of $\hat{\mathbb{E}}[\cdot|\Omega_t]$ that hold for each $X, Y \in L_{ip}(\Omega)$:

(i) If $X \geq Y$, then $\hat{\mathbb{E}}[X|\Omega_t] \geq \hat{\mathbb{E}}[Y|\Omega_t]$.

(ii) $\hat{\mathbb{E}}[\eta|\Omega_t] = \eta$, for each $t \in [0, \infty)$ and $\eta \in L_{ip}(\Omega_t)$.

(iii) $\hat{\mathbb{E}}[X|\Omega_t] - \hat{\mathbb{E}}[Y|\Omega_t] \leq \hat{\mathbb{E}}[X - Y|\Omega_t]$.

(iv) $\hat{\mathbb{E}}[\eta X|\Omega_t] = \eta^+ \hat{\mathbb{E}}[X|\Omega_t] + \eta^- \hat{\mathbb{E}}[-X|\Omega_t]$ for each $\eta \in L_{ip}(\Omega_t)$.

(v)

$\hat{\mathbb{E}}[\hat{\mathbb{E}}[X|\Omega_t]|\Omega_s] = \hat{\mathbb{E}}[X|\Omega_{t\wedge s}]$, in particular, $\hat{\mathbb{E}}[\hat{\mathbb{E}}[X|\Omega_t]] = \hat{\mathbb{E}}[X]$. □

Proposition. (continue) .

For each $X \in L_{ip}(\Omega^t)$, $\hat{\mathbb{E}}[X|\Omega_t] = \hat{\mathbb{E}}[X]$, where $L_{ip}(\Omega^t)$ is the linear space of random variables with the form

$$\varphi(B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \cdots, B_{t_{n+1}} - B_{t_n}),$$
$$n = 1, 2, \cdots, \ \varphi \in C_{Lip}(\mathbb{R}^{d \times n}), \ t_1, \cdots, t_n, t_{n+1} \in [t, \infty).$$

□

Remark.

(ii) and (iii) imply

$$\hat{\mathbb{E}}[X + \eta|\Omega_t] = \hat{\mathbb{E}}[X|\Omega_t] + \eta \quad \text{for } \eta \in L_{ip}(\Omega_t).$$

□

We now consider the completion of sublinear expectation space $(\Omega, L_{ip}(\Omega), \hat{\mathbb{E}})$.

We denote by $L_G^p(\Omega)$, $p \geq 1$, the completion of $L_{ip}(\Omega)$ under the norm $\|X\|_p := (\hat{\mathbb{E}}[|X|^p])^{1/p}$. Similarly, we can define $L_G^p(\Omega_T)$, $L_G^p(\Omega_T^t)$ and $L_G^p(\Omega^t)$. It is clear that for each $0 \leq t \leq T < \infty$, $L_G^p(\Omega_t) \subseteq L_G^p(\Omega_T) \subseteq L_G^p(\Omega)$.

$\hat{\mathbb{E}}[\cdot]$ can be continuously extended to a sublinear expectation on $(\Omega, L_G^1(\Omega))$. For each fixed $t \leq T < \infty$, the conditional G-expectation $\hat{\mathbb{E}}[\cdot|\Omega_t] : L_{ip}(\Omega_T) \to L_{ip}(\Omega_t)$ is a continuous mapping under $\|\cdot\|$. Indeed, we have

$$\hat{\mathbb{E}}[X|\Omega_t] - \hat{\mathbb{E}}[Y|\Omega_t] \leq \hat{\mathbb{E}}[X - Y|\Omega_t] \leq \hat{\mathbb{E}}[|X - Y||\Omega_t],$$

then

$$|\hat{\mathbb{E}}[X|\Omega_t] - \hat{\mathbb{E}}[Y|\Omega_t]| \leq \hat{\mathbb{E}}[|X - Y||\Omega_t].$$

We thus obtain

$$\left\|\hat{\mathbb{E}}[X|\Omega_t] - \hat{\mathbb{E}}[Y|\Omega_t]\right\| \leq \|X - Y\|.$$

It follows that $\hat{\mathbb{E}}[\cdot|\Omega_t]$ can be also extended as a continuous mapping

$$\hat{\mathbb{E}}[\cdot|\Omega_t] : L_G^1(\Omega_T) \to L_G^1(\Omega_t).$$

If the above T is not fixed, then we can obtain
$\hat{\mathbb{E}}[\cdot|\Omega_t] : L_G^1(\Omega) \to L_G^1(\Omega_t)$.

Remark.

The above proposition also holds for $X, Y \in L_G^1(\Omega)$. But in (iv), $\eta \in L_G^1(\Omega_t)$ should be bounded, since $X, Y \in L_G^1(\Omega)$ does not imply $X \cdot Y \in L_G^1(\Omega)$. □

In particular, we have the following independence:

$$\hat{\mathbb{E}}[X|\Omega_t] = \hat{\mathbb{E}}[X], \ \ \forall X \in L_G^1(\Omega^t).$$

We give the following definition similar to the classical one:

Definition

An n-dimensional random vector $Y \in (L_G^1(\Omega))^n$ is said to be independent from Ω_t for some given t if for each $\varphi \in C_{b.Lip}(\mathbb{R}^n)$ we have

$$\hat{\mathbb{E}}[\varphi(Y)|\Omega_t] = \hat{\mathbb{E}}[\varphi(Y)].$$

Remark.

Just as in the classical situation, the increments of G–Brownian motion $(B_{t+s} - B_t)_{s \geq 0}$ are independent from Ω_t. □

The following property is very useful.

Proposition.

Let $X, Y \in L_G^1(\Omega)$ be such that $\hat{\mathbb{E}}[Y|\Omega_t] = -\hat{\mathbb{E}}[-Y|\Omega_t]$, for some $t \in [0, T]$. Then we have

$$\hat{\mathbb{E}}[X + Y|\Omega_t] = \hat{\mathbb{E}}[X|\Omega_t] + \hat{\mathbb{E}}[Y|\Omega_t].$$

In particular, if $\hat{\mathbb{E}}[Y|\Omega_t] = \hat{\mathbb{E}}_G[-Y|\Omega_t] = 0$, then $\hat{\mathbb{E}}[X + Y|\Omega_t] = \hat{\mathbb{E}}[X|\Omega_t]$. □

Proof.

This follows from the following two inequalities:

$$\hat{\mathbb{E}}[X+Y|\Omega_t] \leq \hat{\mathbb{E}}[X|\Omega_t] + \hat{\mathbb{E}}[Y|\Omega_t],$$
$$\hat{\mathbb{E}}[X+Y|\Omega_t] \geq \hat{\mathbb{E}}[X|\Omega_t] - \hat{\mathbb{E}}[-Y|\Omega_t] = \hat{\mathbb{E}}[X|\Omega_t] + \hat{\mathbb{E}}[Y|\Omega_t].$$

□

Now, we give the representation of G-expectation.

