Euclidean quantum fields as Wilson–Itô diffusions

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We introduce Wilson–Itô diffusions: a class of random fields on $\mathbb{R}^d$ that change continuously along a scale parameter via a Markovian dynamics with *local* coefficients. Described via forward-backward stochastic differential equations, their observables naturally form a prefactorization algebra *à la* Costello–Gwilliam. We argue that this is a new non-perturbative quantization method applicable also to gauge theories and independent of a path-integral formulation. Whenever a path-integral is available this approach reproduces the setting of Wilson–Polchinski flow equations.

I. INTRODUCTION

The Kadanoff–Wilson point of view on quantum field theory is a central idea in modern physics. It dictates that one should develop the fluctuation of quantum or statistical fields along a scale decomposition associated to the tunable precision of the observation device. In this framework a random Euclidean field is described by a stochastic process $(\varphi_a)_{a \geqslant 0}$ where a parametrizes a scale of observation with characteristic length $1/a$ and φ_a is the corresponding observation of the field, which must be thought of as containing fluctuations with spatial scales $\gtrsim 1/a$. It is natural to assume that, as we gradually increase the resolution of our measuring devices, the resulting measurements vary continuously. Therefore we will postulate that the stochastic process φ_a is a *pathwise* continuous function of a. It is also reasonable to assume that φ_a contains all the information gathered by observations with lesser precisions, which implies that it is a Markov process along the scale parameter, and we assume further that $\varphi_a \to \varphi_\infty$ as $a \to \infty$. We denote by $\mathscr{F} = (\mathscr{F}_a)_{a \geqslant 0}$ the filtration generated by $(\varphi_a)_{a \geqslant 0}$ and by $\mathbb{E}_b[\,\cdot\,]$ the operator of conditional expectation given $\mathscr{F}_b$, for any $b \geqslant 0$, with $\mathbb{E} = \mathbb{E}_0$ the full expectation. We assume that $\varphi_0 = 0$ whenever the fields take value in a vector space; otherwise we take φ_0 to be a fixed default (classical) configuration (e.g. a background field). We say that a functional $\varphi \mapsto F(\varphi)$ of the field φ is supported in $U \subseteq \mathbb{R}^d$ if $F(\varphi) = F(\psi)$ for any field ψ such that $\varphi = \psi$ in U. For $\varepsilon > 0$ let $U_\varepsilon := \{x \in \mathbb{R}^d : d(x, U) < \varepsilon\}$ be the ε-enlargement of U.

Definition 1. *An* ***observable*** *O is a stochastic process $(O_a)_{a \geqslant 0}$ which is an $\mathscr{F}$-martingale, i.e.*

$$O_b = \mathbb{E}_b[O_a], \qquad (0 \leqslant b \leqslant a < \infty).$$

An ***observable*** *O is said to be* ***supported on a set*** *U if there exists a family of functionals $(\mathring{O}_a(\,\cdot\,))_{a \geqslant 0}$ each supported on the a^{-1}-enlargement $U_{a^{-1}}$ and such that $R_a^O := O_a - \mathring{O}_a(\varphi_a) \to 0$ as $a \to \infty$. We talk of $\mathring{O}$ as a* ***germ for O****. A* ***local observable (field)*** *is field of observables $x \mapsto O(x)$ such that the observable $O(x)$ is supported on the singleton $\{x\}$ for all $x \in \mathbb{R}^d$.*

The definition of the a^{-1}-enlargement is a matter of convention. An observable O that has a germ

$$\mathring{O}_a(\varphi_a)(x) = \mathcal{O}_a(\varphi_a(x), \ldots, \nabla^k \varphi_a(x)),$$

for some finite k and some function $\mathcal{O}_a$, is local. Germs somehow parametrize the space of observables via concrete functionals of the fields.

One owes to ***Itô*** the fundamental insight that continuous diffusions can be constructed via stochastic differential equations [21]. As such, since we postulated that the scale description $(\varphi_a)_a$ of a random field is a continuous Markov process, it is completely described via its infinitesimal rate of change

$$\mathrm{d}\varphi_a = \varphi_{a+\mathrm{d}a} - \varphi_a = B_a^{\varphi}\mathrm{d}a + \mathrm{d}M_a^{\varphi}$$

which comes in two parts: A *drift* $B_a^{\varphi}\mathrm{d}a$ and a *martingale* part $\mathrm{d}M_a^{\varphi}$. The drift component B_a^{φ} is the part of the rate of change which can be predicted from the large scale observations, while the "innovation" process M is a martingale w.r.t. this filtration, it models the additional information gained by augmenting the resolution. Recall a continuous martingale is described via its quadratic variation process $\langle M\rangle_a$ [20].

