Backward Stochastic Differential Equations with Infinite Time Horizon

Holger Metzler

PhD advisor: Prof. G. Tessitore

Università di Milano-Bicocca

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Outline

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1. General setup and standard results
   - The multi-dimensional nonlinear case
   - The one-dimensional nonlinear case
2. Multi-dimensional linear case

General setup

Throughout this talk, we are given

- a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, carrying a standard d-dimensional Brownian motion $(W_t)_{t\geq 0}$,
- the filtration $(\tilde{\mathcal{F}}_t)$ generated by W,
- the filtration $(\mathcal{F}_t)$, which is $(\tilde{\mathcal{F}}_t)$ augmented by all P-null sets.

$\Longrightarrow$ $(\mathcal{F}_t)$ satisfies the usual conditions

Adapted processes are always assumed to be $(\mathcal{F}_t)$-adapted.

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We denote by $\mathcal{M}^{2,\varrho}(E)$ the Hilbert space of processes X with:

- X is progressively measurable, with values in the Euclidean space E,
- $\mathbb{E}\left[\int_0^\infty e^{\varrho s}\|X_s\|_E^2\, ds\right] < \infty$.

Consider the BSDE with infinite time horizon

$$-dY_t = \psi(t, Y_t, Z_t)dt - Z_t dW_t, \quad t \in [0, T], \ T \geq 0. \qquad (1)$$

- $\psi : \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \times L(\mathbb{R}^d, \mathbb{R}^n) \to \mathbb{R}^n$ is such that $\psi(\cdot, y, z)$ is a progressively measurable process.
- A solution is a couple of progressively measurable processes (Y, Z) with values in $\mathbb{R}^n \times L(\mathbb{R}^d, \mathbb{R}^n)$, such that, for all $t \leq T$ with $t, T \geq 0$,

$$Y_t = Y_T + \int_t^T \psi(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s.$$

Assumption (A1)

(A1) There exist $C \geq 0$, $\gamma \geq 0$ and $\mu \in \mathbb{R}$, such that

(1) ψ is uniformly lipschitz, i.e.

$$|\psi(t,y,z) - \psi(t,y',z')| \leq C|y - y'| + \gamma\|z - z'\|;$$

(2) ψ is monotone in y:

$$\langle y - y', \psi(t,y,z) - \psi(t,y',z)\rangle \leq -\mu|y - y'|^2;$$

(3) There exists $\varrho \in \mathbb{R}$, such that $\varrho > \gamma^2 - 2\mu$, and

$$\mathbb{E}\left[\int_0^\infty e^{\varrho s}|\psi(s,0,0)|^2\, ds\right] \leq C.$$

Set $\lambda := \frac{\gamma^2}{2} - \mu$. This implies $\varrho > 2\lambda$. Darling and Pardoux (1997) established the following result.

Theorem

If (A1) holds then BSDE (1) has a unique solution (Y, Z) in $\mathcal{M}^{2,2\lambda}(\mathbb{R}^n \times L(\mathbb{R}^d, \mathbb{R}^n))$. The solution actually belongs to $\mathcal{M}^{2,\varrho}(\mathbb{R}^n \times L(\mathbb{R}^d, \mathbb{R}^n))$.

The major restriction is the structural condition in part (3) of (A1):

- We want to solve the equation for arbitrary bounded $\psi(\cdot, 0, 0)$.
- So we need $\mu > \frac{1}{2}\gamma^2$.

This condition is not natural in applications and, hence, is very unpleasant.

The one-dimensional case ($n = 1$)

- Significant improvement due to Briand and Hu (1998).
- Solution exists for all $\mu > 0$, if $\psi(\cdot, 0, 0)$ is bounded, i.e.
  (3') $|\psi(t, 0, 0)| \leq K$.
- $\mu > 0$ means, ψ is dissipative with respect to y.

Theorem ($n = 1$)

Assume parts (1) and (2) of (A1) with $\mu > 0$, and (3'). Then BSDE (1) has a solution (Y, Z) which belongs to $\mathcal{M}^{2,-2\mu}(\mathbb{R} \times \mathbb{R}^d)$ and such that Y is a bounded process.

This solution is unique in the class of processes (Y, Z), such that Y is continuous and bounded and Z belongs to $\mathcal{M}^2_{loc}(\mathbb{R}^d)$.

Idea of the proof

1. Consider the equation with finite time horizon $[0, m]$. Call the unique solution (Y_m, Z_m).
2. Establish the a priori bound

$$|Y_m(\theta)| \leq \frac{K}{\mu}, \text{ for all } \theta.$$

3. Use this a priori bound to show that $(Y_m, Z_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{M}^{2,-2\mu}(\mathbb{R} \times \mathbb{R}^d)$.