Theorem

([Denis,Hu,Peng2009], [Hu-Peng2009]) For each continuous monotonic and sublinear function $G : \mathbb{S}(d) \to \mathbb{R}$*, let* $\hat{\mathbb{E}}$ *be the corresponding* G*-expectation on* $(\Omega, L_{ip}(\Omega))$ *and* $B_t(\omega) = \omega_t$ *be the* G*-BM. Then there exists a weakly compact family of probability measures* $\mathcal{P}$ *on* $(\Omega, \mathcal{B}(\Omega))$ *such that*

$$\hat{\mathbb{E}}[X] = \max_{P \in \mathcal{P}} E_P[X] \quad \text{for } X \in L_{ip}(\Omega).$$

Moreover $L_G^p(\Omega)$ *is a strict subset*

$$\mathbb{L}^p(\Omega) = \{X \in L^0(\Omega, \mathcal{B}(\Omega)),\ \|X\|_p = \max_{P \in \mathcal{P}} (E_P[|X|^p])^{1/p} < \infty\}.$$

Let $p \geq 1$ be fixed. We consider the following type of simple processes: for a given partition $\pi_T = \{t_0, \cdots, t_N\}$ of $[0, T]$ we set

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) \mathbf{I}_{[t_k, t_{k+1})}(t),$$

where $\xi_k \in L_G^p(\Omega_{t_k})$, $k = 0, 1, 2, \cdots, N-1$ are given. The collection of these processes is denoted by $M_G^{p,0}(0, T)$.

Definition

We denote by $M_G^p(0,T)$ the completion of $M_G^{p,0}(0,T)$ under the norm

$$\|\eta\|_{M_G^p(0,T)} := \left\{\hat{\mathbb{E}}[\int_0^T |\eta_t|^p dt]\right\}^{1/p}.$$

It is clear that $M_G^p(0,T) \supset M_G^q(0,T)$ for $1 \leq p \leq q$. We also use $M_G^p(0,T;\mathbb{R}^n)$ for all n-dimensional stochastic processes $\eta_t = (\eta_t^1, \cdots, \eta_t^n)$, $t \geq 0$ with $\eta_t^i \in M_G^p(0,T)$, $i = 1, 2, \cdots, n$.

We now give the definition of Itô's integral. For simplicity, we first introduce Itô's integral with respect to 1-dimensional G–Brownian motion. Let $(B_t)_{t\geq 0}$ be a 1-dimensional G–Brownian motion with $G(\alpha) = \frac{1}{2}(\bar{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-)$, where $0 \leq \underline{\sigma} \leq \bar{\sigma} < \infty$.

Definition

For each $\eta \in M_G^{2,0}(0,T)$ of the form

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathbf{I}_{[t_j, t_{j+1})}(t),$$

we define

$$I(\eta) = \int_0^T \eta_t dB_t := \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}).$$

Lemma

From Example –eee1, for each j,

$$\hat{\mathbb{E}}[\int_0^T \eta_t dB_t|\Omega_s] = \int_0^s \eta_t dB_t,$$

$$\hat{\mathbb{E}}[\int_0^T \eta_t dB_t|\Omega_0] = \hat{\mathbb{E}}[\int_0^T \eta_t dB_t] = 0,$$

$$\hat{\mathbb{E}}[\left(\int_0^T \eta_t dB_t\right)^2] \le \bar{\sigma}^2 \hat{\mathbb{E}}[\int_0^T \eta_t^2 dt].$$

Definition

We define, for a fixed $\eta \in M_G^2(0, T)$, the stochastic integral

$$\int_0^T \eta_t dB_t := I(\eta).$$

Proposition.

Let $\eta, \theta \in M_G^2(0, T)$ and let $0 \leq s \leq r \leq t \leq T$. Then we have
(i) $\int_s^t \eta_u dB_u = \int_s^r \eta_u dB_u + \int_r^t \eta_u dB_u$.
(ii) $\int_s^t (\alpha \eta_u + \theta_u) dB_u = \alpha \int_s^t \eta_u dB_u + \int_s^t \theta_u dB_u$, if α is bounded and in $L_G^1(\Omega_s)$. (iii) $\hat{\mathbb{E}}[X + \int_r^T \eta_u dB_u | \Omega_s] = \hat{\mathbb{E}}[X|\Omega_s]$ for $X \in L_G^1(\Omega)$. □

Similar to 1-dimensional case, we can define Itô's integral by

$$I(\eta) := \int_0^T \eta_t dB_t^{\mathbf{a}}, \quad \text{for } \eta \in M_G^2(0,T).$$

We still have, for each $\eta \in M_G^2(0,T)$,

$$\hat{\mathbb{E}}[\int_0^T \eta_t dB_t^{\mathbf{a}}] = 0,$$
$$\hat{\mathbb{E}}[(\int_0^T \eta_t dB_t^{\mathbf{a}})^2] \leq \sigma_{\mathbf{aa}^T}^2 \hat{\mathbb{E}}[\int_0^T \eta_t^2 dt].$$

Furthermore, the above properties still hold for the integra.

quadratic variation process

$$\langle B \rangle_t := \lim_{\mu(\pi_t^N)\to 0} \sum_{j=0}^{N-1} (B_{t_{j+1}^N} - B_{t_j^N})^2 = B_t^2 - 2\int_0^t B_s dB_s.$$

By the above construction, $(\langle B\rangle_t)_{t\geq 0}$ is an increasing process with $\langle B\rangle_0 = 0$. We call it the of the G–Brownian motion B. It characterizes the part of statistic uncertainty of G–Brownian motion. It is important to keep in mind that $\langle B\rangle_t$ is not a deterministic process unless $\underline{\sigma} = \bar{\sigma}$, i.e., when $(B_t)_{t\geq 0}$ is a classical Brownian motion. In fact we have the following lemma.

Lemma

For each $0 \leq s \leq t < \infty$, *we have*

$$\hat{\mathbb{E}}[\langle B\rangle_t - \langle B\rangle_s |\Omega_s] = \bar{\sigma}^2(t-s),$$
$$\hat{\mathbb{E}}[-(\langle B\rangle_t - \langle B\rangle_s)|\Omega_s] = -\underline{\sigma}^2(t-s).$$

A very interesting point of the quadratic variation process $\langle B \rangle$ is, just like the G–Brownian motion B itself, the increment $\langle B \rangle_{s+t} - \langle B \rangle_s$ is independent from Ω_s and identically distributed with $\langle B \rangle_t$. In fact we have

Lemma

For each fixed $s,t \geq 0$, $\langle B \rangle_{s+t} - \langle B \rangle_s$ *is identically distributed with* $\langle B \rangle_t$ *and independent from* $\langle B \rangle_{t_1}, \cdots, \langle B \rangle_{t_n}$, *for* $t_1, \cdots, t_n \in [0, T]$.

Lemma

We have
$$\hat{\mathbb{E}}[\langle B\rangle_t^2] \leq 10\bar{\sigma}^4 t^2.$$

Proposition.

