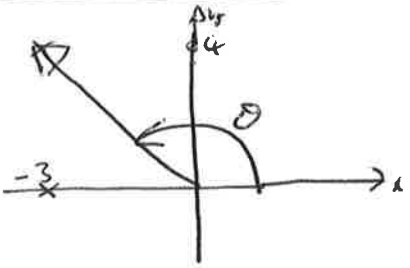
Solutions aux exercices (sélection) § 3.4.

I. $x = -3,\ y = 4$



$\rho = \sqrt{x^2+y^2} = \sqrt{9+16} = 5$

$\theta = \arctan \frac{y}{x} = \arctan\left(\frac{-4}{3}\right)$

Alors $\arctan \frac{4}{3} \sim 53°$, d'où $\arctan\left(\frac{-4}{3}\right) \sim 180° - 53° = 127°$

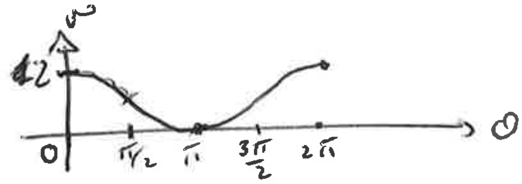
II $\rho = \sqrt{x^2+y^2+z^2}$, $\lambda = \arctan \frac{y}{x}$, $\theta = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$

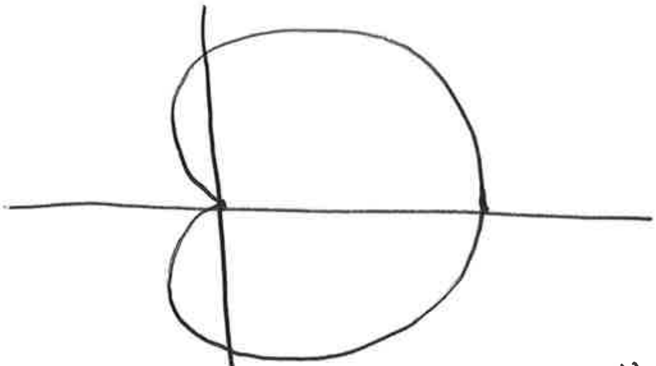
On a besoin d'une calculatrice – il faut faire attention au quadrant du point, car x, y, z sont tous négatifs, d'où

$\pi < \lambda < \frac{3\pi}{2}$, en effet $\lambda = \arctan \frac{y}{x} = \arctan \frac{247804}{245622}$

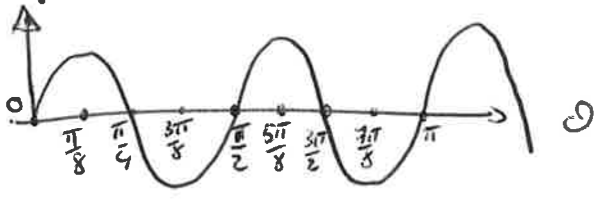
qu'on calcule, qui va donner quelque chose entre 0 et $\pi/2$, puis on ajoute π.

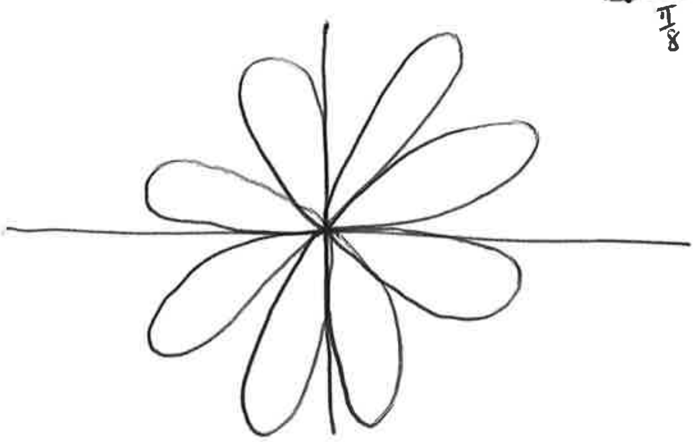
III $\rho = 1 + \cos\theta$



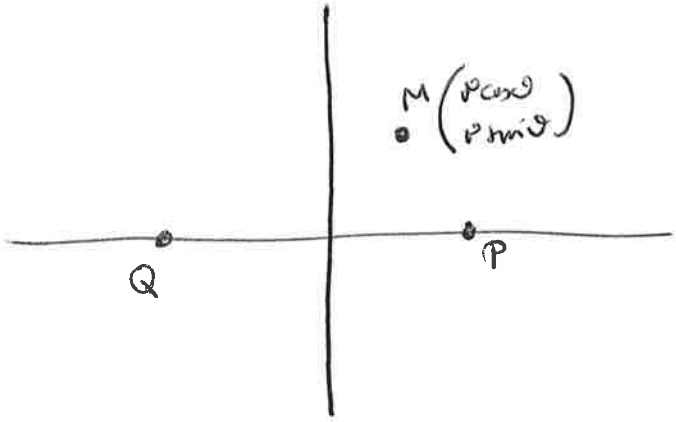


IV Fait en classe





V.