The crucial part is to establish the a priori bound.

The a priori bound

- Linearise ψ to

$$\psi(s, Y_m, Z_m) = \alpha_m(s)Y_m(s) + \beta_m(s)Z_m(s) + \psi(s,0,0)$$

with $\alpha_m(s) \leq -\mu$ and β_m bounded.

- (Y_m, Z_m) solves the equation

$$Y_m(t) = \int_t^m [\alpha_m(s)Y_m(s) + \beta_m(s)Z_m(s) + \psi(s,0,0)]\, ds - \int_t^m Z_m(s)\, dW_s.$$

- Introduce

$$R_m(t) := exp\left(\int_\theta^t \alpha_m(s)\,ds\right),$$

$$W_m(t) := W(t) - \int_0^t \beta_m(s)\,ds.$$

- Note that

$$R_m(s) \leq e^{-\mu(s-\theta)}$$

and

$$\int_\theta^\infty R_m(s)\,ds \leq \frac{1}{\mu}.$$

- Apply Itô's formula to the process $R_m Y_m$:

$$Y_m(\theta) = R_m(m) Y_m(m) + \int_\theta^m R_m(s)\psi(s,0,0)\,ds - \int_\theta^m R_m(s) Z_m(s)\,dW_m(s).$$

- Take into account that $Y_m(m) = 0$:

$$Y_m(\theta) = \int_\theta^m R_m(s)\psi(s,0,0)\,ds - \int_\theta^m R_m(s) Z_m(s)\,dW_m(s).$$

Using Girsanov's theorem, we can consider W_m as a Brownian motion with respect to an equivalent measure $\mathbb{Q}_m$ and hence, we get, $\mathbb{Q}_m$-a.s.,

$$\begin{aligned} |Y_m(\theta)| &= \mathbb{E}^{\mathbb{Q}_m}\left[|Y_m(\theta)| \mid \mathcal{F}_\theta\right] \\ &\leq \mathbb{E}^{\mathbb{Q}_m}\left[\int_\theta^\infty |\psi(s,0,0)| R_m(s)\, ds \mid \mathcal{F}_\theta\right] \\ &\leq \frac{K}{\mu}. \end{aligned}$$

In the end, this estimate assures also the boundedness of the limit process Y.

Problem for $n > 1$

If Y is a multi-dimensional process ($n > 1$), we cannot use this Girsanov trick, because each coordinate needs its own transformation, and these transformations are not consistent among each other.

So we are restricted to the case $\mu > \frac{1}{2}\gamma^2$, whereas the case $\mu > 0$ could have multiple interesting applications, e.g. in stochastic differential games or for homogenisation of PDEs.

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Multi-dimensional linear case

Let us now consider the following equation:

$$-dY_t = [AY_t + \sum_{j=1}^{d} \Gamma^j Z_t^j + f_t]dt - Z_t dW_t, \ t \in [0, T], \ T \geq 0. \quad (2)$$

- $A, \Gamma^j \in \mathbb{R}^{n \times n}$.
- Z_t^j denotes the j-th column vector of $Z_t \in \mathbb{R}^{n \times d}$.
- $f_t \in \mathbb{R}^n$ is bounded by K.
- A is assumed to be dissipative, i.e. there exists $\mu > 0$ such that
$$\langle y - y', A(y - y') \rangle \leq -\mu |y - y'|^2.$$
- The coefficients in equation (2) are non-stochastic and, except f_t, time-independent.

As in the one-dimensional non-linear case, we are interested in progressively measurable solutions (Y, Z), such that Y is bounded. This can be achieved by establishing the above mentioned a priori estimate

$$|Y_m(\theta)| \leq \frac{K}{\mu}.$$

To this end, we consider the dual process to Y_m, denoted by X^x. This process satisfies

$$\begin{cases} dX_t^x = A^* X_t^x dt + \sum_{j=1}^{d} (\Gamma^j)^* X_t^x dW_t^j \\ X_\theta^x = x \in \mathbb{R}^n. \end{cases}$$

By Itô's formula and the Markov property of X^x, we obtain

$$|Y_m(\theta)| \leq \sup_{|x|=1} \mathbb{E}\left[\int_\theta^m \langle X_t^x, f_t\rangle \, dt \mid \mathcal{F}_\theta\right] \leq K \sup_{|x|=1} \mathbb{E} \int_\theta^\infty |X_t^x| \, dt.$$

$\Longrightarrow$ Question of L_1-stability of X^x with $|x| = 1$. We need

$$\mathbb{E} \int_0^\infty |X_t^x| \, dt \leq M.$$

Task: Find appropriate assumptions on Γ^j and μ.

