Nonlinear Expectations and Stochastic Calculus under Uncertainty

—with Robust Central Limit Theorem and G-Brownian Motion

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Ch.I Sublinear Expectations and Risk Measures

Ω: a given set
$\mathcal{H}$: a linear space of real valued functions defined on Ω, s.t.

- a) $c \in \mathcal{H}$ for each constant c,
- b) $X \in \mathcal{H} \implies |X| \in \mathcal{H}$

Definition

A **Sublinear expectation** $\mathbb{E}$ is a functional $\mathbb{E}: \mathcal{H} \to \mathbb{R}$ satisfying

(i) Monotonicity:

$$\mathbb{E}[X] \geq \mathbb{E}[Y] \quad \text{if } X \geq Y.$$

(ii) Constant preserving:

$$\mathbb{E}[c] = c \quad \text{for } c \in \mathbb{R}.$$

(iii) Sub-additivity: For each $X, Y \in \mathcal{H}$,

$$\mathbb{E}[X+Y] \leq \mathbb{E}[X] + \mathbb{E}[Y].$$

(iv) Positive homogeneity:

$$\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X] \quad \text{for } \lambda \geq 0.$$

$(\Omega, \mathcal{H}, \mathbb{E})$: a **sublinear expectation space**.
(i)+(ii): $\mathbb{E}$ is called **nonlinear expectation** .

Definition

Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be two nonlinear expectations defined on $(\Omega, \mathcal{H})$. $\mathbb{E}_1$ is said to be **dominated** by $\mathbb{E}_2$ if

$$\mathbb{E}_1[X] - \mathbb{E}_1[Y] \leq \mathbb{E}_2[X - Y] \quad \text{for } X, Y \in \mathcal{H}. \tag{1}$$

Remark.

From (iii), a sublinear expectation is dominated by itself. In many situations, (iii) is also called the property of self-domination. If the inequality in (iii) becomes equality, then $\mathbb{E}$ is a linear expectation, i.e., $\mathbb{E}$ is a linear functional satisfying (i) and (ii). □

Remark.

(iii)+(iv) is called **sublinearity**. This sublinearity implies

(v) Convexity:

$$\mathbb{E}[\alpha X + (1-\alpha)Y] \leq \alpha\mathbb{E}[X] + (1-\alpha)\mathbb{E}[Y] \quad \text{for } \alpha \in [0,1].$$

If a nonlinear expectation $\mathbb{E}$ satisfies convexity, we call it a **convex expectation**.

□

Remark.

The properties (ii)+(iii) implies

(vi) Cash translatability:

$$\mathbb{E}[X+c] = \mathbb{E}[X] + c \quad \text{for } c \in \mathbb{R}.$$

In fact, we have

$$\mathbb{E}[X] + c = \mathbb{E}[X] - \mathbb{E}[-c] \le \mathbb{E}[X+c] \le \mathbb{E}[X] + \mathbb{E}[c] = \mathbb{E}[X] + c.$$

For property (iv), an equivalence form is

$$\mathbb{E}[\lambda X] = \lambda^{+}\mathbb{E}[X] + \lambda^{-}\mathbb{E}[-X] \quad \text{for } \lambda \in \mathbb{R}.$$

□

In this book, we will systematically study the sublinear expectation spaces. In the following chapters, unless otherwise stated, we consider the following sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$: if $X_1, \cdots, X_n \in \mathcal{H}$ then $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{Lip}(\mathbb{R}^n)$ where $C_{Lip}(\mathbb{R}^n)$ denotes the linear space of functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leq C|x - y| \quad \text{for } x, y \in \mathbb{R}^n.$$

In this case $X = (X_1, \cdots, X_n)$ is called an n-dimensional random vector, denoted by $X \in \mathcal{H}^n$.

It is clear that if $X \in \mathcal{H}$ then $|X|$, $X^m \in \mathcal{H}$.

Here we use $C_{Lip}(\mathbb{R}^n)$ in our framework only for some convenience of techniques. In fact our essential requirement is that $\mathcal{H}$ contains all constants and, moreover, $X \in \mathcal{H}$ implies $|X| \in \mathcal{H}$. In general, $C_{Lip}(\mathbb{R}^n)$ can be replaced by any one of the following spaces of functions defined on $\mathbb{R}^n$.

- $\mathbb{L}^\infty(\mathbb{R}^n)$: the space of bounded Borel-measurable functions;
- $C_b(\mathbb{R}^n)$: the space of bounded and continuous functions;

- $C_b^k(\mathbb{R}^n)$: the space of bounded and k-time continuously differentiable functions with bounded derivatives of all orders less than or equal to k;

- $C_{unif}(\mathbb{R}^n)$: the space of bounded and uniformly continuous functions;

- $C_{b.Lip}(\mathbb{R}^n)$: the space of bounded and Lipschitz continuous functions;

- $L^0(\mathbb{R}^n)$: the space of Borel measurable functions.

- $C_{l.Lip}(\mathbb{R}^n)$: the space of local Lipschitz continuous functions;

Next we give two examples of sublinear expectations.

Example

In a game we select a ball from a box containing W white, B black and Y yellow balls. The owner of the box, who is the banker of the game, does not tell us the exact numbers of W, B and Y. He or she only informs us that $W + B + Y = 100$ and $W = B \in [20, 25]$. Let ξ be a random variable defined by

$$\xi = \begin{cases} 1 & \text{if we get a white ball;} \\ 0 & \text{if we get a yellow ball;} \\ -1 & \text{if we get a black ball.} \end{cases}$$

Problem: how to measure a loss $X = \varphi(\xi)$ for a given function φ on $\mathbb{R}$. We know that the distribution of ξ is

$$\left\{ \begin{matrix} -1 & 0 & 1 \\ \frac{p}{2} & 1-p & \frac{p}{2} \end{matrix} \right\} \text{ with uncertainty: } p \in [\underline{\mu}, \overline{\mu}] = [0.4, 0.5].$$

Thus the **robust expectation** of $X = \varphi(\xi)$ is

$$\mathbb{E}[\varphi(\xi)] := \sup_{P\in\mathcal{P}} E_P[\varphi(\xi)]$$

Example

A more general situation is that the banker of a game can choose among a set of distributions $\{F(\theta, A)\}_{A\in\mathcal{B}(\mathbb{R}),\theta\in\Theta}$ of a random variable ξ . In this situation the robust expectation of a risk position $\varphi(\xi)$ for some $\varphi \in C_{Lip}(\mathbb{R})$ is

$$\mathbb{E}[\varphi(\xi)] := \sup_{\theta\in\Theta} \int_{\mathbb{R}} \varphi(x)F(\theta, dx).$$

Remark.

Prove that a functional $\mathbb{E}$ satisfies sublinearity if and only if it satisfies convexity and positive homogeneity. $\square$

Exercise.

Suppose that all elements in $\mathcal{H}$ are bounded. Prove that the strongest sublinear expectation on $\mathcal{H}$ is

$$\mathbb{E}^{\infty}[X] := X^* = \sup_{\omega\in\Omega} X(\omega).$$

Namely, all other sublinear expectations are dominated by $\mathbb{E}^{\infty}[\cdot]$. $\square$

A sublinear expectation can be expressed as a supremum of linear expectations.

Theorem

Let $\mathbb{E}$ a sublinear functional defined on $(\Omega, \mathcal{H})$. Then $\exists$ a family of linear functionals $\{E_\theta : \theta \in \Theta\}$ on $(\Omega, \mathcal{H})$ s. t.

$$\mathbb{E}[X] = \max_{\theta \in \Theta} E_\theta[X] \quad \text{for } X \in \mathcal{H}$$

Furthermore, if $\mathbb{E}$ is a sublinear expectation, then each E_θ is a linear expectation.

Proof.

- $\mathcal{Q} = \{E_\theta : \theta \in \Theta\} :=$ $\{\text{the family of all linear functionals dominated by} \mathbb{E}\}$.

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- $E = I$ on L and $E \leq \mathbb{E}$ on $\mathcal{H}$.
- This linear functional is dominated by $\mathbb{E}$ such that $\mathbb{E}[X] = E[X]$.

Proof.

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- The proof is complete.

Remark.

It is important to observe that the above linear expectation E_θ is only "finitely additive". A sufficient condition for the σ-additivity of E_θ is to assume that $\mathbb{E}[X_i] \to 0$ for each sequence $\{X_i\}_{i=1}^{\infty}$ of $\mathcal{H}$ such that $X_i(\omega) \downarrow 0$ for each ω. In this case, it is clear that $E_\theta[X_i] \to 0$. Thus we can apply the well-known Daniell-Stone Theorem (see Theorem –TheoremDL in Appendix B) to find a σ-additive probability measure P_θ on $(\Omega, \sigma(\mathcal{H}))$ such that

$$E_\theta[X] = \int_\Omega X(\omega) dP_\theta, \quad X \in \mathcal{H}.$$

The corresponding model uncertainty of probabilities is the subset $\{P_\theta : \theta \in \Theta\}$, and the corresponding uncertainty of distributions for an n-dimensional random vector X in $\mathcal{H}$ is $\{F_X(\theta, A) := P_\theta(X \in A) : A \in \mathcal{B}(\mathbb{R}^n)\}$. □

In many situation, we may concern the probability uncertainty, and the probability maybe only finitely additive. So next we will give another version of the above representation theorem.

Let $\mathcal{P}_f$ be the collection of all finitely additive probability measures on $(\Omega, \mathcal{F})$, we consider $\mathbb{L}_0^\infty(\Omega, \mathcal{F})$ the collection of risk positions with finite values, which consists risk positions X of the form

$$X(\omega) = \sum_{i=1}^{N} x_i \mathbf{I}_{A_i}(\omega),\ x_i \in \mathbb{R},\ A_i \in \mathcal{F}, i = 1, \cdots, N.$$

It is easy to check that, under the norm $\|\cdot\|_\infty$, $\mathbb{L}_0^\infty(\Omega, \mathcal{F})$ is dense in $\mathbb{L}^\infty(\Omega, \mathcal{F})$. For a fixed $Q \in \mathcal{P}_f$ and $X \in \mathbb{L}_0^\infty(\Omega, \mathcal{F})$ we define

$$E_Q[X] = E_Q[\sum_{i=1}^{N} x_i \mathbf{I}_{A_i}(\omega)] := \sum_{i=1}^{N} x_i Q(A_i) = \int_\Omega X(\omega) Q(d\omega).$$

$E_Q : \mathbb{L}_0^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ is a linear functional. It is easy to check that E_Q satisfies (i) monotonicity and (ii) constant preserving. It is also continuous under $\|X\|_\infty$.

$$|E_Q[X]| \leq \sup_{\omega \in \Omega} |X(\omega)| = \|X\|_\infty .$$

Since $\mathbb{L}_0^\infty$ is dense in $\mathbb{L}^\infty$ we then can extend E_Q from $\mathbb{L}_0^\infty$ to a linear continuous functional on $\mathbb{L}^\infty(\Omega, \mathcal{F})$.

Proposition.

The linear functional $E_Q[\cdot] : \mathbb{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ satisfies (i) and (ii). Inversely each linear functional $\eta(\cdot) : \mathbb{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ satisfying (i) and (ii) induces a finitely additive probability measure via $Q_\eta(A) = \eta(\mathbf{I}_A)$, $A \in \mathcal{F}$. The corresponding expectation is η itself

$$\eta(X) = \int_\Omega X(\omega) Q_\eta(d\omega).$$

□

Theorem

A sublinear expectation $\mathbb{E}$ *has the following representation: there exists a subset* $\mathcal{Q} \subset \mathcal{P}_f$*, such that*

$$\mathbb{E}[X] = \sup_{Q \in \mathcal{Q}} E_Q[X] \quad \text{for } X \in \mathcal{H}.$$

Proof.

By Theorem –t1, we have

$$\mathbb{E}[X] = \sup_{\theta \in \Theta} E_\theta[X] \ \text{ for } X \in \mathcal{H},$$

where E_θ is a linear expectation on $\mathcal{H}$ for fixed $\theta \in \Theta$.
We can define a new sublinear expectation on $\mathbb{L}^\infty(\Omega, \sigma(\mathcal{H}))$ by

$$\tilde{\mathbb{E}}_\theta[X] := \inf\{E_\theta[Y]; Y \geq X, Y \in \mathcal{H}\}.$$

□

Proof.

It is not difficult to check that $\tilde{\mathbb{E}}_\theta$ is a sublinear expectation on $\mathbb{L}^\infty(\Omega, \sigma(\mathcal{H}))$, where $\sigma(\mathcal{H})$ is the smallest σ-algebra generated by $\mathcal{H}$. We also have $E_\theta \leq \tilde{\mathbb{E}}_\theta$ on $\mathcal{H}$, by Hahn-Banach theorem, E_θ can be extended from $\mathcal{H}$ to $\mathbb{L}^\infty(\Omega, \sigma(\mathcal{H}))$, by Proposition –prop1, there exists $Q \in \mathcal{P}_f$, such that

$$E_\theta[X] = E_Q[X] \text{ for } X \in \mathcal{H}.$$

So there exists $\mathcal{Q} \subset \mathcal{P}_f$, such that

$$\mathbb{E}[X] = \sup_{Q \in \mathcal{Q}} E_Q[X] \text{ for } X \in \mathcal{H}.$$

□

Exercise.

Prove that $\tilde{\mathbb{E}}_\theta$ is a sublinear expectation. □

Definition

Let X_1 and X_2 be two n–dimensional random vectors defined on nonlinear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$, respectively. They are called **identically distributed**, denoted by $X_1 \overset{d}{=} X_2$, if

$$\mathbb{E}_1[\varphi(X_1)] = \mathbb{E}_2[\varphi(X_2)] \quad \text{for } \varphi \in C_{Lip}(\mathbb{R}^n).$$

It is clear that $X_1 \overset{d}{=} X_2$ if and only if their distributions coincide. We say that the distribution of X_1 is stronger than that of X_2 if $\mathbb{E}_1[\varphi(X_1)] \geq \mathbb{E}_2[\varphi(X_2)]$, for each $\varphi \in C_{Lip}(\mathbb{R}^n)$.

Remark.

In the case of sublinear expectations, $X_1 \overset{d}{=} X_2$ implies that the uncertainty subsets of distributions of X_1 and X_2 are the same, e.g., in the framework of Remark –r1,

$$\{F_{X_1}(\theta_1, \cdot) : \theta_1 \in \Theta_1\} = \{F_{X_2}(\theta_2, \cdot) : \theta_2 \in \Theta_2\}.$$

Similarly if the distribution of X_1 is stronger than that of X_2, then

$$\{F_{X_1}(\theta_1, \cdot) : \theta_1 \in \Theta_1\} \supset \{F_{X_2}(\theta_2, \cdot) : \theta_2 \in \Theta_2\}.$$

□

The distribution of $X \in \mathcal{H}$ has the following four typical parameters:

$$\bar{\mu} := \mathbb{E}[X], \;\; \underline{\mu} := -\mathbb{E}[-X], \qquad \bar{\sigma}^2 := \mathbb{E}[X^2], \;\; \underline{\sigma}^2 := -\mathbb{E}[-X^2].$$

The intervals $[\underline{\mu}, \bar{\mu}]$ and $[\underline{\sigma}^2, \bar{\sigma}^2]$ characterize the **mean-uncertainty** and the **variance-uncertainty** of X respectively.
A natural question is: can we find a family of distribution measures to represent the above sublinear distribution of X? The answer is affirmative:

Lemma

Let $(\Omega, \mathcal{H}, \mathbb{E})$ *be a sublinear expectation space. Let* $X \in \mathcal{H}^d$ *be given. Then for each sequence* $\{\varphi_n\}_{n=1}^{\infty} \subset C_{Lip}(\mathbb{R}^d)$ *satisfying* $\varphi_n \downarrow 0$*, we have* $\mathbb{E}[\varphi_n(X)] \downarrow 0$*.*

Proof.

For each fixed $N > 0$,

$$\varphi_n(x) \leq k_n^N + \varphi_1(x)\mathbf{I}_{[|x|>N]} \leq k_n^N + \frac{\varphi_1(x)|x|}{N} \text{ for each } x \in \mathbb{R}^{d\times m},$$

where $k_n^N = \max_{|x|\leq N} \varphi_n(x)$. We then have

$$\mathbb{E}[\varphi_n(X)] \leq k_n^N + \frac{1}{N^\delta}\mathbb{E}[\varphi_1(X)|X|^\delta].$$

It follows from $\varphi_n \downarrow 0$ that $k_n^N \downarrow 0$. Thus we have $\lim_{n\to\infty} \mathbb{E}[\varphi_n(X)] \leq \frac{C}{N}\mathbb{E}[\varphi_1(X)|X|]$. Since N can be arbitrarily large, we get $\mathbb{E}[\varphi_n(X)] \downarrow 0$. □

Lemma

Let $(\Omega, \mathcal{H}, \mathbb{E})$ *be a sublinear expectation space and let* $\mathbb{F}_X[\varphi] := \mathbb{E}[\varphi(X)]$ *be the sublinear distribution of* $X \in \mathcal{H}^d$. *Then there exists a family of probability measures* $\{F_\theta\}_{\theta \in \Theta}$ *defined on* $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ *such that*

$$\mathbb{F}_X[\varphi] = \sup_{\theta \in \Theta} \int_{\mathbb{R}^d} \varphi(x) F_\theta(dx), \quad \varphi \in C_{l,Lip}(\mathbb{R}^d). \tag{2}$$

Proof.

By the representation theorem, for the sublinear expectation $\mathbb{F}_X[\varphi]$ defined on $(\mathbb{R}^d, C_{Lip}(\mathbb{R}^n))$, there exists a family of linear expectations $\{f_\theta\}_{\theta\in\Theta}$ on $(\mathbb{R}^d, C_{Lip}(\mathbb{R}^n))$ such that

$$\mathbb{F}_X[\varphi] = \sup_{\theta\in\Theta} f_\theta[\varphi], \ \varphi \in C_{Lip}(\mathbb{R}^n).$$

By the above lemma, for each sequence $\{\varphi_n\}_{n=1}^\infty$ in $C_{b.Lip}(\mathbb{R}^n)$ such that $\varphi_n \downarrow 0$ on $\mathbb{R}^d$, $\mathbb{F}_X[\varphi_n] \downarrow 0$, thus $f_\theta[\varphi_n] \downarrow 0$ for each $\theta \in \Theta$. It follows from Daniell-Stone Theorem (see Theorem –TheoremDL in Appendix B) that, for each $\theta \in \Theta$, there exists a unique probability measure $F_\theta(\cdot)$ on $(\mathbb{R}^d, \sigma(C_{b.Lip}(\mathbb{R}^d)) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, such that $f_\theta[\varphi] = \int_{\mathbb{R}^d} \varphi(x) F_\theta(dx)$. Thus we have (–Distr). □

Remark.

The above lemma tells us that in fact the sublinear distribution $\mathbb{F}_X$ of X characterizes the uncertainty of distribution of X which is an subset of distributions $\{F_\theta\}_{\theta\in\Theta}$. □

The following property is very useful in our sublinear expectation theory.

Proposition.

Let $(\Omega, \mathcal{H}, \mathbb{E})$ be a sublinear expectation space and X, Y be two random variables such that $\mathbb{E}[Y] = -\mathbb{E}[-Y]$, i.e., Y has no mean-uncertainty. Then we have

$$\mathbb{E}[X + \alpha Y] = \mathbb{E}[X] + \alpha \mathbb{E}[Y] \quad \text{for } \alpha \in \mathbb{R}.$$

In particular, if $\mathbb{E}[Y] = \mathbb{E}[-Y] = 0$, then $\mathbb{E}[X + \alpha Y] = \mathbb{E}[X]$. □

We have

$$\mathbb{E}[\alpha Y] = \alpha^+\mathbb{E}[Y] + \alpha^-\mathbb{E}[-Y] = \alpha^+\mathbb{E}[Y] - \alpha^-\mathbb{E}[Y] = \alpha\mathbb{E}[Y] \text{ for } \alpha \in \mathbb{R}.$$

Thus

$$\mathbb{E}[X + \alpha Y] \leq \mathbb{E}[X] + \mathbb{E}[\alpha Y] = \mathbb{E}[X] + \alpha\mathbb{E}[Y] = \mathbb{E}[X] - \mathbb{E}[-\alpha Y] \leq \mathbb{E}[X +$$

A more general form of the above proposition is:

Proposition.

We make the same assumptions as the previous proposition. Let $\tilde{\mathbb{E}}$ be a nonlinear expectation on $(\Omega, \mathcal{H})$ dominated by the sublinear expectation $\mathbb{E}$ in the sense of (**??**). If $\mathbb{E}[Y] = \mathbb{E}[-Y]$, then we have

$$\tilde{\mathbb{E}}[\alpha Y] = \alpha \tilde{\mathbb{E}}[Y] = \alpha \mathbb{E}[Y], \quad \alpha \in \mathbb{R} \tag{3}$$

as well as

$$\tilde{\mathbb{E}}[X + \alpha Y] = \tilde{\mathbb{E}}[X] + \alpha \tilde{\mathbb{E}}[Y], \; X \in \mathcal{H}, \; \alpha \in \mathbb{R}. \tag{4}$$

In particular

$$\tilde{\mathbb{E}}[X + c] = \tilde{\mathbb{E}}[X] + c, \text{ for } c \in \mathbb{R}. \tag{5}$$

□

Proof.

We have

$$-\tilde{\mathbb{E}}[Y] = \tilde{\mathbb{E}}[0] - \tilde{\mathbb{E}}[Y] \le \mathbb{E}[-Y] = -\mathbb{E}[Y] \le -\tilde{\mathbb{E}}[Y]$$

and

$$\begin{aligned} \mathbb{E}[Y] &= -\mathbb{E}[-Y] \le -\tilde{\mathbb{E}}[-Y] \\ &= \tilde{\mathbb{E}}[0] - \tilde{\mathbb{E}}[-Y] \le \mathbb{E}[Y]. \end{aligned}$$

From the above relations we have $\tilde{\mathbb{E}}[Y] = \mathbb{E}[Y] = -\tilde{\mathbb{E}}[-Y]$ and thus (–el.3.1). Still by the domination,

$$\begin{aligned} &\tilde{\mathbb{E}}[X + \alpha Y] - \tilde{\mathbb{E}}[X] \le \mathbb{E}[\alpha Y], \\ &\tilde{\mathbb{E}}[X] - \tilde{\mathbb{E}}[X + \alpha Y] \le \mathbb{E}[-\alpha Y] = -\mathbb{E}[\alpha Y]. \end{aligned}$$

Thus (–el.3.2) holds. □

Definition

A sequence of n-dimensional random vectors $\{\eta_i\}_{i=1}^{\infty}$ defined on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is said to **converge in distribution** (or **converge in law**) under $\mathbb{E}$ if for each $\varphi \in C_{b.Lip}(\mathbb{R}^n)$, the sequence $\{\mathbb{E}[\varphi(\eta_i)]\}_{i=1}^{\infty}$ converges.

The following result is easy to check.

Proposition.

Let $\{\eta_i\}_{i=1}^{\infty}$ converge in law in the above sense. Then the mapping $\mathbb{F}[\cdot]: C_{b.Lip}(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$\mathbb{F}[\varphi] := \lim_{i\to\infty} \mathbb{E}[\varphi(\eta_i)] \quad \text{for } \varphi \in C_{b.Lip}(\mathbb{R}^n)$$

is a sublinear expectation defined on $(\mathbb{R}^n, C_{b.Lip}(\mathbb{R}^n))$. □

The following notion of independence plays a key role in the nonlinear expectation theory.

Definition

In a nonlinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, a random vector $Y \in \mathcal{H}^n$ is said to be **independent** from another random vector $X \in \mathcal{H}^m$ under $\mathbb{E}[\cdot]$ if for each test function $\varphi \in C_{Lip}(\mathbb{R}^{m+n})$ we have

$$\mathbb{E}[\varphi(X, Y)] = \mathbb{E}[\mathbb{E}[\varphi(x, Y)]_{x=X}].$$

Remark.

In particular, for a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, Y is independent from X means that the uncertainty of distributions $\{F_Y(\theta, \cdot) : \theta \in \Theta\}$ of Y does not change after the realization of $X = x$. In other words, the "conditional sublinear expectation" of Y with respect to X is $\mathbb{E}[\varphi(x, Y)]_{x=X}$. In the case of linear expectation, this notion of independence is just the classical one. □

Remark.

It is important to note that under sublinear expectations the condition "Y is independent from X" does not imply automatically that "X is independent from Y". □

Example

We consider a case where $\mathbb{E}$ is a sublinear expectation and $X, Y \in \mathcal{H}$ are identically distributed with $\mathbb{E}[X] = \mathbb{E}[-X] = 0$ and $\overline{\sigma}^2 = \mathbb{E}[X^2] > \underline{\sigma}^2 = -\mathbb{E}[-X^2]$. We also assume that $\mathbb{E}[|X|] = \mathbb{E}[X^+ + X^-] > 0$, thus $\mathbb{E}[X^+] = \frac{1}{2}\mathbb{E}[|X| + X] = \frac{1}{2}\mathbb{E}[|X|] > 0$. In the case where Y is independent from X, we have

$$\mathbb{E}[XY^2] = \mathbb{E}[X^+\overline{\sigma}^2 - X^-\underline{\sigma}^2] = (\overline{\sigma}^2 - \underline{\sigma}^2)\mathbb{E}[X^+] > 0.$$

But if X is independent from Y, we have

$$\mathbb{E}[XY^2] = 0.$$

The independence property of two random vectors X, Y involves only the "joint distribution" of (X, Y). The following result tells us how to construct random vectors with given "marginal distributions" and with a specific direction of independence.

Definition.

Let $(\Omega_i, \mathcal{H}_i, \mathbb{E}_i)$, $i = 1,2$ be two sublinear (resp. nonlinear) expectation spaces. We denote

$$\mathcal{H}_1 \otimes \mathcal{H}_2 := \{Z(\omega_1, \omega_2) = \varphi(X(\omega_1), Y(\omega_2)) : (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2, \\ (X, Y) \in \mathcal{H}_1^m \times \mathcal{H}_2^n, \ \varphi \in C_{Lip}(\mathbb{R}^{m+n})\},$$

and, for each random variable of the above form
$Z(\omega_1, \omega_2) = \varphi(X(\omega_1), Y(\omega_2))$,

$$(\mathbb{E}_1 \otimes \mathbb{E}_2)[Z] := \mathbb{E}_1[\bar{\varphi}(X)], \quad \text{where } \bar{\varphi}(x) := \mathbb{E}_2[\varphi(x, Y)], \ x \in \mathbb{R}^m.$$

□

Definition (continue) .

It is easy to check that the triple $(\Omega_1 \times \Omega_2, \mathcal{H}_1 \otimes \mathcal{H}_2, \mathbb{E}_1 \otimes \mathbb{E}_2)$ forms a sublinear (resp. nonlinear) expectation space. We call it the **product space** of sublinear (resp. nonlinear) expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$. In this way, we can define the product space

$$(\prod_{i=1}^{n} \Omega_i, \bigotimes_{i=1}^{n} \mathcal{H}_i, \bigotimes_{i=1}^{n} \mathbb{E}_i)$$

of given sublinear (resp. nonlinear) expectation spaces $(\Omega_i, \mathcal{H}_i, \mathbb{E}_i)$, $i = 1, 2, \cdots, n$. In particular, when $(\Omega_i, \mathcal{H}_i, \mathbb{E}_i) = (\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ we have the product space of the form $(\Omega_1^n, \mathcal{H}_1^{\otimes n}, \mathbb{E}_1^{\otimes n})$. □

Let $X, \bar{X}$ be two n-dimensional random vectors on a sublinear (resp. nonlinear) expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. $\bar{X}$ is called an independent copy of X if $\bar{X} \overset{d}{=} X$ and $\bar{X}$ is independent from X.

The following property is easy to check.

Proposition.

Let X_i be an n_i-dimensional random vector on sublinear (resp. nonlinear) expectation space $(\Omega_i, \mathcal{H}_i, \mathbb{E}_i)$ for $i = 1, \cdots, n$, respectively. We denote

$$Y_i(\omega_1, \cdots, \omega_n) := X_i(\omega_i), \quad i = 1, \cdots, n.$$

Then Y_i, $i = 1, \cdots, n$, are random vectors on $(\prod_{i=1}^n \Omega_i, \bigotimes_{i=1}^n \mathcal{H}_i, \bigotimes_{i=1}^n \mathbb{E}_i)$. Moreover we have $Y_i \stackrel{d}{=} X_i$ and Y_{i+1} is independent from $(Y_1, \cdots, Y_i)$, for each i.

Furthermore, if $(\Omega_i, \mathcal{H}_i, \mathbb{E}_i) = (\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $X_i \stackrel{d}{=} X_1$, for all i, then we also have $Y_i \stackrel{d}{=} Y_1$. In this case Y_i is said to be an **independent copy** of Y_1 for $i = 2, \cdots, n$. □

Remark.

In the above construction the integer n can be also infinite. In this case each random variable $X \in \bigotimes_{i=1}^{\infty} \mathcal{H}_i$ belongs to $(\prod_{i=1}^{k} \Omega_i, \bigotimes_{i=1}^{k} \mathcal{H}_i, \bigotimes_{i=1}^{k} \mathbb{E}_i)$ for some positive integer $k < \infty$ and

$$\bigotimes_{i=1}^{\infty} \mathbb{E}_i[X] := \bigotimes_{i=1}^{k} \mathbb{E}_i[X].$$

□

Remark.

The situation "Y is independent from X" often appears when Y occurs after X, thus a robust expectation should take the information of X into account. □

Exercise.

Suppose $X, Y \in \mathcal{H}^d$ and Y is an independent copy of X. Prove that for each $a \in \mathbb{R}, b \in \mathbb{R}^d$,$a + \langle b, Y\rangle$ is an independent copy of $a + \langle b, X\rangle$. □

In a sublinear expectation space we have:

Example

We consider a situation where two random variables X and Y in $\mathcal{H}$ are identically distributed and their common distribution is

$$\mathbb{F}_X[\varphi] = \mathbb{F}_Y[\varphi] = \sup_{\theta\in\Theta} \int_{\mathbb{R}} \varphi(y) F(\theta, dy) \quad \text{for } \varphi \in C_{Lip}(\mathbb{R}),$$

where for each $\theta \in \Theta$, $\{F(\theta, A)\}_{A\in\mathcal{B}(\mathbb{R})}$ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. In this case, "$Y$ is independent from X" means that the joint distribution of X and Y is

$$\mathbb{F}_{X,Y}[\psi] = \sup_{\theta_1\in\Theta} \int_{\mathbb{R}} \left[\sup_{\theta_2\in\Theta} \int_{\mathbb{R}} \psi(x, y) F(\theta_2, dy) \right] F(\theta_1, dx) \quad \text{for } \psi \in C_{Lip}(\mathbb{R}^2).$$

Exercise.