Enters ***Wilson***. We inject in this standard framework the spatial random structure of Euclidean quantum fields by imposing that the dynamics is completely described by '*local*' coefficients. We do so by postulating that there exists a local observable field $(f_a)_{a\geqslant 0}$ which models the ***microscopic force*** and that, inspired by the Kadanoff-Wilson block averaging procedure, the drift B_a^{φ} at scale a is determined by an averaging of the force field f_a over a region of size $1/a$. Therefore we introduce a "block averaging" operator $C_a = C_a(\varphi_a)$, which could be random and depend on φ_a. We require C_a to have support on a ball of radius $1/a$, that $a\mapsto C_a(\psi)$ varies smoothly for all ψ, that C_a be symmetric and positive definite and such that $C_\infty = 1$. For example, one can take

$$(C_a h)(x) = \int a^d \chi((x-y)a)h(y)\mathrm{d}y, \qquad x\in\mathbb{R}^d, \tag{1}$$

where χ is a smooth, radially symmetric, positive function of unit integral. In particular note that $C_0 = 0$ on all sufficiently nice functions h. Since the drift will be integrated along the scales, the local averaging has to be done in such a way not to over-count the contributions of the microscopic force. We denote by $\dot{C}_a := \partial_a C_a$ the scale-derivative of this averaging and let $B_a^{\varphi} := \dot{C}_a f_a$. This fixes the previsible part of the stochastic dynamics as a function of the microscopic force. To complete our description we need to specify also the quadratic variation of the martingale part. Using the same principles we can assume

$$\mathrm{d}\langle M\rangle_a := \dot{C}_a^{1/2}\sigma_a^2\dot{C}_a^{1/2}\,\mathrm{d}a$$

for a microscopic positive "diffusivity" $(\sigma_a^2)_a$ which is a local observable field of positive scalars. In this way the local diffusion on scales is determined by the datum of two local observable fields $(f_a,\sigma_a^2)_a$ and a family of averaging operators $(C_a)_a$. In particular, by standard results there exists, possibly on an extended probability space, a cylindrical $\mathscr{F}$-Brownian motion

$$\mathbb{E}[W_b(x)W_a(y)] = (b\wedge a)\delta(x-y),$$

such that

$$M_a = \int_0^a \dot{C}_b^{1/2}\,\sigma_b\mathrm{d}W_b$$

where $\sigma_b := (\sigma_b^2)_b^{1/2}$.

Definition 2. *A **Wilson–Itô diffusion** is a continuous stochastic process $(\varphi_a)_{a\geqslant 0}$ taking values in the set of smooth functions on $\mathbb{R}^d$ with the following properties.*

*a) **Dynamics**. There is an effective force $(f_a)_a$ and an effective diffusivity $(\sigma_a^2)_a$ such that $(\varphi_a)_{a\geqslant 0}$ is a Markovian Itô diffusion*

$$\mathrm{d}\varphi_a = \dot{C}_a f_a\mathrm{d}a + \dot{C}_a^{1/2}\sigma_a\mathrm{d}W_a. \tag{2}$$

*b) **Locality**. The effective force f and the effective diffusivity σ^2 are local observable fields.*

*We call equation (2) a **Wilson–Itô differential equation** (WIDE). A **Wilson–Itô field** is the random field φ_∞ obtained as the terminal value of a Wilson–Itô diffusion $(\varphi_a)_{a\geqslant 0}$.*

The main goal of this paper is to propose the hypothesis that Euclidean quantum field theories can be identified with Wilson–Itô fields. This provides a new framework, independent of the path-integral formalism, to study Euclidean quantum fields.

a) This description emerges from simple and natural assumptions and covers in principle much more than those theories that can be reached perturbatively from a Gaussian functional integral.

b) The continuity of the process $(\varphi_a)_a$ with respect to a gives it the structure of an Itô diffusion. When σ_a and C_a are deterministic (hence σ_a is constant by the martingale property of σ_a^2), and without any assumption of a perturbative regime, the Wilson-Itô random field φ_∞ comes with an associated Gaussian field

$$X_a^C := \int_0^a \dot{C}_b^{1/2}\,\sigma_b\,\mathrm{d}W_b,$$

and a coupling to it. In that case, X_∞^C is a white noise

$$\mathbb{E}[X_\infty^C(x)X_\infty^C(y)] = \sigma^2 C_\infty(x-y) = \sigma^2\delta(x-y).$$