Let $(b_t)_{t\geq 0}$ be a process on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ such that
(i) $b_0 = 0$;
(ii) For each $t, s \geq 0$, $b_{t+s} - b_t$ is identically distributed with b_s and independent from $(b_{t_1}, b_{t_2}, \cdots, b_{t_n})$ for each $n \in \mathbb{N}$ and $0 \leq t_1, \cdots, t_n \leq t$;
(iii) $\lim_{t\downarrow 0} \hat{\mathbb{E}}[b_t^2]t^{-1} = 0$.
Then b_t is maximal distributed with

$$\hat{\mathbb{E}}[\varphi(b_t)] = \max_{\underline{\mu}\leq v\leq\overline{\mu}} \varphi(vt)$$

$\overline{\mu} = \hat{\mathbb{E}}[b_1]$ and $\underline{\mu} = -\hat{\mathbb{E}}[-b_1]$. □

Theorem

$\langle B\rangle_t$ *is* $M([\underline{\sigma}^2 t, \bar{\sigma}^2 t])$*-distributed, i.e., for each* $\varphi \in C_{Lip}(\mathbb{R})$,

$$\hat{\mathbb{E}}[\varphi(\langle B\rangle_t)] = \sup_{\underline{\sigma}^2 \le v \le \bar{\sigma}^2} \varphi(vt).$$

Corollary

For each $0 \leq t \leq T < \infty$*, we have*

$$\underline{\sigma}^2(T-t) \leq \langle B \rangle_T - \langle B \rangle_t \leq \bar{\sigma}^2(T-t) \;\; in \; L_G^1(\Omega).$$

Corollary

We have, for each $t, s \geq 0$, $n \in \mathbb{N}$,

$$\hat{\mathbb{E}}[(\langle B\rangle_{t+s} - \langle B\rangle_s)^n|\Omega_s] = \hat{\mathbb{E}}[\langle B\rangle_t^n] = \bar{\sigma}^{2n}t^n$$

and

$$\hat{\mathbb{E}}[-(\langle B\rangle_{t+s} - \langle B\rangle_s)^n|\Omega_s] = \hat{\mathbb{E}}[-\langle B\rangle_t^n] = -\underline{\sigma}^{2n}t^n.$$

We now consider a general form of G–Itô's formula. Consider

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \eta_s d\langle B\rangle_s + \int_0^t \beta_s dB_s.$$

Theorem

Let Φ be a C^2-function on $\mathbb{R}$ such that $\partial^2_{xx}\Phi$ satisfy polynomial growth condition for $\mu, \nu = 1, \cdots, n$. Let α, β and η be bounded processes in $M^2_G(0,T)$. Then for each $t \geq 0$ we have in $L^2_G(\Omega_t)$

$$\Phi(X_t) - \Phi(X_s) = \int_s^t \partial_{x^\nu}\Phi(X_u)\beta_u dB_u + \int_s^t \partial_x\Phi(X_u)\alpha_u du$$
$$+ \int_s^t [\partial_x\Phi(X_u)\eta_u + \frac{1}{2}\partial^2_{xx}\Phi(X_u)\beta_u^2] d\langle B^i, B^j\rangle_u .$$

Problem

How to work with τ-technique with stopping times τ? [Li-X.-P.2010], [Song2010].

Let $G : \mathbb{R}^d \times \mathbb{S}(d) \to \mathbb{R}$ be a given continuous sublinear function monotonic in $A \in \mathbb{S}(d)$. Then there exists a bounded, convex and closed subset $\Sigma \subset \mathbb{R}^d \times \mathbb{S}_+(d)$ such that

$$G(p, A) = \sup_{(q,B)\in\Sigma} [\frac{1}{2}\mathrm{tr}[AB] + \langle p, q\rangle] \quad \text{for } (p, A) \in \mathbb{R}^d \times \mathbb{S}(d).$$

There exists a pair of d-dimensional random vectors (X, Y) which is G-distributed.

We now give the definition of the generalized G-Brownian motion.

Definition

A d-dimensional process $(B_t)_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a **generalized G-Brownian motion** if the following properties are satisfied:
(i) $B_0(\omega) = 0$;
(ii) For each $t, s \geq 0$, the increment $B_{t+s} - B_t$ identically distributed with $\sqrt{s}X + sY$ and is independent from $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$, where (X, Y) is G-distributed.

The following theorem gives a characterization of the generalized G-Brownian motion.

Theorem

Let $(B_t)_{t\geq 0}$ be a d-dimensional process defined on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ such that
(i) $B_0(\omega) = 0$;
(ii) *For each $t, s \geq 0$, $B_{t+s} - B_t$ and B_s are identically distributed and $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$.*
(iii) $\lim_{t\downarrow 0} \hat{\mathbb{E}}[|B_t|^3]t^{-1} = 0$.
Then $(B_t)_{t\geq 0}$ is a generalized G-Brownian motion with
$G(p, A) = \lim_{\delta\downarrow 0} \hat{\mathbb{E}}[\langle p, B_\delta\rangle + \frac{1}{2}\langle AB_\delta, B_\delta\rangle]\delta^{-1}$ *for* $(p, A) \in \mathbb{R}^d \times \mathbb{S}(d)$.

We can also define a G-Brownian motion on a nonlinear expectation space $(\Omega, \mathcal{H}, \widetilde{\mathbb{E}})$.

Definition

A d-dimensional process $(B_t)_{t\geq 0}$ on a **nonlinear expectation** space $(\Omega, \mathcal{H}, \widetilde{\mathbb{E}})$ is called a (nonlinear) $\widetilde{G}$-**Brownian motion** if the following properties are satisfied:
(i) $B_0(\omega) = 0$;
(ii) For each $t, s \geq 0$, the increment $B_{t+s} - B_t$ identically distributed with B_s and is independent from $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$;
(iii) $\lim_{t\downarrow 0} \hat{\mathbb{E}}[|B_t|^3]t^{-1} = 0$.

The following theorem gives a characterization of the nonlinear $\widetilde{G}$-Brownian motion, and give us the generator $\widetilde{G}$ of our $\widetilde{G}$-Brownian motion.

Theorem

Let $\widetilde{\mathbb{E}}$ be a nonlinear expectation and $\hat{\mathbb{E}}$ be a sublinear expectation defined on $(\Omega, \mathcal{H})$. Let $\widetilde{\mathbb{E}}[X] - \widetilde{\mathbb{E}}[Y] \leq \hat{\mathbb{E}}[X - Y]$, $X, Y \in \mathcal{H}$. Let $(B_t, b_t)_{t\geq 0}$ be a given $\widetilde{G}$-Brownian motion on $(\Omega, \mathcal{H}, \widetilde{\mathbb{E}})$ such that $\hat{\mathbb{E}}[B_t] = \hat{\mathbb{E}}[-B_t] = 0$ and $\lim_{t\to 0} \hat{\mathbb{E}}[|b_t|^2]/t = 0$. Then, for $\varphi \in C_{b.Lip}(\mathbb{R}^{2d})$, the function

$$\tilde{u}(t,x,y) := \widetilde{\mathbb{E}}[\varphi(x + B_t, y + b_t)], \quad (t,x,y) \in [0,\infty) \times \mathbb{R}^{2d}$$

solves the PDE:

$$\partial_t \tilde{u} - \widetilde{G}(D_y \tilde{u}, D_x^2 \tilde{u}) = 0, \quad u|_{t=0} = \varphi.$$

where

$$\widetilde{G}(p,A) = \widetilde{\mathbb{E}}[\langle p, b_1\rangle + \frac{1}{2}\langle AB_1, B_1\rangle], \ (p,A) \in \mathbb{R}^d \times \mathbb{S}(d).$$

It is easy to check that $\hat{\mathbb{E}}[\cdot]$ (resp. $\widetilde{\mathbb{E}}$) consistently defines a sublinear (resp. nonlinear) expectation and $\widetilde{\mathbb{E}}[\cdot]$ on $(\Omega, L_{ip}(\Omega))$. Moreover $(B_t, b_t)_{t\geq 0}$ is a G-Brownian motion under $\hat{\mathbb{E}}$ and a $\widetilde{G}$-Brownian motion under $\widetilde{\mathbb{E}}$.

