Brownian Bridge on Stochastic Interval

Definition, First Properties and Applications

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Abstract

In this work we give the definition of a stochastic process β named *Information process*. This process is a Brownian bridge between 0 and 0 on a stochastic interval $[0, \tau]$. The objective is to model the information regarding a *default time*.

Key words:

- Brownian bridge
- totally inaccessible stopping time
- local time
- Credit Risk

Agenda

1. Motivation
2. Definition and Basic Properties
3. Conditional Expectations
4. Local Time of β and classification of τ
5. First Application to Credit Risk
6. Conclusion and Further Development

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Motivation

In Credit Risk literature there are two main class of models:

- Structural Models
- Reduced-form Models (*intensity based approach* and *hazard process approach*)

Brody, Hughston and Macrina in 2007 have introduced a new class of models called *Information-based* whose aim is to avoid some of the problems that are present in previous approaches without losing the advantages.

Structural Models

Information ($\mathbb{F}$) concerning the default time τ is equal to the information generated by some value-process Y observable on the market:

$$Y_t = y_0 + \nu t + \sigma W_t, \quad y_0, \nu > 0$$

$$\mathbb{F} = \mathbb{F}^W$$

$$\tau \triangleq \inf \{ t \in \mathbb{R}_+ \ : \ Y_t = 0 \}$$

- The default time τ is an $\mathbb{F}$-predictable stopping time.
- (+) Approach referring to economic fundamentals. Valuation and hedging are easy.
- (-) In reality the value process is not observable. Possibility of null spreads for short maturities.

Reduced-form models

- Hazard-process approach: $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}\,,\ \mathcal{H}_t \triangleq \sigma(t\wedge\tau)_+,$

$$\mathbb{F} = \mathbb{H} \vee \tilde{\mathbb{F}}$$

- Intensity-based approach: $\exists\, \lambda = \{\lambda_t\}_{t\geq 0}$ non-negative, $\mathbb{F}$-adapted such that

$$M_t = \mathbb{I}_{\{t\geq\tau\}} - \int_0^{t\wedge\tau} \lambda_s ds$$

  is $\mathbb{F}$-martingale

- The default time τ is an $\mathbb{F}$-totally inaccessible stopping time.
- (+) The default occurs by "surprise".
- (-) Difficult pricing formulas. Necessity of some highly-technical assumptions.

Information based approach

- Explicit model of the information: $\xi_t = \sigma t H_T + \beta_{tT}$

$$\mathbb{F} = \mathbb{F}^{\xi}$$

  where $H_T \sim B(1, p)$
- (+) Easy pricing formulas.
- (-) No default time.

Objective

Our approach aims to model the information on the default time allowing for tractable pricing formulas and preserving the "surprise" of the credit event.

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Assumption and definition

$(\Omega, \mathcal{F}, \mathbf{P})$ complete probability space, $\mathcal{N}_P$ the collection of the **P**-null sets.
$W = \{W_t\}_{t \geq 0}$ is a standard BM. $\tau : \Omega \to (0, +\infty)$ random variable.
$F(t) \triangleq \mathbf{P}\{\tau \leq t\}$.

Assumption

τ is independent of W.

Definition

The process $\beta = \{\beta_t\}_{t \geq 0}$ will be called *Information process* :

$$\beta_t \triangleq W_t - \frac{t}{\tau \vee t} W_{\tau \vee t} \tag{1}$$

Properties

$\mathbb{F}^{\beta} = \left(\mathcal{F}_t^{\beta}\right)_{t \geq 0}$ will denote the smallest filtration satisfying the usual condition (right-continuity and completeness) and containing the natural filtration of β.

Proposition

- τ is an $\mathbb{F}^{\beta}$-stopping time .
- For all $t > 0$, $\{\beta_t = 0\} = \{\tau \geq t\}$, **P**-a.s.
- β is an $\mathbb{F}^{\beta}$-Markov process.

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Some notation

$\beta^r = \{\beta_t^r\}_{0 \leq t \leq r}$ Brownian bridge between 0 and 0 on $[0, r]$

Density of β_t^r

$$\varphi_t(r, x) \triangleq \sqrt{\frac{r}{2\pi t (r-t)}} \exp\left[-\frac{x^2 r}{2t(r-t)}\right], \; r > t > 0, \; x \in \mathbb{R}$$

Density of β_u^r given β_t^r

$$f_{\beta_t}(x, u, r) \triangleq \sqrt{\frac{r-t}{2\pi (r-u)(u-t)}} \exp\left[-\frac{\left(x - \frac{r-u}{r-t}\beta_t^r\right)^2}{2\frac{r-u}{r-t}(u-t)}\right], \; u \in (t, r)$$

Conditional Expectation (1/2)

Theorem

Let $t > 0$, $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ *a Borel function such that* $\mathbf{E}\left[\left|g\left(\tau\right)\right|\right] < +\infty$. *Then,* $\mathbf{P}$*-almost surely on* $\{\tau > t\}$

$$\mathbf{E}\left[g\left(\tau\right)\mathbb{I}_{\{\tau>t\}}|\mathcal{F}_t^\beta\right] = \frac{\int_t^{+\infty} g\left(r\right)\varphi_t\left(t,\beta_t\right)dF(r)}{\int_t^{+\infty}\varphi_t\left(t,\beta_t\right)dF(r)}\mathbb{I}_{\{\tau>t\}} \tag{2}$$

$$\mathbf{P}\left\{\tau > u|\mathcal{F}_t^\beta\right\}\mathbb{I}_{\{\tau>t\}} = \frac{\int_u^{+\infty}\varphi_t\left(t,\beta_t\right)dF(r)}{\int_t^{+\infty}\varphi_t\left(t,\beta_t\right)dF(r)}\mathbb{I}_{\{\tau>t\}} \tag{3}$$