Therefore one expects that $\varphi_\infty = X_\infty^C + \int_0^\infty \dot{C}_a f_a \mathrm{d}a$ is, in general, only a distribution of very low regularity.

c) Eq. (2) makes sense for fields defined on and/or taking values in manifolds or vector bundles and for which the path-integral formalism is less clear to apply. Our formalism is non-perturbative and trades the use of functional integrals against Itô calculus.

d) Let $A = A(a)$ be a possibly random, adapted, increasing change of scale such that $A(0) = 0$ and $A(\infty) = \infty$, and let $\mathrm{d}\tilde{W}_a := A'(a)^{-1/2}\mathrm{d}W_{A(a)}$, $\tilde{C}_a = C_{A(a)}$, $\tilde{f}_a := f_{A(a)}$ and $\tilde{\sigma}_a := \sigma_{A(a)}$. Then we have

$$\mathrm{d}(\varphi_{A(a)}) = \partial_a \tilde{C}_a \tilde{f}_a \mathrm{d}a + (\partial_a \tilde{C}_a)^{1/2}\tilde{\sigma}_a \mathrm{d}\tilde{W}_a,$$

which shows that Wilson-Itô diffusion are covariant wrt. random changes of spatial scales. In particular, this justifies that the diffusion has to be averaged with $\dot{C}_b^{1/2}$. Note that observables are also covariant: If $(O_a)_a$ is an observable for $(\varphi_a)_a$ then $(O_{A(a)})_a$ is an observable for the diffusion $(\varphi_{A(a)})_a$.

e) In general only the law of the terminal value φ_∞ and the averages of observables are the physical content of a Wilson-Itô diffusion.

Dyson–Schwinger equations and martingale problems – The law of the process $(\varphi_a)_{a\geqslant 0}$ is determined by a ***martingale problem***: for all sufficiently nice scale-dependent test functions $F(a,\psi)$, the process

$$M_a^F := F(a,\varphi_a) - \int_0^a (\mathscr{L}_b F)(b,\varphi_b)\mathrm{d}b$$

is an $\mathscr{F}$-martingale where

$$\begin{aligned}\mathscr{L}_b F(b,\varphi) \;:=\; & \partial_b F(b,\varphi) + \mathrm{D}F(b,\varphi)\,\dot{C}_b f_b \\ & + \frac{1}{2}\mathrm{Tr}\left[\dot{C}_b^{1/2}\sigma_b^2\dot{C}_b^{1/2}\mathrm{D}^2F(b,\varphi)\right]\end{aligned} \quad (3)$$

is the **generator** of the Wilson–Ito diffusion. Here we denote by $\mathrm{D} := \delta/\delta\varphi$ the derivative of a functional $\varphi \mapsto F(\varphi)$ with respect to the field φ and by Tr the trace operator.

II. FORWARD-BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

Some observables can be obtained via the conditional expectation of a function of φ_∞, i.e. let $O_a = \mathbb{E}_a[F(\varphi_\infty)]$. This kind of martingale is called *closed*, i.e. it admits a "terminal value" $O_\infty = F(\varphi_\infty)$ from which it can be reconstucted. In the case when F is a linear functional we have

$$O_a = \mathbb{E}_a[F(\varphi_\infty)] = F(\mathbb{E}_a[\varphi_\infty]),$$

assuming suitable integrability conditions here and below. Using the Wilson-Itô dynamics, and since $(f_a)_a$ is a martingale, we have

$$\begin{aligned}\mathbb{E}_a[\varphi_\infty] &= \varphi_a + \mathbb{E}_a\int_a^\infty \dot{C}_b f_b \mathrm{d}b = \varphi_a + \int_a^\infty \dot{C}_b f_a \mathrm{d}b \\ &= \varphi_a + C_{\infty,a} f_a\end{aligned} \quad (4)$$

where $C_{\infty,a} := C_\infty - C_a = 1 - C_a$. In general we are not allowed to form non-linear *local* functions $F(\varphi_\infty)$ of the distribution φ_∞: The locality condition for an observable is non-trivial and usually requires renormalization. As a consequence, we do not expect non-linear local observables to be closed martingales. Ignoring for the moment this difficulty, we note that when $F(\varphi_\infty)$ is well-defined the observables $(O_a)_a$ are closed martingales and they satisfy some *backward stochastic differential equations* (BSDEs)

$$\mathrm{d}O_a = Z_a^O \mathrm{d}W_a,$$

for a pair of adapted processes (O_a, Z_a^O) with *terminal* condition $O_\infty = F(\varphi_\infty)$.

