On Some Discontinuous Control Problems

Dan Goreac[1]

Université Paris-Est Marne-la-Vallée

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[1](joint work with Oana-Silvia Serea (CMAP))

$$\begin{cases} dX_s^{t,x,u} = b\left(s, X_s^{t,x,u}, u_s\right) ds \left\{ +\sigma\left(s, X_s^{t,x,u}, u_s\right) dW_s \right\}, \ t \leq s \leq T, \\ X_t^{t,x,u} = x \in \mathbb{R}^N, \end{cases}$$

h semicontinuous,

$$V\left(t, x\right) = \inf \mathbb{E}\left[h\left(X_T^{t,x,u}\right)\right],$$

Plan

- Deterministic framework
- Stochastic framework
- References

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- admissible control $u \in \mathcal{U}$: Lebesque-measurable, U-valued
- $$\begin{cases} dx_t^{t_0,x_0,u} = b\left(t, x_t^{t_0,x_0,u}, u_t\right) dt, & t_0 \leq t \leq T, \\ x_{t_0}^{t_0,x_0,u} = x_0 \in \mathbb{R}^N, & \end{cases} \tag{1}$$

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- if h is u.s.c., $V=\Lambda$

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- $(0, 0) \in cl\left(R(T, t_0)(0, 0)\right)$ and $(0, 0) \notin R(T, t_0)(0, 0) \Longrightarrow$

$$\inf_{u \in \mathcal{U}} h\left(x_T^{t_0, 0, 0, u(\cdot)}, y_T^{t_0, 0, 0, u(\cdot)}\right) = 1 \neq V(t_0, 0, 0).$$

- $$V(t_0, x_0) = \inf \{ h(x) : x \in cl(R(T, t_0) x_0) \} \qquad \text{(V1)}$$

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- $$\begin{cases} \partial_t V(t, x) + \min_{u \in U} \langle \partial_x V(t, x), f(t, x, u) \rangle = 0, \\ \text{if } t \in (0, T), \ x \in \mathbb{R}^N, \end{cases} \tag{HJ Mayer}$$

Main Result

Theorem

- *(a) h l.s.c., V is the smallest l.s.c. supersolution of (HJ Mayer) s.t.* $V(T,\cdot) \geq h(\cdot)$.

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- *(b) h u.s.c., V is the largest u.s.c. subsolution of (HJ Mayer) s.t.* $V(T,\cdot) \leq h(\cdot)$.
- *(c) h is bounded,*

$$V = \inf\left\{ \begin{array}{c} \varphi : \varphi \text{ l.s.c. subsolution of (HJ Mayer) s.t.} \\ \varphi(T,\cdot) \geq h(\cdot) \end{array} \right\} \text{ and}$$

$$V = \sup\left\{ \begin{array}{c} \varphi : \varphi \text{ u.s.c. subsolution of (HJ Mayer) s.t.} \\ \varphi(T,\cdot) \leq h(\cdot) \end{array} \right\}.$$

Idea of the proof of (a)

Lemma

If φ is a l.s.c. supersolution of (HJ Mayer), s.t. $\varphi(T,\cdot) \geq h(\cdot)$, then

$$\varphi(t_0,x_0) \geq \inf\{\varphi(T,x) : x \in cl(R(T,t_0)x_0)\},$$

$\forall (t_0,x_0) \in (0,T) \times \mathbb{R}^N$.

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- $h_n(x) = \inf_{y \in \mathbb{R}^N} (h(y) + n|y-x|)$,

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- $h_n(x) = \inf_{y \in \mathbb{R}^N} (h(y) + n|y - x|)$,
- $V^n(t_0, x_0) = \inf_{u \in \mathcal{U}} h_n\left(x_T^{t_0, x_0, u}\right)$, $W = \sup_n V^n$

Idea of the proof of (b)

Lemma

If φ is an u.s.c. subsolution of (HJ Mayer), s.t. $\varphi(T,x) \leq h(x)$, $\forall x \in \mathbb{R}^N$, *then*

$$\varphi(t_0,x_0) \leq \varphi(T,x),$$

$\forall\ (t_0,x_0) \in (0,T) \times \mathbb{R}^N$, $x \in R(T,t_0)\,x_0$.