$$MP = \sqrt{(1-r\cos\theta)^2 + r^2\sin^2\theta} = \sqrt{1-2r\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta} = \sqrt{1+r^2-2r\cos\theta}$$

$$MQ = \sqrt{(1+r\cos\theta)^2 + r^2\sin^2\theta} = \sqrt{1+r^2+2r\cos\theta}$$

$$MP \times MQ = 1 \iff \sqrt{1+r^2-2r\cos\theta}\sqrt{1+r^2+2r\cos\theta} = 1$$

$$\iff \sqrt{(1+r^2-2r\cos\theta)(1+r^2+2r\cos\theta)} = 1$$

$$\iff \sqrt{(1+r^2)^2 - 4r^2\cos^2\theta}$$

$$\iff \sqrt{r^4+2r^2-4r^2\cos^2\theta+1} = 1$$

On note que $1-2\cos^2\theta = -\cos 2\theta$

On prend le carré de chaque partie :

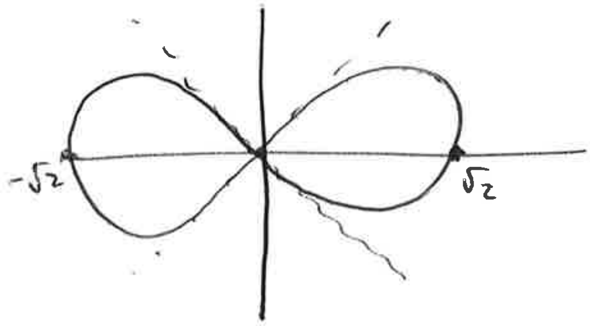
$$r^4+1-2r^2\cos 2\theta = 1$$

$$\iff r^4 - 2r^2\cos 2\theta = 0 \iff r^2 = 2\cos 2\theta$$

Il faut alors que $\cos 2\theta \geq 0$ c'est à dire $-\frac{\pi}{2} \leq 2\theta \leq \frac{\pi}{2}$

$$\iff -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$$

Pour chaque θ entre $-\pi/4$ et $\frac{\pi}{4}$ il y a deux valeurs de $r = \sqrt{2\cos 2\theta}$



VI (a) Coord cylindriques : $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$d_{AB} = \sqrt{(x_A-x_B)^2+(y_A-y_B)^2+(z_A-z_B)^2} = \sqrt{(r_A\cos\theta_A - r_B\cos\theta_B)^2 + (r_A\sin\theta_A - r_B\sin\theta_B)^2 + (z_A-z_B)^2}$$

$$= \sqrt{\left\{ r_A^2\cos^2\theta_A - 2r_Ar_B\cos\theta_A\cos\theta_B + r_B^2\cos^2\theta_B + r_A^2\sin^2\theta_A - 2r_Ar_B\sin\theta_A\sin\theta_B + r_B^2\sin^2\theta_B + (z_A-z_B)^2 \right\}}$$

$$= \sqrt{\left\{ r_A^2 + r_B^2 - 2r_Ar_B(\cos\theta_A\cos\theta_B + \sin\theta_A\sin\theta_B) + (z_A-z_B)^2 \right\}}$$

$$= \sqrt{r_A^2 + r_B^2 - 2r_Ar_B\cos(\theta_A-\theta_B) + (z_A-z_B)^2}$$

(b) similaire.

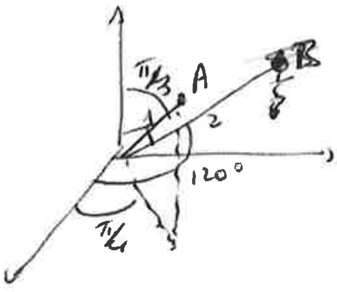
VII. $(r_A, \theta_A, \lambda_A) = (1, \frac{\pi}{2}, \pi)$, $(r_B, \theta_B, \lambda_B) = (8, \frac{\pi}{4}, \frac{5\pi}{4})$

$$\begin{aligned} x &= r\sin\theta\cos\lambda \\ y &= r\sin\theta\sin\lambda \\ z &= r\cos\theta \end{aligned} \quad\Bigg|\quad \begin{aligned} x_A &= 1\sin\tfrac{\pi}{2}\cos\pi = 1\times1\times(-1) = -1 \\ y_A &= 1\sin\tfrac{\pi}{2}\sin\pi = 1\times1\times0 = 0 \\ z_A &= 1\cos\tfrac{\pi}{2} = 1\times0 = 0 \end{aligned}$$

$$\begin{aligned} x_B &= 8\sin\tfrac{\pi}{4}\cos\tfrac{5\pi}{4} = 8\times\tfrac{1}{\sqrt2}\times\left(-\tfrac{1}{\sqrt2}\right) = -4 \\ y_B &= 8\sin\tfrac{\pi}{4}\sin\tfrac{5\pi}{4} = 8\times\tfrac{1}{\sqrt2}\times\left(-\tfrac{1}{\sqrt2}\right) = -4 \\ z_B &= 8\cos\tfrac{\pi}{4} = 8\times\tfrac{1}{\sqrt2} = \tfrac{8}{\sqrt2} \end{aligned}$$