Let $(\Omega, \mathcal{H}, \mathbb{E})$ be a sublinear expectation space. Prove that if $\mathbb{E}[\varphi(X)] = \mathbb{E}[\varphi(Y)]$ for any $\varphi \in C_{b,Lip}$, then it still holds for any $\varphi \in C_{l,Lip}$. That is, we can replace $\varphi \in C_{l,Lip}$ in Definition –d1 by $\varphi \in C_{b,Lip}$. □

Let $(\Omega, \mathcal{H}, \mathbb{E})$ be a sublinear expectation space. We have the following useful inequalities.

We first give the following well-known inequalities.

Lemma

For $r > 0$ and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$|a+b|^r \leq \max\{1, 2^{r-1}\}(|a|^r + |b|^r) \quad \text{for } a, b \in \mathbb{R}, \tag{6}$$

$$|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}. \tag{7}$$

Proposition.

For each $X, Y \in \mathcal{H}$, we have

$$\mathbb{E}[|X+Y|^r] \leq 2^{r-1}(\mathbb{E}[|X|^r] + \mathbb{E}[|Y|^r]), \tag{8}$$

$$\mathbb{E}[|XY|] \leq (\mathbb{E}[|X|^p])^{1/p} \cdot (\mathbb{E}[|Y|^q])^{1/q}, \tag{9}$$

$$(\mathbb{E}[|X+Y|^p])^{1/p} \leq (\mathbb{E}[|X|^p])^{1/p} + (\mathbb{E}[|Y|^p])^{1/p}, \tag{10}$$

where $r \geq 1$ and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.
In particular, for $1 \leq p < p'$, we have $(\mathbb{E}[|X|^p])^{1/p} \leq (\mathbb{E}[|X|^{p'}])^{1/p'}$. □

Proof.

The inequality (–ee04.5) follows from (–ee04.3).
For the case $\mathbb{E}[|X|^p] \cdot \mathbb{E}[|Y|^q] > 0$, we set

$$\xi = \frac{X}{(\mathbb{E}[|X|^p])^{1/p}}, \quad \eta = \frac{Y}{(\mathbb{E}[|Y|^q])^{1/q}}.$$

By (–ee04.4) we have

$$\mathbb{E}[|\xi\eta|] \leq \mathbb{E}[\frac{|\xi|^p}{p} + \frac{|\eta|^q}{q}] \leq \mathbb{E}[\frac{|\xi|^p}{p}] + \mathbb{E}[\frac{|\eta|^q}{q}]$$
$$= \frac{1}{p} + \frac{1}{q} = 1.$$

Thus (–ee04.6) follows. □

Proof (continue) .

For the case $\mathbb{E}[|X|^p] \cdot \mathbb{E}[|Y|^q] = 0$, we consider $\mathbb{E}[|X|^p] + \varepsilon$ and $\mathbb{E}[|Y|^q] + \varepsilon$ for $\varepsilon > 0$. Applying the above method and letting $\varepsilon \to 0$, we get (–ee04.6).

We now prove (–ee04.7). We only consider the case $\mathbb{E}[|X+Y|^p] > 0$.

$$\begin{aligned}\mathbb{E}[|X+Y|^p] &= \mathbb{E}[|X+Y| \cdot |X+Y|^{p-1}] \\ &\leq \mathbb{E}[|X| \cdot |X+Y|^{p-1}] + \mathbb{E}[|Y| \cdot |X+Y|^{p-1}] \\ &\leq (\mathbb{E}[|X|^p])^{1/p} \cdot (\mathbb{E}[|X+Y|^{(p-1)q}])^{1/q} \\ &\quad + (\mathbb{E}[|Y|^p])^{1/p} \cdot (\mathbb{E}[|X+Y|^{(p-1)q}])^{1/q}.\end{aligned}$$

Since $(p-1)q = p$, we have (–ee04.7).

By(–ee04.6), it is easy to deduce that $(\mathbb{E}[|X|^p])^{1/p} \leq (\mathbb{E}[|X|^{p'}])^{1/p'}$ for $1 \leq p < p'$. □

For each fixed $p \geq 1$, we observe that $\mathcal{H}_0^p = \{X \in \mathcal{H}, \mathbb{E}[|X|^p] = 0\}$ is a linear subspace of $\mathcal{H}$. Taking $\mathcal{H}_0^p$ as our null space, we introduce the quotient space $\mathcal{H}/\mathcal{H}_0^p$. Observing that, for every $\{X\} \in \mathcal{H}/\mathcal{H}_0^p$ with a representation $X \in \mathcal{H}$, we can define an expectation $\mathbb{E}[\{X\}] := \mathbb{E}[X]$ which is still a sublinear expectation. We set $\|X\|_p := (\mathbb{E}[|X|^p])^{\frac{1}{p}}$. By Proposition **??**, it is easy to check that $\|\cdot\|_p$ forms a Banach norm on $\mathcal{H}/\mathcal{H}_0^p$. We extend $\mathcal{H}/\mathcal{H}_0^p$ to its completion $\hat{\mathcal{H}}_p$ under this norm, then $(\hat{\mathcal{H}}_p, \|\cdot\|_p)$ is a Banach space. In particular, when $p = 1$, we denote it by $(\hat{\mathcal{H}}, \|\cdot\|)$.

For each $X \in \mathcal{H}$, the mappings

$$X^+(\omega) : \mathcal{H} \to \mathcal{H} \quad \text{and} \quad X^-(\omega) : \mathcal{H} \to \mathcal{H}$$

satisfy

$$|X^+ - Y^+| \leq |X - Y| \quad \text{and} \quad |X^- - Y^-| = |(-X)^+ - (-Y)^+| \leq |X - Y|.$$

Thus they are both contraction mappings under $\|\cdot\|_p$ and can be continuously extended to the Banach space $(\hat{\mathcal{H}}_p, \|\cdot\|_p)$.

We can define the partial order "$\geq$" in this Banach space.

Definition

An element X in $(\hat{\mathcal{H}}, \|\cdot\|)$ is said to be nonnegative, or $X \geq 0$, $0 \leq X$, if $X = X^{+}$. We also denote by $X \geq Y$, or $Y \leq X$, if $X - Y \geq 0$.

It is easy to check that $X \geq Y$ and $Y \geq X$ imply $X = Y$ on $(\hat{\mathcal{H}}_p, \|\cdot\|_p)$.
For each $X, Y \in \mathcal{H}$, note that

$$|\mathbb{E}[X] - \mathbb{E}[Y]| \leq \mathbb{E}[|X - Y|] \leq ||X - Y||_p.$$

We then can define

Definition

The sublinear expectation $\mathbb{E}[\cdot]$ can be continuously extended to $(\hat{\mathcal{H}}_p, \|\cdot\|_p)$ on which it is still a sublinear expectation. We still denote by $(\Omega, \hat{\mathcal{H}}_p, \mathbb{E})$. Let $(\Omega, \mathcal{H}, \mathbb{E}_1)$ be a nonlinear expectation space. $\mathbb{E}_1$ is said to be dominated by $\mathbb{E}$ if

$$\mathbb{E}_1[X] - \mathbb{E}_1[Y] \leq \mathbb{E}[X - Y] \quad \text{for } X, Y \in \mathcal{H}.$$

From this we can easily deduce that $|\mathbb{E}_1[X] - \mathbb{E}_1[Y]| \leq \mathbb{E}[|X - Y|]$, thus the nonlinear expectation $\mathbb{E}_1[\cdot]$ can be continuously extended to $(\hat{\mathcal{H}}_p, \|\cdot\|_p)$ on which it is still a nonlinear expectation. We still denote by $(\Omega, \hat{\mathcal{H}}_p, \mathbb{E}_1)$.

Remark.

It is important to note that $X_1, \cdots, X_n \in \hat{\mathcal{H}}$ does not imply $\varphi(X_1, \cdots, X_n) \in \hat{\mathcal{H}}$ for each $\varphi \in C_{Lip}(\mathbb{R}^n)$. Thus, when we talk about the notions of distributions, independence and product spaces on $(\Omega, \hat{\mathcal{H}}, \mathbb{E})$, the space $C_{Lip}(\mathbb{R}^n)$ is replaced by $C_{b.Lip}(\mathbb{R}^n)$ unless otherwise stated. □

Exercise.

Prove that the inequalities (–ee04.5),(–ee04.6),(–ee04.7) still hold for $(\Omega, \hat{\mathcal{H}}, \mathbb{E})$. □

Let the pair $(\Omega, \mathcal{H})$ be such that Ω is a set of scenarios and $\mathcal{H}$ is the collection of all possible risk positions in a financial market.
If $X \in \mathcal{H}$, then for each constant c, $X \vee c$, $X \wedge c$ are all in $\mathcal{H}$. One typical example in finance is that X is the tomorrow's price of a stock. In this case, any European call or put options with strike price K of forms $(S-K)^+$, $(K-S)^+$ are in $\mathcal{H}$.

A risk supervisor is responsible for taking a rule to tell traders, securities companies, banks or other institutions under his supervision, which kind of risk positions is unacceptable and thus a minimum amount of risk capitals should be deposited to make the positions acceptable. The collection of acceptable positions is defined by

$$\mathcal{A} = \{X \in \mathcal{H} : X \text{ is acceptable}\}.$$

This set has meaningful properties in economy.

Definition

A set $\mathcal{A}$ is called a **coherent acceptable set** if it satisfies

(i) Monotonicity:

$$X \in \mathcal{A},\ Y \geq X \text{ imply } Y \in \mathcal{A}.$$

(ii) $0 \in \mathcal{A}$ but $-1 \notin \mathcal{A}$.

(iii) Positive homogeneity

$$X \in \mathcal{A} \text{ implies } \lambda X \in \mathcal{A} \text{ for } \lambda \geq 0.$$

(iv) Convexity:

$$X, Y \in \mathcal{A} \text{ imply } \alpha X + (1-\alpha) Y \in \mathcal{A} \text{ for } \alpha \in [0,1].$$

Remark.

(iii)+(iv) imply

(v) Sublinearity:

$$X, Y \in \mathcal{A} \Rightarrow \mu X + \nu Y \in \mathcal{A} \text{ for } \mu, \nu \geq 0.$$

□

Remark.

If the set $\mathcal{A}$ only satisfies (i),(ii) and (iv), then $\mathcal{A}$ is called a **convex acceptable set**. □

In this section we mainly study the coherent case. Once the rule of the acceptable set is fixed, the minimum requirement of risk deposit is then automatically determined.

Definition

Given a coherent acceptable set $\mathcal{A}$, the functional $\rho(\cdot)$ defined by

$$\rho(X) = \rho_{\mathcal{A}}(X) := \inf\{m \in \mathbb{R} : \ m + X \in \mathcal{A}\}, \quad X \in \mathcal{H}$$

is called the **coherent risk measure** related to $\mathcal{A}$.

It is easy to see that

$$\rho(X+\rho(X))=0.$$

Proposition.

$\rho(\cdot)$ is a coherent risk measure satisfying four properties:

(i) Monotonicity: If $X \geq Y$ then $\rho(X) \leq \rho(Y)$.
(ii) Constant preserving: $\rho(1)=-\rho(-1)=-1$.
(iii) Sub-additivity: For each $X, Y \in \mathcal{H}$, $\rho(X+Y) \leq \rho(X)+\rho(Y)$.
(iv) Positive homogeneity: $\rho(\lambda X)=\lambda\rho(X)$ for $\lambda \geq 0$. □

Proof.

(i), (ii) are obvious.

We now prove (iii). Indeed,

$$\begin{aligned}\rho(X+Y) =& \inf\{m \in \mathbb{R} : \ m+(X+Y) \in \mathcal{A}\} \\ =& \inf\{m+n : m, n \in \mathbb{R}, \ (m+X)+(n+Y) \in \mathcal{A}\} \\ \leq& \inf\{m \in \mathbb{R} : \ m+X \in \mathcal{A}\} + \inf\{n \in \mathbb{R} : \ n+Y \in \mathcal{A}\} \\ =& \rho(X)+\rho(Y).\end{aligned}$$

To prove (iv), in fact the case $\lambda = 0$ is trivial; when $\lambda > 0$,

$$\begin{aligned}\rho(\lambda X) &= \inf\{m \in \mathbb{R} : \ m+\lambda X \in \mathcal{A}\} \\ &= \lambda \inf\{n \in \mathbb{R} : \ n+X \in \mathcal{A}\} = \lambda\rho(X),\end{aligned}$$

where $n = m/\lambda$. □

Obviously, if $\mathbb{E}$ is a sublinear expectation, we define $\rho(X) := \mathbb{E}[-X]$, then ρ is a coherent risk measure. Inversely, if ρ is a coherent risk measure, we define $\mathbb{E}[X] := \rho(-X)$, then $\mathbb{E}$ is a sublinear expectation.

Exercise.

Let $\rho(\cdot)$ be a coherent risk measure. Then we can inversely define

$$\mathcal{A}_\rho := \{X \in \mathcal{H} : \rho(X) \leq 0\}.$$

Prove that $\mathcal{A}_\rho$ is a coherent acceptable set. □

The sublinear expectation is also called the upper expectation (see Huber (1981) [**?**] in robust statistics), or the upper prevision in the theory of imprecise probabilities (see Walley (1991) [**?**] and a rich literature provided in the Notes of this book). To our knowledge, the Representation Theorem –t1 was firstly obtained for the case where Ω is a finite set by [**?**], and this theorem was rediscovered independently by Artzner, Delbaen, Eber and Heath (1999) [**?**] and then by Delbaen (2002) [**?**] for the general Ω.

A typical example of dynamic nonlinear expectation, called g–expectation (small g), was introduced in Peng (1997) [?] in the framework of backward stochastic differential equations. Readers are referred to Briand, Coquet, Hu, Mémin and Peng (2000) [?], Chen (1998) [?], Chen and Epstein (2002) [?], Chen, Kulperger and Jiang (2003) [?], Chen and Peng (1998) [?] and (2000) [?], Coquet, Hu, Mémin and Peng (2001) [?] (2002) [?] , Jiang (2004) [?], Jiang and Chen (2004) [?, ?], Peng (1999) [?] and (2004) [?], Peng and Xu (2003) [?] and Rosazza (2006) [?] for the further development of this theory.

It seems that the notions of distributions and independence under nonlinear expectations were new. We think that these notions are perfectly adapted for the further development of dynamic nonlinear expectations. For other types of the related notions of distributions and independence under nonlinear expectations or non-additive probabilities, we refer to the Notes of the book [**?**] and the references listed in Marinacci (1999) [**?**] and Maccheroni and Marinacci (2005) [**?**]. Coherent risk measures can be also regarded as sublinear expectations defined on the space of risk positions in financial market. This notion was firstly introduced in [**?**]. Readers can be referred also to the well-known book of Föllmer and Schied (2004) [**?**] for the systematical presentation of coherent risk measures and convex risk measures. For the dynamic risk measure in continuous time, see [**?**] or [**?**], Barrieu and El Karoui (2004) [**?**] using g-expectations. Super-hedging and super pricing (see El Karoui and Quenez (1995) [**?**] and El Karoui, Peng and Quenez (1997) [**?**]) are also closely related to this formulation.

Law of Large Numbers and Central Limit Theorem

(label)ch2 In this chapter, we first introduce two types of fundamentally important distributions, namely, maximal distribution and G-normal distribution, in the theory of sublinear expectations. The former corresponds to constants and the latter corresponds to normal distribution in classical probability theory. We then present the law of large numbers (LLN) and central limit theorem (CLT) under sublinear expectations. It is worth pointing out that the limit in LLN is a maximal distribution and the limit in CLT is a G-normal distribution.

We will firstly define a special type of very simple distributions which are frequently used in practice, known as "worst case risk measure".

Definition

(**maximal distribution**) (label)Prop-G1 copy(1)A d-dimensional random vector $\eta = (\eta_1, \cdots, \eta_d)$ on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called **maximal distributed** if there exists a bounded, closed and convex subset $\Gamma \subset \mathbb{R}^d$ such that

$$\mathbb{E}[\varphi(\eta)] = \max_{y \in \Gamma} \varphi(y).$$

Remark.

Here Γ gives the degree of uncertainty of η. It is easy to check that this maximal distributed random vector η satisfies

$$a\eta + b\bar{\eta} \stackrel{d}{=} (a+b)\eta \quad \text{for } a, b \geq 0,$$

where $\bar{\eta}$ is an independent copy of η. We will see later that in fact this relation characterizes a maximal distribution. Maximal distribution is also called "worst case risk measure" in finance. □

Remark.

When $d = 1$ we have $\Gamma = [\underline{\mu}, \overline{\mu}]$, where $\overline{\mu} = \mathbb{E}[\eta]$ and $\underline{\mu} = -\mathbb{E}[-\eta]$. The distribution of η is

$$\mathbb{F}_\eta[\varphi] = \mathbb{E}[\varphi(\eta)] = \sup_{\underline{\mu} \leq y \leq \bar{\mu}} \varphi(y) \quad \text{for } \varphi \in C_{Lip}(\mathbb{R}).$$

$\square$

Recall a well-known characterization: $X \overset{d}{=} N(0, \Sigma)$ if and only if

$$(label)ch2e1aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2}X \quad \text{for } a, b \geq 0, \tag{11}$$

where $\bar{X}$ is an independent copy of X. The covariance matrix Σ is defined by $\Sigma = E[XX^T]$. We now consider the so called G-normal distribution in probability model uncertainty situation. The existence, uniqueness and characterization will be given later.

Definition

(G-**normal distribution**) (label)Def-Gnormal copy(2) A d-dimensional random vector $X = (X_1, \cdots, X_d)^T$ on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called (centralized) G-**normal distributed** if $X_i^2 \in \mathcal{H}$ for $i = 1, \cdots, d$ and

$$aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2} X \quad \text{for } a, b \geq 0,$$

where $\bar{X}$ is an independent copy of X.

Remark.

Noting that $\mathbb{E}[X+\bar{X}] = 2\mathbb{E}[X]$ and $\mathbb{E}[X+\bar{X}] = \mathbb{E}[\sqrt{2}X] = \sqrt{2}\mathbb{E}[X]$, we then have $\mathbb{E}[X] = 0$. Similarly, we can prove that $\mathbb{E}[-X] = 0$. Namely, X has no mean-uncertainty. □

The following property is easy to prove by the definition.

Proposition.

(label)GCDLet X be G-normal distributed. Then for each $A \in \mathbb{R}^{m \times d}$, AX is also G-normal distributed. In particular, for each $\mathbf{a} \in \mathbb{R}^d$, $\langle \mathbf{a}, X \rangle$ is a 1-dimensional G-normal distributed random variable, but its inverse is not true (see Exercise –ex1). □

We denote by $\mathbb{S}(d)$(label)sd the collection of all $d \times d$ symmetric matrices. Let X be G-normal distributed and η be maximal distributed d-dimensional random vectors on $(\Omega, \mathcal{H}, \mathbb{E})$. The following function is very important to characterize their distributions:

$$G(p, A) := \mathbb{E}[\frac{1}{2}\langle AX, X\rangle + \langle p, \eta\rangle], \quad (p, A) \in \mathbb{R}^d \times \mathbb{S}(d).(label)e313 \tag{12}$$

It is easy to check that G is a sublinear function monotonic in $A \in \mathbb{S}(d)$ in the following sense: for each $p, \bar{p} \in \mathbb{R}^d$ and $A, \bar{A} \in \mathbb{S}(d)$

$$\begin{cases} G(p+\bar{p}, A+\bar{A}) & \leq G(p, A) + G(\bar{p}, \bar{A}), \\ G(\lambda p, \lambda A) & = \lambda G(p, A), \quad \forall \lambda \geq 0, \quad (label)e314 \\ G(p, A) & \geq G(p, \bar{A}), \quad \text{if } A \geq \bar{A}. \end{cases} \tag{13}$$

Clearly, G is also a continuous function.

By Theorem –t1 in Chapter –ch1, there exists a bounded and closed subset $\Gamma \subset \mathbb{R}^d \times \mathbb{R}^{d\times d}$ such that

$$(label)ch2e2G(p,A) = \sup_{(q,Q)\in\Gamma} [\frac{1}{2}\mathrm{tr}[AQQ^T] + \langle p,q\rangle] \text{ for } (p,A) \in \mathbb{R}^d \times \mathbb{S}(d). \quad (14)$$

We have the following result, which will be proved in the next section.

Proposition.

(label)Prop-Gnorm copy(1) Let $G : \mathbb{R}^d \times \mathbb{S}(d) \to \mathbb{R}$ be a given sublinear and continuous function, monotonic in $A \in \mathbb{S}(d)$ in the sense of (–e314). Then there exists a G-normal distributed d-dimensional random vector X and a maximal distributed d-dimensional random vector η on some sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ satisfying (–e313) and

$$(aX + b\bar{X}, a^2\eta + b^2\bar{\eta}) \overset{d}{=} (\sqrt{a^2 + b^2}X, (a^2 + b^2)\eta), \text{ for } a, b \geq 0, (label)e311 \quad (15)$$

where $(\bar{X}, \bar{\eta})$ is an independent copy of (X, η). □

Definition

The pair (X,η) satisfying (–e311) is called G-**distributed**.

Remark.

In fact, if the pair (X,η) satisfies (–e311), then

$$aX + b\bar{X} \overset{d}{=} \sqrt{a^2+b^2}X, \ \ a\eta + b\bar{\eta} \overset{d}{=} (a+b)\eta \quad \text{for } a,b \geq 0.$$

Thus X is G-normal and η is maximal distributed. □

The above pair (X, η) is characterized by the following parabolic partial differential equation (PDE for short) defined on $[0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$:

$$\partial_t u - G(D_y u, D_x^2 u) = 0, (label) ee03 \tag{16}$$

with Cauchy condition $u|_{t=0} = \varphi$, where $G : \mathbb{R}^d \times \mathbb{S}(d) \to \mathbb{R}$ is defined by (–e313) and $D^2 u = (\partial^2_{x_i x_j} u)_{i,j=1}^d$, $Du = (\partial_{x_i} u)_{i=1}^d$. The PDE (–ee03) is called a G-**equation**.

In this book we will mainly use the notion of viscosity solution to describe the solution of this PDE. For reader's convenience, we give a systematical introduction of the notion of viscosity solution and its related properties used in this book (see Appendix C, Section 1-3). It is worth to mention here that for the case where G is non-degenerate, the viscosity solution of the G-equation becomes a classical $C^{1,2}$ solution (see Appendix C, Section 4). Readers without knowledge of viscosity solutions can simply understand solutions of the G-equation in the classical sense along the whole book.

Proposition.

For the pair (X,η) satisfying (–e311) and a function $\varphi \in C_{Lip}(\mathbb{R}^d \times \mathbb{R}^d)$, we define

$$u(t,x,y) := \mathbb{E}[\varphi(x+\sqrt{t}X, y+t\eta)],\ (t,x,y) \in [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d.$$

Then we have

$$u(t+s,x,y) = \mathbb{E}[u(t,x+\sqrt{s}X, y+s\eta)], \quad s \geq 0.(label)e315 \qquad (17)$$

□

Proposition (continue).

We also have the estimates: for each $T > 0$, there exist constants $C, k > 0$ such that, for all $t, s \in [0, T]$ and $x, \bar{x}, y, \bar{y} \in \mathbb{R}^d$,

$$|u(t,x,y) - u(t,\bar{x},\bar{y})| \leq C(1 + |x|^k + |y|^k + |\bar{x}|^k + |\bar{y}|^k)(|x - \bar{x}| + |y - \bar{y}|)(\tag{18}$$

and

$$|u(t,x,y) - u(t+s,x,y)| \leq C(1 + |x|^k + |y|^k)(s + |s|^{1/2}).(label)e05 \tag{19}$$

Moreover, u is the unique viscosity solution, continuous in the sense of (–e04) and (–e05), of the PDE (–ee03). □

Proof.

Since

$$\begin{aligned} u(t,x,y) - u(t,\bar{x},\bar{y}) &= \mathbb{E}[\varphi(x+\sqrt{t}X, y+t\eta)] - \mathbb{E}[\varphi(\bar{x}+\sqrt{t}X, \bar{y}+t\eta)] \\ &\leq \mathbb{E}[\varphi(x+\sqrt{t}X, y+t\eta) - \varphi(\bar{x}+\sqrt{t}X, \bar{y}+t\eta)] \\ &\leq \mathbb{E}[C_1(1+|X|^k+|\eta|^k+|x|^k+|y|^k+|\bar{x}|^k+|\bar{y}|^k)] \\ &\quad \times (|x-\bar{x}|+|y-\bar{y}|) \\ &\leq C(1+|x|^k+|y|^k+|\bar{x}|^k+|\bar{y}|^k)(|x-\bar{x}|+|y-\bar{y}|), \end{aligned}$$

we have (–e04).

□

Proof (continue).

Let $(\bar{X},\bar{\eta})$ be an independent copy of (X,η). By (–e311),

$$\begin{aligned}u(t+s,x,y) &= \mathbb{E}[\varphi(x+\sqrt{t+s}X,y+(t+s)\eta)] \\ &= \mathbb{E}[\varphi(x+\sqrt{s}X+\sqrt{t}\bar{X},y+s\eta+t\bar{\eta})] \\ &= \mathbb{E}[\mathbb{E}[\varphi(x+\sqrt{s}\widetilde{x}+\sqrt{t}\bar{X},y+s\widetilde{y}+t\bar{\eta})]_{(\widetilde{x},\widetilde{y})=(X,\eta)}] \\ &= \mathbb{E}[u(t,x+\sqrt{s}X,y+s\eta)],\end{aligned}$$

we thus obtain (–e315). From this and (–e04) it follows that

$$\begin{aligned}&u(t+s,x,y)-u(t,x,y) = \mathbb{E}[u(t,x+\sqrt{s}X,y+s\eta)-u(t,x,y)] \\ &\leq \mathbb{E}[C_1(1+|x|^k+|y|^k+|X|^k+|\eta|^k)(\sqrt{s}|X|+s|\eta|)],\end{aligned}$$

thus we obtain (–e05). □

Proof. (continue) .

Now, for a fixed $(t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$, let $\psi \in C_b^{2,3}([0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d)$ be such that $\psi \geq u$ and $\psi(t,x,y) = u(t,x,y)$. By (–e315) and Taylor's expansion, it follows that, for $\delta \in (0,t)$,

$$
\begin{aligned}
0 &\leq \mathbb{E}[\psi(t-\delta, x+\sqrt{\delta}X, y+\delta\eta) - \psi(t,x,y)] \\
&\leq \bar{C}(\delta^{3/2}+\delta^2) - \partial_t\psi(t,x,y)\delta \\
&\quad + \mathbb{E}[\langle D_x\psi(t,x,y), X\rangle \sqrt{\delta} + \langle D_y\psi(t,x,y), \eta\rangle \delta + \frac{1}{2}\langle D_x^2\psi(t,x,y)X, X\rangle \delta \\
&= -\partial_t\psi(t,x,y)\delta + \mathbb{E}[\langle D_y\psi(t,x,y), \eta\rangle + \frac{1}{2}\langle D_x^2\psi(t,x,y)X, X\rangle]\delta + \bar{C}(\delta^{3/2} \\
&= -\partial_t\psi(t,x,y)\delta + \delta G(D_y\psi, D_x^2\psi)(t,x,y) + \bar{C}(\delta^{3/2}+\delta^2),
\end{aligned}
$$

□

Proof. (continue) .

from which it is easy to check that

$$[\partial_t \psi - G(D_y \psi, D_x^2 \psi)](t, x, y) \leq 0.$$

Thus u is a viscosity subsolution of (–ee03). Similarly we can prove that u is a viscosity supersolution of (–ee03). □

Corollary

(label)gG-P1coro copy(1) If both (X, η) *and* $(\bar{X}, \bar{\eta})$ *satisfy* (–e311) *with the same* G, *i.e.,*

$$G(p, A) := \mathbb{E}[\frac{1}{2}\langle AX, X\rangle + \langle p, \eta\rangle] = \mathbb{E}[\frac{1}{2}\langle A\bar{X}, \bar{X}\rangle + \langle p, \bar{\eta}\rangle] \quad \text{for } (p, A) \in \mathbb{R}^d$$

then $(X, \eta) \stackrel{d}{=} (\bar{X}, \bar{\eta})$. *In particular,* $X \stackrel{d}{=} -X$.

Proof.

For each $\varphi \in C_{Lip}(\mathbb{R}^d \times \mathbb{R}^d)$, we set

$$u(t,x,y) := \mathbb{E}[\varphi(x+\sqrt{t}X, y+t\eta)],$$
$$\bar{u}(t,x,y) := \mathbb{E}[\varphi(x+\sqrt{t}\bar{X}, y+t\bar{\eta})], \ (t,x,y) \in [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d.$$

By Proposition –gG-P1 copy(1), both u and $\bar{u}$ are viscosity solutions of the G-equation (–ee03) with Cauchy condition $u|_{t=0} = \bar{u}|_{t=0} = \varphi$. It follows from the uniqueness of the viscosity solution that $u \equiv \bar{u}$. In particular,

$$\mathbb{E}[\varphi(X,\eta)] = \mathbb{E}[\varphi(\bar{X},\bar{\eta})].$$

Thus $(X,\eta) \overset{d}{=} (\bar{X},\bar{\eta})$. □

Corollary

(label)GvLet (X,η) *satisfy* (–e311). *For each* $\psi \in C_{Lip}(\mathbb{R}^d)$ *we define*

$$v(t,x) := \mathbb{E}[\psi((x+\sqrt{t}X+t\eta)], \ (t,x) \in [0,\infty) \times \mathbb{R}^d. (label)e320 \quad (20)$$

Then v *is the unique viscosity solution of the following parabolic PDE:*

$$\partial_t v - G(D_x v, D_x^2 v) = 0, \quad v|_{t=0} = \psi.(label)e318 \quad (21)$$

Moreover, we have $v(t,x+y) \equiv u(t,x,y)$, *where* u *is the solution of the PDE* (–ee03) *with initial condition* $u(t,x,y)|_{t=0} = \psi(x+y)$.

Example.