Idea of the proof of (b)

Lemma

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$\forall\ (t_0,x_0) \in (0,T) \times \mathbb{R}^N,\ x \in R(T,t_0)\,x_0.$

- sup-convolution

Idea of the proof of (c) 1

- Choose $x_\varepsilon \in cl\left(R\left(T, t_0\right) x_0\right)$ s.t.

$$h\left(x_\varepsilon\right) < V\left(t_0, x_0\right) + \varepsilon.$$

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- Define $h_\varepsilon : \mathbb{R}^N \longrightarrow \mathbb{R}$ l.s.c.

$$h_\varepsilon(x) = \begin{cases} h\left(x_\varepsilon\right), & \text{if } x = x_\varepsilon, \\ \sup_x h(x), & \text{otherwise.} \end{cases}$$

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- Value function

$$V_\varepsilon\left(t, x\right) = \inf\left\{h_\varepsilon\left(y\right) : y \in cl\left(R\left(T, t\right)x\right)\right\},$$

satisfies: $V_\varepsilon\left(t_0, x_0\right) = h\left(x_\varepsilon\right) \leq V\left(t_0, x_0\right) + \varepsilon$

Idea of the proof of (c) 2

- Define $g : \mathbb{R}^N \longrightarrow \mathbb{R}$ u.s.c.

$$g(x) = \begin{cases} V(t_0, x_0), & \text{if } x \in cl(R(T, t_0) x_0), \\ \inf_{y \in \mathbb{R}^N} h(y), & \text{otherwise.} \end{cases}$$

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$$V_g(t, x) = \inf\{g(y) : y \in cl(R(T, t)x)\},$$

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$$V_g(t, x) = \inf \{ g(y) : y \in cl(R(T, t) x) \},$$

- $V_g(T, \cdot) \leq g(\cdot) \leq h(\cdot)$ and $V_g(t_0, x_0) = V(t_0, x_0)$

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- $$\begin{cases} dX_s^{t,x,u} = b\left(s, X_s^{t,x,u}, u_s\right) ds + \sigma\left(s, X_s^{t,x,u}, u_s\right) dW_s, \ t \leq s \leq T, \\ X_t^{t,x,u} = x \in \mathbb{R}^N, \end{cases}$$
  (2)

Main assumptions

- $b : \mathbb{R} \times \mathbb{R}^N \times U \longrightarrow \mathbb{R}^N$, $\sigma : \mathbb{R} \times \mathbb{R}^N \times U \longrightarrow \mathbb{R}^{N \times d}$

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- (ii) $\exists c > 0$ s.t.
  $|b(t,x,u) - b(t,y,u)| + |\sigma(t,x,u) - \sigma(t,y,u)| \leq c|x-y|$
  and
  $|b(t,x,u) - b(s,x,u)| + |\sigma(t,x,u) - \sigma(s,x,u)| \leq c|t-s|^{\frac{\delta_0}{2}}$,
  $\forall (t,s,x,y,u) \in [0,T]^2 \times \mathbb{R}^{2N} \times U$.

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- $$\begin{cases} -\partial_t V_h(t,x) + H\left(x, DV_h(t,x), D^2 V_h(t,x)\right) = 0, \\ \text{for all } (t,x) \in (0,T) \times \mathbb{R}^N, \qquad \text{(HJB)} \\ V_h(T,\cdot) = h(\cdot) \text{ on } \mathbb{R}^N, \end{cases},$$
  $H(t,x,p,A) = \sup_{u \in U}\left\{-\frac{1}{2} Tr\left(\sigma\sigma^*(t,x,u) A\right) - \langle b(t,x,u), p\rangle\right\}$,

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- $\gamma_{t,x,u}(A \times B \times C \times D) =$
  $\frac{1}{T-t}\mathbb{E}\left[\int_t^T 1_{A\times B\times C}\left(\left(s, X_s^{t,x,u}, u_s\right)\right) ds\right] \mathbb{P}\left(X_T^{t,x,u} \in D\right),$