$\left(\cos\frac{5\pi}{4} = \cos\left(\pi+\frac{\pi}{4}\right) = \cos\pi\cos\frac{\pi}{4} - \sin\pi\sin\frac{\pi}{4} = -1\times\frac{1}{\sqrt2}\right)$

VIII



Soit α l'angle AOB

Alors $\cos\alpha = \dfrac{\vec{OA}}{\|\vec{OA}\|} \cdot \dfrac{\vec{OB}}{\|\vec{OB}\|}$

On remarque que $\|\vec{OA}\| = r_A$, $\|\vec{OB}\| = r_B$

d'où $\dfrac{\vec{OA}}{\|\vec{OA}\|} = \frac{1}{r_A}(r_A\sin\theta_A\cos\lambda_A, r_A\sin\theta_A\sin\lambda_A, r_A\cos\theta_A) = (\sin\theta_A\cos\lambda_A, \sin\theta_A\sin\lambda_A, \cos\theta_A)$

de même $\dfrac{\vec{OB}}{\|\vec{OB}\|} = (\sin\theta_B\cos\lambda_B, \sin\theta_B\sin\lambda_B, \cos\theta_B)$

$$\cos\alpha = \left(\sin\tfrac{\pi}{3}\cos\tfrac{\pi}{4}, \sin\tfrac{\pi}{3}\sin\tfrac{\pi}{4}, \cos\tfrac{\pi}{3}\right)\cdot\left(\sin\tfrac{\pi}{2}\cos\tfrac{2\pi}{3}, \sin\tfrac{\pi}{2}\sin\tfrac{2\pi}{3}, \cos\tfrac{\pi}{2}\right)$$

$$= \left(\tfrac{\sqrt3}{2}\tfrac{1}{\sqrt2}, \tfrac{\sqrt3}{2}\tfrac{1}{\sqrt2}, \tfrac12\right)\cdot\left(1\left(-\tfrac12\right), 1\times\tfrac{\sqrt3}{2}, 0\right) = -\frac{\sqrt3}{4\sqrt2} + \frac{3}{4\sqrt2} = \frac{3-\sqrt3}{4\sqrt2}$$

$$\alpha = \arccos\left(\frac{3-\sqrt3}{4\sqrt2}\right)$$

§4·6

I et II Voir solution à la question 4 de l'épreuve d'entraînement.

III

$$AB = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ -3 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 0-1+0 & 1+1+0 & 3-0+0 \\ 0+0+1 & 2+0+1 & 6+0+2 \\ 0-1+1 & -3+1+1 & -9+0+2 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 3 \\ 1 & 3 & 8 \\ 0 & -1 & -7 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ -3 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 0+2-9 & 0+0-3 & 0+1+3 \\ 1-2+0 & -1+0+0 & 0-1+0 \\ 1+2-6 & -1+0-2 & 0+1+2 \end{pmatrix} = \begin{pmatrix} -7 & -3 & 4 \\ -1 & -1 & -1 \\ -3 & -3 & 3 \end{pmatrix}$$

$$AB-BA = \begin{pmatrix} -1 & 2 & 3 \\ 1 & 3 & 8 \\ 0 & -1 & -7 \end{pmatrix} - \begin{pmatrix} -7 & -3 & 4 \\ -1 & -1 & -1 \\ -3 & -3 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 5 & -1 \\ 2 & 4 & 9 \\ 3 & 2 & -10 \end{pmatrix}$$

$\mathrm{trace}(AB-BA) = 6+4-10 = 0$, ce qui démontre que $\mathrm{trace}\, AB = \mathrm{trace}\, BA$ (même si $AB \neq BA$)

V

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & -1 & 0 \\ 5 & 1 & 1 \end{pmatrix} \; ; \; \det A = \det\begin{pmatrix} -2 & 0 & 0 \\ 1 & -1 & 0 \\ 5 & 1 & 1 \end{pmatrix} \quad (\text{ligne 1} - \text{ligne 3})$$

$$= -2\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} = -2 \times (-1) = 2$$

$$\mathrm{com}(A) = \begin{pmatrix} \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 5 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 5 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 5 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 5 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -1-0 & -(1-0) & 1+5 \\ -(1-1) & 3-5 & 3-5 \\ 0+1 & -(0-1) & -3-1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 6 \\ 0 & -2 & +2 \\ 1 & 1 & -4 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det A}\,\mathrm{com}(A)^t = \frac{1}{2}\begin{pmatrix} -1 & 0 & 1 \\ -1 & -2 & 1 \\ 6 & +2 & -4 \end{pmatrix}$$

Les équations de la question IV s'écrivent $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

d'où $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 0 & 1 \\ -1 & -2 & 1 \\ 6 & +2 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1+0+2 \\ -1+2+2 \\ 6-2-8 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 3/2 \\ -2 \end{pmatrix}$

on vérifie : équ1 : $3 \times \frac{1}{2} + \frac{3}{2} - 2 = 3-2 = 1$ OK

équ2 : $x - y = \frac{1}{2} - \frac{3}{2} = -1$ OK

équ3 : $\frac{5}{2} + \frac{3}{2} - 2 = 4-2 = 2$ OK.