Let X be G-normal distributed. The distribution of X is characterized by

$$u(t,x) = \mathbb{E}[\varphi(x+\sqrt{t}X)], \quad \varphi \in C_{Lip}(\mathbb{R}^d).$$

In particular, $\mathbb{E}[\varphi(X)] = u(1,0)$, where u is the solution of the following parabolic PDE defined on $[0,\infty) \times \mathbb{R}^d$:

$$(label)e03\partial_t u - G(D^2 u) = 0, \quad u|_{t=0} = \varphi, \tag{22}$$

where $G = G_X(A) : \mathbb{S}(d) \to \mathbb{R}$ is defined by

$$G(A) := \frac{1}{2}\mathbb{E}[\langle AX, X\rangle], \quad A \in \mathbb{S}(d).$$

The parabolic PDE (–e03) is called a G-**heat equation**. □

Example (continue) .

It is easy to check that G is a sublinear function defined on $\mathbb{S}(d)$. By Theorem –t1 in Chapter –ch1, there exists a bounded, convex and closed subset $\Theta \subset \mathbb{S}(d)$ such that

$$(label) GaChII \frac{1}{2}\mathbb{E}[\langle AX, X\rangle] = G(A) = \frac{1}{2}\sup_{Q\in\Theta} tr[AQ], \quad A \in \mathbb{S}(d). \quad (23)$$

Since $G(A)$ is monotonic: $G(A_1) \geq G(A_2)$, for $A_1 \geq A_2$, it follows that

$$\Theta \subset \mathbb{S}_+(d) = \{\theta \in \mathbb{S}(d) : \theta \geq 0\} = \{BB^T : B \in \mathbb{R}^{d\times d}\},$$

where $\mathbb{R}^{d\times d}$(label)splusd is the set of all $d \times d$ matrices. If Θ is a singleton: $\Theta = \{Q\}$, then X is classical zero-mean normal distributed with covariance Q. In general, Θ characterizes the covariance uncertainty of X. We denote $X \stackrel{d}{=} N(\{0\} \times \Theta)$ (Recall equation (–ch2e2), we can set $(q, Q) \in \{0\} \times \Theta$). □

Example (continue) .

When $d=1$, we have $X \overset{d}{=} N(\{0\}\times[\underline{\sigma}^2,\bar{\sigma}^2])$ (We also denoted by $X \overset{d}{=} N(0,[\underline{\sigma}^2,\bar{\sigma}^2])$), where $\bar{\sigma}^2=\mathbb{E}[X^2]$ and $\underline{\sigma}^2=-\mathbb{E}[-X^2]$. The corresponding G-heat equation is

$$\partial_t u-\frac{1}{2}(\bar{\sigma}^2(\partial^2_{xx}u)^+-\underline{\sigma}^2(\partial^2_{xx}u)^-)=0,\ u|_{t=0}=\varphi.$$

For the case $\underline{\sigma}^2>0$, this equation is also called the Barenblatt equation. □

In the following two typical situations, the calculation of $\mathbb{E}[\varphi(X)]$ is very easy:

- For each **convex** function φ, we have

$$\mathbb{E}[\varphi(X)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\overline{\sigma}^2 y) \exp(-\frac{y^2}{2}) dy.$$

Indeed, for each fixed $t \geq 0$, it is easy to check that the function $u(t,x) := \mathbb{E}[\varphi(x + \sqrt{t}X)]$ is convex in x:

$$\begin{aligned} u(t, \alpha x + (1-\alpha)y) &= \mathbb{E}[\varphi(\alpha x + (1-\alpha)y + \sqrt{t}X)] \\ &\leq \alpha \mathbb{E}[\varphi(x + \sqrt{t}X)] + (1-\alpha)\mathbb{E}[\varphi(x + \sqrt{t}X)] \\ &= \alpha u(t,x) + (1-\alpha)u(t,x). \end{aligned}$$

It follows that $(\partial^2_{xx} u)^- \equiv 0$ and thus the above G-heat equation becomes

$$\partial_t u = \frac{\overline{\sigma}^2}{2} \partial^2_{xx} u, \quad u|_{t=0} = \varphi.$$

- For each **concave** function φ, we have

$$\mathbb{E}[\varphi(X)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\underline{\sigma}^2 y) \exp(-\frac{y^2}{2}) dy.$$

In particular,

$$\mathbb{E}[X] = \mathbb{E}[-X] = 0, \quad \mathbb{E}[X^2] = \overline{\sigma}^2, \quad -\mathbb{E}[-X^2] = \underline{\sigma}^2$$

and

$$\mathbb{E}[X^4] = 3\overline{\sigma}^4, \; -\mathbb{E}[-X^4] = 3\underline{\sigma}^4 \; .$$

Example.

(label)example116 Let η be maximal distributed, the distribution of η is characterized by the following parabolic PDE defined on $[0, \infty) \times \mathbb{R}^d$:

$$\partial_t u - g(Du) = 0, (label)e01 \quad u|_{t=0} = \varphi, \tag{24}$$

where $g = g_\eta(p) : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$g_\eta(p) := \mathbb{E}[\langle p, \eta\rangle], \quad p \in \mathbb{R}^d.$$

□

Example (continue) .

It is easy to check that g_η is a sublinear function defined on $\mathbb{R}^d$. By Theorem –t1 in Chapter –ch1, there exists a bounded, convex and closed subset $\bar{\Theta} \subset R^d$ such that

$$g(p) = \sup_{q \in \bar{\Theta}} \langle p, q \rangle , \qquad p \in \mathbb{R}^d.(label)e35 \tag{25}$$

□

Example (continue) .

By this characterization, we can prove that the distribution of η is given by

$$\hat{\mathbb{F}}_\eta[\varphi] = \mathbb{E}[\varphi(\eta)] = \sup_{v\in\bar{\Theta}} \varphi(v) = \sup_{v\in\bar{\Theta}} \int_{\mathbb{R}^d} \varphi(x)\delta_v(dx), \ \ \varphi \in C_{Lip}(\mathbb{R}^d), (label) \tag{26}$$

where δ_v is Dirac measure. Namely it is the maximal distribution with the uncertainty subset of probabilities as Dirac measures concentrated at $\bar{\Theta}$. We denote $\eta \overset{d}{=} N(\bar{\Theta} \times \{0\})$ (Recall equation (–ch2e2), we can set $(q, Q) \in \bar{\Theta} \times \{0\}$).

□

Example (continue) .

In particular, for $d = 1$,

$$g_\eta(p) := \mathbb{E}[p\eta] = \bar{\mu}p^+ - \underline{\mu}p^-, \quad p \in \mathbb{R},$$

where $\bar{\mu} = \mathbb{E}[\eta]$ and $\underline{\mu} = -\hat{\mathbb{E}}[-\eta]$. The distribution of η is given by (–e003). We denote $\eta \overset{d}{=} N([\underline{\mu}, \bar{\mu}] \times \{0\})$. □

Exercise.

(label)ex1 We consider $X = (X_1, X_2)$, where $X_1 \overset{d}{=} N(\{0\} \times [\underline{\sigma}^2, \overline{\sigma}^2])$ with $\overline{\sigma} > \underline{\sigma}$, X_2 is an independent copy of X_1. Show that
(1) For each $a \in \mathbb{R}^2$, $\langle a, X \rangle$ is a 1-dimensional G-normal distributed random variable.
(2) X is not G-normal distributed. □

Exercise.

Let X be G-normal distributed. For each $\varphi \in C_{Lip}(\mathbb{R}^d)$, we define a function

$$u(t,x) := \mathbb{E}[\varphi(x+\sqrt{t}X)],\ (t,x) \in [0,\infty) \times \mathbb{R}^d.$$

Show that u is the unique viscosity solution of the PDE (–e03) with Cauchy condition $u|_{t=0} = \varphi$. □

Exercise.

(label)ex2 Let η be maximal distributed. For each $\varphi \in C_{Lip}(\mathbb{R}^d)$, we define a function

$$u(t,y) := \mathbb{E}[\varphi(y+t\eta)],\ (t,y) \in [0,\infty) \times \mathbb{R}^d.$$

Show that u is the unique viscosity solution of the PDE (–e01) with Cauchy condition $u|_{t=0} = \varphi$. □

(label)c2s2 In this section, we give the proof of the existence of G-distributed random variables, namely, the proof of Proposition –Prop-Gnorm copy(1).

Let $G : \mathbb{R}^d \times \mathbb{S}(d) \to \mathbb{R}$ be a given sublinear function monotonic in $A \in \mathbb{S}(d)$ in the sense of (–e314). We now construct a pair of d-dimensional random vectors (X, η) on some sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ satisfying (–e313) and (–e311).

For each $\varphi \in C_{Lip}(\mathbb{R}^{2d})$, let $u = u^{\varphi}$ be the unique viscosity solution of the G-equation (–ee03) with $u^{\varphi}|_{t=0} = \varphi$. We take $\widetilde{\Omega} = \mathbb{R}^{2d}$, $\widetilde{\mathcal{H}} = C_{Lip}(\mathbb{R}^{2d})$ and $\widetilde{\omega} = (x, y) \in \mathbb{R}^{2d}$. The corresponding sublinear expectation $\widetilde{\mathbb{E}}[\cdot]$ is defined by $\widetilde{\mathbb{E}}[\xi] = u^{\varphi}(1, 0, 0)$, for each $\xi \in \widetilde{\mathcal{H}}$ of the form $\xi(\widetilde{\omega}) = (\varphi(x, y))_{(x,y)\in\mathbb{R}^{2d}} \in C_{Lip}(\mathbb{R}^{2d})$. The monotonicity and sub-additivity of u^{φ} with respect to φ are known in the theory of viscosity solution. For reader's convenience we provide a new and simple proof in Appendix C (see Corollary –Comparison and Corollary –Domination). The constant preserving and positive homogeneity of $\widetilde{\mathbb{E}}[\cdot]$ are easy to check. Thus the functional $\widetilde{\mathbb{E}}[\cdot] : \widetilde{\mathcal{H}} \to \mathbb{R}$ forms a sublinear expectation.

We now consider a pair of d-dimensional random vectors $(\widetilde{X},\widetilde{\eta})(\widetilde{\omega})=(x,y)$. We have

$$\widetilde{\mathbb{E}}[\varphi(\widetilde{X},\widetilde{\eta})]=u^{\varphi}(1,0,0)\quad \text{for } \varphi\in C_{Lip}(\mathbb{R}^{2d}).$$

In particular, just setting $\varphi_0(x,y)=\frac{1}{2}\langle Ax,x\rangle+\langle p,y\rangle$, we can check that

$$u^{\varphi_0}(t,x,y)=G(p,A)t+\frac{1}{2}\langle Ax,x\rangle+\langle p,y\rangle\,.$$

We thus have

$$\widetilde{\mathbb{E}}[\frac{1}{2}\left\langle A\widetilde{X},\widetilde{X}\right\rangle+\langle p,\widetilde{\eta}\rangle]=u^{\varphi_0}(1,0,0)=G(p,A),\ \ (p,A)\in\mathbb{R}^d\times\mathbb{S}(d).$$

We construct a product space

$$(\Omega, \mathcal{H}, \mathbb{E}) = (\widetilde{\Omega} \times \widetilde{\Omega}, \widetilde{\mathcal{H}} \otimes \widetilde{\mathcal{H}}, \widetilde{\mathbb{E}} \otimes \widetilde{\mathbb{E}}),$$

and introduce two pairs of random vectors

$$(X, \eta)(\widetilde{\omega}_1, \widetilde{\omega}_2) = \widetilde{\omega}_1, \ (\bar{X}, \bar{\eta})(\widetilde{\omega}_1, \widetilde{\omega}_2) = \widetilde{\omega}_2, \quad (\widetilde{\omega}_1, \widetilde{\omega}_2) \in \widetilde{\Omega} \times \widetilde{\Omega}.$$

By Proposition –pp1 in Chapter –ch1, $(X,\eta) \overset{d}{=} (\widetilde{X},\widetilde{\eta})$ and $(\bar{X},\bar{\eta})$ is an independent copy of (X,η).
We now prove that the distribution of (X,η) satisfies condition (–e311).
For each $\varphi \in C_{Lip}(\mathbb{R}^{2d})$ and for each fixed $\lambda > 0$, $(\bar{x},\bar{y}) \in \mathbb{R}^{2d}$, since the function v defined by $v(t,x,y) := u^{\varphi}(\lambda t, \bar{x} + \sqrt{\lambda}x, \bar{y} + \lambda y)$ solves exactly the same equation (–ee03), but with Cauchy condition

$$v|_{t=0} = \varphi(\bar{x} + \sqrt{\lambda} \times \cdot, \bar{y} + \lambda \times \cdot).$$

Thus

$$\mathbb{E}[\varphi(\bar{x} + \sqrt{\lambda}X, \bar{y} + \lambda\eta)] = v(1,0,0) = u^{\varphi}(\lambda, \bar{x}, \bar{y}).$$

By the definition of $\mathbb{E}$, for each $t>0$ and $s>0$,

$$\begin{aligned}\mathbb{E}[\varphi(\sqrt{t}X+\sqrt{s}\bar{X},t\eta+s\bar{\eta})] &= \mathbb{E}[\mathbb{E}[\varphi(\sqrt{t}x+\sqrt{s}\bar{X},ty+s\bar{\eta})]_{(x,y)=(X,\eta)}] \\ &= \mathbb{E}[u^{\varphi}(s,\sqrt{t}X,t\eta)] = u^{u^{\varphi}(s,\cdot,\cdot)}(t,0,0) \\ &= u^{\varphi}(t+s,0,0) \\ &= \mathbb{E}[\varphi(\sqrt{t+s}X,(t+s)\eta)].\end{aligned}$$

Namely $(\sqrt{t}X+\sqrt{s}\bar{X},t\eta+s\bar{\eta}) \overset{d}{=} (\sqrt{t+s}X,(t+s)\eta)$. Thus the distribution of (X,η) satisfies condition (–e311).

Remark.

From now on, when we mention the sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, we suppose that there exists a pair of random vectors (X, η) on $(\Omega, \mathcal{H}, \mathbb{E})$ such that (X, η) is G-distributed. $\square$

Exercise.

(label)exxee1 Prove that $\mathbb{E}[X^3] > 0$ for $X \overset{d}{=} N(\{0\} \times [\underline{\sigma}^2, \bar{\sigma}^2])$ with $\underline{\sigma}^2 < \bar{\sigma}^2$.

It is worth to point that $\mathbb{E}[\varphi(X)]$ not always equal to $\sup_{\underline{\sigma}^2 \leq \sigma \leq \bar{\sigma}^2} E_\sigma[\varphi(X)]$ for $\varphi \in C_{l,Lip}(\mathbb{R})$, where E_σ denotes the linear expectation corresponding to the normal distributed density function $N(0, \sigma^2)$. □

Theorem

*(**Law of large numbers**) Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of $\mathbb{R}^d$-valued random variables on a sublinear expectation space $\Omega, \mathcal{H}, \mathbb{E})$. We assume that $Y_{i+1} \overset{d}{=} Y_i$ and Y_{i+1} is independent from $\{Y_1, \cdots, Y_i\}$ for each $i = 1, 2, \cdots$. Then $\{\bar{S}_n\}_{n=1}^{\infty}$ defined by*

$$\bar{S}_n := \frac{1}{n}\sum_{i=1}^{n} Y_i$$

converges in law to a maximal distribution:

$$\lim_{n\to\infty} \mathbb{E}[\varphi(\bar{S}_n)] = \mathbb{E}[\varphi(\eta)], \quad (label)e325 \tag{27}$$

for all $\varphi \in C(\mathbb{R}^d)$ with linear growth condition $(|\varphi(x)| \leq C(1 + |x|))$, where η is maximal distributed with

$$g(p) := \mathbb{E}[\langle p, Y_1\rangle], \quad p \in \mathbb{R}^d.$$

Remark.

When $d = 1$, the sequence $\{\bar{S}_n\}_{n=1}^{\infty}$ converges in law to $N([\underline{\mu}, \bar{\mu}] \times \{0\})$, where $\bar{\mu} = \mathbb{E}[Y_1]$ and $\underline{\mu} = -\mathbb{E}[-Y_1]$. For the general case, the sum $\frac{1}{n}\sum_{i=1}^{n} Y_i$ converges in law to $N(\bar{\Theta} \times \{0\})$, where $\bar{\Theta} \subset \mathbb{R}^d$ is the bounded, convex and closed subset defined in Example –example116. If we take in particular $\varphi(y) = d_{\bar{\Theta}}(y) = \inf\{|x - y| : x \in \bar{\Theta}\}$, then by (–e325) we have the following generalized law of large numbers:

$$\lim_{n\to\infty} \mathbb{E}[d_{\bar{\Theta}}(\frac{1}{n}\sum_{i=1}^{n} Y_i)] = \sup_{\theta\in\bar{\Theta}} d_{\bar{\Theta}}(\theta) = 0.(label)e319 \tag{28}$$

If Y_i has no mean-uncertainty, or in other words, $\bar{\Theta}$ is a singleton: $\bar{\Theta} = \{\bar{\theta}\}$, then (–e319) becomes

$$\lim_{n\to\infty} \mathbb{E}[|\frac{1}{n}\sum_{i=1}^{n} Y_i - \bar{\theta}|] = 0.$$

(Central limit theorem with zero-mean).

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of $\mathbb{R}^d$-valued random variables on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. We assume that $X_{i+1} \overset{d}{=} X_i$ and X_{i+1} is independent from $\{X_1, \cdots, X_i\}$ for each $i = 1, 2, \cdots$. We further assume that

$$\mathbb{E}[X_1] = \mathbb{E}[-X_1] = 0.$$

Then the sequence $\{\bar{S}_n\}_{n=1}^{\infty}$ defined by

$$\bar{S}_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$$

converges in law to X, i.e.,

$$\lim_{n\to\infty} \mathbb{E}[\varphi(\bar{S}_n)] = \mathbb{E}[\varphi(X)],$$

for all functions $\varphi \in C(\mathbb{R}^d)$ satisfying linear growth condition, □

(Central limit theorem with zero-mean) (continue) .

where X is a G-normal distributed random vector and the corresponding sublinear function $G : \mathbb{S}(d) \to \mathbb{R}$ is defined by

$$G(A) := \mathbb{E}[\frac{1}{2} \langle AX_1, X_1 \rangle], \ \ A \in \mathbb{S}(d).$$

□

Remark.

When $d = 1$, the sequence $\{\bar{S}_n\}_{n=1}^{\infty}$ converges in law to $N(\{0\} \times [\underline{\sigma}^2, \overline{\sigma}^2])$, where $\overline{\sigma}^2 = \mathbb{E}[X_1^2]$ and $\underline{\sigma}^2 = -\mathbb{E}[-X_1^2]$. In particular, if $\overline{\sigma}^2 = \underline{\sigma}^2$, then it becomes a classical central limit theorem. □

The following theorem is a nontrivial generalization of the above two theorems.

(Central Limit Theorem with law of large numbers).

(label)CLT Let $\{(X_i, Y_i)\}_{i=1}^{\infty}$ be a sequence of $\mathbb{R}^d \times \mathbb{R}^d$-valued random vectors on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. We assume that $(X_{i+1}, Y_{i+1}) \overset{d}{=} (X_i, Y_i)$ and (X_{i+1}, Y_{i+1}) is independent from $\{(X_1, Y_1), \cdots, (X_i, Y_i)\}$ for each $i = 1, 2, \cdots$. We further assume that

$$\mathbb{E}[X_1] = \mathbb{E}[-X_1] = 0.$$

□

Central Limit Theorem with law of large numbers.

Then the sequence $\{\bar{S}_n\}_{n=1}^{\infty}$ defined by

$$\bar{S}_n := \sum_{i=1}^{n} \left(\frac{X_i}{\sqrt{n}} + \frac{Y_i}{n}\right)$$

converges in law to $X + \eta$, i.e.,

$$\lim_{n\to\infty} \mathbb{E}[\varphi(\bar{S}_n)] = \mathbb{E}[\varphi(X+\eta)], \quad (label)e12 \tag{29}$$

for all functions $\varphi \in C(\mathbb{R}^d)$ satisfying a linear growth condition, where the pair (X, η) is G-distributed. The corresponding sublinear function $G : \mathbb{R}^d \times \mathbb{S}(d) \to \mathbb{R}$ is defined by

$$G(p, A) := \mathbb{E}[\langle p, Y_1\rangle + \frac{1}{2}\langle AX_1, X_1\rangle], \ \ A \in \mathbb{S}(d), \quad p \in \mathbb{R}^d.$$

Thus $\mathbb{E}[\varphi(X+\eta)]$ can be calculated by Corollary –Gv. □

The following result is equivalent to the above central limit theorem.

Theorem

(label)CLT1We make the same assumptions as in Theorem –CLT. Then for each function $\varphi \in C(\mathbb{R}^d \times \mathbb{R}^d)$ *satisfying linear growth condition, we have*

$$\lim_{n\to\infty} \mathbb{E}[\varphi(\sum_{i=1}^{n} \frac{X_i}{\sqrt{n}}, \sum_{i=1}^{n} \frac{Y_i}{n})] = \mathbb{E}[\varphi(X,\eta)].$$

Proof.

It is easy to prove Theorem –CLT by Theorem –CLT1. To prove Theorem –CLT1 from Theorem –CLT, it suffices to define a pair of $2d$-dimensional random vectors

$$\bar{X}_i = (X_i, 0), \quad \bar{Y}_i = (0, Y_i) \text{ for } i = 1, 2, \cdots .$$

We have

$$\lim_{n\to\infty} \mathbb{E}[\varphi(\sum_{i=1}^n \frac{X_i}{\sqrt{n}}, \sum_{i=1}^n \frac{Y_i}{n})] = \lim_{n\to\infty} \mathbb{E}[\varphi(\sum_{i=1}^n (\frac{\bar{X}_i}{\sqrt{n}} + \frac{\bar{Y}_i}{n}))] = \mathbb{E}[\varphi(\bar{X} + \eta)]$$
$$= \mathbb{E}[\varphi(X, \eta)]$$

with $\bar{X} = (X, 0)$ and $\bar{\eta} = (0, \eta)$. □

To prove Theorem –CLT, we need the following norms to measure the regularity of a given real functions u defined on $Q = [0, T] \times \mathbb{R}^d$:

$$\|u\|_{C^{0,0}(Q)} = \sup_{(t,x)\in Q} |u(t,x)|,$$

$$\|u\|_{C^{1,1}(Q)} = \|u\|_{C^{0,0}(Q)} + \|\partial_t u\|_{C^{0,0}(Q)} + \sum_{i=1}^{d} \|\partial_{x_i} u\|_{C^{0,0}(Q)},$$

$$\|u\|_{C^{1,2}(Q)} = \|u\|_{C^{1,1}(Q)} + \sum_{i,j=1}^{d} \left\|\partial_{x_i x_j} u\right\|_{C^{0,0}(Q)}.$$

For given constants $\alpha, \beta \in (0,1)$, we denote

$$\|u\|_{C^{\alpha,\beta}(Q)} = \sup_{\substack{x,y\in\mathbb{R}^d,\ x\neq y \\ s,t\in[0,T],s\neq t}} \frac{|u(s,x)-u(t,y)|}{|r-s|^{\alpha}+|x-y|^{\beta}},$$

$$\|u\|_{C^{1+\alpha,1+\beta}(Q)} = \|u\|_{C^{\alpha,\beta}(Q)} + \|\partial_t u\|_{C^{\alpha,\beta}(Q)} + \sum_{i=1}^{d} \|\partial_{x_i} u\|_{C^{\alpha,\beta}(Q)},$$

$$\|u\|_{C^{1+\alpha,2+\beta}(Q)} = \|u\|_{C^{1+\alpha,1+\beta}(Q)} + \sum_{i,j=1}^{d} \left\|\partial_{x_i x_j} u\right\|_{C^{\alpha,\beta}(Q)}.$$

If, for example, $\|u\|_{C^{1+\alpha,2+\beta}(Q)} < \infty$, then u is said to be a $C^{1+\alpha,2+\beta}$-function on Q.

We need the following lemma.

Lemma

(label)Lem-CLTWe assume the same assumptions as in Theorem –CLT. We further assume that there exists a constant $\beta > 0$ such that, for each A, $\bar{A} \in \mathbb{S}(d)$ with $A \geq \bar{A}$, we have

$$\mathbb{E}[\langle AX_1, X_1 \rangle] - \mathbb{E}[\langle \bar{A}X_1, X_1 \rangle] \geq \beta tr[A - \bar{A}].(label)Ellip \tag{30}$$

Then our main result (–e12) *holds.*

Proof.

We first prove (–e12) for $\varphi \in C_{b.Lip}(\mathbb{R}^d)$. For a small but fixed $h > 0$, let V be the unique viscosity solution of

$$\partial_t V + G(DV, D^2 V) = 0,\ (t,x) \in [0, 1+h) \times \mathbb{R}^d,\ \ V|_{t=1+h} = \varphi.(label)e14 \tag{31}$$

Since (X, η) satisfies (–e311), we have

$$V(h,0) = \mathbb{E}[\varphi(X+\eta)],\ \ V(1+h, x) = \varphi(x).(label)equ-h \tag{32}$$

Since (–e14) is a uniformly parabolic PDE and G is a convex function, by the interior regularity of V (see Appendix C), we have

$$\|V\|_{C^{1+\alpha/2, 2+\alpha}([0,1]\times\mathbb{R}^d)} < \infty \text{ for some } \alpha \in (0,1).$$

□

Proof (continue).

We set $\delta = \frac{1}{n}$ and $S_0 = 0$. Then

$$V(1, \bar{S}_n) - V(0,0) = \sum_{i=0}^{n-1} \{V((i+1)\delta, \bar{S}_{i+1}) - V(i\delta, \bar{S}_i)\}$$

$$= \sum_{i=0}^{n-1} \{[V((i+1)\delta, \bar{S}_{i+1}) - V(i\delta, \bar{S}_{i+1})] + [V(i\delta, \bar{S}_{i+1}) - V(i\delta, \bar{S}_i)]\}$$

$$= \sum_{i=0}^{n-1} \{I_\delta^i + J_\delta^i\}$$

with, by Taylor's expansion,

$$J_\delta^i = \partial_t V(i\delta, \bar{S}_i)\delta + \frac{1}{2}\left\langle D^2 V(i\delta, \bar{S}_i) X_{i+1}, X_{i+1} \right\rangle \delta + \left\langle DV(i\delta, \bar{S}_i), X_{i+1}\sqrt{\delta} + Y\right.$$

□

Proof (continue).

$$I_\delta^i = \delta \int_0^1 [\partial_t V((i+\beta)\delta, \bar{S}_{i+1}) - \partial_t V(i\delta, \bar{S}_{i+1})] d\beta + [\partial_t V(i\delta, \bar{S}_{i+1}) - \partial_t V(i\delta$$

$$+ \left\langle D^2 V(i\delta, \bar{S}_i) X_{i+1}, Y_{i+1} \right\rangle \delta^{3/2} + \frac{1}{2} \left\langle D^2 V(i\delta, \bar{S}_i) Y_{i+1}, Y_{i+1} \right\rangle \delta^2$$

$$+ \int_0^1 \int_0^1 \left\langle \Theta_{\beta\gamma}^i (X_{i+1}\sqrt{\delta} + Y_{i+1}\delta), X_{i+1}\sqrt{\delta} + Y_{i+1}\delta \right\rangle \gamma d\beta d\gamma$$

with

$$\Theta_{\beta\gamma}^i = D^2 V(i\delta, \bar{S}_i + \gamma\beta(X_{i+1}\sqrt{\delta} + Y_{i+1}\delta)) - D^2 V(i\delta, \bar{S}_i).$$

□

Proof (continue).

Thus

$$\mathbb{E}[\sum_{i=0}^{n-1} J_\delta^i] - \mathbb{E}[-\sum_{i=0}^{n-1} I_\delta^i] \leq \mathbb{E}[V(1,\bar{S}_n)] - V(0,0) \leq \mathbb{E}[\sum_{i=0}^{n-1} J_\delta^i] + \mathbb{E}[\sum_{i=0}^{n-1} I_\delta^i].(lab \quad (33)$$

We now prove that $\mathbb{E}[\sum_{i=0}^{n-1} J_\delta^i] = 0$. For J_δ^i, note that

$$\mathbb{E}[\left\langle DV(i\delta,\bar{S}_i), X_{i+1}\sqrt{\delta}\right\rangle] = \mathbb{E}[-\left\langle DV(i\delta,\bar{S}_i), X_{i+1}\sqrt{\delta}\right\rangle] = 0,$$

then, from the definition of the function G, we have

$$\mathbb{E}[J_\delta^i] = \mathbb{E}[\partial_t V(i\delta,\bar{S}_i) + G(DV(i\delta,\bar{S}_i), D^2 V(i\delta,\bar{S}_i))]\delta.$$

□

Proof (continue).

Combining the above two equalities with $\partial_t V + G(DV, D^2V) = 0$ as well as the independence of (X_{i+1}, Y_{i+1}) from $\{(X_1, Y_1), \cdots, (X_i, Y_i)\}$, it follows that

$$\mathbb{E}[\sum_{i=0}^{n-1} J_\delta^i] = \mathbb{E}[\sum_{i=0}^{n-2} J_\delta^i] = \cdots = 0.$$

Thus (–c2ee15) can be rewritten as

$$-\mathbb{E}[-\sum_{i=0}^{n-1} I_\delta^i] \leq \mathbb{E}[V(1, \bar{S}_n)] - V(0,0) \leq \mathbb{E}[\sum_{i=0}^{n-1} I_\delta^i].$$

□

Proof (continue).

But since both $\partial_t V$ and $D^2 V$ are uniformly $\frac{\alpha}{2}$-hölder continuous in t and α-hölder continuous in x on $[0,1] \times \mathbb{R}^d$, we then have

$$|I_\delta^i| \leq C\delta^{1+\alpha/2}(1 + |X_{i+1}|^{2+\alpha} + |Y_{i+1}|^{2+\alpha}).$$

It follows that

$$\mathbb{E}[|I_\delta^i|] \leq C\delta^{1+\alpha/2}(1 + \mathbb{E}[|X_1|^{2+\alpha} + |Y_1|^{2+\alpha}]).$$

□

Proof (continue).

Thus

$$-C(\frac{1}{n})^{\alpha/2}(1+\mathbb{E}[|X_1|^{2+\alpha}+|Y_1|^{2+\alpha}]) \leq \mathbb{E}[V(1,\bar{S}_n)] - V(0,0)$$
$$\leq C(\frac{1}{n})^{\alpha/2}(1+\mathbb{E}[|X_1|^{2+\alpha}+|Y_1|^{2+\alpha}]$$

As $n \longrightarrow \infty$, we have

$$\lim_{n\to\infty} \mathbb{E}[V(1,\bar{S}_n)] = V(0,0).(label)equ-0 \quad (34)$$

On the other hand, for each $t, t' \in [0, 1+h]$ and $x \in \mathbb{R}^d$, we have

$$|V(t,x) - V(t',x)| \leq C(\sqrt{|t-t'|} + |t-t'|).$$

□

Proof (continue).