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- $\int_{[t,T]\times\mathbb{R}^N\times U\times\mathbb{R}^N} \left(|y|^2 + |z|^2\right) \gamma_{t,x,u}(ds, dy, dv, dz) \leq C_0\left(|x|^2 + 1\right).$

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- $\gamma_{t,x,u} \in \mathcal{P}\left([t,T] \times \mathbb{R}^N \times U \times \mathbb{R}^N\right) : \forall \phi \in C_b^{1,2}\left([0,T]\times\mathbb{R}^N\right)$,
$$\int_{[t,T]\times\mathbb{R}^N\times U\times\mathbb{R}^N} \left[\begin{array}{c} (T-t)\mathcal{L}^v\phi(s,y) \\ +\phi(t,x) - \phi(T,z) \end{array}\right] \gamma(ds,dy,dv,dz) = 0$$
(Itô's formula).

(Finite horizon) Occupational measures 2

- $J_h(t,x,u) = \mathbb{E}\left[h\left(X_T^{t,x,u}\right)\right] = \int_{\mathbb{R}^N} h(z)\,\gamma_{t,x,u}\left([t,T]\times\mathbb{R}^N\times U, dz\right).$

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- $\Theta(t,x) =$

$$\left\{\begin{array}{c} \gamma \in \mathcal{P}\left([t,T]\times\mathbb{R}^N\times U\times\mathbb{R}^N\right) : \forall\phi\in C_b^{1,2}\left([0,T]\times\mathbb{R}^N\right), \\ \int_{[t,T]\times\mathbb{R}^N\times U\times\mathbb{R}^N}\left[\begin{array}{c}(T-t)\,\mathcal{L}^v\phi(s,y)\\ +\phi(t,x)-\phi(T,z)\end{array}\right]\gamma(ds,dy,dv,dz)=0, \\ \int_{[t,T]\times\mathbb{R}^N\times U\times\mathbb{R}^N}\left(|y|^2+|z|^2\right)\gamma(ds,dy,dv,dz)\le C_0\left(|x|^2+1\right) \end{array}\right.$$

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- $\Theta(t,x) =$
$$\left\{\begin{array}{c} \gamma\in\mathcal{P}\left([t,T]\times\mathbb{R}^N\times U\times\mathbb{R}^N\right) : \forall\phi\in C_b^{1,2}\left([0,T]\times\mathbb{R}^N\right), \\ \int_{[t,T]\times\mathbb{R}^N\times U\times\mathbb{R}^N}\left[\begin{array}{c}(T-t)\,\mathcal{L}^v\phi(s,y)\\ +\phi(t,x)-\phi(T,z)\end{array}\right]\gamma(ds,dy,dv,dz)=0, \\ \int_{[t,T]\times\mathbb{R}^N\times U\times\mathbb{R}^N}\left(|y|^2+|z|^2\right)\gamma(ds,dy,dv,dz)\le C_0\left(|x|^2+1\right)\end{array}\right.$$
- $\mathcal{L}^v\phi(s,y) = \frac{1}{2}Tr\left[(\sigma\sigma^*)(s,y,v)\,D^2\phi(s,y)\right] + \langle b(s,y,v), D\phi(s,y)\rangle + \partial_t\phi(s,y),$

Linearized formulation

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- dual formulation:

$$\eta^*(t,x) = \sup \left\{ \begin{array}{c} \eta \in \mathbb{R} : \exists \phi \in C_b^{1,2}\left([0,T] \times \mathbb{R}^N\right) \text{ s.t.} \\ \forall (s,y,v,z) \in [t,T] \times \mathbb{R}^N \times V \times \mathbb{R}^N, \\ \eta \le (T-t)\mathcal{L}^v \phi(s,y) + h(z) - \phi(T,z) + \phi(t,x), \end{array} \right\},$$

$(t,x) \in [0,T) \times \mathbb{R}^N$.