Proposition.

For each $X, Y \in L_{ip}(\Omega)$:
(i) If $X \geq Y$, then $\widetilde{\mathbb{E}}[X|\Omega_t] \geq \widetilde{\mathbb{E}}[Y|\Omega_t]$.
(ii) $\widetilde{\mathbb{E}}[X+\eta|\Omega_t] = \widetilde{\mathbb{E}}[X|\Omega_t] + \eta$, for each $t \geq 0$ and $\eta \in L_{ip}(\Omega_t)$.
(iii) $\widetilde{\mathbb{E}}[X|\Omega_t] - \widetilde{\mathbb{E}}[Y|\Omega_t] \leq \hat{\mathbb{E}}[X - Y|\Omega_t]$.
(iv) $\widetilde{\mathbb{E}}[\widetilde{\mathbb{E}}[X|\Omega_t]|\Omega_s] = \widetilde{\mathbb{E}}[X|\Omega_{t\wedge s}]$, in particular, $\widetilde{\mathbb{E}}[\widetilde{\mathbb{E}}[X|\Omega_t]] = \widetilde{\mathbb{E}}[X]$.
(v) For each $X \in L_{ip}(\Omega^t)$, $\widetilde{\mathbb{E}}[X|\Omega_t] = \widetilde{\mathbb{E}}[X]$, where $L_{ip}(\Omega^t)$ is the linear space of random variables with the form

$$\varphi(W_{t_2} - W_{t_1}, W_{t_3} - W_{t_2}, \cdots, W_{t_n} - W_{t_{n-1}}), \; t_1, \cdots, t_n \in [t, \infty).$$

□

Since $\hat{\mathbb{E}}$ can be considered as a special nonlinear expectation of $\widetilde{\mathbb{E}}$ dominated by its self, thus $\hat{\mathbb{E}}[\cdot|\Omega_t]$ also satisfies above properties (i)–(v).

Moreover

Proposition.

The conditional sublinear expectation $\hat{\mathbb{E}}[\cdot|\Omega_t]$ satisfies (i)-(v). Moreover $\hat{\mathbb{E}}[\cdot|\Omega_t]$ itself is sublinear, i.e.,

(vi) $\hat{\mathbb{E}}[X|\Omega_t] - \hat{\mathbb{E}}[Y|\Omega_t] \leq \hat{\mathbb{E}}[X - Y|\Omega_t]$, .

(vii) $\hat{\mathbb{E}}[\eta X|\Omega_t] = \eta^+\hat{\mathbb{E}}[X|\Omega_t] + \eta^-\hat{\mathbb{E}}[-X|\Omega_t]$ for each $\eta \in L_{ip}(\Omega_t)$.

□

We now consider the completion of sublinear expectation space $(\Omega, L_{ip}(\Omega), \hat{\mathbb{E}})$.

We denote by $L_G^p(\Omega)$, $p \geq 1$, the completion of $L_{ip}(\Omega)$ under the norm $\|X\|_p := (\hat{\mathbb{E}}[|X|^p])^{1/p}$. Similarly, we can define $L_G^p(\Omega_T)$, $L_G^p(\Omega_T^t)$ and $L_G^p(\Omega^t)$. It is clear that for each $0 \leq t \leq T < \infty$, $L_G^p(\Omega_t) \subseteq L_G^p(\Omega_T) \subseteq L_G^p(\Omega)$.

$\hat{\mathbb{E}}[\cdot]$ can be continuously extended to $(\Omega, L_G^1(\Omega))$. Moreover, since $\widetilde{\mathbb{E}}$ is dominated by $\hat{\mathbb{E}}$, thus $(\Omega, L_G^1(\Omega), \hat{\mathbb{E}})$ forms a sublinear expectation space and $(\Omega, L_G^1(\Omega), \widetilde{\mathbb{E}})$ forms a nonlinear expectation space.

We now consider the extension of conditional G-expectation. For each fixed $t \leq T$, the conditional G-expectation $\hat{\mathbb{E}}[\cdot|\Omega_t] : L_{ip}(\Omega_T) \to L_{ip}(\Omega_t)$ is a continuous mapping under $\|\cdot\|$. Indeed, we have

$$\widetilde{\mathbb{E}}[X|\Omega_t] - \widetilde{\mathbb{E}}[Y|\Omega_t] \leq \hat{\mathbb{E}}[X - Y|\Omega_t] \leq \hat{\mathbb{E}}[|X - Y||\Omega_t],$$

then

$$|\widetilde{\mathbb{E}}[X|\Omega_t] - \widetilde{\mathbb{E}}[Y|\Omega_t]| \leq \hat{\mathbb{E}}[|X - Y||\Omega_t].$$

We thus obtain

$$\left\|\widetilde{\mathbb{E}}[X|\Omega_t]-\widetilde{\mathbb{E}}[Y|\Omega_t]\right\| \leq \|X-Y\|.$$

It follows that $\widetilde{\mathbb{E}}[\cdot|\Omega_t]$ can be also extended as a continuous mapping

$$\widetilde{\mathbb{E}}[\cdot|\Omega_t]: L_G^1(\Omega_T) \to L_G^1(\Omega_t).$$

If the above T is not fixed, then we can obtain
$\widetilde{\mathbb{E}}[\cdot|\Omega_t]: L_G^1(\Omega) \to L_G^1(\Omega_t)$.

Remark.

The above proposition also holds for $X, Y \in L_G^1(\Omega)$. But in (iv), $\eta \in L_G^1(\Omega_t)$ should be bounded, since $X, Y \in L_G^1(\Omega)$ does not imply $X \cdot Y \in L_G^1(\Omega)$. □

Notes and Comments

Bachelier (1900) [**?**] proposed Brownian motion as a model for fluctuations of the stock market, Einstein (1905) [**?**] used Brownian motion to give experimental confirmation of the atomic theory, and Wiener (1923) [**?**] gave a mathematically rigorous construction of Brownian motion. Here we follow Kolmogorov's idea (1956) [40] to construct G-Brownian motion by introducing infinite dimensional function space and the corresponding family of infinite dimensional sublinear distributions, instead of linear distributions in [40].