Thus $|V(0,0) - V(h,0)| \leq C(\sqrt{h} + h)$ and, by (–equ-0),

$$|\mathbb{E}[V(1,\bar{S}_n)] - \mathbb{E}[\varphi(\bar{S}_n)]| = |\mathbb{E}[V(1,\bar{S}_n)] - \mathbb{E}[V(1+h,\bar{S}_n)]| \leq C(\sqrt{h}+h).$$

It follows from (–equ-h) and (–equ-0) that

$$\limsup_{n\to\infty} |\mathbb{E}[\varphi(\bar{S}_n)] - \mathbb{E}[\varphi(X+\eta)]| \leq 2C(\sqrt{h}+h).$$

Since h can be arbitrarily small, we have

$$\lim_{n\to\infty} \mathbb{E}[\varphi(\bar{S}_n)] = \mathbb{E}[\varphi(X+\eta)].$$

□

Remark.

From the proof we can check that the main assumption of identical distribution of $\{X_i, Y_i\}_{i=1}^{\infty}$ can be weaken to

$$\mathbb{E}[\langle p, Y_i\rangle + \frac{1}{2}\langle AX_i, X_i\rangle] = G(p, A), \quad i = 1, 2, \cdots, \; p \in \mathbb{R}^d, \; A \in \mathbb{S}(d).$$

Another essential condition is $\mathbb{E}[|X_i|^{2+\delta}] + \mathbb{E}[|Y_i|^{1+\delta}] \leq C$ for some $\delta > 0$. We do not need the condition $\mathbb{E}[|X_i|^n] + \mathbb{E}[|Y_i|^n] < \infty$ for each $n \in \mathbb{N}$. □

We now give the proof of Theorem –CLT. **Proof of Theorem –CLT**. For the case when the uniform elliptic condition (–Ellip) does not hold, we first introduce a perturbation to prove the above convergence for $\varphi \in C_{b.Lip}(\mathbb{R}^d)$. According to Definition –dd1 and Proposition –pp1 in Chap I, we can construct a sublinear expectation space $(\bar{\Omega}, \bar{\mathcal{H}}, \bar{\mathbb{E}})$ and a sequence of three random vectors $\{(\bar{X}_i, \bar{Y}_i, \bar{\kappa}_i)\}_{i=1}^{\infty}$ such that, for each $n = 1, 2, \cdots$, $\{(\bar{X}_i, \bar{Y}_i)\}_{i=1}^{n} \stackrel{d}{=} \{(X_i, Y_i)\}_{i=1}^{n}$ and $(\bar{X}_{n+1}, \bar{Y}_{n+1}, \bar{\kappa}_{n+1})$ is independent from $\{(\bar{X}_i, \bar{Y}_i, \bar{\kappa}_i)\}_{i=1}^{n}$ and, moreover,

$$\bar{\mathbb{E}}[\psi(\bar{X}_i, \bar{Y}_i, \bar{\kappa}_i)] = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathbb{E}[\psi(X_i, Y_i, x)] e^{-|x|^2/2} dx \text{ for } \psi \in C_{Lip}(\mathbb{R}^{3\times c}$$

We then use the perturbation $\bar{X}_i^{\varepsilon} = \bar{X}_i + \varepsilon \bar{\kappa}_i$ for a fixed $\varepsilon > 0$. It is easy to see that the sequence $\{(\bar{X}_i^{\varepsilon}, \bar{Y}_i)\}_{i=1}^{\infty}$ satisfies all conditions in the above CLT, in particular,

$$G_{\varepsilon}(p, A) := \bar{\mathbb{E}}[\frac{1}{2} \langle A\bar{X}_1^{\varepsilon}, \bar{X}_1^{\varepsilon} \rangle + \langle p, \bar{Y}_1 \rangle] = G(p, A) + \frac{\varepsilon^2}{2} \mathrm{tr}[A].$$

Thus it is strictly elliptic.

We then can apply Lemma –Lem-CLT to

$$\bar{S}_n^\varepsilon := \sum_{i=1}^n \left(\frac{\bar{X}_i^\varepsilon}{\sqrt{n}} + \frac{\bar{Y}_i}{n}\right) = \sum_{i=1}^n \left(\frac{\bar{X}_i}{\sqrt{n}} + \frac{\bar{Y}_i}{n}\right) + \varepsilon J_n, \quad J_n = \sum_{i=1}^n \frac{\bar{\kappa}_i}{\sqrt{n}}$$

and obtain

$$\lim_{n\to\infty} \bar{\mathbb{E}}[\varphi(\bar{S}_n^\varepsilon)] = \bar{\mathbb{E}}[\varphi(\bar{X} + \bar{\eta} + \varepsilon\bar{\kappa})],$$

where $((\bar{X}, \bar{\kappa}), (\bar{\eta}, 0))$ is $\bar{G}$-distributed under $\bar{\mathbb{E}}[\cdot]$ and

$$\bar{G}(\bar{p}, \bar{A}) := \bar{\mathbb{E}}[\frac{1}{2}\left\langle \bar{A}(\bar{X}_1, \bar{\kappa}_1)^T, (\bar{X}_1, \bar{\kappa}_1)^T \right\rangle + \left\langle \bar{p}, (\bar{Y}_1, 0)^T \right\rangle], \quad \bar{A} \in \mathbb{S}(2d), \quad \bar{p} \in$$

By Proposition –GCD, it is easy to prove that $(\bar{X}+\varepsilon\bar{\kappa},\bar{\eta})$ is G_ε-distributed and $(\bar{X},\bar{\eta})$ is G-distributed. But we have

$$\begin{aligned}|\mathbb{E}[\varphi(\bar{S}_n)]-\bar{\mathbb{E}}[\varphi(\bar{S}_n^\varepsilon)]| &= |\bar{\mathbb{E}}[\varphi(\bar{S}_n^\varepsilon-\varepsilon J_n)]-\bar{\mathbb{E}}[\varphi(\bar{S}_n^\varepsilon)]| \\ &\leq \varepsilon C\bar{\mathbb{E}}[|J_n|]\leq C'\varepsilon\end{aligned}$$

and similarly,

$|\mathbb{E}[\varphi(X+\eta)] - \bar{\mathbb{E}}[\varphi(\bar{X}+\bar{\eta}+\varepsilon\bar{\kappa})]| = |\bar{\mathbb{E}}[\varphi(\bar{X}+\bar{\eta})] - \bar{\mathbb{E}}[\varphi(\bar{X}+\bar{\eta}+\varepsilon\bar{\kappa})]| \leq C\varepsilon$. Since ε can be arbitrarily small, it follows that

$$\lim_{n\to\infty} \mathbb{E}[\varphi(\bar{S}_n)] = \mathbb{E}[\varphi(X+\eta)] \text{ for } \varphi \in C_{b.Lip}(\mathbb{R}^d).$$

On the other hand, it is easy to check that $\sup_n \mathbb{E}[|\bar{S}_n|^2] + \mathbb{E}[|X+\eta|^2] < \infty$. We then can apply the following lemma to prove that the above convergence holds for $\varphi \in C(\mathbb{R}^d)$ with linear growth condition. The proof is complete. □

Lemma

Let $(\Omega, \mathcal{H}, \mathbb{E})$ *and* $(\widetilde{\Omega}, \widetilde{\mathcal{H}}, \widetilde{\mathbb{E}})$ *be two sublinear expectation spaces and let* $Y_n \in \mathcal{H}$ *and* $Y \in \widetilde{\mathcal{H}}$, $n = 1, 2, \cdots$, *be given. We assume that, for a given* $p \geq 1$, $\sup_n \mathbb{E}[|Y_n|^p] + \widetilde{\mathbb{E}}[|Y|^p] < \infty$. *If the convergence* $\lim_{n\to\infty} \mathbb{E}[\varphi(Y_n)] = \widetilde{\mathbb{E}}[\varphi(Y)]$ *holds for each* $\varphi \in C_{b.Lip}(\mathbb{R}^d)$, *then it also holds for all functions* $\varphi \in C(\mathbb{R}^d)$ *with the growth condition* $|\varphi(x)| \leq C(1 + |x|^{p-1})$.

Proof.

We first prove that the above convergence holds for $\varphi \in C_b(\mathbb{R}^d)$ with a compact support. In this case, for each $\varepsilon > 0$, we can find a $\bar{\varphi} \in C_{b.Lip}(\mathbb{R}^d)$ such that $\sup_{x\in\mathbb{R}^d} |\varphi(x) - \bar{\varphi}(x)| \leq \frac{\varepsilon}{2}$. We have

$$|\mathbb{E}[\varphi(Y_n)] - \widetilde{\mathbb{E}}[\varphi(Y)]| \leq |\mathbb{E}[\varphi(Y_n)] - \mathbb{E}[\bar{\varphi}(Y_n)]| + |\widetilde{\mathbb{E}}[\varphi(Y)] - \widetilde{\mathbb{E}}[\bar{\varphi}(Y)]|$$
$$+ |\mathbb{E}[\bar{\varphi}(Y_n)] - \widetilde{\mathbb{E}}[\bar{\varphi}(Y)]| \leq \varepsilon + |\mathbb{E}[\bar{\varphi}(Y_n)] - \widetilde{\mathbb{E}}[\bar{\varphi}(Y)]|.$$

□

Proof. (continue).

Thus $\limsup_{n\to\infty} |\mathbb{E}[\varphi(Y_n)] - \widetilde{\mathbb{E}}[\varphi(Y)]| \leq \varepsilon$. The convergence must hold since ε can be arbitrarily small.

Now let φ be an arbitrary $C(\mathbb{R}^d)$-function with growth condition $|\varphi(x)| \leq C(1+|x|^{p-1})$. For each $N>0$ we can find $\varphi_1, \varphi_2 \in C(\mathbb{R}^d)$ such that $\varphi = \varphi_1 + \varphi_2$ where φ_1 has a compact support and $\varphi_2(x) = 0$ for $|x| \leq N$, and $|\varphi_2(x)| \leq |\varphi(x)|$ for all x. It is clear that

$$|\varphi_2(x)| \leq \frac{2C(1+|x|^p)}{N} \quad \text{for } x \in \mathbb{R}^d.$$

□

Proof. (continue).

Thus

$$
\begin{aligned}
|\mathbb{E}[\varphi(Y_n)] - \widetilde{\mathbb{E}}[\varphi(Y)]| &= |\mathbb{E}[\varphi_1(Y_n) + \varphi_2(Y_n)] - \widetilde{\mathbb{E}}[\varphi_1(Y) + \varphi_2(Y)]| \\
&\leq |\mathbb{E}[\varphi_1(Y_n)] - \widetilde{\mathbb{E}}[\varphi_1(Y)]| + \mathbb{E}[|\varphi_2(Y_n)|] + \widetilde{\mathbb{E}}[|\varphi_2(\\
&\leq |\mathbb{E}[\varphi_1(Y_n)] - \widetilde{\mathbb{E}}[\varphi_1(Y)]| + \frac{2C}{N}(2 + \mathbb{E}[|Y_n|^p] + \widetilde{\mathbb{E}}[\\
&\leq |\mathbb{E}[\varphi_1(Y_n)] - \widetilde{\mathbb{E}}[\varphi_1(Y)]| + \frac{\bar{C}}{N},
\end{aligned}
$$

where $\bar{C} = 2C(2 + \sup_n \mathbb{E}[|Y_n|^p] + \widetilde{\mathbb{E}}[|Y|^p])$. We thus have $\limsup_{n\to\infty} |\mathbb{E}[\varphi(Y_n)] - \widetilde{\mathbb{E}}[\varphi(Y)]| \leq \frac{\bar{C}}{N}$. Since N can be arbitrarily large, $\mathbb{E}[\varphi(Y_n)]$ must converge to $\widetilde{\mathbb{E}}[\varphi(Y)]$. □

Exercise.

Let $X_i \in \mathcal{H}, i = 1, 2, \cdots$, be such that X_{i+1} is independent from $\{X_1, \cdots, X_i\}$, for each $i = 1, 2, \cdots$. We further assume that

$$\mathbb{E}[X_i] = -\mathbb{E}[-X_i] = 0,$$

$$\lim_{i\to\infty} \mathbb{E}[X_i^2] = \overline{\sigma}^2 < \infty, \lim_{i\to\infty} -\mathbb{E}[-X_i^2] = \underline{\sigma}^2,$$

$$\mathbb{E}[|X_i|^{2+\delta}] \leq M \text{ for some } \delta > 0 \text{ and a constant } M.$$

Prove that the sequence $\{\bar{S}_n\}_{n=1}^{\infty}$ defined by

$$\bar{S}_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$$

converges in law to X, i.e.,
$\lim_{n\to\infty} \mathbb{E}[\varphi(\bar{S}_n)] = \mathbb{E}[\varphi(X)]$ for $\varphi \in C_{b,lip}(\mathbb{R})$, where $X \sim N(\{0\} \times [\underline{\sigma}^2, \overline{\sigma}^2])$. In particular, if $\overline{\sigma}^2 = \underline{\sigma}^2$, it becomes a classical central limit theorem. □

The contents of this chapter are mainly from Peng (2008) [?] (see also Peng (2007) [?]).
The notion of G-normal distribution was firstly introduced by Peng (2006) [?] for 1-dimensional case, and Peng (2008) [?] for multi-dimensional case.
In the classical situation, a distribution satisfying equation (–ch2e1) is said to be stable (see Lévy (1925) [?] and (1965) [?]). In this sense, our G-normal distribution can be considered as the most typical stable distribution under the framework of sublinear expectations.

Marinacci (1999) [**?**] used different notions of distributions and independence via capacity and the corresponding Choquet expectation to obtain a law of large numbers and a central limit theorem for non-additive probabilities (see also Maccheroni and Marinacci (2005) [**?**]). But since a sublinear expectation can not be characterized by the corresponding capacity, our results can not be derived from theirs. In fact, our results show that the limit in CLT, under uncertainty, is a G-normal distribution in which the distribution uncertainty is not just the parameter of the classical normal distributions (see Exercise –exxee1).

The notion of viscosity solutions plays a basic role in the definition and properties of G-normal distribution and maximal distribution. This notion was initially introduced by Crandall and Lions (1983) [?]. This is a fundamentally important notion in the theory of nonlinear parabolic and elliptic PDEs. Readers are referred to Crandall, Ishii and Lions (1992) [?] for rich references of the beautiful and powerful theory of viscosity solutions. For books on the theory of viscosity solutions and the related HJB equations, see Barles (1994) [?], Fleming and Soner (1992) [?] as well as Yong and Zhou (1999) [?].

We note that, for the case when the uniform elliptic condition holds, the viscosity solution (–e320) becomes a classical $C^{1+\frac{\alpha}{2},2+\alpha}$-solution (see Krylov (1987) [?] and the recent works in Cabre and Caffarelli (1997) [?] and Wang (1992) [?]). In 1-dimensional situation, when $\underline{\sigma}^2 > 0$, the G-equation becomes the following Barenblatt equation:

$$\partial_t u + \gamma|\partial_t u| = \triangle u, \ |\gamma| < 1.$$

This equation was first introduced by Barenblatt (1979) [?] (see also Avellaneda, Levy and Paras (1995) [?]).

(label)ch3

The aim of this chapter is to introduce the concept of G-Brownian motion, to study its properties and to construct Itô's integral with respect to G-Brownian motion. We emphasize here that our definition of G-Brownian motion is consistent with the classical one in the sense that if there is no volatility uncertainty. Our G-Brownian motion also has independent increments with identical G-normal distributions. G-Brownian motion has a very rich and interesting new structure which non-trivially generalizes the classical one. We thus can establish the related stochastic calculus, especially Itô's integrals and the related quadratic variation process. A very interesting new phenomenon of our G-Brownian motion is that its quadratic process also has independent increments which are identically distributed. The corresponding G-Itô's formula is obtained.

Definition

Let $(\Omega, \mathcal{H}, \mathbb{E})$ be a sublinear expectation space. $(X_t)_{t \geq 0}$ is called a d-dimensional **stochastic process** if for each $t \geq 0$, X_t is a d-dimensional random vector in $\mathcal{H}$.

Let $G(\cdot) : \mathbb{S}(d) \to \mathbb{R}$ be a given monotonic and sublinear function. By Theorem –t1 in Chapter –ch1, there exists a bounded, convex and closed subset $\Sigma \subset \mathbb{S}_+(d)$ such that

$$G(A) = \frac{1}{2} \sup_{B \in \Sigma} (A, B), \qquad A \in \mathbb{S}(d).$$

By Section –c2s2 in Chapter –ch2, we know that the G-normal distribution $N(\{0\} \times \Sigma)$ exists.
We now give the definition of G-Brownian motion.

Definition

A d-dimensional process $(B_t)_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called a G–**Brownian motion** (label)bbbif the following properties are satisfied:
(i) $B_0(\omega) = 0$;
(ii) For each $t, s \geq 0$, the increment $B_{t+s} - B_t$ is $N(\{0\} \times s\Sigma)$-distributed and is independent from $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$.

Remark.

We can prove that, for each $t_0 > 0$, $(B_{t+t_0} - B_{t_0})_{t\geq 0}$ is a G-Brownian motion. For each $\lambda > 0$, $(\lambda^{-\frac{1}{2}} B_{\lambda t})_{t\geq 0}$ is also a G-Brownian motion. This is the scaling property of G-Brownian motion, which is the same as that of the classical Brownian motion. □

We will denote in the rest of this book

$$B_t^{\mathbf{a}} = \langle \mathbf{a}, B_t \rangle \quad \text{for each } \mathbf{a} = (a_1, \cdots, a_d)^T \in \mathbb{R}^d.$$

By the above definition we have the following proposition which is important in stochastic calculus.

Proposition.

Let $(B_t)_{t\geq 0}$ be a d-dimensional G-Brownian motion on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Then $(B_t^{\mathbf{a}})_{t\geq 0}$ is a 1-dimensional $G_{\mathbf{a}}$-Brownian motion for each $\mathbf{a} \in \mathbb{R}^d$, where
$G_{\mathbf{a}}(\alpha) = \frac{1}{2}(\sigma^2_{\mathbf{aa}^T}\alpha^+ - \sigma^2_{-\mathbf{aa}^T}\alpha^-)$, $\sigma^2_{\mathbf{aa}^T} = 2G(\mathbf{aa}^T) = \mathbb{E}[\langle \mathbf{a}, B_1\rangle^2]$,
$\sigma^2_{-\mathbf{aa}^T} = -2G(-\mathbf{aa}^T) = -\mathbb{E}[-\langle \mathbf{a}, B_1\rangle^2]$.
In particular, for each $t, s \geq 0$,
$B^{\mathbf{a}}_{t+s} - B^{\mathbf{a}}_t \overset{d}{=} N(\{0\} \times [s\sigma^2_{-\mathbf{aa}^T}, s\sigma^2_{\mathbf{aa}^T}])$. □

Proposition.

For each convex function φ , we have

$$\mathbb{E}[\varphi(B^{\mathbf{a}}_{t+s} - B^{\mathbf{a}}_t)] = \frac{1}{\sqrt{2\pi s\sigma^2_{\mathbf{a}\mathbf{a}^T}}} \int_{-\infty}^{\infty} \varphi(x) \exp(-\frac{x^2}{2s\sigma^2_{\mathbf{a}\mathbf{a}^T}}) dx.$$

For each concave function φ and $\sigma^2_{-\mathbf{a}\mathbf{a}^T} > 0$, we have

$$\mathbb{E}[\varphi(B^{\mathbf{a}}_{t+s} - B^{\mathbf{a}}_t)] = \frac{1}{\sqrt{2\pi s\sigma^2_{-\mathbf{a}\mathbf{a}^T}}} \int_{-\infty}^{\infty} \varphi(x) \exp(-\frac{x^2}{2s\sigma^2_{-\mathbf{a}\mathbf{a}^T}}) dx.$$

□

Proposition (continue).

In particular, we have

$$\mathbb{E}[(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})^2] = \sigma^2_{\mathbf{a}\mathbf{a}^T}(t-s), \quad \mathbb{E}[(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})^4] = 3\sigma^4_{\mathbf{a}\mathbf{a}^T}(t-s)^2,$$
$$\mathbb{E}[-(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})^2] = -\sigma^2_{-\mathbf{a}\mathbf{a}^T}(t-s), \quad \mathbb{E}[-(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})^4] = -3\sigma^4_{-\mathbf{a}\mathbf{a}^T}(t-s)^2.$$

□

The following theorem gives a characterization of G-Brownian motion.

Theorem

Let $(B_t)_{t\geq 0} = (B_t^1, \cdots, B_t^d)_{t\geq 0}$ be a d-dimensional process defined on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ such that $|B_t^i|^k \in \mathcal{H}$, for $i = 1, \cdots, d$ and $k = 1, 2, 3$, and such that
(i) $B_0(\omega) = 0$;
(ii) *For each $t, s \geq 0$, $B_{t+s} - B_t$ and B_s are identically distributed and $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$.*
(iii) $\mathbb{E}[B_t] = \mathbb{E}[-B_t] = 0$ *and* $\lim_{t\downarrow 0} \mathbb{E}[|B_t|^3] t^{-1} = 0$.
Then $(B_t)_{t\geq 0}$ is a G-Brownian motion with $G(A) = \frac{1}{2}\mathbb{E}[\langle AB_1, B_1\rangle]$, $A \in \mathbb{S}(d)$.

Proof.

We only need to prove that B_1 is G-normal distributed and $B_t \overset{d}{=} \sqrt{t}B_1$. We first prove that

$$\mathbb{E}[\langle AB_t, B_t\rangle] = 2G(A)t, \quad A \in \mathbb{S}(d).$$

For each given $A \in \mathbb{S}(d)$, we set $b(t) = \mathbb{E}[\langle AB_t, B_t\rangle]$. Then $b(0) = 0$ and $|b(t)| \leq |A|(\mathbb{E}[|B_t|^3])^{2/3} \to 0$ as $t \to 0$. Since for each $t, s \geq 0$,

$$\begin{aligned} b(t+s) &= \mathbb{E}[\langle AB_{t+s}, B_{t+s}\rangle] = \hat{\mathbb{E}}[\langle A(B_{t+s} - B_s + B_s), B_{t+s} - B_s + B_s\rangle] \\ &= \mathbb{E}[\langle A(B_{t+s} - B_s), (B_{t+s} - B_s)\rangle + \langle AB_s, B_s\rangle + 2\langle A(B_{t+s} - B_s), B \\ &= b(t) + b(s), \end{aligned}$$

we have $b(t) = b(1)t = 2G(A)t$.

We now prove that B_1 is G-normal distributed and $B_t \overset{d}{=} \sqrt{t}B_1$. For this, we just need to prove that, for each fixed $\varphi \in C_{b.Lip}(\mathbb{R}^d)$, the function

$$u(t,x) := \mathbb{E}[\varphi(x + B_t)], \quad (t,x) \in [0,\infty) \times \mathbb{R}^d$$

Proof. (continue).

We first prove that u is Lipschitz in x and $\frac{1}{2}$-Hölder continuous in t. In fact, for each fixed t, $u(t,\cdot) \in C_{b.Lip}(\mathbb{R}^d)$ since

$$\begin{aligned}|u(t,x)-u(t,y)| &= |\mathbb{E}[\varphi(x+B_t)]-\mathbb{E}[\varphi(y+B_t)]| \\ &\leq \mathbb{E}[|\varphi(x+B_t)-\varphi(y+B_t)|] \\ &\leq C|x-y|,\end{aligned}$$

where C is Lipschitz constant of φ.

□

Proof. (continue).

For each $\delta \in [0, t]$, since $B_t - B_\delta$ is independent from B_δ, we also have

$$\begin{aligned} u(t,x) &= \mathbb{E}[(x + B_\delta + (B_t - B_\delta)] \\ &= \mathbb{E}[\mathbb{E}[\varphi(y + (B_t - B_\delta))]_{y=x+B_\delta}], \end{aligned}$$

hence

$$u(t,x) = \mathbb{E}[u(t-\delta, x + B_\delta)].(label)Dyna \tag{36}$$

□

Proof. (continue).

Thus

$$
\begin{aligned}
|u(t,x)-u(t-\delta,x)| &= |\mathbb{E}[u(t-\delta,x+B_\delta)-u(t-\delta,x)]| \\
&\le \mathbb{E}[|u(t-\delta,x+B_\delta)-u(t-\delta,x)|] \\
&\le \mathbb{E}[C|B_\delta|] \le C\sqrt{2G(I)}\sqrt{\delta}.
\end{aligned}
$$

To prove that u is a viscosity solution of (–G-heat-BM), we fix $(t,x)\in(0,\infty)\times\mathbb{R}^d$ and let $v\in C_b^{2,3}([0,\infty)\times\mathbb{R}^d)$ be such that $v\ge u$ and $v(t,x)=u(t,x)$. From (–Dyna) we have

$$
v(t,x)=\mathbb{E}[u(t-\delta,x+B_\delta)]\le\mathbb{E}[v(t-\delta,x+B_\delta)].
$$

□

Proof. (continue).

Therefore by Taylor's expansion,

$$
\begin{aligned}
0 &\leq \mathbb{E}[v(t-\delta, x+B_\delta) - v(t,x)] \\
&= \mathbb{E}[v(t-\delta, x+B_\delta) - v(t, x+B_\delta) + (v(t, x+B_\delta) - v(t,x))] \\
&= \mathbb{E}[-\partial_t v(t,x)\delta + \langle Dv(t,x), B_\delta\rangle + \frac{1}{2}\langle D^2 v(t,x)B_\delta, B_\delta\rangle + I_\delta] \\
&\leq -\partial_t v(t,x)\delta + \frac{1}{2}\mathbb{E}[\langle D^2 v(t,x)B_\delta, B_\delta\rangle] + \mathbb{E}[I_\delta] \\
&= -\partial_t v(t,x)\delta + G(D^2 v(t,x))\delta + \mathbb{E}[I_\delta],
\end{aligned}
$$

where

$$
\begin{aligned}
I_\delta = &\int_0^1 -[\partial_t v(t-\beta\delta, x+B_\delta) - \partial_t v(t,x)]\delta d\beta \\
&+ \int_0^1\int_0^1 \langle (D^2 v(t, x+\alpha\beta B_\delta) - D^2 v(t,x))B_\delta, B_\delta\rangle \alpha d\beta d\alpha.
\end{aligned}
$$

With the assumption (iii) we can check that $\lim_{\delta\downarrow 0}\mathbb{E}[|I_\delta|]\delta^{-1} = 0$, from

Exercise.

Let B_t be a 1-dimensional Brownian motion, and $B_1 \overset{d}{=} N(\{0\} \times [\underline{\sigma}^2, \overline{\sigma}^2])$. Prove that for each $m \in \mathbb{N}$,

$$\hat{\mathbb{E}}[|B_t|^m] = \begin{cases} 2(m-1)!!\overline{\sigma}^m t^{\frac{m}{2}} / \sqrt{2\pi} & m \text{ is odd,} \\ (m-1)!!\overline{\sigma}^m t^{\frac{m}{2}} & m \text{ is even.} \end{cases}$$

□

In the rest of this book, we denote by $\Omega = C_0^d(\mathbb{R}^+)$ the space of all $\mathbb{R}^d$–valued continuous paths $(\omega_t)_{t\in\mathbb{R}^+}$, with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1,\omega^2) := \sum_{i=1}^{\infty} 2^{-i}[(\max_{t\in[0,i]} |\omega_t^1 - \omega_t^2|) \wedge 1].$$

For each fixed $T \in [0,\infty)$, we set $\Omega_T := \{\omega_{\cdot\wedge T} : \omega \in \Omega\}$. We will consider the canonical process $B_t(\omega) = \omega_t$, $t \in [0,\infty)$, for $\omega \in \Omega$.

For each fixed $T \in [0, \infty)$, we set

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1 \wedge T}, \cdots, B_{t_n \wedge T}) : n \in \mathbb{N}, t_1, \cdots, t_n \in [0, \infty), \varphi \in C_{Lip}(\mathbb{R}$$

It is clear that $L_{ip}(\Omega_t) \subseteq L_{ip}(\Omega_T)$, for $t \leq T$. We also set

$$L_{ip}(\Omega) := \bigcup_{n=1}^{\infty} L_{ip}(\Omega_n).$$

Remark.

It is clear that $C_{Lip}(\mathbb{R}^{d\times n})$, $L_{ip}(\Omega_T)$ and $L_{ip}(\Omega)$ are vector lattices. Moreover, note that $\varphi,\psi\in C_{Lip}(\mathbb{R}^{d\times n})$ imply $\varphi\cdot\psi\in C_{Lip}(\mathbb{R}^{d\times n})$, then X, $Y\in L_{ip}(\Omega_T)$ imply $X\cdot Y\in L_{ip}(\Omega_T)$. In particular, for each $t\in[0,\infty)$, $B_t\in L_{ip}(\Omega)$. □

Let $G(\cdot):\mathbb{S}(d)\to\mathbb{R}$ be a given monotonic and sublinear function. In the following, we want to construct a sublinear expectation on $(\Omega, L_{ip}(\Omega))$ such that the canonical process $(B_t)_{t\geq 0}$ is a G-Brownian motion. For this, we first construct a sequence of d-dimensional random vectors $(\xi_i)_{i=1}^{\infty}$ on a sublinear expectation space $(\widetilde{\Omega},\widetilde{\mathcal{H}},\widetilde{\mathbb{E}})$ such that ξ_i is G-normal distributed and ξ_{i+1} is independent from $(\xi_1,\cdots,\xi_i)$ for each $i=1,2,\cdots$.

We now introduce a sublinear expectation $\hat{\mathbb{E}}$ defined on $L_{ip}(\Omega)$ via the following procedure: for each $X \in L_{ip}(\Omega)$ with

$$X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})$$

for some $\varphi \in C_{Lip}(\mathbb{R}^{d\times n})$ and $0 = t_0 < t_1 < \cdots < t_n < \infty$, we set

$$\hat{\mathbb{E}}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})]$$

$$:= \widetilde{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0}\xi_1, \cdots, \sqrt{t_n - t_{n-1}}\xi_n)].$$

The related conditional expectation of $X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})$ under Ω_{t_j} is defined by

$$\hat{\mathbb{E}}[X|\Omega_{t_j}] = \hat{\mathbb{E}}[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})|\Omega_{t_j}](label)Condition \tag{37}$$

$$:= \psi(B_{t_1}, \cdots, B_{t_j} - B_{t_{j-1}}),$$

where

$$\psi(x_1, \cdots, x_j) = \widetilde{\mathbb{E}}[\varphi(x_1, \cdots, x_j, \sqrt{t_{j+1} - t_j}\xi_{j+1}, \cdots, \sqrt{t_n - t_{n-1}}\xi_n)].$$

It is easy to check that $\hat{\mathbb{E}}[\cdot]$ consistently defines a sublinear expectation on $L_{ip}(\Omega)$ and $(B_t)_{t\geq 0}$ is a G-Brownian motion. Since $L_{ip}(\Omega_T)\subseteq L_{ip}(\Omega)$, $\hat{\mathbb{E}}[\cdot]$ is also a sublinear expectation on $L_{ip}(\Omega_T)$.