A procedure to construct local observables starts with some approximate local observable given by a function $\mathring{O}^{a_0}(\varphi_\infty)$ localized at scale a_0^{-1} and setting

$$O_a^{a_0} := \mathbb{E}_a[\mathring{O}^{a_0}(\varphi_\infty)].$$

One can then study the convergence of the family of the *non-local* observables $(O_a^{a_0})_{0\leqslant a\leqslant a_0}$ as $a_0 \to \infty$. Provided the functions $\mathring{O}^{a_0}(\varphi_\infty)$ contains appropriate (diverging) renormalizations one is able to show that the observables O^{a_0} converge to a local observable as $a_0 \to \infty$.

This approach leads naturally to the analysis of a general class of *forward-backward stochastic differential equation* (FBSDEs) of the form

$$\mathrm{d}\phi_a = \dot{C}_a \mathbb{E}_a[\tilde{f}(\phi_\infty)]\mathrm{d}a + \dot{C}_a^{1/2}\mathbb{E}_a[\tilde{\sigma}^2(\phi_\infty)]^{1/2}\mathrm{d}W_a \quad (5)$$

for some approximately local functionals $f_a = \mathbb{E}_a[\tilde{f}(\phi_\infty)]$, $\sigma_a = \mathbb{E}_a[\tilde{\sigma}^2(\phi_\infty)]^{1/2}$. (Recall $\dot{C}_a$ may depend on ϕ_a in a general setting. Think e.g. of $\tilde{f}(\phi_\infty)$ as a function of a regularized version of ϕ_∞. Note also that we use the letter ϕ for these 'approximate' dynamics while we use φ for the exact dyanmics.) These FBSDE are not proper Wilson-Itô diffusions, since their coefficients are not local. BSDEs and FBSDEs are well studied in the mathematical literature (see e.g. [17]) and we dispose also of numerical methods to approximate their solutions. In relation to the numerical aspects, note that the formalism allows to replace $\mathbb{R}^d$ by a finite discrete lattice $(\varepsilon\mathbb{Z}\cap[-L,L])^d$ of size ε.

A. Linear-like force

To give a first example consider the case of an approximate force $\tilde{f}$ with a linear component and a constant diffusivity

$$\tilde{f}(\phi_\infty) = \alpha(-A\phi_\infty + h(\phi_\infty)), \qquad \tilde{\sigma}(\phi_\infty) = \alpha^{1/2},$$

for some positive constant α, some positive linear operator A and some additional force component $h(\phi_\infty)$. For a local operator A the linear functional $A\phi_\infty$ is always well-defined in the space of distributions, so it defines a local force field. We assume here that the averaging operator C_a is deterministic and field-independent. Think of the case where $A = m^2 - \Delta$, $h = 0$ and C_a is given by Eq. (1). Using (4) we have

$$\mathbb{E}_a[\phi_\infty] = \phi_a - \alpha C_{\infty,a} A\mathbb{E}_a[\phi_\infty] + \alpha C_{\infty,a}\mathbb{E}_a[h(\phi_\infty)]$$

Solving for $\mathbb{E}_a[\phi_\infty]$ and letting $\psi_a := (1+\alpha C_{\infty,a}A)^{-1}\phi_a$, we have $\psi_\infty = \phi_\infty$ and

$$\mathrm{d}\psi_a = \dot{Q}_a\mathbb{E}_a[h(\psi_\infty)]\mathrm{d}a + \dot{Q}_a^{1/2}\mathrm{d}W_a, \tag{6}$$

where $\dot{Q}_a := \partial_a(A^{-1}(1+\alpha C_{\infty,a}A)^{-1})$. This computation shows that the linear component in the force can always be integrated and gives rise to a modified FBSDE where the local averaging operator $\dot{C}_a$ has been replaced by the operator $\dot{Q}_a$. Note in particular that the Gaussian field

$$X_a^Q := \int_0^a \dot{Q}_c^{1/2}\mathrm{dW}_c, \tag{7}$$

has covariance $Q_a - Q_0$ and that X_∞^Q has covariance $\alpha(1+\alpha A)^{-1}$. One gets back the operator A^{-1} in the large α limit, in which case X_∞^Q is a massive GFF in the model situation. We can invert the transformation and go from a FBSDE of the form (6) back to the FBSDE (5) setting

$$1 - C_a := \int_a^\infty Q_\infty^2 Q_b^{-2}\dot{Q}_b\mathrm{d}b$$

and

$$A = (Q_\infty - Q_0)^{-1} - \frac{1}{\alpha}.$$

B. Gradient diffusions

Assume further that the additional force component h is given by an effective UV-regularized potential V_∞ as

$$h = -DV_\infty(\psi_\infty),$$

and let

$$V_a(\varphi) := \log\frac{\mathbb{E}\big[e^{V_\infty(\varphi + X_\infty^Q - X_a^Q)}\big]}{\mathbb{E}\big[e^{V_\infty(X_\infty^Q)}\big]}.$$