The notions of G-Brownian motion and the related stochastic calculus of Itô's type were firstly introduced by Peng (2006) [61] for 1-dimensional case and then in (2008) [65] for multi-dimensional situation. It is very interesting that Denis and Martini (2006) [23] studied super-pricing of contingent claims under model uncertainty of volatility. They have introduced a norm on the space of continuous paths $\Omega = C([0, T])$ which corresponds to our L_G^2-norm and developed a stochastic integral. There is no notion of nonlinear expectation and the related nonlinear distribution, such as G-expectation, conditional G-expectation, the related G-normal distribution and the notion of independence in their paper. But on the other hand, powerful tools in capacity theory enable them to obtain pathwise results for random variables and stochastic processes through the language of "quasi-surely" (see e.g. Dellacherie (1972) [18], Dellacherie and Meyer (1978 and 1982) [19], Feyel and de La Pradelle (1989) [25]) in place of "almost surely" in classical probability theory.

The main motivations of G-Brownian motion were the pricing and risk measures under volatility uncertainty in financial markets (see Avellaneda, Levy and Paras (1995) [4] and Lyons (1995) [46]). It was well-known that under volatility uncertainty the corresponding uncertain probabilities are singular from each other. This causes a serious problem for the related path analysis to treat, e.g., path-dependent derivatives, under a classical probability space. Our G-Brownian motion provides a powerful tool to such type of problems.

Our new Itô's calculus for G-Brownian motion is of course inspired from Itô's groundbreaking work since 1942 [37] on stochastic integration, stochastic differential equations and stochastic calculus through interesting books cited in Chapter –ch5. Itô's formula given by Theorem –Thm6.5 is from Peng [61], [65]. Gao (2009)[30] proved a more general Itô's formula for G-Brownian motion. An interesting problem is: can we get an Itô's formula in which the conditions correspond the classical one? Recently Li and Peng have solved this problem in [45].

Using nonlinear Markovian semigroup known as Nisio's semigroup (see Nisio (1976) [50]), Peng (2005) [59] studied the processes with Markovian properties under a nonlinear expectation.

The Notion of G-martingales

We now give the notion of G–martingales.

Definition

A process $(M_t)_{t\geq 0}$ is called a G–**martingale** (respectively, G–**supermartingale**, G–**submartingale**) if for each $t \in [0,\infty)$, $M_t \in L_G^1(\Omega_t)$ and for each $s \in [0,t]$, we have

$$\hat{\mathbb{E}}[M_t|\Omega_s] = M_s \quad (\text{respectively, } \leq M_s, \ \geq M_s).$$

Example

For each fixed $X \in L_G^1(\Omega)$, it is clear that $(\hat{\mathbb{E}}[X|\Omega_t])_{t\geq 0}$ is a G–martingale.

Problem

Doob-Meyer decomposition: a G-supermartingale $(X_t)_{t\geq 0}$ can be decomposed as $X_t = M_t - A_t$.

(recall Doob-Meyer decoposition of a g-supermartingale [Peng 1999 PTRF].

Example

Both B_t and $-B_t$ are G–martingales. $\langle B \rangle_t - \overline{\sigma}^2 t$ is a G–martingale since

$$\begin{aligned}\hat{\mathbb{E}}[\langle B \rangle_t - \overline{\sigma}^2 t | \Omega_s] &= \hat{\mathbb{E}}[\langle B \rangle_s - \overline{\sigma}^2 t + (\langle B \rangle_t - \langle B \rangle_s) | \Omega_s] \\ &= \langle B \rangle_s - \overline{\sigma}^2 t + \hat{\mathbb{E}}[\langle B \rangle_t - \langle B \rangle_s] \\ &= \langle B \rangle_s - \overline{\sigma}^2 s.\end{aligned}$$

$-\langle B^{\mathbf{a}} \rangle_t + \underline{\sigma}^2 t$ is also a G–submartingale.

In general, we have the following important property.

Proposition.

Let $M_0 \in \mathbb{R}$, $\varphi = (\varphi^j)_{j=1}^d \in M_G^2(0,T;\mathbb{R}^d)$ and $\eta = (\eta^{ij})_{i,j=1}^d \in M_G^1(0,T;\mathbb{S}(d))$ be given and let

$$M_t = M_0 + \int_0^t \varphi_u^j dB_u^j + \int_0^t \eta_u^{ij} d\left\langle B^i, B^j\right\rangle_u - \int_0^t 2G(\eta_u)du \quad \text{for } t \in [0,T].$$

Then M is a G–martingale. □

Remark.

It is worth to mention that for a G–martingale M, in general, $-M$ is not a G–martingale. But in Proposition –ch5p1, when $\eta \equiv 0$, $-M$ is still a G–martingale. □

G-martingale representation theorem: is still a largely open problem.

- Xu and Zhang (2009,SPA), a martingale representation for 'symmetric' G-martingale process.

G-martingale representation theorem: is still a largely open problem.

- Xu and Zhang (2009,SPA), a martingale representation for 'symmetric' G-martingale process.
- More general case: [Soner, Touzi, Zhang (arxiv)], [Song (arxiv)].

Here we present the formulation of this G-martingale representation theorem under a very strong assumption.
$G:\mathbb{S}(d)\rightarrow\mathbb{R}$ satisfying,

$$G(A)-G(\bar{A})\geq\beta\mathrm{tr}[A-\bar{A}],\ \ \forall A,\bar{A}\in\mathbb{S}(d),\ A\geq\bar{A}.$$

Lemma

Let $\xi = \varphi(B_T - B_{t_1})$, $\varphi \in C_{b.Lip}(\mathbb{R}^d)$. *Then:*

$$\xi = \hat{\mathbb{E}}[\xi] + \int_{t_1}^{T} \langle \beta_t, dB_t \rangle + \int_{t_1}^{T} (\eta_t, d\langle B \rangle_t) - \int_{t_1}^{T} 2G(\eta_t) dt.$$

Proof.

$u(t,x) = \hat{\mathbb{E}}[\varphi(x + B_T - B_t)]$ solves:

$$\partial_t u + G(D^2 u) = 0 \quad (t,x) \in [0,T] \times \mathbb{R}^d, \ u(T,x) = \varphi(x).$$

Krylov's interior estimate, for $\varepsilon > 0$,

$$\|u\|_{C^{1+\alpha/2,2+\alpha}([0,T-\varepsilon]\times\mathbb{R}^d)} < \infty.$$

Itô's formula to $u(t, B_t - B_{t_1})$ on $[t_1, T-\varepsilon]$,

$$\begin{aligned} \xi &= \hat{\mathbb{E}}[\xi] + \int_{t_1}^T \partial_t u dt + \int_{t_1}^T \langle Du, dB_t \rangle + \frac{1}{2}\int_{t_1}^T (D^2 u, d\langle B \rangle_t) \\ &= \hat{\mathbb{E}}[\xi] + \int_{t_1}^T \langle Du, dB_t \rangle + \frac{1}{2}\int_{t_1}^T (D^2 u, d\langle B \rangle_t) - \int_{t_1}^T G(D^2 u) dt. \end{aligned}$$

□

The representation theorem of $\xi = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_N} - B_{t_{N-1}})$.