Definition

The sublinear expectation $\hat{\mathbb{E}}[\cdot]$: $L_{ip}(\Omega) \to \mathbb{R}$ defined through the above procedure is called a G–**expectation**.(label)ge The corresponding canonical process $(B_t)_{t\geq 0}$ on the sublinear expectation space $(\Omega, L_{ip}(\Omega), \hat{\mathbb{E}})$ is called a G–Brownian motion.

In the rest of this book, when we talk about G–Brownian motion, we mean that the canonical process $(B_t)_{t\geq 0}$ is under G-expectation.

Proposition.

(label)Prop-1-9-1 We list the properties of $\hat{\mathbb{E}}[\cdot|\Omega_t]$ that hold for each $X, Y \in L_{ip}(\Omega)$:

(i) If $X \geq Y$, then $\hat{\mathbb{E}}[X|\Omega_t] \geq \hat{\mathbb{E}}[Y|\Omega_t]$.

(ii) $\hat{\mathbb{E}}[\eta|\Omega_t] = \eta$, for each $t \in [0, \infty)$ and $\eta \in L_{ip}(\Omega_t)$.

(iii) $\hat{\mathbb{E}}[X|\Omega_t] - \hat{\mathbb{E}}[Y|\Omega_t] \leq \hat{\mathbb{E}}[X - Y|\Omega_t]$.

(iv) $\hat{\mathbb{E}}[\eta X|\Omega_t] = \eta^+\hat{\mathbb{E}}[X|\Omega_t] + \eta^-\hat{\mathbb{E}}[-X|\Omega_t]$ for each $\eta \in L_{ip}(\Omega_t)$.

(v) $\hat{\mathbb{E}}[\hat{\mathbb{E}}[X|\Omega_t]|\Omega_s] = \hat{\mathbb{E}}[X|\Omega_{t\wedge s}]$, in particular, $\hat{\mathbb{E}}[\hat{\mathbb{E}}[X|\Omega_t]] = \hat{\mathbb{E}}[X]$.

□

Proposition. (continue) .

For each $X \in L_{ip}(\Omega^t)$, $\hat{\mathbb{E}}[X|\Omega_t] = \hat{\mathbb{E}}[X]$, where $L_{ip}(\Omega^t)$ is the linear space of random variables with the form

$$\varphi(B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \cdots, B_{t_{n+1}} - B_{t_n}),$$
$$n = 1, 2, \cdots, \ \varphi \in C_{Lip}(\mathbb{R}^{d\times n}), \ t_1, \cdots, t_n, t_{n+1} \in [t, \infty).$$

□

Remark.

(ii) and (iii) imply

$$\hat{\mathbb{E}}[X + \eta|\Omega_t] = \hat{\mathbb{E}}[X|\Omega_t] + \eta \quad \text{for } \eta \in L_{ip}(\Omega_t).$$

□

We now consider the completion of sublinear expectation space $(\Omega, L_{ip}(\Omega), \hat{\mathbb{E}})$.
We denote by $L_G^p(\Omega)$, $p \geq 1$, the completion of $L_{ip}(\Omega)$ under the norm $\|X\|_p := (\hat{\mathbb{E}}[|X|^p])^{1/p}$. Similarly, we can define $L_G^p(\Omega_T)$, $L_G^p(\Omega_T^t)$ and $L_G^p(\Omega^t)$. It is clear that for each $0 \leq t \leq T < \infty$,
$L_G^p(\Omega_t) \subseteq L_G^p(\Omega_T) \subseteq L_G^p(\Omega)$.

According to Sec.–c1s5 in Chap.–ch1, $\hat{\mathbb{E}}[\cdot]$ can be continuously extended to a sublinear expectation on $(\Omega, L_G^1(\Omega))$ still denoted by $\hat{\mathbb{E}}[\cdot]$. We now consider the extension of conditional expectations. For each fixed $t \le T$, the conditional G-expectation $\hat{\mathbb{E}}[\cdot|\Omega_t] : L_{ip}(\Omega_T) \to L_{ip}(\Omega_t)$ is a continuous mapping under $\|\cdot\|$. Indeed, we have

$$\hat{\mathbb{E}}[X|\Omega_t] - \hat{\mathbb{E}}[Y|\Omega_t] \le \hat{\mathbb{E}}[X - Y|\Omega_t] \le \hat{\mathbb{E}}[|X - Y||\Omega_t],$$

then

$$|\hat{\mathbb{E}}[X|\Omega_t] - \hat{\mathbb{E}}[Y|\Omega_t]| \le \hat{\mathbb{E}}[|X - Y||\Omega_t].$$

We thus obtain

$$\left\|\hat{\mathbb{E}}[X|\Omega_t] - \hat{\mathbb{E}}[Y|\Omega_t]\right\| \leq \|X - Y\|.$$

It follows that $\hat{\mathbb{E}}[\cdot|\Omega_t]$ can be also extended as a continuous mapping

$$\hat{\mathbb{E}}[\cdot|\Omega_t] : L_G^1(\Omega_T) \to L_G^1(\Omega_t).$$

If the above T is not fixed, then we can obtain
$\hat{\mathbb{E}}[\cdot|\Omega_t] : L_G^1(\Omega) \to L_G^1(\Omega_t)$.

Remark.

The above proposition also holds for $X, Y \in L_G^1(\Omega)$. But in (iv), $\eta \in L_G^1(\Omega_t)$ should be bounded, since $X, Y \in L_G^1(\Omega)$ does not imply $X \cdot Y \in L_G^1(\Omega)$. □

In particular, we have the following independence:

$$\hat{\mathbb{E}}[X|\Omega_t] = \hat{\mathbb{E}}[X], \quad \forall X \in L_G^1(\Omega^t).$$

We give the following definition similar to the classical one:

Definition

An n-dimensional random vector $Y \in (L_G^1(\Omega))^n$ is said to be independent from Ω_t for some given t if for each $\varphi \in C_{b.Lip}(\mathbb{R}^n)$ we have

$$\hat{\mathbb{E}}[\varphi(Y)|\Omega_t] = \hat{\mathbb{E}}[\varphi(Y)].$$

Remark.

Just as in the classical situation, the increments of G–Brownian motion $(B_{t+s} - B_t)_{s\geq 0}$ are independent from Ω_t. □

The following property is very useful.

Proposition.

(label)d-E-x+yLet $X, Y \in L_G^1(\Omega)$ be such that $\hat{\mathbb{E}}[Y|\Omega_t] = -\hat{\mathbb{E}}[-Y|\Omega_t]$, for some $t \in [0, T]$. Then we have

$$\hat{\mathbb{E}}[X + Y|\Omega_t] = \hat{\mathbb{E}}[X|\Omega_t] + \hat{\mathbb{E}}[Y|\Omega_t].$$

In particular, if $\hat{\mathbb{E}}[Y|\Omega_t] = \hat{\mathbb{E}}_G[-Y|\Omega_t] = 0$, then $\hat{\mathbb{E}}[X + Y|\Omega_t] = \hat{\mathbb{E}}[X|\Omega_t]$. □

Proof.

This follows from the following two inequalities:

$$\hat{\mathbb{E}}[X+Y|\Omega_t] \leq \hat{\mathbb{E}}[X|\Omega_t] + \hat{\mathbb{E}}[Y|\Omega_t],$$
$$\hat{\mathbb{E}}[X+Y|\Omega_t] \geq \hat{\mathbb{E}}[X|\Omega_t] - \hat{\mathbb{E}}[-Y|\Omega_t] = \hat{\mathbb{E}}[X|\Omega_t] + \hat{\mathbb{E}}[Y|\Omega_t].$$

□

Example

For each fixed $\mathbf{a} \in \mathbb{R}^d, s \leq t$, we have

$$\hat{\mathbb{E}}[B_t^{\mathbf{a}} - B_s^{\mathbf{a}}|\Omega_s] = 0, \quad \hat{\mathbb{E}}[-(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})|\Omega_s] = 0,$$
$$\hat{\mathbb{E}}[(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})^2|\Omega_s] = \sigma^2_{\mathbf{aa}^T}(t-s), \quad \hat{\mathbb{E}}[-(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})^2|\Omega_s] = -\sigma^2_{-\mathbf{aa}^T}(t-s),$$
$$\hat{\mathbb{E}}[(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})^4|\Omega_s] = 3\sigma^4_{\mathbf{aa}^T}(t-s)^2, \quad \hat{\mathbb{E}}[-(B_t^{\mathbf{a}} - B_s^{\mathbf{a}})^4|\Omega_s] = -3\sigma^4_{-\mathbf{aa}^T}(t-s$$

where $\sigma^2_{\mathbf{aa}^T} = 2G(\mathbf{aa}^T)$ and $\sigma^2_{-\mathbf{aa}^T} = -2G(-\mathbf{aa}^T)$.

Example

(label)eee1For each $\mathbf{a} \in \mathbb{R}^d$, $n \in \mathbb{N}$, $0 \leq t \leq T$, $X \in L_G^1(\Omega_t)$ and $\varphi \in C_{Lip}(\mathbb{R})$, we have

$$\begin{aligned}\hat{\mathbb{E}}[X\varphi(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})|\Omega_t] &= X^+\hat{\mathbb{E}}[\varphi(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})|\Omega_t] + X^-\hat{\mathbb{E}}[-\varphi(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})|\Omega_t] \\ &= X^+\hat{\mathbb{E}}[\varphi(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})] + X^-\hat{\mathbb{E}}[-\varphi(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})].\end{aligned}$$

Example (continue).

In particular, we have

$$\hat{\mathbb{E}}[X(B^{\mathbf{a}}_T - B^{\mathbf{a}}_t)|\Omega_t] = X^+\hat{\mathbb{E}}[(B^{\mathbf{a}}_T - B^{\mathbf{a}}_t)] + X^-\hat{\mathbb{E}}[-(B^{\mathbf{a}}_T - B^{\mathbf{a}}_t)] = 0.$$

This, together with Proposition –d-E-x+y, yields

$$\hat{\mathbb{E}}[Y + X(B^{\mathbf{a}}_T - B^{\mathbf{a}}_t)|\Omega_t] = \hat{\mathbb{E}}[Y|\Omega_t], \quad Y \in L^1_G(\Omega).$$

□

Example (continue).

We also have

$$\hat{\mathbb{E}}[X(B^{\mathbf{a}}_T - B^{\mathbf{a}}_t)^2|\Omega_t] = X^+\hat{\mathbb{E}}[(B^{\mathbf{a}}_T - B^{\mathbf{a}}_t)^2] + X^-\hat{\mathbb{E}}[-(B^{\mathbf{a}}_T - B^{\mathbf{a}}_t)^2]$$
$$= [X^+\sigma^2_{\mathbf{aa}^T} - X^-\sigma^2_{-\mathbf{aa}^T}](T-t)$$

and

$$\hat{\mathbb{E}}[X(B^{\mathbf{a}}_T - B^{\mathbf{a}}_t)^{2n-1}|\Omega_t] = X^+\hat{\mathbb{E}}[(B^{\mathbf{a}}_T - B^{\mathbf{a}}_t)^{2n-1}] + X^-\hat{\mathbb{E}}[-(B^{\mathbf{a}}_T - B^{\mathbf{a}}_t)^{2n-1}]$$
$$= |X|\hat{\mathbb{E}}[(B^{\mathbf{a}}_{T-t})^{2n-1}].$$

□

Example

(label)Exam-B2Since

$$\hat{\mathbb{E}}[2B^{\mathbf{a}}_s(B^{\mathbf{a}}_t - B^{\mathbf{a}}_s)|\Omega_s] = \hat{\mathbb{E}}[-2B^{\mathbf{a}}_s(B^{\mathbf{a}}_t - B^{\mathbf{a}}_s)|\Omega_s] = 0,$$

we have

$$\begin{aligned}\hat{\mathbb{E}}[(B^{\mathbf{a}}_t)^2 - (B^{\mathbf{a}}_s)^2|\Omega_s] &= \hat{\mathbb{E}}[(B^{\mathbf{a}}_t - B^{\mathbf{a}}_s + B^{\mathbf{a}}_s)^2 - (B^{\mathbf{a}}_s)^2|\Omega_s] \\ &= \hat{\mathbb{E}}[(B^{\mathbf{a}}_t - B^{\mathbf{a}}_s)^2 + 2(B^{\mathbf{a}}_t - B^{\mathbf{a}}_s)B^{\mathbf{a}}_s|\Omega_s] \\ &= \sigma^2_{\mathbf{aa}^T}(t-s).\end{aligned}$$

Exercise.

Show that if $X \in Lip(\Omega_T)$ and $\hat{\mathbb{E}}[X] = -\hat{\mathbb{E}}[-X]$, then $\hat{\mathbb{E}}[X] = E_P[X]$, where P is a Wiener measure on Ω. □

Exercise.

Show that if $X \in Lip(\Omega_T)$ and $\hat{\mathbb{E}}[X] = -\hat{\mathbb{E}}[-X]$, then $\hat{\mathbb{E}}[X] = E_P[X]$, where P is a Wiener measure on Ω. □

Exercise.

(label)d-Exm-GBM-14a copy(1) For each $s, t \geq 0$, we set $B_t^s := B_{t+s} - B_s$. Let $\eta = (\eta_{ij})_{i,j=1}^d \in L_G^1(\Omega_s; \mathbb{S}(d))$. Prove that

$$\hat{\mathbb{E}}[\langle \eta B_t^s, B_t^s \rangle | \Omega_s] = 2G(\eta)t.$$

□

Definition

(label)d-Def-4For $T \in \mathbb{R}^+$, a partition π_T of $[0, T]$ is a finite ordered subset $\pi_T = \{t_0, t_1, \cdots, t_N\}$ such that $0 = t_0 < t_1 < \cdots < t_N = T$.

$$\mu(\pi_T) := \max\{|t_{i+1} - t_i| \ : \ i = 0, 1, \cdots, N-1\}.$$

We use $\pi_T^N = \{t_0^N, t_1^N, \cdots, t_N^N\}$ to denote a sequence of partitions of $[0, T]$ such that $\lim_{N\to\infty} \mu(\pi_T^N) = 0$.

Let $p \geq 1$ be fixed. We consider the following type of simple processes: for a given partition $\pi_T = \{t_0, \cdots, t_N\}$ of $[0, T]$ we set

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) \mathbf{I}_{[t_k, t_{k+1})}(t),$$

where $\xi_k \in L_G^p(\Omega_{t_k})$, $k = 0, 1, 2, \cdots, N-1$ are given. The collection of these processes is denoted by $M_G^{p,0}(0, T)$(label)mp0.

Definition

(label)d-Def-5For an $\eta \in M_G^{p,0}(0,T)$ with $\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) \mathbf{I}_{[t_k, t_{k+1})}(t)$, the related **Bochner integral** is

$$\int_0^T \eta_t(\omega) dt := \sum_{k=0}^{N-1} \xi_k(\omega)(t_{k+1} - t_k).$$

For each $\eta \in M_G^{p,0}(0,T)$, we set

$$\widetilde{\mathbb{E}}_T[\eta] := \frac{1}{T}\hat{\mathbb{E}}[\int_0^T \eta_t dt] = \frac{1}{T}\hat{\mathbb{E}}[\sum_{k=0}^{N-1} \xi_k(\omega)(t_{k+1}-t_k)].$$

It is easy to check that $\widetilde{\mathbb{E}}_T : M_G^{p,0}(0,T) \to \mathbb{R}$ forms a sublinear expectation. We then can introduce a natural norm $\|\eta\|_{M_G^p(0,T)}$, under which, $M_G^{p,0}(0,T)$ can be extended to $M_G^p(0,T)$ which is a Banach space.

Definition

For each $p \geq 1$, we denote by $M_G^p(0,T)$ the completion of $M_G(label)mgp^{p,0}(0,T)$ under the norm

$$\|\eta\|_{M_G^p(0,T)} := \left\{ \hat{\mathbb{E}}[\int_0^T |\eta_t|^p dt] \right\}^{1/p}.$$

It is clear that $M_G^p(0,T) \supset M_G^q(0,T)$ for $1 \le p \le q$. We also use $M_G^p(0,T;\mathbb{R}^n)$ for all n-dimensional stochastic processes $\eta_t = (\eta_t^1, \cdots, \eta_t^n)$, $t \ge 0$ with $\eta_t^i \in M_G^p(0,T)$, $i = 1, 2, \cdots, n$.

We now give the definition of Itô's integral. For simplicity, we first introduce Itô's integral with respect to 1-dimensional G–Brownian motion. Let $(B_t)_{t\geq 0}$ be a 1-dimensional G–Brownian motion with $G(\alpha)=\frac{1}{2}(\bar{\sigma}^2\alpha^+-\underline{\sigma}^2\alpha^-)$, where $0\leq\underline{\sigma}\leq\bar{\sigma}<\infty$.

Definition

For each $\eta \in M_G^{2,0}(0,T)$ of the form

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)\mathbf{I}_{[t_j,t_{j+1})}(t),$$

we define

$$I(\eta) = \int_0^T \eta_t dB_t := \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j}).$$

Lemma

(label)bddThe mapping $I : M_G^{2,0}(0,T) \to L_G^2(\Omega_T)$ *is a continuous linear mapping and thus can be continuously extended to* $I : M_G^2(0,T) \to L_G^2(\Omega_T)$. *We have*

$$\hat{\mathbb{E}}[\int_0^T \eta_t dB_t] = 0, \quad (label)e1 \tag{38}$$

$$\hat{\mathbb{E}}[(\int_0^T \eta_t dB_t)^2] \leq \bar{\sigma}^2 \hat{\mathbb{E}}[\int_0^T \eta_t^2 dt].(label)e2 \tag{39}$$

Proof.

From Example –eee1, for each j,

$$\hat{\mathbb{E}}[\xi_j(B_{t_{j+1}} - B_{t_j})|\Omega_{t_j}] = \hat{\mathbb{E}}[-\xi_j(B_{t_{j+1}} - B_{t_j})|\Omega_{t_j}] = 0.$$

We have

$$\begin{aligned}\hat{\mathbb{E}}[\int_0^T \eta_t dB_t] &= \hat{\mathbb{E}}[\int_0^{t_{N-1}} \eta_t dB_t + \xi_{N-1}(B_{t_N} - B_{t_{N-1}})] \\ &= \hat{\mathbb{E}}[\int_0^{t_{N-1}} \eta_t dB_t + \hat{\mathbb{E}}[\xi_{N-1}(B_{t_N} - B_{t_{N-1}})|\Omega_{t_{N-1}}]] \\ &= \hat{\mathbb{E}}[\int_0^{t_{N-1}} \eta_t dB_t].\end{aligned}$$

Then we can repeat this procedure to obtain (–e1).

□

Proof. (continue).

We now give the proof of (–e2). Firstly, from Example –eee1, we have

$$
\begin{aligned}
\hat{\mathbb{E}}[(\int_0^T \eta_t dB_t)^2] &= \hat{\mathbb{E}}[\left(\int_0^{t_{N-1}} \eta_t dB_t + \xi_{N-1}(B_{t_N} - B_{t_{N-1}})\right)^2] \\
&= \hat{\mathbb{E}}[\left(\int_0^{t_{N-1}} \eta_t dB_t\right)^2 + \xi_{N-1}^2(B_{t_N} - B_{t_{N-1}})^2 \\
&\quad + 2\left(\int_0^{t_{N-1}} \eta_t dB_t\right)\xi_{N-1}(B_{t_N} - B_{t_{N-1}})] \\
&= \hat{\mathbb{E}}[\left(\int_0^{t_{N-1}} \eta_t dB_t\right)^2 + \xi_{N-1}^2(B_{t_N} - B_{t_{N-1}})^2] \\
&= \cdots = \hat{\mathbb{E}}[\sum_{i=0}^{N-1} \xi_i^2 (B_{t_{i+1}} - B_{t_i})^2].
\end{aligned}
$$

□

Proof. (continue).

Then, for each $i = 0, 1, \cdots, N-1$, we have

$$
\begin{aligned}
&\hat{\mathbb{E}}[\xi_i^2(B_{t_{i+1}} - B_{t_i})^2 - \overline{\sigma}^2\xi_i^2(t_{i+1} - t_i)] \\
=&\hat{\mathbb{E}}[\hat{\mathbb{E}}[\xi_i^2(B_{t_{i+1}} - B_{t_i})^2 - \overline{\sigma}^2\xi_i^2(t_{i+1} - t_j)|\Omega_{t_i}]] \\
=&\hat{\mathbb{E}}[\overline{\sigma}^2\xi_i^2(t_{i+1} - t_i) - \overline{\sigma}^2\xi_i^2(t_{i+1} - t_i)] = 0.
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
&\hat{\mathbb{E}}[(\int_0^T \eta_t dB_t)^2] = \hat{\mathbb{E}}[\sum_{i=0}^{N-1} \xi_i^2(B_{t_{i+1}} - B_{t_i})^2] \\
\leq&\hat{\mathbb{E}}[\sum_{i=0}^{N-1} \xi_i^2(B_{t_{i+1}} - B_{t_i})^2 - \sum_{i=0}^{N-1} \overline{\sigma}^2\xi_i^2(t_{i+1} - t_i)] + \hat{\mathbb{E}}[\sum_{i=0}^{N-1} \overline{\sigma}^2\xi_i^2(t_{i+1} - t_i)] \\
\leq& \sum_{i=0}^{N-1} \hat{\mathbb{E}}[\xi_i^2(B_{t_{i+1}} - B_{t_i})^2 - \overline{\sigma}^2\xi_i^2(t_{i+1} - t_i)] + \hat{\mathbb{E}}[\sum_{i=0}^{N-1} \overline{\sigma}^2\xi_i^2(t_{i+1} - t_i)] \\
=&\hat{\mathbb{E}}[\sum_{i=0}^{N-1} \overline{\sigma}^2\xi_i^2(t_{i+1} - t_i)] = \bar{\sigma}^2\hat{\mathbb{E}}[\int_0^T \eta_t^2 dt].
\end{aligned}
$$

Definition

We define, for a fixed $\eta \in M_G^2(0,T)$, the stochastic integral

$$\int_0^T \eta_t dB_t := I(\eta).$$

It is clear that (–e1) and (–e2) still hold for $\eta \in M_G^2(0,T)$.

We list some main properties of Itô's integral of G–Brownian motion. We denote, for some $0 \leq s \leq t \leq T$,

$$\int_s^t \eta_u dB_u := \int_0^T \mathbf{I}_{[s,t]}(u)\eta_u dB_u.$$

Proposition.

(label)Prop-IntegLet $\eta, \theta \in M_G^2(0,T)$ and let $0 \leq s \leq r \leq t \leq T$. Then we have
(i) $\int_s^t \eta_u dB_u = \int_s^r \eta_u dB_u + \int_r^t \eta_u dB_u$.
(ii) $\int_s^t (\alpha\eta_u + \theta_u) dB_u = \alpha \int_s^t \eta_u dB_u + \int_s^t \theta_u dB_u$, if α is bounded and in $L_G^1(\Omega_s)$. (iii) $\hat{\mathbb{E}}[X + \int_r^T \eta_u dB_u | \Omega_s] = \hat{\mathbb{E}}[X|\Omega_s]$ for $X \in L_G^1(\Omega)$. □

We now consider the multi-dimensional case. Let $G(\cdot):\mathbb{S}(d)\longrightarrow\mathbb{R}$ be a given monotonic and sublinear function and let $(B_t)_{t\geq 0}$ be a d-dimensional G–Brownian motion. For each fixed $\mathbf{a}\in\mathbb{R}^d$, we still use $B_t^{\mathbf{a}}:=\langle \mathbf{a},B_t\rangle$. Then $(B_t^{\mathbf{a}})_{t\geq 0}$ is a 1-dimensional $G_{\mathbf{a}}$–Brownian motion with $G_{\mathbf{a}}(\alpha)=\frac{1}{2}(\sigma^2_{\mathbf{aa}^T}\alpha^+-\sigma^2_{-\mathbf{aa}^T}\alpha^-)$, where $\sigma^2_{\mathbf{aa}^T}=2G(\mathbf{aa}^T)$ and $\sigma^2_{-\mathbf{aa}^T}=-2G(-\mathbf{aa}^T)$.

Similar to 1-dimensional case, we can define Itô's integral by

$$I(\eta) := \int_0^T \eta_t dB_t^{\mathbf{a}}, \quad \text{for } \eta \in M_G^2(0,T).$$

We still have, for each $\eta \in M_G^2(0,T)$,

$$\hat{\mathbb{E}}[\int_0^T \eta_t dB_t^{\mathbf{a}}] = 0,$$
$$\hat{\mathbb{E}}[(\int_0^T \eta_t dB_t^{\mathbf{a}})^2] \leq \sigma_{\mathbf{aa}^T}^2 \hat{\mathbb{E}}[\int_0^T \eta_t^2 dt].$$

Furthermore, Proposition –Prop-Integ still holds for the integral with respect to $B_t^{\mathbf{a}}$.

Exercise.

Prove that, for a fixed $\eta \in M_G^2(0,T)$,

$$\underline{\sigma}^2 \hat{\mathbb{E}}[\int_0^T \eta_t^2 dt] \leq \hat{\mathbb{E}}[(\int_0^T \eta_t dB_t)^2] \leq \overline{\sigma}^2 \hat{\mathbb{E}}[\int_0^T \eta_t^2 dt],$$

where $\overline{\sigma}^2 = \hat{\mathbb{E}}[B_1^2]$ and $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-B_1^2]$.

□

Exercise.

Prove that, for each $\eta \in M_G^p(0,T)$, we have

$$\hat{\mathbb{E}}[\int_0^T |\eta_t|^p dt] \leq \int_0^T \hat{\mathbb{E}}[|\eta_t|^p] dt.$$

□

We first consider the quadratic variation process of 1-dimensional G–Brownian motion $(B_t)_{t\geq 0}$ with $B_1 \overset{d}{=} N(\{0\} \times [\underline{\sigma}^2, \bar{\sigma}^2])$. Let π_t^N, $N = 1, 2, \cdots$, be a sequence of partitions of $[0, t]$. We consider

$$B_t^2 = \sum_{j=0}^{N-1} (B^2_{t^N_{j+1}} - B^2_{t^N_j})$$
$$= \sum_{j=0}^{N-1} 2B_{t^N_j}(B_{t^N_{j+1}} - B_{t^N_j}) + \sum_{j=0}^{N-1} (B_{t^N_{j+1}} - B_{t^N_j})^2.$$

As $\mu(\pi_t^N) \to 0$, the first term of the right side converges to $2\int_0^t B_s dB_s$ in $L_G^2(\Omega)$. The second term must be convergent. We denote its limit by $\langle B \rangle_t$, i.e.,

$$\langle B \rangle_t := \lim_{\mu(\pi_t^N)\to 0} \sum_{j=0}^{N-1} (B_{t^N_{j+1}} - B_{t^N_j})^2 = B_t^2 - 2\int_0^t B_s dB_s.(label)quadra-def \tag{40}$$

By the above construction, $(\langle B \rangle_t)_{t \geq 0}$ is an increasing process with $\langle B \rangle_0 = 0$. We call it the **quadratic variation process**(label)qvb of the G–Brownian motion B. It characterizes the part of statistic uncertainty of G–Brownian motion. It is important to keep in mind that $\langle B \rangle_t$ is not a deterministic process unless $\underline{\sigma} = \bar{\sigma}$, i.e., when $(B_t)_{t \geq 0}$ is a classical Brownian motion. In fact we have the following lemma.

Lemma

(label)Lem-Q1For each $0 \leq s \leq t < \infty$*, we have*

$$\hat{\mathbb{E}}[\langle B\rangle_t - \langle B\rangle_s |\Omega_s] = \bar{\sigma}^2(t-s), \quad (label)quadra \tag{41}$$

$$\hat{\mathbb{E}}[-(\langle B\rangle_t - \langle B\rangle_s)|\Omega_s] = -\underline{\sigma}^2(t-s).(label)quadra1 \tag{42}$$

Proof.

By the definition of $\langle B \rangle$ and Proposition –Prop-Integ (iii),

$$\hat{\mathbb{E}}[\langle B\rangle_t - \langle B\rangle_s |\Omega_s] = \hat{\mathbb{E}}[B_t^2 - B_s^2 - 2\int_s^t B_u dB_u|\Omega_s]$$
$$= \hat{\mathbb{E}}[B_t^2 - B_s^2|\Omega_s] = \bar{\sigma}^2(t-s).$$

The last step follows from Example –Exam-B2. We then have (–quadra). The equality (–quadra1) can be proved analogously with the consideration of $\hat{\mathbb{E}}[-(B_t^2 - B_s^2)|\Omega_s] = -\underline{\sigma}^2(t-s)$. □

A very interesting point of the quadratic variation process $\langle B\rangle$ is, just like the G–Brownian motion B itself, the increment $\langle B\rangle_{s+t}-\langle B\rangle_s$ is independent from Ω_s and identically distributed with $\langle B\rangle_t$. In fact we have

Lemma

(label)Lem-Qua2For each fixed $s,t \geq 0$, $\langle B\rangle_{s+t} - \langle B\rangle_s$ *is identically distributed with* $\langle B\rangle_t$ *and independent from* Ω_s.

Proof.