Linearized formulation

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$$\eta^*(t,x) = \sup\left\{\begin{array}{c} \eta\in\mathbb{R} : \exists\phi\in C_b^{1,2}\left([0,T]\times\mathbb{R}^N\right) \text{ s.t.} \\ \forall (s,y,v,z)\in[t,T]\times\mathbb{R}^N\times V\times\mathbb{R}^N, \\ \eta\leq (T-t)\,\mathcal{L}^v\phi(s,y)+h(z)-\phi(T,z)+\phi(t,x), \end{array}\right\},$$

$(t,x)\in[0,T)\times\mathbb{R}^N.$

- In infinite horizon (discounted) setting : Buckdahn, G., Quincampoix (preprint)

Linearized formulation

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Theorem

h Lipschitz, bounded $\Longrightarrow V_h(t,x) = h^*(t,x) = \eta^*(t,x),$
$\forall (t,x) \in [0,T) \times \mathbb{R}^N$.

Idea of the proof

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- $\eta \leq \left(T-t\right)\mathcal{L}^v\phi\left(s,y\right)+h\left(z\right)-\phi\left(T,z\right)+\phi\left(t,x\right)$ integrate w.r.t. $\gamma \in \Theta\left(\ t,x\right) \Longrightarrow h^*\left(t,x\right) \geq \eta^*\left(t,x\right)$.
- approximate V_h by smooth subsolutions V^ε; $V^\varepsilon(t,x)-C\varepsilon \leq \eta^*\left(t,x\right)$ then $\varepsilon \rightarrow 0$ to get $\eta^*\left(t,x\right) \geq V_h\left(t,x\right)$

Lower semicontinuous case

- $V_h(t,x) = \inf_{\gamma \in \Theta(t,x)} \int_{\mathbb{R}^N} h(z)\, \gamma\left([t,T], \mathbb{R}^N, U, dz\right)$,
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Theorem

V_h is the smallest lower semicontinuous viscosity supersolution and

$$V_h(t,x) = \eta^*(t,x),$$

$\forall\ (t,x) \in [0,T) \times \mathbb{R}^N$.

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  $m \to \infty$, $n \to \infty$.

Upper semicontinuous case

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Theorem

V_h is the largest upper semicontinuous viscosity subsolution of (HJB).

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- $\gamma \in \Theta(t,x), \ V^n(t,x) \leq \int_{\mathbb{R}^N} h_n(z)\, \gamma\left([t,T], \mathbb{R}^N, U, dz \right),$ pass $n \to \infty$.

What about the dual formulation? 1

- $\begin{cases} dX_s^{t,x} = 0, \text{ for } 0 \leq t \leq s \leq T = 1, \\ X_t^{t,x} = x \in \mathbb{R}. \end{cases}$, $h(\cdot) = 1_{\{0\}}(\cdot)$.

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If h is u.s.c., then $V_h\left(t,x\right) = V_h^w\left(t,x\right)$, $\left(t,x\right) \in \left[0, T\right] \times \mathbb{R}^N$.

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- use l.s.c. of h and convexity to get $V_h(t,x) \geq V_h^w(t,x)$

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- Thus,
  $\inf_{u \in \mathcal{U}} h\left(x_1^{t_0,0,0,u(\cdot)}, y_1^{t_0,0,0,u(\cdot)}\right) = 1 > 0 = V_h^w(t_0, 0, 0)$,

References 1

- H. Frankowska, *Lower semicontinuous solutions of Hamilton-Jacobi-Bellman equations*, SIAM J. Control Optimization, 31(1):257–272 (1993).
- S. Plaskacz, M. Quincampoix, *Value function for differential games and control systems with discontinuous terminal cost*, SIAM J. Control Optim., 39(5):1485-1498 (2001).
- O.S. Serea. Discontinuous differential games and control systems with supremum cost, J. Math. Anal. Appl., 270(2):519–542 (2002).

References 2

- Gaitsgory, V., *On a representation of the limit occupational measures set of a control system with applications to singularly perturbed control systems*, SIAM J. Control Optim., 43(1), pp. 325–340 (2004).
- Gaitsgory, V., Quincampoix, M., *Linear programming approach to deterministic infinite horizon optimal control problems with discounting*, SIAM J. Control Optimization, 48(4), pp. 2480-2512 (2009).
- Buckdahn, R., Goreac, D., Quincampoix, M., *Stochastic Optimal Control and Linear Programming Approach, (*preprint 2009*).*

Thank you !