Theorem

Let $\xi = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_N} - B_{t_{N-1}})$, $\varphi \in C_{b.Lip}(\mathbb{R}^{d\times N})$, $0 \leq t_1 < t_2 < \cdots < t_N = T < \infty$. *Then:*

$$\xi = \hat{\mathbb{E}}[\xi] + \int_0^T \langle \beta_t, dB_t \rangle + \int_0^T (\eta_t, d\langle B \rangle_t) - \int_0^T 2G(\eta_t) dt.$$

Proof.

Case $\xi = \varphi(B_{t_1}, B_T - B_{t_1})$. $(x, y) \in \mathbb{R}^{2d}$,

$$u(t,x,y) = \hat{\mathbb{E}}[\varphi(x, y + B_T - B_t)];\ \varphi_1(x) = \hat{\mathbb{E}}[\varphi(x, B_T - B_{t_1})].$$

$\bar{\xi} := \varphi(x, B_T - B_{t_1})$. By the above Lemma,

$$\bar{\xi} = \varphi_1(x) + \int_{t_1}^T \langle D_y u(t, x, B_t - B_{t_1}), dB_t\rangle + \frac{1}{2}\int_{t_1}^T (D_y^2 u, d\langle B\rangle_t) - \int_{t_1}^T G(D_y^2 u) dt.$$

$x = B_{t_1}$:
$\xi = \varphi_1(B_{t_1}) + \int_{t_1}^T \langle \beta_t, dB_t\rangle + \frac{1}{2}\int_{t_1}^T (\eta_t, d\langle B\rangle_t) - \int_{t_1}^T G(\eta_t) dt.$ □

G-martingale representation theorem.

Theorem

Let $(M_t)_{t\in[0,T]}$ be a G-martingale with $M_T = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_N} - B_{t_{N-1}})$, $\varphi \in C_{b.Lip}(\mathbb{R}^{d\times N})$, $0 \le t_1 < t_2 < \cdots < t_N = T < \infty$. Then

$$M_t = \hat{\mathbb{E}}[M_T] + \int_0^t \langle \beta_s, dB_s\rangle + \int_0^t (\eta_s, d\langle B\rangle_s) - \int_0^t 2G(\eta_s)ds, \ t \le T.$$

Proof.

For M_T, by Theorem –ch5t1, we have

$$M_T = \hat{\mathbb{E}}[M_T] + \int_0^T \langle \beta_s, dB_s \rangle + \int_0^T (\eta_s, d\langle B \rangle_s) - \int_0^T 2G(\eta_s)ds.$$

Take $\hat{\mathbb{E}}[\cdot|\Omega_t]$ on both sides. □

Notes and Comments

This chapter is mainly from Peng (2007) [63].

Peng (1997) [54] introduced a filtration consistent (or time consistent, or dynamic) nonlinear expectation, called g-expectation, via BSDE, and then in (1999) [56] for some basic properties of the g-martingale such as nonlinear Doob-Meyer decomposition theorem, see also Briand, Coquet, Hu, Mémin and Peng (2000) [8], Chen, Kulperger and Jiang (2003) [**?**], Chen and Peng (1998) [**?**] and (2000) [10], Coquet, Hu, Mémin and Peng (2001) [**?**], and (2002) [13], Peng (1999) [56], (2004) [**?**], Peng and Xu (2003) [**?**], Rosazza (2006) [68].

Our conjecture is that all properties obtained for g-martingales must has its correspondence for G-martingale. But this conjecture is still far from being complete. Here we present some properties of G-martingales.

The problem G-martingale representation theorem has been raised as a problem in Peng (2007) [63]. In Section 2, we only give a result with very regular random variables. Some very interesting developments to this important problem can be found in Soner, Touzi and Zhang (2009) [69] and Song (2009) [71].

Under the framework of g-expectation, Chen, Kulperger and Jiang (2003) [**?**], Hu (2005) [**?**], Jiang and Chen (2004) [**?**] investigate the Jensen's inequality for g-expectation. Recently, Jia and Peng (2007) [39] introduced the notion of g-convex function and obtained many interesting properties. Certainly a G-convex function concerns fully nonlinear situations.

In this chapter, we denote by $\bar{M}_G^p(0,T;\mathbb{R}^n)$, $p \geq 1$, the completion of $M_G^{p,0}(0,T;\mathbb{R}^n)$ under the norm $(\int_0^T \hat{\mathbb{E}}[|\eta_t|^p]dt)^{1/p}$. It is not hard to prove that $\bar{M}_G^p(0,T;\mathbb{R}^n) \subseteq M_G^p(0,T;\mathbb{R}^n)$. We consider all the problems in the space $\bar{M}_G^p(0,T;\mathbb{R}^n)$, and the sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is fixed.

We consider the following SDE driven by a d-dimensional G-Brownian motion:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t h_{ij}(s, X_s)d\left\langle B^i, B^j \right\rangle_s + \int_0^t \sigma_j(s, X_s)dB_s^j.$$

$X_0 \in \mathbb{R}^n$: a given constant,
$b(\cdot, x)$, $h_{ij}(\cdot, x)$, $\sigma_j(\cdot, x) \in \bar{M}_G^2(0, T; \mathbb{R}^n)$: given s.t.

$$|\phi(t, x) - \phi(t, x')| \leq K|x - x'|, \quad \text{for } \phi = b, h_{ij}, \sigma_j$$

The solution is a process $X \in \bar{M}_G^2(0, T; \mathbb{R}^n)$ satisfying the SDE.

Theorem

There exists a unique solution $X \in \bar{M}_G^2(0, T; \mathbb{R}^n)$ *of the stochastic differential equation* (SDE).

We consider the following type of BSDE:

$$Y_t = \hat{\mathbb{E}}[\xi + \int_t^T f(s, Y_s)ds + \int_t^T h_{ij}(s, Y_s)d\left\langle B^i, B^j \right\rangle_s |\Omega_t], \quad t \in [0, T],$$

$\xi \in L^1_G(\Omega_T; \mathbb{R}^n)$: given, f, h_{ij} are given functions satisfying $f(\cdot, y)$, $h_{ij}(\cdot, y) \in \bar{M}^1_G(0, T; \mathbb{R}^n)$: given s.t.

$$|\phi(t, y) - \phi(t, y')| \leq K|y - y'|, \quad \phi = f, h_{ij}$$

The solution is a process $Y \in \bar{M}^1_G(0, T; \mathbb{R}^n)$ satisfying the above BSDE.

Theorem

There exists a unique solution $(Y_t)_{t\in[0,T]} \in \bar{M}_G^1(0,T;\mathbb{R}^n)$ *of the backward stochastic differential equation* (BSDE).