The results follow directly from

$$\begin{aligned}\langle B\rangle_{s+t}-\langle B\rangle_{s}&=B_{s+t}^{2}-2\int_{0}^{s+t}B_{r}dB_{r}-[B_{s}^{2}-2\int_{0}^{s}B_{r}dB_{r}]\\&=(B_{s+t}-B_{s})^{2}-2\int_{s}^{s+t}(B_{r}-B_{s})d(B_{r}-B_{s})\\&=\langle B^{s}\rangle_{t},\end{aligned}$$

where $\langle B^s\rangle$ is the quadratic variation process of the G–Brownian motion $B_t^s=B_{s+t}-B_s,\ t\geq 0$. □

We now define the integral of a process $\eta \in M_G^1(0,T)$ with respect to $\langle B \rangle$. We first define a mapping:

$$Q_{0,T}(\eta) = \int_0^T \eta_t d\langle B \rangle_t := \sum_{j=0}^{N-1} \xi_j(\langle B \rangle_{t_{j+1}} - \langle B \rangle_{t_j}) : M_G^{1,0}(0,T) \to L_G^1(\Omega_T).$$

Lemma

(label)Lem-Q2For each $\eta \in M_G^{1,0}(0,T)$,

$$\hat{\mathbb{E}}[|Q_{0,T}(\eta)|] \leq \bar{\sigma}^2 \hat{\mathbb{E}}[\int_0^T |\eta_t| dt]. \ (label)dA \tag{43}$$

Thus $Q_{0,T} : M_G^{1,0}(0,T) \to L_G^1(\Omega_T)$ *is a continuous linear mapping. Consequently,* $Q_{0,T}$ *can be uniquely extended to* $M_G^1(0,T)$*. We still denote this mapping by*

$$\int_0^T \eta_t d\langle B\rangle_t := Q_{0,T}(\eta) \quad \text{for } \eta \in M_G^1(0,T).$$

We still have

$$\hat{\mathbb{E}}[|\int_0^T \eta_t d\langle B\rangle_t|] \leq \bar{\sigma}^2 \hat{\mathbb{E}}[\int_0^T |\eta_t| dt] \quad \text{for } \eta \in M_G^1(0,T).(label)qua-ine \tag{44}$$

Proof.

Firstly, for each $j = 1, \cdots, N-1$, we have

$$\begin{aligned}
&\hat{\mathbb{E}}[|\xi_j|(\langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j}) - \overline{\sigma}^2|\xi_j|(t_{j+1} - t_j)] \\
=&\hat{\mathbb{E}}[\hat{\mathbb{E}}[|\xi_j|(\langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j})|\Omega_{t_j}] - \overline{\sigma}^2|\xi_j|(t_{j+1} - t_j)] \\
=&\hat{\mathbb{E}}[|\xi_j|\overline{\sigma}^2(t_{j+1} - t_j) - \overline{\sigma}^2|\xi_j|(t_{j+1} - t_j)] = 0.
\end{aligned}$$

Then (–dA) can be checked as follows:

$$\begin{aligned}
&\hat{\mathbb{E}}[|\sum_{j=0}^{N-1} \xi_j(\langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j})|] \le \hat{\mathbb{E}}[\sum_{j=0}^{N-1} |\xi_j|\,\langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j}] \\
\le&\hat{\mathbb{E}}[\sum_{j=0}^{N-1} |\xi_j|[(\langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j}) - \overline{\sigma}^2(t_{j+1} - t_j)]] + \hat{\mathbb{E}}[\overline{\sigma}^2 \sum_{j=0}^{N-1} |\xi_j|(t_{j+1} - t_j)] \\
\le& \sum_{j=0}^{N-1} \hat{\mathbb{E}}[|\xi_j|[(\langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j}) - \overline{\sigma}^2(t_{j+1} - t_j)]] + \hat{\mathbb{E}}[\overline{\sigma}^2 \sum_{j=0}^{N-1} |\xi_j|(t_{j+1} - t_j)] \\
=&\hat{\mathbb{E}}[\overline{\sigma}^2 \sum_{j=0}^{N-1} |\xi_j|(t_{j+1} - t_j)] = \overline{\sigma}^2\hat{\mathbb{E}}[\int_0^T |\eta_t|dt].
\end{aligned}$$

Proposition.

(label)Prop-tempLet $0 \leq s \leq t$, $\xi \in L_G^2(\Omega_s)$, $X \in L_G^1(\Omega)$. Then

$$\hat{\mathbb{E}}[X + \xi(B_t^2 - B_s^2)] = \hat{\mathbb{E}}[X + \xi(B_t - B_s)^2]$$
$$= \hat{\mathbb{E}}[X + \xi(\langle B \rangle_t - \langle B \rangle_s)].$$

□

Proof.

By (–quadra-def) and Proposition –Prop-Integ (iii), we have

$$\hat{\mathbb{E}}[X + \xi(B_t^2 - B_s^2)] = \hat{\mathbb{E}}[X + \xi(\langle B \rangle_t - \langle B \rangle_s + 2\int_s^t B_u dB_u)]$$
$$= \hat{\mathbb{E}}[X + \xi(\langle B \rangle_t - \langle B \rangle_s)].$$

We also have

$$\hat{\mathbb{E}}[X + \xi(B_t^2 - B_s^2)] = \hat{\mathbb{E}}[X + \xi((B_t - B_s)^2 + 2(B_t - B_s)B_s)]$$
$$= \hat{\mathbb{E}}[X + \xi(B_t - B_s)^2].$$

We have the following isometry.

Proposition.

Let $\eta \in M_G^2(0,T)$. Then

$$\hat{\mathbb{E}}[(\int_0^T \eta_t dB_t)^2] = \hat{\mathbb{E}}[\int_0^T \eta_t^2 d\left\langle B\right\rangle_t].(label)isometry \qquad (45)$$

□

Proof.

We first consider $\eta \in M_G^{2,0}(0,T)$ of the form

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathbf{I}_{[t_j,t_{j+1})}(t)$$

and then $\int_0^T \eta_t dB_t = \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j})$. From Proposition –Prop-Integ, we get

$$\hat{\mathbb{E}}[X + 2\xi_j(B_{t_{j+1}} - B_{t_j})\xi_i(B_{t_{i+1}} - B_{t_i})] = \hat{\mathbb{E}}[X] \text{ for } X \in L_G^1(\Omega),\ i \neq j.$$

□

Proof (continue).

Thus

$$\hat{\mathbb{E}}[(\int_0^T \eta_t dB_t)^2] = \hat{\mathbb{E}}[(\sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}))^2] = \hat{\mathbb{E}}[\sum_{j=0}^{N-1} \xi_j^2 (B_{t_{j+1}} - B_{t_j})^2].$$

From this and Proposition –Prop-temp, it follows that

$$\hat{\mathbb{E}}[(\int_0^T \eta_t dB_t)^2] = \hat{\mathbb{E}}[\sum_{j=0}^{N-1} \xi_j^2 (\langle B \rangle_{t_{j+1}} - \langle B \rangle_{t_j})] = \hat{\mathbb{E}}[\int_0^T \eta_t^2 d\langle B \rangle_t].$$

Thus (–isometry) holds for $\eta \in M_G^{2,0}(0,T)$. We can continuously extend the above equality to the case $\eta \in M_G^2(0,T)$ and get (–isometry). □

We now consider the multi-dimensional case. Let $(B_t)_{t\geq 0}$ be a d-dimensional G–Brownian motion. For each fixed $\mathbf{a} \in \mathbb{R}^d$, $(B_t^{\mathbf{a}})_{t\geq 0}$ is a 1-dimensional $G_{\mathbf{a}}$–Brownian motion. Similar to 1-dimensional case, we can define

$$\langle B^{\mathbf{a}}\rangle_t := \lim_{\mu(\pi_t^N)\to 0} \sum_{j=0}^{N-1} (B^{\mathbf{a}}_{t^N_{j+1}} - B^{\mathbf{a}}_{t^N_j})^2 = (B^{\mathbf{a}}_t)^2 - 2\int_0^t B^{\mathbf{a}}_s dB^{\mathbf{a}}_s,$$

where $\langle B^{\mathbf{a}}\rangle$ is called the **quadratic variation process** of $B^{\mathbf{a}}$. The above results also hold for $\langle B^{\mathbf{a}}\rangle$.

In particular,

$$\hat{\mathbb{E}}[|\int_0^T \eta_t d\langle B^{\mathbf{a}}\rangle_t|] \leq \sigma^2_{\mathbf{a}\mathbf{a}^T}\hat{\mathbb{E}}[\int_0^T |\eta_t| dt] \quad \text{for } \eta \in M_G^1(0,T)$$

and

$$\hat{\mathbb{E}}[(\int_0^T \eta_t dB_t^{\mathbf{a}})^2] = \hat{\mathbb{E}}[\int_0^T \eta_t^2 d\langle B^{\mathbf{a}}\rangle_t] \quad \text{for } \eta \in M_G^2(0,T).$$

Let $\mathbf{a} = (a_1, \cdots, a_d)^T$ and $\bar{\mathbf{a}} = (\bar{a}_1, \cdots, \bar{a}_d)^T$ be two given vectors in $\mathbb{R}^d$. We then have their quadratic variation processes of $\langle B^{\mathbf{a}} \rangle$ and $\langle B^{\bar{\mathbf{a}}} \rangle$. We can define their **mutual variation process** by

$$\begin{aligned}\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}} \rangle_t &:= \frac{1}{4}[\langle B^{\mathbf{a}} + B^{\bar{\mathbf{a}}} \rangle_t - \langle B^{\mathbf{a}} - B^{\bar{\mathbf{a}}} \rangle_t] \\ &= \frac{1}{4}[\langle B^{\mathbf{a}+\bar{\mathbf{a}}} \rangle_t - \langle B^{\mathbf{a}-\bar{\mathbf{a}}} \rangle_t].\end{aligned}$$

Since $\left\langle B^{\mathbf{a}-\bar{\mathbf{a}}}\right\rangle = \left\langle B^{\bar{\mathbf{a}}-\mathbf{a}}\right\rangle = \left\langle -B^{\mathbf{a}-\bar{\mathbf{a}}}\right\rangle$, we see that $\left\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\right\rangle_t = \left\langle B^{\bar{\mathbf{a}}}, B^{\mathbf{a}}\right\rangle_t$. In particular, we have $\langle B^{\mathbf{a}}, B^{\mathbf{a}}\rangle = \langle B^{\mathbf{a}}\rangle$. Let π_t^N, $N = 1, 2, \cdots$, be a sequence of partitions of $[0, t]$.

We observe that

$$\sum_{k=0}^{N-1}(B^{\mathbf{a}}_{t^N_{k+1}}-B^{\mathbf{a}}_{t^N_k})(B^{\bar{\mathbf{a}}}_{t^N_{k+1}}-B^{\bar{\mathbf{a}}}_{t^N_k})=\frac{1}{4}\sum_{k=0}^{N-1}[(B^{\mathbf{a}+\bar{\mathbf{a}}}_{t_{k+1}}-B^{\mathbf{a}+\bar{\mathbf{a}}}_{t_k})^2-(B^{\mathbf{a}-\bar{\mathbf{a}}}_{t_{k+1}}-B^{\mathbf{a}-\bar{\mathbf{a}}}_{t_k})^2].$$

Thus as $\mu(\pi^N_t)\to 0$ we have

$$\lim_{N\to\infty}\sum_{k=0}^{N-1}(B^{\mathbf{a}}_{t^N_{k+1}}-B^{\mathbf{a}}_{t^N_k})(B^{\bar{\mathbf{a}}}_{t^N_{k+1}}-B^{\bar{\mathbf{a}}}_{t^N_k})=\left\langle B^{\mathbf{a}},B^{\bar{\mathbf{a}}}\right\rangle_t.$$

We also have

$$
\begin{aligned}
\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}} \rangle_t &= \frac{1}{4}[\langle B^{\mathbf{a}+\bar{\mathbf{a}}} \rangle_t - \langle B^{\mathbf{a}-\bar{\mathbf{a}}} \rangle_t] \\
&= \frac{1}{4}[(B_t^{\mathbf{a}+\bar{\mathbf{a}}})^2 - 2\int_0^t B_s^{\mathbf{a}+\bar{\mathbf{a}}} dB_s^{\mathbf{a}+\bar{\mathbf{a}}} - (B_t^{\mathbf{a}-\bar{\mathbf{a}}})^2 + 2\int_0^t B_s^{\mathbf{a}-\bar{\mathbf{a}}} dB_s^{\mathbf{a}-\bar{\mathbf{a}}}] \\
&= B_t^{\mathbf{a}} B_t^{\bar{\mathbf{a}}} - \int_0^t B_s^{\mathbf{a}} dB_s^{\bar{\mathbf{a}}} - \int_0^t B_s^{\bar{\mathbf{a}}} dB_s^{\mathbf{a}}.
\end{aligned}
$$

Now for each $\eta \in M_G^1(0,T)$, we can consistently define

$$\int_0^T \eta_t d\left\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\right\rangle_t = \frac{1}{4}\int_0^T \eta_t d\left\langle B^{\mathbf{a}+\bar{\mathbf{a}}}\right\rangle_t - \frac{1}{4}\int_0^T \eta_t d\left\langle B^{\mathbf{a}-\bar{\mathbf{a}}}\right\rangle_t .$$

Lemma

(label)d-Lem-mutualLet $\eta^N \in M_G^{2,0}(0,T)$, $N = 1, 2, \cdots$, *be of the form*

$$\eta_t^N(\omega) = \sum_{k=0}^{N-1} \xi_k^N(\omega) \mathbf{I}_{[t_k^N, t_{k+1}^N)}(t)$$

with $\mu(\pi_T^N) \to 0$ *and* $\eta^N \to \eta$ *in* $M_G^2(0,T)$, *as* $N \to \infty$. *Then we have the following convergence in* $L_G^2(\Omega_T)$:

$$\sum_{k=0}^{N-1} \xi_k^N (B^{\mathbf{a}}_{t_{k+1}^N} - B^{\mathbf{a}}_{t_k^N})(B^{\bar{\mathbf{a}}}_{t_{k+1}^N} - B^{\bar{\mathbf{a}}}_{t_k^N}) \to \int_0^T \eta_t d\left\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}} \right\rangle_t.$$

Proof.

Since

$$\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_{t_{k+1}^N} - \langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_{t_k^N} = (B^{\mathbf{a}}_{t_{k+1}^N} - B^{\mathbf{a}}_{t_k^N})(B^{\bar{\mathbf{a}}}_{t_{k+1}^N} - B^{\bar{\mathbf{a}}}_{t_k^N}) - \int_{t_k^N}^{t_{k+1}^N} (B^{\mathbf{a}}_s - B^{\mathbf{a}}_{t_k^N}) dB^{\bar{\mathbf{a}}}_s - \int_{t_k^N}^{t_{k+1}^N} (B^{\bar{\mathbf{a}}}_s - B^{\bar{\mathbf{a}}}_{t_k^N}) dB^{\mathbf{a}}_s$$

□

Proof. (continue).

we only need to prove

$$\hat{\mathbb{E}}[\sum_{k=0}^{N-1}(\xi_k^N)^2(\int_{t_k^N}^{t_{k+1}^N}(B_s^{\mathbf{a}}-B_{t_k^N}^{\mathbf{a}})dB_s^{\bar{\mathbf{a}}})^2]\to 0.$$

For each $k=1,\cdots,N-1$, we have

$$\begin{aligned}
&\hat{\mathbb{E}}[(\xi_k^N)^2(\int_{t_k^N}^{t_{k+1}^N}(B_s^{\mathbf{a}}-B_{t_k^N}^{\mathbf{a}})dB_s^{\bar{\mathbf{a}}})^2-C(\xi_k^N)^2(t_{k+1}^N-t_k^N)^2]\\
=&\hat{\mathbb{E}}[\hat{\mathbb{E}}[(\xi_k^N)^2(\int_{t_k^N}^{t_{k+1}^N}(B_s^{\mathbf{a}}-B_{t_k^N}^{\mathbf{a}})dB_s^{\bar{\mathbf{a}}})^2|\Omega_{t_k^N}]-C(\xi_k^N)^2(t_{k+1}^N-t_k^N)^2]\\
\leq&\hat{\mathbb{E}}[C(\xi_k^N)^2(t_{k+1}^N-t_k^N)^2-C(\xi_k^N)^2(t_{k+1}^N-t_k^N)^2]=0,
\end{aligned}$$

where $C=\bar{\sigma}^2_{\mathbf{a}\mathbf{a}^T}\bar{\sigma}^2_{\bar{\mathbf{a}}\bar{\mathbf{a}}^T}/2$.

□

Proof. (continue).

Thus we have

$$
\begin{aligned}
&\hat{\mathbb{E}}[\sum_{k=0}^{N-1}(\xi_k^N)^2(\int_{t_k^N}^{t_{k+1}^N}(B_s^{\mathbf{a}}-B_{t_k^N}^{\mathbf{a}})dB_s^{\bar{\mathbf{a}}})^2] \\
\leq &\hat{\mathbb{E}}[\sum_{k=0}^{N-1}(\xi_k^N)^2[(\int_{t_k^N}^{t_{k+1}^N}(B_s^{\mathbf{a}}-B_{t_k^N}^{\mathbf{a}})dB_s^{\bar{\mathbf{a}}})^2-C(t_{k+1}^N-t_k^N)^2]] \\
&+\hat{\mathbb{E}}[\sum_{k=0}^{N-1}C(\xi_k^N)^2(t_{k+1}^N-t_k^N)^2] \\
\leq &\sum_{k=0}^{N-1}\hat{\mathbb{E}}[(\xi_k^N)^2[(\int_{t_k^N}^{t_{k+1}^N}(B_s^{\mathbf{a}}-B_{t_k^N}^{\mathbf{a}})dB_s^{\bar{\mathbf{a}}})^2-C(t_{k+1}^N-t_k^N)^2]] \\
&+\hat{\mathbb{E}}[\sum_{k=0}^{N-1}C(\xi_k^N)^2(t_{k+1}^N-t_k^N)^2] \\
\leq &\hat{\mathbb{E}}[\sum_{k=0}^{N-1}C(\xi_k^N)^2(t_{k+1}^N-t_k^N)^2]\leq C\mu(\pi_T^N)\hat{\mathbb{E}}[\int_0^T|\eta_t^N|^2dt],
\end{aligned}
$$

Exercise.

Let B_t be a 1-dimensional G-Brownian motion and φ be a bounded and Lipschitz function on $\mathbb{R}$. Show that

$$\lim_{N\to\infty}\hat{\mathbb{E}}[|\sum_{k=0}^{N-1}\varphi(B_{t_k^N})[(B_{t_{k+1}^N}-B_{t_k^N})^2-(\langle B\rangle_{t_{k+1}^N}-\langle B\rangle_{t_k^N})]|]=0,$$

where $t_k^N=kT/N, k=0,2,\cdots,N-1$. □

Exercise.

Prove that, for a fixed $\eta\in M_G^1(0,T)$,

$$\underline{\sigma}^2\hat{\mathbb{E}}[\int_0^T|\eta_t|dt]\leq\hat{\mathbb{E}}[\int_0^T|\eta_t|d\langle B\rangle_t]\leq\overline{\sigma}^2\hat{\mathbb{E}}[\int_0^T|\eta_t|dt],$$

where $\overline{\sigma}^2=\hat{\mathbb{E}}[B_1^2]$ and $\underline{\sigma}^2=-\hat{\mathbb{E}}[-B_1^2]$. □

In this section, we first consider the 1-dimensional G–Brownian motion $(B_t)_{t\geq 0}$ with $B_1 \overset{d}{=} N(\{0\} \times [\underline{\sigma}^2, \bar{\sigma}^2])$.
The quadratic variation process $\langle B \rangle$ of G-Brownian motion B is a very interesting process. We have seen that the G-Brownian motion B is a typical process with variance uncertainty but without mean-uncertainty. In fact, $\langle B \rangle$ is concentrated all uncertainty of the G-Brownian motion B. Moreover, $\langle B \rangle$ itself is a typical process with mean-uncertainty. This fact will be applied to measure the mean-uncertainty of risk positions.

Lemma

We have

$$\hat{\mathbb{E}}[\langle B\rangle_t^2] \leq 10\bar{\sigma}^4 t^2.(label)Qua2 \qquad (46)$$

Proof.

Indeed,

$$
\begin{aligned}
\hat{\mathbb{E}}[\langle B\rangle_t^2] &= \hat{\mathbb{E}}[(B_t{}^2 - 2\int_0^t B_u dB_u)^2] \\
&\le 2\hat{\mathbb{E}}[B_t^4] + 8\hat{\mathbb{E}}[(\int_0^t B_u dB_u)^2] \\
&\le 6\bar{\sigma}^4 t^2 + 8\bar{\sigma}^2\hat{\mathbb{E}}[\int_0^t B_u{}^2 du] \\
&\le 6\bar{\sigma}^4 t^2 + 8\bar{\sigma}^2 \int_0^t \hat{\mathbb{E}}[B_u{}^2] du \\
&= 10\bar{\sigma}^4 t^2.
\end{aligned}
$$

□

Proposition.

(label)c3p1 Let $(b_t)_{t\geq 0}$ be a process on a sublinear expectation space $(\Omega,\mathcal{H},\hat{\mathbb{E}})$ such that
(i) $b_0 = 0$;
(ii) For each $t, s \geq 0$, $b_{t+s} - b_t$ is identically distributed with b_s and independent from $(b_{t_1}, b_{t_2}, \cdots, b_{t_n})$ for each $n \in \mathbb{N}$ and $0 \leq t_1, \cdots, t_n \leq t$;
(iii) $\lim_{t\downarrow 0} \hat{\mathbb{E}}[b_t^2]t^{-1} = 0$.
Then b_t is $N([\underline{\mu}t, \overline{\mu}t] \times \{0\})$-distributed with $\overline{\mu} = \hat{\mathbb{E}}[b_1]$ and $\underline{\mu} = -\hat{\mathbb{E}}[-b_1]$. □

Proof.

We first prove that

$$\hat{\mathbb{E}}[b_t] = \overline{\mu}t \text{ and } -\hat{\mathbb{E}}[-b_t] = \underline{\mu}t.$$

We set $\varphi(t) := \hat{\mathbb{E}}[b_t]$. Then $\varphi(0) = 0$ and $\lim_{t\downarrow 0}\varphi(t)$ =0. Since for each $t, s \geq 0$,

$$\begin{aligned}\varphi(t+s) &= \hat{\mathbb{E}}[b_{t+s}] = \hat{\mathbb{E}}[(b_{t+s} - b_s) + b_s] \\ &= \varphi(t) + \varphi(s).\end{aligned}$$

Thus $\varphi(t)$ is linear and uniformly continuous in t, which means that $\hat{\mathbb{E}}[b_t] = \overline{\mu}t$. Similarly $-\hat{\mathbb{E}}[-b_t] = \underline{\mu}t$. □

We now prove that b_t is $N([\underline{\mu}t, \overline{\mu}t] \times \{0\})$-distributed. By Exercise –ex2 in Chap.–ch2, we just need to prove that for each fixed $\varphi \in C_{b.Lip}(\mathbb{R})$, the function

$$u(t,x) := \hat{\mathbb{E}}[\varphi(x+b_t)], \quad (t,x) \in [0,\infty) \times \mathbb{R}$$

is the viscosity solution of the following parabolic PDE:

$$\partial_t u - g(\partial_x u) = 0, \qquad u|_{t=0} = \varphi(label) G - mean \tag{47}$$

with $g(a) = \overline{\mu}a^+ - \underline{\mu}a^-$.

We first prove that u is Lipschitz in x and $\frac{1}{2}$-Hölder continuous in t. In fact, for each fixed t, $u(t,\cdot)\in C_{b.Lip}(\mathbb{R})$ since

$$\begin{aligned}|\hat{\mathbb{E}}[\varphi(x+b_t)]-\hat{\mathbb{E}}[\varphi(y+b_t)]| &\leq \hat{\mathbb{E}}[|\varphi(x+b_t)-\varphi(y+b_t)|] \\ &\leq C|x-y|.\end{aligned}$$

For each $\delta \in [0,t]$, since $b_t - b_\delta$ is independent from b_δ, we have

$$\begin{aligned} u(t,x) &= \hat{\mathbb{E}}[(x + b_\delta + (b_t - b_\delta)] \\ &= \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(y + (b_t - b_\delta))]_{y=x+b_\delta}], \end{aligned}$$

hence

$$u(t,x) = \hat{\mathbb{E}}[u(t-\delta, x + b_\delta)].(label)eq4.21 \tag{48}$$

Thus

$$\begin{aligned} |u(t,x) - u(t-\delta,x)| &= |\hat{\mathbb{E}}[u(t-\delta,x+b_\delta) - u(t-\delta,x)]| \\ &\leq \hat{\mathbb{E}}[|u(t-\delta,x+b_\delta) - u(t-\delta,x)|] \\ &\leq \hat{\mathbb{E}}[C|b_\delta|] \leq C_1\sqrt{\delta}. \end{aligned}$$

To prove that u is a viscosity solution of the PDE (–G-mean), we fix a point $(t, x) \in (0, \infty) \times \mathbb{R}$ and let $v \in C_b^{2,2}([0, \infty) \times \mathbb{R})$ be such that $v \geq u$ and $v(t, x) = u(t, x)$. From (–eq4.21), we have

$$v(t, x) = \hat{\mathbb{E}}[u(t - \delta, x + b_\delta)] \leq \hat{\mathbb{E}}[v(t - \delta, x + b_\delta)].$$

Therefore, by Taylor's expansion,

$$
\begin{aligned}
0 &\le \hat{\mathbb{E}}[v(t-\delta, x+b_\delta) - v(t,x)] \\
&= \hat{\mathbb{E}}[v(t-\delta, x+b_\delta) - v(t, x+b_\delta) + (v(t, x+b_\delta) - v(t,x))] \\
&= \hat{\mathbb{E}}[-\partial_t v(t,x)\delta + \partial_x v(t,x) b_\delta + I_\delta] \\
&\le -\partial_t v(t,x)\delta + \hat{\mathbb{E}}[\partial_x v(t,x) b_\delta] + \hat{\mathbb{E}}[I_\delta] \\
&= -\partial_t v(t,x)\delta + g(\partial_x v(t,x))\delta + \hat{\mathbb{E}}[I_\delta],
\end{aligned}
$$

where

$$
\begin{aligned}
I_\delta = \delta \int_0^1 &[-\partial_t v(t-\beta\delta, x+b_\delta) + \partial_t v(t,x)] d\beta \\
&+ b_\delta \int_0^1 [\partial_x v(t, x+\beta b_\delta) - \partial_x v(t,x)] d\beta.
\end{aligned}
$$

Proof. (continue).

With the assumption that $\lim_{t\downarrow 0}\hat{\mathbb{E}}[b_t^2]t^{-1}=0$, we can check that

$$\lim_{\delta\downarrow 0}\hat{\mathbb{E}}[|I_\delta|]\delta^{-1}=0,$$

from which we get $\partial_t v(t,x)-g(\partial_x v(t,x))\leq 0$, hence u is a viscosity subsolution of (–G-mean). We can analogously prove that u is also a viscosity supersolution. It follows that b_t is $N([\underline{\mu}t,\overline{\mu}t]\times\{0\})$-distributed. The proof is complete. □

It is clear that $\langle B \rangle$ satisfies all the conditions in the Proposition –c3p1, thus we immediately have

Theorem

$\langle B \rangle_t$ is $N([\underline{\sigma}^2 t, \bar{\sigma}^2 t] \times \{0\})$-distributed, i.e., for each $\varphi \in C_{Lip}(\mathbb{R})$,

$$\hat{\mathbb{E}}[\varphi(\langle B \rangle_t)] = \sup_{\underline{\sigma}^2 \leq v \leq \bar{\sigma}^2} \varphi(vt). \tag{49}$$

Corollary

For each $0 \leq t \leq T < \infty$, *we have*

$$\underline{\sigma}^2(T-t) \leq \langle B \rangle_T - \langle B \rangle_t \leq \bar{\sigma}^2(T-t) \ \ in\ L_G^1(\Omega).$$

Proof.

It is a direct consequence of

$$\hat{\mathbb{E}}[(\langle B\rangle_T - \langle B\rangle_t - \bar{\sigma}^2(T-t))^+] = \sup_{\underline{\sigma}^2 \le v \le \bar{\sigma}^2} (v - \bar{\sigma}^2)^+(T-t) = 0$$

and

$$\hat{\mathbb{E}}[(\langle B\rangle_T - \langle B\rangle_t - \underline{\sigma}^2(T-t))^-] = \sup_{\underline{\sigma}^2 \le v \le \bar{\sigma}^2} (v - \underline{\sigma}^2)^-(T-t) = 0.$$

□

Corollary

(label)Lem-Qua2 copy(1) We have, for each $t, s \geq 0$, $n \in \mathbb{N}$,

$$\hat{\mathbb{E}}[(\langle B \rangle_{t+s} - \langle B \rangle_s)^n | \Omega_s] = \hat{\mathbb{E}}[\langle B \rangle_t^n] = \bar{\sigma}^{2n} t^n \quad (50)$$

and

$$\hat{\mathbb{E}}[-(\langle B \rangle_{t+s} - \langle B \rangle_s)^n | \Omega_s] = \hat{\mathbb{E}}[-\langle B \rangle_t^n] = -\underline{\sigma}^{2n} t^n. \quad (51)$$

We now consider the multi-dimensional case. For notational simplicity, we denote by $B^i := B^{\mathbf{e}_i}$ the i-th coordinate of the G–Brownian motion B, under a given orthonormal basis $(\mathbf{e}_1, \cdots, \mathbf{e}_d)$ of $\mathbb{R}^d$. We denote

$$(\langle B \rangle_t)_{ij} := \langle B^i, B^j \rangle_t .$$

Then $\langle B \rangle_t$, $t \geq 0$, is an $\mathbb{S}(d)$-valued process.

Since

$$\hat{\mathbb{E}}[\langle AB_t, B_t\rangle] = 2G(A)t \quad \text{for } A \in \mathbb{S}(d),$$

we have

$$\begin{aligned}
\hat{\mathbb{E}}[(\langle B\rangle_t, A)] &= \hat{\mathbb{E}}[\sum_{i,j=1}^{d} a_{ij} \left\langle B^i, B^j \right\rangle_t] \\
&= \hat{\mathbb{E}}[\sum_{i,j=1}^{d} a_{ij}(B_t^i B_t^j - \int_0^t B_s^i dB_s^j - \int_0^t B_s^j dB_s^i)] \\
&= \hat{\mathbb{E}}[\sum_{i,j=1}^{d} a_{ij} B_t^i B_t^j] = 2G(A)t \quad \text{for } A \in \mathbb{S}(d),
\end{aligned}$$

where $(a_{ij})_{i,j=1}^d = A$.

Now we set, for each $\varphi \in C_{Lip}(\mathbb{S}(d))$,

$$v(t, X) := \hat{\mathbb{E}}[\varphi(X + \langle B \rangle_t)], \quad (t, X) \in [0, \infty) \times \mathbb{S}(d).$$

Let $\Sigma \subset \mathbb{S}_+(d)$ be the bounded, convex and closed subset such that

$$G(A) = \frac{1}{2} \sup_{B \in \Sigma} (A, B), \qquad A \in \mathbb{S}(d).$$

Proposition.