Lyapunov approach

Try to find "Lyapunov" function $v \in C^2(\mathbb{R}^n)$ with

(1) $v \geq 0$,
(2) $v(x) \leq c|x|$, for some $c > 0$,
(3) $[\mathcal{L}v](x) \leq -\delta|x|$, for some $\delta > 0$.

Here $\mathcal{L}$ is the Kolmogorov operator of X^x, i.e.

$$dv(X_t^x) = [\mathcal{L}v](X_t^x)dt + \text{"martingale part"}.$$

This approach was used by Ichikawa (1984) to show stability properties of strongly continuous semigroups.

Itô's formula and the Markov property of X^x give us

$$\mathbb{E}[v(X_t^x) - v(X_\theta^x)] = \mathbb{E} \int_\theta^t [\mathcal{L}v](X_s^x)\, ds$$
$$\leq -\delta\, \mathbb{E} \int_\theta^t |X_s^x|\, ds.$$

By showing $\mathbb{E}[v(X_t^x)] \to 0$ as $t \to \infty$, we obtain

$$\mathbb{E} \int_\theta^\infty |X_s^x|\, ds \leq \frac{1}{\delta}\, \mathbb{E}[v(X_\theta^x)] \leq \frac{c}{\delta}\, |x|$$
$$\leq \frac{c}{\delta} =: M.$$

How to find a Lyapunov function?

- First idea: $v(x) = |x|$.
- Problem: v is not C^2, hence Itô's formula inapplicable.

- Second idea: Define, for $\varepsilon > 0$,

$$v_\varepsilon(x) = \sqrt{|x|^2 + \varepsilon}\,.$$

- $v_\varepsilon(x) \to |x|$.

How to proceed?

- Calculate $[\mathcal{L}v_\varepsilon](x)$.
- Choose μ large enough, such that the coefficient in front of $|x|^4$ is negative. This choice will depend on Γ^j.
- Find appropriate $\kappa_\varepsilon > 0$, $\kappa_\varepsilon \to 0$ and split the integral on the RHS:

$$\mathbb{E}v_\varepsilon(X_t^x) - \mathbb{E}v_\varepsilon(X_\theta^x) = \mathbb{E}\int_\theta^t [\mathcal{L}v_\epsilon](X_s^x)\,ds$$

$$= \mathbb{E}\int_\theta^t [\mathcal{L}v_\epsilon](X_s^x)\mathbb{1}_{\{|X_s^x|\geq\kappa_\varepsilon\}}\,ds + \mathbb{E}\int_\theta^t [\mathcal{L}v_\epsilon](X_s^x)\mathbb{1}_{\{|X_s^x|<\kappa_\varepsilon\}}\,ds$$

- Obtain with $\varepsilon \to 0$

$$\mathbb{E}|X_t| - \mathbb{E}|X_\theta| \leq -\delta\,\mathbb{E}\int_\theta^t |X_s|\,ds.$$

- Apply Gronwall's lemma to $\Phi(t) := \mathbb{E}|X_t|$.

$$\Longrightarrow \quad \lim_{t\to\infty} \mathbb{E}|X_t| = 0$$

$$\Longrightarrow \quad \mathbb{E}\int_\theta^\infty |X_t| \leq \frac{1}{\delta} =: M$$

So X^x is L_1-stable and equation (2) admits a bounded solution.

Simple example

Assume $\Gamma = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}$ and $\gamma := \max\{|\gamma_1|, |\gamma_2|\}$.

- $[\mathcal{L} v_\varepsilon](x) \leq \frac{\left[\frac{1}{8}(\gamma_1 - \gamma_2)^2 - \mu\right]|x|^4 + \frac{1}{2}\varepsilon\gamma^2|x|^2}{(|x|^2+\varepsilon)^{\frac{3}{2}}}$
- For $\mu > \frac{1}{8}(\gamma_1 - \gamma_2)^2$ is X^x L_1-stable, and equation (2) has a bounded solution.

- The general result from the first part requires the much stronger assumption

$$\mu > \frac{1}{2}\|\Gamma\|^2 = \frac{1}{2}(\gamma_1^2 + \gamma_2^2).$$

L_2-stability is strictly stronger than L_1-stability.

Example

We take $n = d = 1$ and consider the following equation:

$$\begin{cases} dX_t = -\mu X_t dt + \gamma X_t dW_t \\ X_0 = 1. \end{cases}$$

The solution is a geometric Brownian motion

$$X_t = e^{-\mu t} e^{\gamma W_t - \frac{1}{2}\gamma^2 t}$$

and

$$\mathbb{E}|X_t| = e^{-\mu t}, \qquad \mathbb{E}|X_t|^2 = e^{-2\mu t} e^{\gamma^2 t}.$$

So X is L_1-stable for each $\mu > 0$, but L_2-stable only for $\mu > \frac{1}{2}\gamma^2$.

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