Problem

General BSDE

$$-dY_t = f(t, Y_t, Z_t, \eta_t)dt - Z_t dB_t - \eta_t d\langle B\rangle_t + 2G(\eta_t)dt$$
$$Y_T = \xi \in L_G^2(\Omega_T).$$

Consider the following SDE:

$$\begin{cases} dX_s^{t,\xi} = b(X_s^{t,\xi})ds + h_{ij}(X_s^{t,\xi})d\left\langle B^i, B^j\right\rangle_s + \sigma_j(X_s^{t,\xi})dB_s^j, \ s \in [t,T], \\ \ X_t^{t,\xi} = \xi, \end{cases}$$

$\xi \in L_G^2(\Omega_t;\mathbb{R}^n)$: given,
$b,\ h_{ij},\ \sigma_j : \mathbb{R}^n \to \mathbb{R}^n$ are given Lipschitz functions.

We then consider associated BSDE:

$$Y_s^{t,\xi} = \hat{\mathbb{E}}[\Phi(X_T^{t,\xi}) + \int_s^T f(X_r^{t,\xi}, Y_r^{t,\xi})dr + \int_s^T g_{ij}(X_r^{t,\xi}, Y_r^{t,\xi})d\left\langle B^i, B^j \right\rangle_r |\Omega_s]$$

$\Phi : \mathbb{R}^n \to \mathbb{R}$: given Lipschitz function
f, $g_{ij} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$: given Lipschitz functions

For each $A \in \mathbb{S}(n)$, $p \in \mathbb{R}^n$, $r \in \mathbb{R}$, we set

$$F(A, p, r, x) := G(B(A, p, r, x)) + \langle p, b(x) \rangle + f(x, r),$$

where

$$B_{ij}(A, p, r, x) := \langle A\sigma_i(x), \sigma_j(x) \rangle + \langle p, h_{ij}(x) + h_{ji}(x) \rangle + g_{ij}(x, r) + g_{ji}(x, r).$$

Theorem

$u(t,x)$ is a viscosity solution of the following PDE:

$$\begin{cases} \partial_t u + F(D^2u, Du, u, x) = 0, \\ u(T,x) = \Phi(x). \end{cases}$$

Notes and Comments

This chapter is mainly from Peng (2007) [63].

There are many excellent books on Itô's stochastic calculus and stochastic differential equations founded by Itô's original paper [37], as well as on martingale theory. Readers are referred to Chung and Williams (1990) [**?**], Dellacherie and Meyer (1978 and 1982) [19], He, Wang and Yan (1992) [**?**], Itô and McKean (1965) [**?**], Ikeda and Watanabe (1981) [**?**], Kallenberg (2002) [**?**], Karatzas and Shreve (1988) [**?**], Øksendal (1998) [**?**], Protter (1990) [**?**], Revuz and Yor (1999)[**?**] and Yong and Zhou (1999) [**?**].

Linear backward stochastic differential equation (BSDE) was first introduced by Bismut in (1973) [?] and (1978) [?]. Bensoussan developed this approach in (1981) [?] and (1982) [?]. The existence and uniqueness theorem of a general nonlinear BSDE, was obtained in 1990 in Pardoux and Peng [?]. The present version of the proof was based on El Karoui, Peng and Quenez (1997) [24], which is also a very good survey on BSDE theory and its applications, specially in finance. Comparison theorem of BSDEs was obtained in Peng (1992) [52] for the case when g is a C^1-function and then in [24] when g is Lipschitz. Nonlinear Feynman-Kac formula for BSDE was introduced by Peng (1991) [51] and (1992) [53]. Here we obtain the corresponding Feynman-Kac formula under the framework of G-expectation. We also refer to Yong and Zhou (1999) [?], as well as in Peng (1997) [55] (in 1997, in Chinese) and (2004) [57] for systematic presentations of BSDE theory. For contributions in the developments of this theory, readers can be referred to the literatures listing in the Notes and Comments in Chap. –ch1.

$\Omega = C^d([0,\infty))$: a complete separable metric space;
$\mathcal{B}(\Omega)$: the Borel σ-algebra of Ω;
$\mathcal{M}$: the collection of all probability measures on $(\Omega, \mathcal{B}(\Omega))$.

Now, we give the representation of G-expectation.

Theorem

For each continuous monotonic and sublinear function $G:\mathbb{S}(d)\to\mathbb{R}$, let $\hat{\mathbb{E}}$ be the corresponding G-expectation on $(\Omega, L_{ip}(\Omega))$. Then there exists a weakly compact family of probability measures $\mathcal{P}$ on $(\Omega,\mathcal{B}(\Omega))$ such that

$$\hat{\mathbb{E}}[X]=\max_{P\in\mathcal{P}} E_P[X] \quad \textit{for } X\in L_{ip}(\Omega).$$

Contents

We denote

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

One can easily verify the following theorem.

Theorem

The set function $c(\cdot)$ is a Choquet capacity, i.e. (see [12, 18]),

1. $0 \leq c(A) \leq 1, \quad \forall A \subset \Omega$.
2. *If* $A \subset B$, *then* $c(A) \leq c(B)$.
3. *If* $(A_n)_{n=1}^{\infty}$ *is a sequence in* $\mathcal{B}(\Omega)$, *then* $c(\cup A_n) \leq \sum c(A_n)$.
4. *If* $(A_n)_{n=1}^{\infty}$ *is an increasing sequence in* $\mathcal{B}(\Omega)$: $A_n \uparrow A = \cup A_n$, *then* $c(\cup A_n) = \lim_{n\to\infty} c(A_n)$.

Definition

We use the standard capacity-related vocabulary: a set A is **polar** if $c(A) = 0$ and a property holds "**quasi-surely**" (q.s.)"qs if it holds outside a polar set.

Following [36] (see also [20, **?**]) the upper expectation of $\mathcal{P}$ is defined as follows: for each $X \in L^0(\Omega)$ such that $E_P[X]$ exists for each $P \in \mathcal{P}$,

$$\mathbb{E}[X] = \mathbb{E}^{\mathcal{P}}[X] := \sup_{P \in \mathcal{P}} E_P[X].$$

It is easy to verify

Theorem

The upper expectation $\mathbb{E}[\cdot]$ *of the family* $\mathcal{P}$ *is a sublinear expectation on* $B_b(\Omega)$ *as well as on* $C_b(\Omega)$*, i.e.,*

1. *for all* X, Y *in* $B_b(\Omega)$, $X \geq Y \Longrightarrow \mathbb{E}[X] \geq \mathbb{E}[Y]$.
2. *for all* X, Y *in* $B_b(\Omega)$, $\mathbb{E}[X+Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.
3. *for all* $\lambda \geq 0$, $X \in B_b(\Omega)$, $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$.
4. *for all* $c \in \mathbb{R}$, $X \in B_b(\Omega)$, $\mathbb{E}[X + c] = \mathbb{E}[X] + c$.