(label)p1 The function v solves the following first order PDE:

$$\partial_t v - 2G(Dv) = 0,\ v|_{t=0} = \varphi,$$

where $Dv = (\partial_{x_{ij}} v)_{i,j=1}^d$. We also have

$$v(t,X) = \sup_{\Lambda\in\Sigma} \varphi(X + t\Lambda).$$

□

Sketch of the Proof. We have

$$v(t+\delta, X) = \hat{\mathbb{E}}[\varphi(X + \langle B \rangle_\delta + \langle B \rangle_{t+\delta} - \langle B \rangle_\delta)]$$
$$= \hat{\mathbb{E}}[v(t, X + \langle B \rangle_\delta)].$$

The rest part of the proof is similar to the 1-dimensional case. □

Corollary

We have

$$\langle B \rangle_t \in t\Sigma := \{t \times \gamma : \gamma \in \Sigma\},$$

or equivalently, $d_{t\Sigma}(\langle B \rangle_t) = 0$, *where*
$d_U(X) = \inf\{\sqrt{(X - Y, X - Y)} : Y \in U\}$.

Proof.

Since

$$\hat{\mathbb{E}}[d_{t\Sigma}(\langle B \rangle_t)] = \sup_{\Lambda \in \Sigma} d_{t\Sigma}(t\Lambda) = 0,$$

it follows that $d_{t\Sigma}(\langle B \rangle_t) = 0$. □

Exercise.

Complete the proof of Proposition –p1. □

In this section, we give Itô's formula for a "G-Itô process" X. For simplicity, we first consider the case of the function Φ is sufficiently regular.

Lemma

(label)d-Lem-26Let $\Phi \in C^2(\mathbb{R}^n)$ *with* $\partial_{x^\nu}\Phi$, $\partial^2_{x^\mu x^\nu}\Phi \in C_{b.Lip}(\mathbb{R}^n)$ *for* $\mu, \nu = 1, \cdots, n$. *Let* $s \in [0, T]$ *be fixed and let* $X = (X^1, \cdots, X^n)^T$ *be an* n*–dimensional process on* $[s, T]$ *of the form*

$$X_t^\nu = X_s^\nu + \alpha^\nu(t-s) + \eta^{\nu ij}(\langle B^i, B^j \rangle_t - \langle B^i, B^j \rangle_s) + \beta^{\nu j}(B_t^j - B_s^j),$$

where, for $\nu = 1, \cdots, n$, $i, j = 1, \cdots, d$, α^ν, $\eta^{\nu ij}$ *and* $\beta^{\nu j}$ *are bounded elements in* $L_G^2(\Omega_s)$ *and* $X_s = (X_s^1, \cdots, X_s^n)^T$ *is a given random vector in* $L_G^2(\Omega_s)$.

Lemma (continue).

Then we have, in $L_G^2(\Omega_t)$,

$$\Phi(X_t) - \Phi(X_s) = \int_s^t \partial_{x^\nu}\Phi(X_u)\beta^{\nu j} dB_u^j + \int_s^t \partial_{x^\nu}\Phi(X_u)\alpha^\nu du(label)d - B - \tag{52}$$

$$+ \int_s^t [\partial_{x^\nu}\Phi(X_u)\eta^{\nu ij} + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u)\beta^{\mu i}\beta^{\nu j}] d\left\langle B^i, B^j \right\rangle_u .$$

Here we use the , i.e., the above repeated indices μ, ν, i and j imply the summation. □

Proof.

For each positive integer N, we set $\delta = (t-s)/N$ and take the partition

$$\pi^N_{[s,t]} = \{t^N_0, t^N_1, \cdots, t^N_N\} = \{s, s+\delta, \cdots, s+N\delta = t\}.$$

We have

$$\Phi(X_t) - \Phi(X_s) = \sum_{k=0}^{N-1} [\Phi(X_{t^N_{k+1}}) - \Phi(X_{t^N_k})](label)d - Ito \tag{53}$$

$$= \sum_{k=0}^{N-1} \{\partial_{x^\nu}\Phi(X_{t^N_k})(X^\nu_{t^N_{k+1}} - X^\nu_{t^N_k})$$

$$+ \frac{1}{2}[\partial^2_{x^\mu x^\nu}\Phi(X_{t^N_k})(X^\mu_{t^N_{k+1}} - X^\mu_{t^N_k})(X^\nu_{t^N_{k+1}} - X^\nu_{t^N_k}) + \eta^N_k]\},$$

where

$$\eta^N_k = [\partial^2_{x^\mu x^\nu}\Phi(X_{t^N_k} + \theta_k(X_{t^N_{k+1}} - X_{t^N_k})) - \partial^2_{x^\mu x^\nu}\Phi(X_{t^N_k})](X^\mu_{t^N_{k+1}} - X^\mu_{t^N_k})(X^\nu_{t^N_{k+1}} - X$$

with $\theta_k \in [0,1]$.

Proof. (Continue).

We have

$$\hat{\mathbb{E}}[|\eta_k^N|^2] = \hat{\mathbb{E}}[|[\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N} + \theta_k(X_{t_{k+1}^N} - X_{t_k^N})) - \partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})]$$
$$\times (X^\mu_{t_{k+1}^N} - X^\mu_{t_k^N})(X^\nu_{t_{k+1}^N} - X^\nu_{t_k^N})|^2]$$
$$\leq c\hat{\mathbb{E}}[|X_{t_{k+1}^N} - X_{t_k^N}|^6] \leq C[\delta^6 + \delta^3],$$

where c is the Lipschitz constant of $\{\partial^2_{x^\mu x^\nu}\Phi\}_{\mu,\nu=1}^n$ and C is a constant independent of k. Thus

$$\hat{\mathbb{E}}[|\sum_{k=0}^{N-1}\eta_k^N|^2] \leq N\sum_{k=0}^{N-1}\hat{\mathbb{E}}[|\eta_k^N|^2] \to 0.$$

□

Proof. (Continue).

The rest terms in the summation of the right side of (–d-Ito) are $\xi_t^N + \zeta_t^N$ with

$$\xi_t^N = \sum_{k=0}^{N-1} \{\partial_{x^\nu} \Phi(X_{t_k^N})[\alpha^\nu(t_{k+1}^N - t_k^N) + \eta^{\nu ij}(\langle B^i, B^j\rangle_{t_{k+1}^N} - \langle B^i, B^j\rangle_{t_k^N})$$

$$+ \beta^{\nu j}(B^j_{t_{k+1}^N} - B^j_{t_k^N})] + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})\beta^{\mu i}\beta^{\nu j}(B^i_{t_{k+1}^N} - B^i_{t_k^N})(B^j_{t_{k+1}^N} - B^j_{t_k^N})$$

□

Proof. (Continue).

and

$$\zeta_t^N = \frac{1}{2}\sum_{k=0}^{N-1} \partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})\{[\alpha^\mu(t_{k+1}^N - t_k^N) + \eta^{\mu ij}(\langle B^i, B^j\rangle_{t_{k+1}^N} - \langle B^i, B^j\rangle_{t_k^N})]$$
$$\times [\alpha^\nu(t_{k+1}^N - t_k^N) + \eta^{\nu lm}(\left\langle B^l, B^m\right\rangle_{t_{k+1}^N} - \left\langle B^l, B^m\right\rangle_{t_k^N})]$$
$$+ 2[\alpha^\mu(t_{k+1}^N - t_k^N) + \eta^{\mu ij}(\langle B^i, B^j\rangle_{t_{k+1}^N} - \langle B^i, B^j\rangle_{t_k^N})]\beta^{\nu l}(B^l_{t_{k+1}^N} - B^l_{t_k^N})\}$$

□

Proof. (Continue).

We observe that, for each $u \in [t_k^N, t_{k+1}^N)$,

$$\begin{aligned}&\hat{\mathbb{E}}[|\partial_{x^\nu}\Phi(X_u) - \sum_{k=0}^{N-1}\partial_{x^\nu}\Phi(X_{t_k^N})\mathbf{I}_{[t_k^N, t_{k+1}^N)}(u)|^2] \\ &= \hat{\mathbb{E}}[|\partial_{x^\nu}\Phi(X_u) - \partial_{x^\nu}\Phi(X_{t_k^N})|^2] \\ &\leq c^2\hat{\mathbb{E}}[|X_u - X_{t_k^N}|^2] \leq C[\delta + \delta^2],\end{aligned}$$

where c is the Lipschitz constant of $\{\partial_{x^\nu}\Phi\}_{\nu=1}^n$ and C is a constant independent of k. Thus $\sum_{k=0}^{N-1}\partial_{x^\nu}\Phi(X_{t_k^N})\mathbf{I}_{[t_k^N, t_{k+1}^N)}(\cdot)$ converges to $\partial_{x^\nu}\Phi(X_\cdot)$ in $M_G^2(0,T)$. Similarly, $\sum_{k=0}^{N-1}\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})\mathbf{I}_{[t_k^N, t_{k+1}^N)}(\cdot)$ converges to $\partial^2_{x^\mu x^\nu}\Phi(X_\cdot)$ in $M_G^2(0,T)$.

□

Proof. (Continue).

From Lemma –d-Lem-mutual as well as the definitions of the integrations of dt, dB_t and $d\langle B\rangle_t$, the limit of ζ_t^N in $L_G^2(\Omega_t)$ is just the right hand side of (–d-B-Ito). By the next Remark we also have $\zeta_t^N \longrightarrow 0$ in $L_G^2(\Omega_t)$. We then have proved (–d-B-Ito). □

Remark.

To prove $\zeta_t^N \to 0$ in $L_G^2(\Omega_t)$, we use the following estimates: for $\psi^N \in M_G^{2,0}(0,T)$ with $\psi_t^N = \sum_{k=0}^{N-1} \xi_{t_k}^N \mathbf{I}_{[t_k^N, t_{k+1}^N)}(t)$, and $\pi_T^N = \{t_0^N, \cdots, t_N^N\}$ such that $\lim_{N\to\infty} \mu(\pi_T^N) = 0$ and $\hat{\mathbb{E}}[\sum_{k=0}^{N-1} |\xi_{t_k}^N|^2 (t_{k+1}^N - t_k^N)] \le C$, for all $N = 1, 2, \cdots$, we have $\hat{\mathbb{E}}[|\sum_{k=0}^{N-1} \xi_k^N (t_{k+1}^N - t_k^N)^2|^2] \to 0$ and, for any fixed $\mathbf{a}, \bar{\mathbf{a}} \in \mathbb{R}^d$,

$$\hat{\mathbb{E}}[|\sum_{k=0}^{N-1} \xi_k^N (\langle B^{\mathbf{a}} \rangle_{t_{k+1}^N} - \langle B^{\mathbf{a}} \rangle_{t_k^N})^2|^2] \le C\hat{\mathbb{E}}[\sum_{k=0}^{N-1} |\xi_k^N|^2 (\langle B^{\mathbf{a}} \rangle_{t_{k+1}^N} - \langle B^{\mathbf{a}} \rangle_{t_k^N})^3]$$

$$\le C\hat{\mathbb{E}}[\sum_{k=0}^{N-1} |\xi_k^N|^2 \sigma_{\mathbf{aa}^T}^6 (t_{k+1}^N - t_k^N)^3] \to 0,$$

□

Remark. (continue).

$$\hat{\mathbb{E}}[|\sum_{k=0}^{N-1}\xi_k^N(\langle B^{\mathbf{a}}\rangle_{t_{k+1}^N}-\langle B^{\mathbf{a}}\rangle_{t_k^N})(t_{k+1}^N-t_k^N)|^2]$$
$$\leq C\hat{\mathbb{E}}[\sum_{k=0}^{N-1}|\xi_k^N|^2(t_{k+1}^N-t_k^N)(\langle B^{\mathbf{a}}\rangle_{t_{k+1}^N}-\langle B^{\mathbf{a}}\rangle_{t_k^N})^2]$$
$$\leq C\hat{\mathbb{E}}[\sum_{k=0}^{N-1}|\xi_k^N|^2\sigma_{\mathbf{aa}^T}^4(t_{k+1}^N-t_k^N)^3]\to 0,$$

as well as

$$\hat{\mathbb{E}}[|\sum_{k=0}^{N-1}\xi_k^N(t_{k+1}^N-t_k^N)(B_{t_{k+1}^N}^{\mathbf{a}}-B_{t_k^N}^{\mathbf{a}})|^2]$$
$$\leq C\hat{\mathbb{E}}[\sum_{k=0}^{N-1}|\xi_k^N|^2(t_{k+1}^N-t_k^N)|B_{t_{k+1}^N}^{\mathbf{a}}-B_{t_k^N}^{\mathbf{a}}|^2]$$
$$\leq C\hat{\mathbb{E}}[\sum_{k=0}^{N-1}|\xi_k^N|^2\sigma_{\mathbf{aa}^T}^2(t_{k+1}^N-t_k^N)^2]\to 0$$

Remark. (continue).

and

$$\hat{\mathbb{E}}[|\sum_{k=0}^{N-1}\xi_k^N(\langle B^{\mathbf{a}}\rangle_{t_{k+1}^N}-\langle B^{\mathbf{a}}\rangle_{t_k^N})(B^{\bar{\mathbf{a}}}_{t_{k+1}^N}-B^{\bar{\mathbf{a}}}_{t_k^N})|^2]$$

$$\leq C\hat{\mathbb{E}}[\sum_{k=0}^{N-1}|\xi_k^N|^2(\langle B^{\mathbf{a}}\rangle_{t_{k+1}^N}-\langle B^{\mathbf{a}}\rangle_{t_k^N})|B^{\bar{\mathbf{a}}}_{t_{k+1}^N}-B^{\bar{\mathbf{a}}}_{t_k^N}|^2]$$

$$\leq C\hat{\mathbb{E}}[\sum_{k=0}^{N-1}|\xi_k^N|^2\sigma^2_{\mathbf{a}\mathbf{a}^T}\sigma^2_{\bar{\mathbf{a}}\bar{\mathbf{a}}^T}(t_{k+1}^N-t_k^N)^2]\to 0.$$

□

We now consider a general form of G–Itô's formula. Consider

$$X_t^\nu = X_0^\nu + \int_0^t \alpha_s^\nu ds + \int_0^t \eta_s^{\nu ij} d\left\langle B^i, B^j \right\rangle_s + \int_0^t \beta_s^{\nu j} dB_s^j.$$

Proposition.

(label)d-Prop-ItoLet $\Phi \in C^2(\mathbb{R}^n)$ with $\partial_{x^\nu}\Phi,\ \partial^2_{x^\mu x^\nu}\Phi \in C_{b.Lip}(\mathbb{R}^n)$ for $\mu, \nu = 1, \cdots, n$. Let α^ν, $\beta^{\nu j}$ and $\eta^{\nu ij}$, $\nu = 1, \cdots, n$, $i, j = 1, \cdots, d$ be bounded processes in $M_G^2(0, T)$. Then for each $t \geq 0$ we have, in $L_G^2(\Omega_t)$

$$\Phi(X_t) - \Phi(X_s) = \int_s^t \partial_{x^\nu}\Phi(X_u)\beta_u^{\nu j} dB_u^j + \int_s^t \partial_{x^\nu}\Phi(X_u)\alpha_u^\nu du (label) d - Ito - \tag{54}$$

$$+ \int_s^t [\partial_{x^\nu}\Phi(X_u)\eta_u^{\nu ij} + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u)\beta_u^{\mu i}\beta_u^{\nu j}] d\left\langle B^i, B^j \right\rangle_u.$$

□

Proof.

We first consider the case where α, η and β are step processes of the form

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) \mathbf{I}_{[t_k, t_{k+1})}(t).$$

From the above lemma, it is clear that (–d-Ito-form1) holds true. Now let

$$X_t^{\nu,N} = X_0^\nu + \int_0^t \alpha_s^{\nu,N} ds + \int_0^t \eta_s^{\nu ij,N} d\left\langle B^i, B^j \right\rangle_s + \int_0^t \beta_s^{\nu j,N} dB_s^j,$$

where α^N, η^N and β^N are uniformly bounded step processes that converge to α, η and β in $M_G^2(0,T)$ as $N \to \infty$, respectively. □

Proof. (continue).

From Lemma –d-Lem-26,

$$\Phi(X_t^N) - \Phi(X_s^N) = \int_s^t \partial_{x^\nu}\Phi(X_u^N)\beta_u^{\nu j,N} dB_u^j + \int_s^t \partial_{x^\nu}\Phi(X_u^N)\alpha_u^{\nu,N} du(label)d \tag{55}$$

$$+ \int_s^t [\partial_{x^\nu}\Phi(X_u^N)\eta_u^{\nu ij,N} + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u^N)\beta_u^{\mu i,N}\beta_u^{\nu j,N}]d\langle B^i,$$

Since

$$\begin{aligned}&\hat{\mathbb{E}}[|X_t^{\nu,N} - X_t^\nu|^2] \\ \leq & C\hat{\mathbb{E}}[\int_0^T [(\alpha_s^{\nu,N} - \alpha_s^\nu)^2 + |\eta_s^{\nu,N} - \eta_s^\nu|^2 + |\beta_s^{\nu,N} - \beta_s^\nu|^2]ds],\end{aligned}$$

where C is a constant independent of N.

□

Proof. (continue).

We can prove that, in $M_G^2(0,T)$,

$$\partial_{x^\nu}\Phi(X_\cdot^N)\eta_\cdot^{\nu ij,N} \to \partial_{x^\nu}\Phi(X_\cdot)\eta_\cdot^{\nu ij},$$
$$\partial^2_{x^\mu x^\nu}\Phi(X_\cdot^N)\beta_\cdot^{\mu i,N}\beta_\cdot^{\nu j,N} \to \partial^2_{x^\mu x^\nu}\Phi(X_\cdot)\beta_\cdot^{\mu i}\beta_\cdot^{\nu j},$$
$$\partial_{x^\nu}\Phi(X_\cdot^N)\alpha_\cdot^{\nu,N} \to \partial_{x^\nu}\Phi(X_\cdot)\alpha_\cdot^{\nu},$$
$$\partial_{x^\nu}\Phi(X_\cdot^N)\beta_\cdot^{\nu j,N} \to \partial_{x^\nu}\Phi(X_\cdot)\beta_\cdot^{\nu j}.$$

We then can pass to limit as $N \to \infty$ in both sides of (–d-N-Ito) to get (–d-Ito-form1). □

In order to consider the general Φ, we first prove a useful inequality. For the G-expectation $\hat{\mathbb{E}}$, we have the following representation (see Chap.–ch6):

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X] \quad \text{for } X \in L_G^1(\Omega), \tag{56}$$

where $\mathcal{P}$ is a weakly compact family of probability measures on $(\Omega, \mathcal{B}(\Omega))$.

Proposition.

(label)B-D-G Let $\beta \in M_G^p(0,T)$ with $p \geq 2$ and let $\mathbf{a} \in \mathbb{R}^d$ be fixed. Then we have $\int_0^T \beta_t dB_t^{\mathbf{a}} \in L_G^p(\Omega_T)$ and

$$(label)ebdg\hat{\mathbb{E}}[|\int_0^T \beta_t dB_t^{\mathbf{a}}|^p] \leq C_p \hat{\mathbb{E}}[|\int_0^T \beta_t^2 d\langle B^{\mathbf{a}}\rangle_t|^{p/2}]. \quad (57)$$

□

Proof.

It suffices to consider the case where β is a step process of the form

$$\beta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) \mathbf{I}_{[t_k, t_{k+1})}(t).$$

For each $\xi \in L_{ip}(\Omega_t)$ with $t \in [0, T]$, we have

$$\hat{\mathbb{E}}[\xi \int_t^T \beta_s dB_s^{\mathbf{a}}] = 0.$$

From this we can easily get $E_P[\xi \int_t^T \beta_s dB_s^{\mathbf{a}}] = 0$ for each $P \in \mathcal{P}$, which implies that $(\int_0^t \beta_s dB_s^{\mathbf{a}})_{t \in 0, T]}$ is a P-martingale. Similarly we can prove that

$$M_t := (\int_0^t \beta_s dB_s^{\mathbf{a}})^2 - \int_0^t \beta_s^2 d\langle B^{\mathbf{a}} \rangle_s, \quad t \in [0, T]$$

is a P-martingale for each $P \in \mathcal{P}$. □

Proof. (continue).

By the Burkholder-Davis-Gundy inequalities, we have

$$E_P[|\int_0^T \beta_t dB_t^{\mathbf{a}}|^p] \leq C_p E_P[|\int_0^T \beta_t^2 d\langle B^{\mathbf{a}}\rangle_t|^{p/2}] \leq C_p \hat{\mathbb{E}}[|\int_0^T \beta_t^2 d\langle B^{\mathbf{a}}\rangle_t|^{p/2}],$$

where C_p is a universal constant independent of P. Thus we get (–ebdg). □

We now give the general G–Itô's formula.

Theorem

(label)Thm6.5 Let Φ be a C^2-function on $\mathbb{R}^n$ such that $\partial^2_{x^\mu x^\nu}\Phi$ satisfy polynomial growth condition for $\mu,\nu = 1,\cdots,n$. Let α^ν, $\beta^{\nu j}$ and $\eta^{\nu ij}$, $\nu = 1,\cdots,n$, $i,j = 1,\cdots,d$ be bounded processes in $M^2_G(0,T)$. Then for each $t \geq 0$ we have in $L^2_G(\Omega_t)$

$$\Phi(X_t) - \Phi(X_s) = \int_s^t \partial_{x^\nu}\Phi(X_u)\beta^{\nu j}_u dB^j_u + \int_s^t \partial_{x^\nu}\Phi(X_u)\alpha^\nu_u du (label)e629 \tag{58}$$

$$+ \int_s^t [\partial_{x^\nu}\Phi(X_u)\eta^{\nu ij}_u + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u)\beta^{\mu i}_u\beta^{\nu j}_u] d\left\langle B^i, B^j \right\rangle_u .$$

Proof.

By the assumptions on Φ, we can choose a sequence of functions $\Phi_N \in C_0^2(\mathbb{R}^n)$ such that

$$|\Phi_N(x) - \Phi(x)| + |\partial_{x^\nu}\Phi_N(x) - \partial_{x^\nu}\Phi(x)| + |\partial^2_{x^\mu x^\nu}\Phi_N(x) - \partial^2_{x^\mu x^\nu}\Phi(x)| \leq \frac{C_1}{N}$$

where C_1 and k are positive constants independent of N. Obviously, Φ_N satisfies the conditions in Proposition –d-Prop-Ito, therefore,

$$\Phi_N(X_t) - \Phi_N(X_s) = \int_s^t \partial_{x^\nu}\Phi_N(X_u)\beta_u^{\nu j} dB_u^j + \int_s^t \partial_{x^\nu}\Phi_N(X_u)\alpha_u^\nu du(label)e6 \quad (59)$$

$$+ \int_s^t [\partial_{x^\nu}\Phi_N(X_u)\eta_u^{\nu ij} + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi_N(X_u)\beta_u^{\mu i}\beta_u^{\nu j}] d\left\langle B^i, B^j\right.$$

□

Proof. (continue).

For each fixed $T > 0$, by Proposition –B-D-G, there exists a constant C_2 such that

$$\hat{\mathbb{E}}[|X_t|^{2k}] \leq C_2 \quad \text{for } t \in [0, T].$$

Thus we can prove that $\Phi_N(X_t) \to \Phi(X_t)$ in $L_G^2(\Omega_t)$ and in $M_G^2(0, T)$,

$$\partial_{x^\nu} \Phi_N(X_\cdot)\eta_\cdot^{\nu ij} \to \partial_{x^\nu} \Phi(X_\cdot)\eta_\cdot^{\nu ij},$$

$$\partial^2_{x^\mu x^\nu} \Phi_N(X_\cdot)\beta_\cdot^{\mu i}\beta_\cdot^{\nu j} \to \partial^2_{x^\mu x^\nu} \Phi(X_\cdot)\beta_\cdot^{\mu i}\beta_\cdot^{\nu j},$$

$$\partial_{x^\nu} \Phi_N(X_\cdot)\alpha_\cdot^{\nu} \to \partial_{x^\nu} \Phi(X_\cdot)\alpha_\cdot^{\nu},$$

$$\partial_{x^\nu} \Phi_N(X_\cdot)\beta_\cdot^{\nu j} \to \partial_{x^\nu} \Phi(X_\cdot)\beta_\cdot^{\nu j}.$$

We then can pass to limit as $N \to \infty$ in both sides of (–e630) to get (–e629). □

Corollary

Let Φ be a polynomial and $\mathbf{a}, \mathbf{a}^{\nu} \in \mathbb{R}^d$ be fixed for $\nu = 1, \cdots, n$. Then we have

$$\Phi(X_t) - \Phi(X_s) = \int_s^t \partial_{x^\nu} \Phi(X_u) dB_u^{\mathbf{a}^\nu} + \frac{1}{2} \int_s^t \partial^2_{x^\mu x^\nu} \Phi(X_u) d \left\langle B^{\mathbf{a}^\mu}, B^{\mathbf{a}^\nu} \right\rangle_u,$$

where $X_t = (B_t^{\mathbf{a}^1}, \cdots, B_t^{\mathbf{a}^n})^T$. In particular, we have, for $k = 2, 3, \cdots$,

$$(B_t^{\mathbf{a}})^k = k \int_0^t (B_s^{\mathbf{a}})^{k-1} dB_s^{\mathbf{a}} + \frac{k(k-1)}{2} \int_0^t (B_s^{\mathbf{a}})^{k-2} d \langle B^{\mathbf{a}} \rangle_s.$$

If $\hat{\mathbb{E}}$ becomes a linear expectation, then the above G–Itô's formula is the classical one.
Let $G : \mathbb{R}^d \times \mathbb{S}(d) \to \mathbb{R}$ be a given continuous sublinear function monotonic in $A \in \mathbb{S}(d)$. Then by Theorem –t1 in Chap.–ch1, there exists a bounded, convex and closed subset $\Sigma \subset \mathbb{R}^d \times \mathbb{S}_+(d)$ such that

$$G(p, A) = \sup_{(q,B)\in\Sigma} [\frac{1}{2}\mathrm{tr}[AB] + \langle p, q\rangle] \quad \text{for } (p, A) \in \mathbb{R}^d \times \mathbb{S}(d).$$

By Chapter –ch2, we know that there exists a pair of d-dimensional random vectors (X, Y) which is G-distributed.

We now give the definition of the generalized G-Brownian motion.

Definition

A d-dimensional process $(B_t)_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a **generalized G-Brownian motion** if the following properties are satisfied:
(i) $B_0(\omega) = 0$;
(ii) For each $t, s \geq 0$, the increment $B_{t+s} - B_t$ identically distributed with $\sqrt{s}X + sY$ and is independent from $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$, where (X, Y) is G-distributed.

The following theorem gives a characterization of the generalized G-Brownian motion.

Theorem

Let $(B_t)_{t\geq 0}$ be a d-dimensional process defined on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ such that
(i) $B_0(\omega) = 0$;
(ii) *For each $t, s \geq 0$, $B_{t+s} - B_t$ and B_s are identically distributed and $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$.*
(iii) $\lim_{t\downarrow 0} \hat{\mathbb{E}}[|B_t|^3]t^{-1} = 0$.
Then $(B_t)_{t\geq 0}$ is a generalized G-Brownian motion with
$G(p, A) = \lim_{\delta\downarrow 0} \hat{\mathbb{E}}[\langle p, B_\delta\rangle + \frac{1}{2}\langle AB_\delta, B_\delta\rangle]\delta^{-1}$ *for* $(p, A) \in \mathbb{R}^d \times \mathbb{S}(d)$.

Proof.

We first prove that $\lim_{\delta\downarrow 0}\hat{\mathbb{E}}[\langle p,B_\delta\rangle+\frac{1}{2}\langle AB_\delta,B_\delta\rangle]\delta^{-1}$ exists. For each fixed $(p,A)\in\mathbb{R}^d\times\mathbb{S}(d)$, we set

$$f(t):=\hat{\mathbb{E}}[\langle p,B_t\rangle+\frac{1}{2}\langle AB_t,B_t\rangle].$$

Since

$$|f(t+h)-f(t)|\le\hat{\mathbb{E}}[(|p|+2|A||B_t|)|B_{t+h}-B_t|+|A||B_{t+h}-B_t|^2]\to 0,$$

we get that $f(t)$ is a continuous function. It is easy to prove that

$$\hat{\mathbb{E}}[\langle q,B_t\rangle]=\hat{\mathbb{E}}[\langle q,B_1\rangle]t\ \text{ for } q\in\mathbb{R}^d.$$

□

Proof. (continue).

Thus for each $t, s > 0$,

$$|f(t+s) - f(t) - f(s)| \leq C\hat{\mathbb{E}}[|B_t|]s,$$

where $C = |A|\hat{\mathbb{E}}[|B_1|]$. By (iii), there exists a constant $\delta_0 > 0$ such that $\hat{\mathbb{E}}[|B_t|^3] \leq t$ for $t \leq \delta_0$. Thus for each fixed $t > 0$ and $N \in \mathbb{N}$ such that $Nt \leq \delta_0$, we have

$$|f(Nt) - Nf(t)| \leq \frac{3}{4}C(Nt)^{4/3}.$$

From this and the continuity of f, it is easy to show that $\lim_{t\downarrow 0} f(t)t^{-1}$ exists. Thus we can get $G(p, A)$ for each $(p, A) \in \mathbb{R}^d \times \mathbb{S}(d)$. It is also easy to check that G is a continuous sublinear function monotonic in $A \in \mathbb{S}(d)$.

□

Proof. (continue).

We only need to prove that, for each fixed $\varphi \in C_{b.Lip}(\mathbb{R}^d)$, the function

$$u(t,x) := \hat{\mathbb{E}}[\varphi(x+B_t)], \quad (t,x) \in [0,\infty) \times \mathbb{R}^d$$

is the viscosity solution of the following parabolic PDE:

$$(label)e731\partial_t u - G(Du, D^2u) = 0, \quad u|_{t=0} = \varphi. \tag{60}$$

We first prove that u is Lipschitz in x and $\frac{1}{2}$-Hölder continuous in t. In fact, for each fixed t, $u(t,\cdot) \in C_{b.Lip}(\mathbb{R}^d)$ since

$$\begin{aligned} |\hat{\mathbb{E}}[\varphi(x+B_t)] - \hat{\mathbb{E}}[\varphi(y+B_t)]| &\leq \hat{\mathbb{E}}[|\varphi(x+B_t) - \varphi(y+B_t)|] \\ &\leq C|x-y|. \end{aligned}$$

□

Proof. (continue).