Conditional Expectation (2/2)

Theorem

Let $u > t > 0$ *and* g *a bounded Borel function defined on* $\mathbb{R}^+ \times \mathbb{R}$ *such that* $\mathbf{E}\left[\left|g\left(\tau, \beta_u\right)\right|\right] < +\infty$. *Then,* $\mathbf{P}$*-almost surely*

$$\mathbf{E}\left[g\left(\tau, \beta_u\right)\left|\mathcal{F}_t^{\beta}\right.\right] = g\left(\tau, 0\right)\mathbb{I}_{\{\tau \leq t\}} + \tag{4}$$

$$+\frac{\int_u^{+\infty}\left(\int_{\mathbb{R}} g\left(r, x\right) f_{\beta_t}\left(x, u, r\right) dx\right)\varphi_t\left(r, \beta_t\right) dF(r)}{\int_t^{+\infty}\varphi_t\left(r, \beta_t\right) dF(r)}\mathbb{I}_{\{\tau > t\}} +$$

$$+\frac{\int_t^u g\left(r, 0\right)\varphi_t\left(r, \beta_t\right) dF(r)}{\int_t^{+\infty}\varphi_t\left(r, \beta_t\right) dF(r)}\mathbb{I}_{\{\tau > t\}}$$

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Main Theorem

Theorem

Suppose $F(t)$ admits a continuous density with respect to the Lebesgue measure: $dF(t) = f(t)dt$. Then τ is a totally inaccessible stopping time with respect to $\mathbb{F}^{\beta}$ and its compensator $K = \{K_t\}_{t\geq 0}$ is given by

$$K_t = \int_0^{\tau\wedge t} \frac{f(r)dl_r}{\int_r^{+\infty} \varphi_r(v,0) f(v)dv} \tag{5}$$

where l_t is the local time at 0 of the process β at time t.

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Example: CDS (1/3).

Following Bielecki, Jeanblanc and Rutkowsky (2007) we consider the case of pricing a Credit Default Swap (CDS) in an elementary market model. $D = \{D_t\}_{0 \le t \le T}$ is the *dividend process* on a certain lifespan $[0, T]$. D is of finite variation, $D_0 = 0$ and $\int_{]t,T]} dD_r$ is $\mathbf{P}$-integrable for any $t \in [0, T]$.

Definition

The ex-dividend price process S of a contract expiring at T and paying dividends according to a process $D = \{D_t\}_{0 \le t \le T}$ equals, for every $t \in [0, T]$

$$S_t = \mathbf{E}\left[\int_{(t,T]} dD_r | \mathcal{F}_t\right]$$

$\mathbb{F} = (\mathcal{F}_t)_{t \ge 0}$ is the market filtration.

Example: CDS (2/3).

Definition

A CDS with a constant rate k and a recovery at default is a defaultable claim $(0, A, Z, \tau)$ where $Z(t) = \delta(t)$ and $A(t) = -kt$ for every $t \in [0, T]$. A function $\delta : [0, T] \to \mathbb{R}$ represents the default protection and k is the CDS rate. $H = \{H_t\}_{t \geq 0}$, $H_t \triangleq \mathbb{I}_{\{t \geq \tau\}}$

Let $s \in [0, T]$ be a fixed date. We consider a stylized T-maturity CDS with a constant spread k and a constant protection δ, initiated at time s and with maturity T. The dividend process $D = \{D_t\}_{0 \leq t \leq T}$ equals

$$D_t = \int_{(s,t]} \delta(r) dH_r - k \int_{(s,t]} (1 - H_r)\, dr$$

Example: CDS (3/3)

Lemma

If $\mathbb{F} = \mathbb{F}^{\beta}$, for $t \in [s, T]$ we have

$$S_t\left(k, \delta, T\right) = \mathbb{I}_{\{\tau > t\}} \left[-\int_t^T \delta(r) d\Psi_t(r) - k \int_t^T \Psi_t(r) dr \right]$$

Where $\Psi_t(r) \triangleq \mathbf{P}\left\{\tau > r | \mathcal{F}_t^{\beta}\right\}$.

Lemma

If $\mathbb{F} = \mathbb{H}$, for $t \in [s, T]$ we have

$$S_t\left(k, \delta, T\right) = \mathbb{I}_{\{\tau > t\}} \left[-\int_t^T \delta(r) dG(r) - k \int_t^T G(r) dr \right]$$

Where $G(r) \triangleq \mathbf{P}\left\{\tau > r\right\}$

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Conclusion and Further Development

- Modeling the information regarding a default time τ with a Brownian bridge on the stochastic interval $[0, \tau]$, allows to reconcile the Information-based approach to Credit-Risk with the reduced-form models.
- Explicit formulas can be obtained and they appear to be an intuitive generalization of some simple models already present in literature.
- Further development concerning the enlargement of a reference filtration $\mathbb{F}$ with $\mathbb{F}^{\beta}$ will be presented in another work.

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