Moreover, it is also easy to check

Theorem

We have

1. *Let* $\mathbb{E}[X_n]$ *and* $\mathbb{E}[\sum_{n=1}^{\infty} X_n]$ *be finite. Then* $\mathbb{E}[\sum_{n=1}^{\infty} X_n] \leq \sum_{n=1}^{\infty} \mathbb{E}[X_n]$.
2. *Let* $X_n \uparrow X$ *and* $\mathbb{E}[X_n]$, $\mathbb{E}[X]$ *be finite. Then* $\mathbb{E}[X_n] \uparrow \mathbb{E}[X]$.

Definition

The functional $\mathbb{E}[\cdot]$ is said to be **regular** if for each $\{X_n\}_{n=1}^{\infty}$ in $C_b(\Omega)$ such that $X_n \downarrow 0$ on Ω, we have $\mathbb{E}[X_n] \downarrow 0$.

Similar to the above Lemma we have:

Theorem

$\mathbb{E}[\cdot]$ *is regular if and only if* $\mathcal{P}$ *is relatively compact.*

Functional spaces

We set, for $p > 0$,

- $\mathcal{L}^p := \{X \in L^0(\Omega) : \mathbb{E}[|X|^p] = \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty\}$;
- $\mathcal{N}^p := \{X \in L^0(\Omega) : \mathbb{E}[|X|^p] = 0\}$;
- $\mathcal{N} := \{X \in L^0(\Omega) : X = 0,\ c\text{-q.s.}\}$.

It is seen that $\mathcal{L}^p$ and $\mathcal{N}^p$ are linear spaces and $\mathcal{N}^p = \mathcal{N}$, for each $p > 0$. We denote $\mathbb{L}^p := \mathcal{L}^p / \mathcal{N}$. As usual, we do not take care about the distinction between classes and their representatives.

Lemma

Let $X \in \mathbb{L}^p$*. Then for each* $\alpha > 0$

$$c(\{|X| > \alpha\}) \leq \frac{\mathbb{E}[|X|^p]}{\alpha^p}.$$

Proposition.

We have

1. For each $p \geq 1$, $\mathbb{L}^p$ is a Banach space under the norm $\|X\|_p := (\mathbb{E}[|X|^p])^{\frac{1}{p}}$.

$\square$

With respect to the distance defined on $\mathbb{L}^p$, $p > 0$, we denote by

- $\mathbb{L}_b^p$ the completion of $B_b(\Omega)$.
- $\mathbb{L}_c^p$ the completion of $C_b(\Omega)$.

By Proposition –Prop3, we have

$$\mathbb{L}_c^p \subset \mathbb{L}_b^p \subset \mathbb{L}^p, \quad p > 0.$$

The following Proposition is obvious and the proof is left to the reader.

Proposition.

Let $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then $X \in \mathbb{L}^p$ and $Y \in \mathbb{L}^q$ implies

$$XY \in \mathbb{L}^1 \text{ and } \mathbb{E}[|XY|] \leq (\mathbb{E}[|X|^p])^{\frac{1}{p}} (\mathbb{E}[|Y|^q])^{\frac{1}{q}};$$

□

Definition

A mapping X on Ω with values in a topological space is said to be quasi-continuous (q.c.) if

$$\forall \varepsilon > 0, \ \exists \text{ open } O \text{ with } c(O) < \varepsilon \text{ s.t. } X|_{O^c} \text{ is continuous.}$$

Definition

We say that $X : \Omega \to \mathbb{R}$ has a quasi-continuous version if there exists a quasi-continuous function $Y : \Omega \to \mathbb{R}$ with $X = Y$ q.s..

The following theorem gives a concrete characterization of the space $\mathbb{L}_c^p$.

Theorem

For each $p \geq 1$, $\mathbb{L}_G^p(\Omega) = \mathbb{L}_c^p(\Omega)$

$$\mathbb{L}_c^p = \{X \in \mathbb{L}^p : X \text{ has a q.-c. version}, \lim_{n\to\infty} \mathbb{E}[|X|^p \mathbf{1}_{\{|X|>n\}}] = 0\}.$$

We immediately have the following corollary.

Corollary

Let $\mathcal{P}$ be weakly compact and let $\{X_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{L}_c^1$ decreasingly converging to 0 q.s.. Then $\mathbb{E}[X_n] \downarrow 0$.

Kolmogorov's criterion

Definition

Let I be a set of indices, $(X_t)_{t\in I}$ and $(Y_t)_{t\in I}$ be two processes indexed by I . We say that Y is a quasi-modification of X if for all $t \in I$, $X_t = Y_t$ q.s..

Remark.

In the above definition, quasi-modification is also called modification in some papers. □

We now give a Kolmogorov criterion for a process indexed by $\mathbb{R}^d$ with $d \in \mathbb{N}$.

Theorem

Let $p > 0$ and $(X_t)_{t\in[0,1]^d}$ be a process such that for all $t \in [0,1]^d$, X_t belongs to $\mathbb{L}^p$. Assume that there exist positive constants c and ε such that

$$\mathbb{E}[|X_t - X_s|^p] \leq c|t-s|^{d+\varepsilon}.$$

Then X admits a modification $\tilde{X}$ such that

$$\mathbb{E}\left[\left(\sup_{s\neq t}\frac{|\tilde{X}_t - \tilde{X}_s|}{|t-s|^\alpha}\right)^p\right] < \infty,$$

for every $\alpha \in [0, \varepsilon/p)$. As a consequence, paths of $\tilde{X}$ are quasi-surely Höder continuous of order α for every $\alpha < \varepsilon/p$ in the sense that there exists a Borel set N of capacity 0 such that for all $w \in N^c$, the map $t \to \tilde{X}(w)$ is Höder continuous of order α for every $\alpha < \varepsilon/p$. Moreover, if $X_t \in \mathbb{L}_c^p$ for each t, then we also have $\tilde{X}_t \in \mathbb{L}_c^p$.

Theorem

Problem

Analysis of $\mathbb{L}^p(\Omega) \supset \mathbb{L}^p_G(\Omega)$

Notes and Comments

The results of this chapter for G-Brownian motions were mainly obtained by Denis, Hu and Peng (2008) [22] (see also Denis and Martini (2006) [23] and the related comments after Chapter III). Hu and Peng (2009) [33] then have introduced an intrinsic and simple approach. This approach can be regarded as a combination and extension of the original Brownian motion construction approach of Kolmogorov (for more general stochastic processes) and a sort of cylinder Lipschitz functions technique already introduced in Chap. –ch3. Section 1 is from [22] and Theorem –Gt34 is firstly obtained in [22], whereas contents of Sections 2 and 3 are mainly from [33].

Choquet capacity was first introduced by Choquet (1953) [12], see also Dellacherie (1972) [18] and the references therein for more properties. The capacitability of Choquet capacity was first studied by Choquet [12] under 2-alternating case, see Dellacherie and Meyer (1978 and 1982) [19], Huber and Strassen (1972) [36] and the references therein for more general case. It seems that the notion of upper expectations was first discussed by Huber (1981) [35] in robust statistics. Recently, it was rediscovered in mathematical finance, especially in risk measure, see Delbaen (1992, 2002) [**?**, 20], Föllmer and Schied (2002, 2004) [**?**] and etc..

Thank you

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