For each $\delta \in [0, t]$, since $B_t - B_\delta$ is independent from B_δ,

$$u(t,x) = \hat{\mathbb{E}}[(x + B_\delta + (B_t - B_\delta)] \\ = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(y + (B_t - B_\delta))]_{y=x+B_\delta}].$$

Hence

$$(label)e732u(t,x) = \hat{\mathbb{E}}[u(t-\delta, x + B_\delta)]. \quad (61)$$

Thus

$$|u(t,x) - u(t-\delta,x)| = |\hat{\mathbb{E}}[u(t-\delta, x+B_\delta) - u(t-\delta,x)]| \\ \leq \hat{\mathbb{E}}[|u(t-\delta,x+B_\delta) - u(t-\delta,x)|] \\ \leq \hat{\mathbb{E}}[C|B_\delta|] \leq C\sqrt{G(0,I)+1}\sqrt{\delta}.$$

□

Proof. (continue).

To prove that u is a viscosity solution of (–e731), we fix a $(t,x) \in (0,\infty) \times \mathbb{R}^d$ and let $v \in C_b^{2,3}([0,\infty) \times \mathbb{R}^d)$ be such that $v \geq u$ and $v(t,x) = u(t,x)$. From (–e732), we have

$$v(t,x) = \hat{\mathbb{E}}[u(t-\delta, x+B_\delta)] \leq \hat{\mathbb{E}}[v(t-\delta, x+B_\delta)].$$

Therefore, by Taylor's expansion,

$$\begin{aligned} 0 &\leq \hat{\mathbb{E}}[v(t-\delta, x+B_\delta) - v(t,x)] \\ &= \hat{\mathbb{E}}[v(t-\delta, x+B_\delta) - v(t, x+B_\delta) + (v(t, x+B_\delta) - v(t,x))] \\ &= \hat{\mathbb{E}}[-\partial_t v(t,x)\delta + \langle Dv(t,x), B_\delta\rangle + \frac{1}{2}\langle D^2 v(t,x) B_\delta, B_\delta\rangle + I_\delta] \\ &\leq -\partial_t v(t,x)\delta + \hat{\mathbb{E}}[\langle Dv(t,x), B_\delta\rangle + \frac{1}{2}\langle D^2 v(t,x) B_\delta, B_\delta\rangle] + \hat{\mathbb{E}}[I_\delta], \end{aligned}$$

where

$$I_\delta = \int_0^1 -[\partial_t v(t-\beta\delta, x+B_\delta) - \partial_t v(t,x)]\delta d\beta$$

Proof. (continue).

With the assumption (iii) we can check that $\lim_{\delta\downarrow 0}\hat{\mathbb{E}}[|I_\delta|]\delta^{-1}=0$, from which we get $\partial_t v(t,x)-G(Dv(t,x),D^2v(t,x))\leq 0$, hence u is a viscosity subsolution of (–e731). We can analogously prove that u is a viscosity supersolution. Thus u is a viscosity solution and $(B_t)_{t\geq 0}$ is a generalized G-Brownian motion. □

In many situations we are interested in a generalized $2d$-dimensional Brownian motion $(B_t, b_t)_{t\geq 0}$ such that $\hat{\mathbb{E}}[B_t] = -\hat{\mathbb{E}}[-B_t] = 0$ and $\hat{\mathbb{E}}[|b_t|^2]/t \to 0$, as $t \downarrow 0$. In this case B is in fact a G-Brownian motion defined on Definition 2.1 of Chapter –ch2.

Moreover the process b satisfies properties of Proposition 5.2. We define $u(t,x,y) = \hat{\mathbb{E}}[\varphi(x + B_t, y + b_t)]$. By the above proposition it follows that u is the solution of the PDE

$$\partial_t u = G(D_y u, D_{xx}^2 u), \quad u|_{t=0} = \varphi \in C_{Lip}(\mathbb{R}^{2d}).$$

where G is a sublinear function of $(p, A) \in \mathbb{R}^d$, defined by

$$G(p, A) := \hat{\mathbb{E}}[\langle p, b_t\rangle + \langle AB_t, B_t\rangle].$$

Here $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathbb{R}^d}$.

We can also define a G-Brownian motion on a nonlinear expectation space $(\Omega, \mathcal{H}, \widetilde{\mathbb{E}})$.

Definition

(label)defIII901A d-dimensional process $(B_t)_{t\geq 0}$ on a nonlinear expectation space $(\Omega, \mathcal{H}, \widetilde{\mathbb{E}})$ is called a (nonlinear) $\widetilde{G}$-**Brownian motion** if the following properties are satisfied:
(i) $B_0(\omega) = 0$;
(ii) For each $t, s \geq 0$, the increment $B_{t+s} - B_t$ identically distributed with B_s and is independent from $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$;
(iii) $\lim_{t\downarrow 0} \hat{\mathbb{E}}[|B_t|^3]t^{-1} = 0$.

The following theorem gives a characterization of the nonlinear $\widetilde{G}$-Brownian motion, and give us the generator $\widetilde{G}$ of our $\widetilde{G}$-Brownian motion.

Theorem

(label)ThmIII8.2 Let $\widetilde{\mathbb{E}}$ be a nonlinear expectation and $\hat{\mathbb{E}}$ be a sublinear expectation defined on $(\Omega, \mathcal{H})$. let $\widetilde{\mathbb{E}}$ be dominated by $\hat{\mathbb{E}}$, namely

$$\widetilde{\mathbb{E}}[X] - \widetilde{\mathbb{E}}[Y] \leq \hat{\mathbb{E}}[X - Y], \quad X, Y \in \mathcal{H}.$$

Let $(B_t, b_t)_{t \geq 0}$ be a given $\mathbb{R}^{2d}$–valued $\widetilde{G}$-Brownian motion on $(\Omega, \mathcal{H}, \widetilde{\mathbb{E}})$ such that $\hat{\mathbb{E}}[B_t] = \hat{\mathbb{E}}[-B_t] = 0$ and $\lim_{t \to 0} \hat{\mathbb{E}}[|b_t|^2]/t = 0$. Then, for each fixed $\varphi \in C_{b.Lip}(\mathbb{R}^{2d})$, the function

$$\tilde{u}(t, x, y) := \widetilde{\mathbb{E}}[\varphi(x + B_t, y + b_t)], \quad (t, x, y) \in [0, \infty) \times \mathbb{R}^{2d}$$

is the viscosity solution of the following parabolic PDE:

$$\partial_t \tilde{u} - \widetilde{G}(D_y \tilde{u}, D_x^2 \tilde{u}) = 0, \quad u|_{t=0} = \varphi.(label)eIII831 \tag{62}$$

where

$$\widetilde{G}(p, A) = \widetilde{\mathbb{E}}[\langle p, b_1 \rangle + \frac{1}{2} \langle A B_1, B_1 \rangle], \ (p, A) \in \mathbb{R}^d \times \mathbb{S}(d).$$

Remark.

Let $G(p,A) := \hat{\mathbb{E}}[\langle p, b_1\rangle + \frac{1}{2}\langle AB_1, B_1\rangle]$. Then the function $\widetilde{G}$ is dominated by the sublinear function G in the following sense:

$$\widetilde{G}(p,A) - \widetilde{G}(p',A') \leq G(p-p', A-A'), \;\; (p,A),\, (p',A') \in \mathbb{R}^d \times \mathbb{S}(d).(label \tag{63}$$

□

Proof of Theorem –ThmIII8.2.

We set

$$f(t) = f_{A,t}(t) := \widetilde{\mathbb{E}}[\langle p, b_t \rangle + \frac{1}{2}\langle AB_t, B_t \rangle], \ t \geq 0.$$

Since

$$|f(t+h) - f(t)| \leq \hat{\mathbb{E}}[(|p| + 2|A||B_t|)|B_{t+h} - B_t| + |A||B_{t+h} - B_t|^2] \to 0,$$

we get that $f(t)$ is a continuous function. Since $\hat{\mathbb{E}}[B_t] = \hat{\mathbb{E}}[-B_t] = 0$, it follows from Proposition –PropI.3.4 that $\widetilde{\mathbb{E}}[X + \langle p, B_t \rangle] = \widetilde{\mathbb{E}}[X]$ for each $X \in \mathcal{H}$ and $p \in \mathbb{R}^d$.

□

Proof. (continue).

Thus

$$
\begin{aligned}
f(t+h) &= \widetilde{\mathbb{E}}[\langle p, b_{t+h} - b_t\rangle + \langle p, b_t\rangle \\
&\quad + \frac{1}{2}\langle AB_{t+h} - B_t, B_{t+h} - B_t\rangle + \frac{1}{2}\langle AB_t, B_t\rangle] \\
&= \widetilde{\mathbb{E}}[\langle p, b_h\rangle + \frac{1}{2}\langle AB_h, B_h\rangle] + \widetilde{\mathbb{E}}[\langle p, b_t\rangle + \frac{1}{2}\langle AB_t, B_t\rangle] \\
&= f(t) + f(h).
\end{aligned}
$$

□

Proof. (continue).

It then follows that $f(t) = f(1)t = \widetilde{G}(A, p)t$. We now prove that the function u is Lipschitz in x and uniformly continuous in t. In fact, for each fixed t, $u(t, \cdot) \in C_{b.Lip}(\mathbb{R}^d)$ since

$$
\begin{aligned}
&|\widetilde{\mathbb{E}}[\varphi(x + B_t, y + b_t)] - \widetilde{\mathbb{E}}[\varphi(x' + B_t, y' + b_t)]| \\
&\leq \hat{\mathbb{E}}[|\varphi(x + B_t, y + b_t) - \varphi(x' + B_t, y' + b_t)|] \leq C(|x - x'| + |y - y'|).
\end{aligned}
$$

For each $\delta \in [0, t]$, since $(B_t - B_\delta, b_t - b_\delta)$ is independent from (B_δ, b_δ),

$$
\begin{aligned}
\tilde{u}(t, x) &= \widetilde{\mathbb{E}}[\varphi(x + B_\delta + (B_t - B_\delta), y + b_\delta + (b_t - b_\delta))] \\
&= \widetilde{\mathbb{E}}[\widetilde{\mathbb{E}}[\varphi(\bar{x} + (B_t - B_\delta), \bar{y} + (b_t - b_\delta))]_{\bar{x} = x + B_\delta, \bar{y} = y + b_\delta}].
\end{aligned}
$$

□

Proof. (continue).

Hence

$$u(t,x) = \hat{\mathbb{E}}[u(t-\delta, x+B_\delta, y+b_\delta)].(label)elll832 \qquad (64)$$

Thus

$$\begin{aligned}|\tilde{u}(t,x,y) - \tilde{u}(t-\delta,x,y)| &= |\tilde{\mathbb{E}}[\tilde{u}(t-\delta, x+B_\delta, y+b_\delta) - \tilde{u}(t-\delta,x,y)]| \\ &\leq \hat{\mathbb{E}}[|\tilde{u}(t-\delta, x+B_\delta, y+b_\delta) - \tilde{u}(t-\delta,x,y)|] \\ &\leq \hat{\mathbb{E}}[C(|B_\delta| + |b_\delta|)].\end{aligned}$$

It follows from (iii) of Definition –deflll901 that $u(t,x,y)$ is continuous in t uniformly in $(t,x) \in [0,\infty) \times \mathbb{R}^{2d}$.

□

Proof. (continue).

To prove that $\tilde{u}$ is a viscosity solution of (–eIII831), we fix a $(t,x,y) \in (0,\infty) \times \mathbb{R}^{2d}$ and let $v \in C_b^{2,3}([0,\infty) \times \mathbb{R}^{2d})$ be such that $v \geq u$ and $v(t,x,y) = u(t,x,y)$. From (–eIII832), we have

$$v(t,x,y) = \widetilde{\mathbb{E}}[u(t-\delta, x+B_\delta, y+b_\delta)] \leq \widetilde{\mathbb{E}}[v(t-\delta, x+B_\delta, y+b_\delta)].$$

□

Proof. (continue).

Therefore, by Taylor's expansion,

$$0 \leq \widetilde{\mathbb{E}}[v(t-\delta, x+B_\delta, y+b_\delta) - v(t,x)]$$
$$= \widetilde{\mathbb{E}}[v(t-\delta, x+B_\delta, y+b_\delta) - v(t, x+B_\delta, y+b_\delta)$$
$$+ (v(t, x+B_\delta, y+b_\delta) - v(t,x,y)]$$
$$= \widetilde{\mathbb{E}}[-\partial_t v(t,x,y)\delta + \langle D_y v(t,x,y), b_\delta\rangle + \langle \partial_x v(t,x,y), B_\delta\rangle + \frac{1}{2}\langle D^2_{xx} v(t,x,$$
$$\leq -\partial_t v(t,x,y)\delta + \widetilde{\mathbb{E}}[\langle D_y v(t,x,y), b_\delta\rangle + \frac{1}{2}\langle D^2_{xx} v(t,x,y)B_\delta, B_\delta\rangle] + \hat{\mathbb{E}}[I_\delta],$$

□

Proof. (continue).

where

$$
\begin{aligned}
I_\delta = & \int_0^1 -[\partial_t v(t-\delta\gamma, x+B_\delta, y+b_\delta) - \partial_t v(t,x,y)]\delta d\gamma \\
& + \int_0^1 \langle \partial_y v(t, x+\gamma B_\delta, y+\gamma b_\delta) - \partial_y v(t,x,y), b_\delta \rangle \, d\gamma \\
& + \int_0^1 \langle \partial_x v(t, x, y+\gamma b_\delta) - \partial_x v(t,x,y), B_\delta \rangle \, d\gamma \\
& + \int_0^1 \int_0^1 \langle (D^2_{xx} v(t, x+\alpha\gamma B_\delta, y+\gamma b_\delta) - D^2_{xx} v(t,x,y)) B_\delta, B_\delta \rangle \gamma d\gamma d\alpha.
\end{aligned}
$$

With the assumption (iii) we can check that $\lim_{\delta\downarrow 0} \hat{\mathbb{E}}[|I_\delta|]\delta^{-1} = 0$, from which we get $\partial_t v(t,x) - G(Dv(t,x), D^2 v(t,x)) \leq 0$, hence u is a viscosity subsolution of (–eIII831). We can analogously prove that u is a viscosity supersolution. Thus u is a viscosity solution. □

Let $G(\cdot) : \mathbb{R}^d \times \mathbb{S}(d) \to \mathbb{R}$ be a given sublinear function monotonic on $A \in \mathbb{S}(d)$ and $\widetilde{G}(\cdot) : \mathbb{R}^d \times \mathbb{S}(d) \to \mathbb{R}$ be a given function dominated by G in the sense of (–eIIIGdom). The construction of a $\mathbb{R}^{2d}$-dimensional $\widetilde{G}$-Brownian motion $(B_t, b_t)_{t \geq 0}$ under a nonlinear expectation $\widetilde{\mathbb{E}}$, dominated by a sublinear expectation $\hat{\mathbb{E}}$ is based on a similar approach introduced in Section 2. In fact we will see that by our construction $(B_t, b_t)_{t \geq 0}$ is also a G-Brownian motion of the sublinear expectation $\hat{\mathbb{E}}$.

We denote by $\Omega = C_0^{2d}(\mathbb{R}^+)$ the space of all $\mathbb{R}^{2d}$–valued continuous paths $(\omega_t)_{t\in\mathbb{R}^+}$. For each fixed $T \in [0,\infty)$, we set $\Omega_T := \{\omega_{\cdot\wedge T} : \omega \in \Omega\}$. We will consider the canonical process $(B_t, b_t)(\omega) = \omega_t$, $t \in [0,\infty)$, for $\omega \in \Omega$. We also follow section 2 to introduce the spaces of random variables $L_{ip}(\Omega_T)$ and $L_{ip}(\Omega)$ so that to define $\hat{\mathbb{E}}$ and $\widetilde{\mathbb{E}}$ on $(\Omega, L_{ip}(\Omega))$.

To this purpose we first construct a sequence of d-dimensional random vectors $(X_i, \eta_i)_{i=1}^{\infty}$ on a sublinear expectation space $(\overline{\Omega}, \overline{\mathcal{H}}, \overline{\mathbb{E}})$ such that (X_i, η_i) is G-distributed and (X_{i+1}, η_{i+1}) is independent from $((X_1, \eta_1), \cdots, (X_i, \eta_i))$ for each $i = 1, 2, \cdots$. By the definition of G-distribution the function

$$u(t, x, y) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X_1, y + t\eta_1)], \quad t \geq 0, \quad x, y \in \mathbb{R}^d$$

is the viscosity solution of the following parabolic PDE, which is the same as equation (–ee03) in Chap.II.

$$\partial_t u - G(D_y u, D_{xx}^2 u) = 0, \quad u|_{t=0} = \varphi \in C_{Lip}(\mathbb{R}^{2d}).$$

We also consider the PDE (for the existence, uniqueness, comparison and domination properties, see Theorem –Com-G in Appendix C).

$$\partial_t \tilde{u} - \widetilde{G}(D_y \tilde{u}, D^2_{xx} \tilde{u}) = 0, \quad \tilde{u}|_{t=0} = \varphi \in C_{Lip}(\mathbb{R}^{2d}),$$

and denote by $\widetilde{P}_t[\varphi](x,y) = \tilde{u}(t,x,y)$. Since $\widetilde{G}$ is dominated by G, it follows from the domination theorem of viscosity solutions, i.e., Theorem –G-Tilde in Appendix C, that, for each $\varphi, \psi \in C_{b,Lip}(\mathbb{R}^{2d})$,

$$\widetilde{P}_t[\varphi](x,y) - \widetilde{P}_t[\psi](x,y) \leq \overline{\mathbb{E}}[(\varphi - \psi)(x + \sqrt{t}X_1, y + t\eta_1)].$$

We now introduce a sublinear expectation $\hat{\mathbb{E}}$ and a nonlinear $\widetilde{\mathbb{E}}$ defined on $L_{ip}(\Omega)$ via the following procedure: for each $X \in L_{ip}(\Omega)$ with

$$X = \varphi(B_{t_1} - B_{t_0}, b_{t_1} - b_{t_0}, \cdots, B_{t_1} - B_{t_0}, b_{t_n} - b_{t_{n-1}})$$

for $\varphi \in C_{Lip}(\mathbb{R}^{2d \times n})$ and $0 = t_0 < t_1 < \cdots < t_n < \infty$, we set

$$\begin{aligned} &\hat{\mathbb{E}}[\varphi(B_{t_1} - B_{t_0}, b_{t_1} - b_{t_0}, \cdots, B_{t_n} - B_{t_{n-1}}, b_{t_n} - b_{t_{n-1}})] \\ := &\overline{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0} X_1, (t_1 - t_0)\eta_1, \cdots, \sqrt{t_n - t_{n-1}} X_n, (t_n - t_{n-1})\eta_n)]. \end{aligned}$$

and

$$\widetilde{\mathbb{E}}[\varphi(B_{t_1} - B_{t_0}, b_{t_1} - b_{t_0}, \cdots, B_{t_n} - B_{t_{n-1}}, b_{t_n} - b_{t_{n-1}})] = \varphi_n(0,0)$$

where $\varphi_n \in C_{b.Lip}(\mathbb{R}^{2d})$ is defined iteratively through

$$\begin{aligned} \varphi_1(x_1, y_1, \cdots, x_{n-1}, y_{n-1}) &= \widetilde{P}_{t_n - t_{n-1}}[\varphi_1(x_1, y_1, \cdots, x_{n-1}, y_{n-1}, \cdot)](0,0), \\ &\vdots \\ \varphi_{n-1}(x_1, y_1) &= \widetilde{P}_{t_2 - t_1}[\varphi_{n-2}(x_1, y_1, \cdot)](0,0), \\ \varphi_n(x_1, y_1) &= \widetilde{P}_{t_2 - t_1}[\varphi_{n-1}(\cdot)](x_1, y_1). \end{aligned}$$

The related conditional expectation of $X = \varphi(B_{t_1} - B_{t_0}, b_{t_1} - b_{t_0}, \cdots, B_{t_n} - B_{t_{n-1}}, b_{t_n} - b_{t_{n-1}})$ under Ω_{t_j} is defined by

$$\hat{\mathbb{E}}[X|\Omega_{t_j}] = \hat{\mathbb{E}}[\varphi(B_{t_1} - B_{t_0}, b_{t_1} - b_{t_0}, \cdots, B_{t_n} - B_{t_{n-1}}, b_{t_n} - b_{t_{n-1}})\ |\Omega_{t_j}](labe \tag{65}$$

$$:= \psi(B_{t_1} - B_{t_0}, b_{t_1} - b_{t_0}, \cdots, B_{t_j} - B_{t_{j-1}}, b_{t_j} - b_{t_{j-1}}),$$

where

$$\psi(x_1, \cdots, x_j) = \overline{\mathbb{E}}[\varphi(x_1, \cdots, x_j, \sqrt{t_{j+1} - t_j}X_{j+1}, (t_1 - t_0)\eta_{j+1}, \cdots, \sqrt{t_n - t_n}$$

Similarly

$$\widetilde{\mathbb{E}}[X|\Omega_{t_j}] = \varphi_{n-j}(B_{t_1} - B_{t_0}, b_{t_1} - b_{t_0}, \cdots, B_{t_j} - B_{t_{j-1}}, b_{t_j} - b_{t_{j-1}}).$$

It is easy to check that $\hat{\mathbb{E}}[\cdot]$ (resp. $\widetilde{\mathbb{E}}$) consistently defines a sublinear (resp. nonlinear) expectation and $\widetilde{\mathbb{E}}[\cdot]$ on $(\Omega, L_{ip}(\Omega))$. Moreover $(B_t, b_t)_{t\geq 0}$ is a G-Brownian motion under $\hat{\mathbb{E}}$ and a $\widetilde{G}$-Brownian motion under $\widetilde{\mathbb{E}}$.

Proposition.

(label)Prop-1-9-2We also list the properties of $\widetilde{\mathbb{E}}[\cdot|\Omega_t]$ that hold for each $X, Y \in L_{ip}(\Omega)$:

(i) If $X \geq Y$, then $\widetilde{\mathbb{E}}[X|\Omega_t] \geq \widetilde{\mathbb{E}}[Y|\Omega_t]$.

(ii) $\widetilde{\mathbb{E}}[X+\eta|\Omega_t] = \widetilde{\mathbb{E}}[X|\Omega_t] + \eta$, for each $t \geq 0$ and $\eta \in L_{ip}(\Omega_t)$.

(iii) $\widetilde{\mathbb{E}}[X|\Omega_t] - \widetilde{\mathbb{E}}[Y|\Omega_t] \leq \hat{\mathbb{E}}[X - Y|\Omega_t]$.

(iv) $\widetilde{\mathbb{E}}[\widetilde{\mathbb{E}}[X|\Omega_t]|\Omega_s] = \widetilde{\mathbb{E}}[X|\Omega_{t\wedge s}]$, in particular, $\widetilde{\mathbb{E}}[\widetilde{\mathbb{E}}[X|\Omega_t]] = \widetilde{\mathbb{E}}[X]$.

(v) For each $X \in L_{ip}(\Omega^t)$, $\widetilde{\mathbb{E}}[X|\Omega_t] = \widetilde{\mathbb{E}}[X]$, where $L_{ip}(\Omega^t)$ is the linear space of random variables with the form

$$\varphi(W_{t_2} - W_{t_1}, W_{t_3} - W_{t_2}, \cdots, W_{t_{n+1}} - W_{t_n}),$$
$$n = 1, 2, \cdots, \ \varphi \in C_{Lip}(\mathbb{R}^{d\times n}),\ t_1, \cdots, t_n, t_{n+1} \in [t, \infty).$$

□

Since $\hat{\mathbb{E}}$ can be considered as a special nonlinear expectation of $\widetilde{\mathbb{E}}$ dominated by its self, thus $\hat{\mathbb{E}}[\cdot|\Omega_t]$ also satisfies above properties (i)–(v).

Moreover

Proposition.

(label)Prop-1-9-1The conditional sublinear expectation $\hat{\mathbb{E}}[\cdot|\Omega_t]$ satisfies (i)-(v). Moreover $\hat{\mathbb{E}}[\cdot|\Omega_t]$ itself is sublinear, i.e.,

(vi) $\hat{\mathbb{E}}[X|\Omega_t] - \hat{\mathbb{E}}[Y|\Omega_t] \leq \hat{\mathbb{E}}[X - Y|\Omega_t]$, .

(vii) $\hat{\mathbb{E}}[\eta X|\Omega_t] = \eta^+\hat{\mathbb{E}}[X|\Omega_t] + \eta^-\hat{\mathbb{E}}[-X|\Omega_t]$ for each $\eta \in L_{ip}(\Omega_t)$.

□

We now consider the completion of sublinear expectation space $(\Omega, L_{ip}(\Omega), \hat{\mathbb{E}})$.

We denote by $L_G^p(\Omega)$, $p \geq 1$, the completion of $L_{ip}(\Omega)$ under the norm $\|X\|_p := (\hat{\mathbb{E}}[|X|^p])^{1/p}$. Similarly, we can define $L_G^p(\Omega_T)$, $L_G^p(\Omega_T^t)$ and $L_G^p(\Omega^t)$. It is clear that for each $0 \leq t \leq T < \infty$, $L_G^p(\Omega_t) \subseteq L_G^p(\Omega_T) \subseteq L_G^p(\Omega)$.

According to Sec.–c1s5 in Chap.–ch1, $\hat{\mathbb{E}}[\cdot]$ can be continuously extended to $(\Omega, L_G^1(\Omega))$. Moreover, since $\widetilde{\mathbb{E}}$ is dominated by $\hat{\mathbb{E}}$, thus by Definition –DefI4.4 in Chap.I, $(\Omega, L_G^1(\Omega), \hat{\mathbb{E}})$ forms a sublinear expectation space and $(\Omega, L_G^1(\Omega), \widetilde{\mathbb{E}})$ forms a nonlinear expectation space.

We now consider the extension of conditional G-expectation. For each fixed $t \leq T$, the conditional G-expectation $\hat{\mathbb{E}}[\cdot|\Omega_t] : L_{ip}(\Omega_T) \to L_{ip}(\Omega_t)$ is a continuous mapping under $\|\cdot\|$. Indeed, we have

$$\widetilde{\mathbb{E}}[X|\Omega_t] - \widetilde{\mathbb{E}}[Y|\Omega_t] \leq \hat{\mathbb{E}}[X - Y|\Omega_t] \leq \hat{\mathbb{E}}[|X - Y||\Omega_t],$$

then

$$|\widetilde{\mathbb{E}}[X|\Omega_t] - \widetilde{\mathbb{E}}[Y|\Omega_t]| \leq \hat{\mathbb{E}}[|X - Y||\Omega_t].$$

We thus obtain

$$\left\|\widetilde{\mathbb{E}}[X|\Omega_t] - \widetilde{\mathbb{E}}[Y|\Omega_t]\right\| \leq \|X - Y\|.$$

It follows that $\widetilde{\mathbb{E}}[\cdot|\Omega_t]$ can be also extended as a continuous mapping

$$\widetilde{\mathbb{E}}[\cdot|\Omega_t] : L_G^1(\Omega_T) \to L_G^1(\Omega_t).$$

If the above T is not fixed, then we can obtain
$\widetilde{\mathbb{E}}[\cdot|\Omega_t] : L_G^1(\Omega) \to L_G^1(\Omega_t)$.

Remark.

The above proposition also holds for $X, Y \in L_G^1(\Omega)$. But in (iv), $\eta \in L_G^1(\Omega_t)$ should be bounded, since $X, Y \in L_G^1(\Omega)$ does not imply $X \cdot Y \in L_G^1(\Omega)$. □

In particular, we have the following independence:

$$\widetilde{\mathbb{E}}[X|\Omega_t] = \widetilde{\mathbb{E}}[X], \ \ \forall X \in L_G^1(\Omega^t).$$

We give the following definition similar to the classical one:

Definition

An n-dimensional random vector $Y \in (L_G^1(\Omega))^n$ is said to be independent from Ω_t for some given t if for each $\varphi \in C_{b.Lip}(\mathbb{R}^n)$ we have

$$\widetilde{\mathbb{E}}[\varphi(Y)|\Omega_t] = \widetilde{\mathbb{E}}[\varphi(Y)].$$

Bachelier (1900) [**?**] proposed Brownian motion as a model for fluctuations of the stock market, Einstein (1905) [**?**] used Brownian motion to give experimental confirmation of the atomic theory, and Wiener (1923) [**?**] gave a mathematically rigorous construction of Brownian motion. Here we follow Kolmogorov's idea (1956) [**?**] to construct G-Brownian motion by introducing infinite dimensional function space and the corresponding family of infinite dimensional sublinear distributions, instead of linear distributions in [**?**].

The notions of G-Brownian motion and the related stochastic calculus of Itô's type were firstly introduced by Peng (2006) [?] for 1-dimensional case and then in (2008) [?] for multi-dimensional situation. It is very interesting that Denis and Martini (2006) [?] studied super-pricing of contingent claims under model uncertainty of volatility. They have introduced a norm on the space of continuous paths $\Omega = C([0, T])$ which corresponds to our L_G^2-norm and developed a stochastic integral. There is no notion of nonlinear expectation and the related nonlinear distribution, such as G-expectation, conditional G-expectation, the related G-normal distribution and the notion of independence in their paper. But on the other hand, powerful tools in capacity theory enable them to obtain pathwise results for random variables and stochastic processes through the language of "quasi-surely" (see e.g. Dellacherie (1972) [?], Dellacherie and Meyer (1978 and 1982) [?], Feyel and de La Pradelle (1989) [?]) in place of "almost surely" in classical probability theory.

The main motivations of G-Brownian motion were the pricing and risk measures under volatility uncertainty in financial markets (see Avellaneda, Levy and Paras (1995) [?] and Lyons (1995) [?]). It was well-known that under volatility uncertainty the corresponding uncertain probabilities are singular from each other. This causes a serious problem for the related path analysis to treat, e.g., path-dependent derivatives, under a classical probability space. Our G-Brownian motion provides a powerful tool to such type of problems.

Our new Itô's calculus for G-Brownian motion is of course inspired from Itô's groundbreaking work since 1942 [**?**] on stochastic integration, stochastic differential equations and stochastic calculus through interesting books cited in Chapter –ch5. Itô's formula given by Theorem –Thm6.5 is from Peng [**?**], [**?**]. Gao (2009)[**?**] proved a more general Itô's formula for G-Brownian motion. An interesting problem is: can we get an Itô's formula in which the conditions correspond the classical one? Recently Li and Peng have solved this problem in [**?**].
Using nonlinear Markovian semigroup known as Nisio's semigroup (see Nisio (1976) [**?**]), Peng (2005) [**?**] studied the processes with Markovian properties under a nonlinear expectation.