Then $(V_a)_a$ is the solution of the Polchinski flow equation [19]

$$\partial_a V_a + \frac{1}{2}\mathrm{D}V_a\dot{Q}_a\mathrm{D}V_a + \frac{1}{2}\dot{Q}_a\mathrm{D}^2V_a = 0. \tag{8}$$

with terminal condition V_∞ for $a\to\infty$. Letting $Z_a := \exp(V_a(\psi_a) - V_0(0))$ and using Itô formula, we have that Z_a is a positive martingale wiht $Z_0 = 1$ and moreover that, under the probability measure $\mathbb{Q}$ defined on $\mathscr{F}_a$ by $\mathrm{d}\mathbb{Q} := Z_a\mathrm{d}\mathbb{P}$, the process ψ_a is a martingale with deterministic quadratic variation $\mathrm{d}\langle\psi\rangle_a = \dot{Q}_a\mathrm{d}a$, so ψ_∞ is under $\mathbb{Q}$ a GFF. Note also that if follows from (8) that $F_a(\psi_a) := 2\mathrm{D}V_a(\psi_a)$ is a martingale under $\mathbb{Q}$. Recalling Eq. (7), we conclude that

$$\begin{aligned}\mathbb{E}_\mathbb{P}[G(\psi_a)] &= \mathbb{E}_\mathbb{Q}[G(\psi_a)e^{V_0(0)-V_a(\psi_a)}] \\ &= \mathbb{E}_\mathbb{P}\big[G(X_a^Q)e^{V_0(0)-V_a(X_a^Q)}\big]\end{aligned} \tag{9}$$

for any function G and any $a \geqslant 0$. This implies that the law ν_∞ of the random field $\psi_\infty = \phi_\infty$ is given by

$$\nu_\infty(\mathrm{d}\psi) = \frac{e^{-V_\infty(\psi)}\mu^{Q_\infty}(\mathrm{d}\psi)}{\int e^{-V_\infty(\psi)}\mu^{Q_\infty}(\mathrm{d}\psi)} \tag{10}$$

where μ^{Q_∞} denotes the Gaussian law of covariance Q_∞.

This shows that Wilson–Itô fields comprise as a particular case the Euclidean quantum fields (10) constructed as perturbations of a Gaussian field. They are obtained solving FBSDEs of the form

$$\mathrm{d}\psi_a = -\dot{Q}_a\mathbb{E}_a[\mathrm{D}V_\infty(\psi_\infty)]\,\mathrm{d}a + \dot{Q}_a^{1/2}\mathrm{d}W_a. \tag{11}$$

which we call ***Polchinski FBSDEs***, since they describe the Polchinski semigroup [8]. Even in this potential framework where there is a formal link with the well-known Polchinski flow equation, the FBSDE (11) does not require the a priori knowledge of the solution of Eq. (8). Indeed we only used it to derive the formal connection while in Eq. (11) only the boundary condition DV_∞ is needed. In general this quantity, which a priori needs an ultraviolet regularization, has to be tuned in order for the FBSDE (11) to reach a well-defined local limit as the regularization is removed.

C. One-dimensional gradient diffusions

Take $d=1$ and let

$$F(\varphi_\infty)(x) = -(-\Delta)\varphi_\infty(x) - \rho(x) v'(\varphi_\infty(x)), \qquad x \in \mathbb{R}$$

where v' is the derivative of a function $v\colon \mathbb{R}\to\mathbb{R}$ smooth and bounded from below and $\rho\colon \mathbb{R}\to\mathbb{R}_+$ is a compactly supported function of the space variable (an IR cutoff). A priori φ_∞ is only a distribution, but the mild formulation of the WIDE shows that $\varphi_\infty=\psi_\infty$, $A=-\Delta$ and

$$\mathrm{d}\psi_a = -\dot{Q}_a \rho \mathbb{E}_a v'(\psi_\infty)\mathrm{d}a + \dot{Q}_a^{1/2}\mathrm{d}W_a. \tag{12}$$

From this equation we see that ψ_∞ is actually comparable in regularity to the $d=1$ massive GFF X_∞ with covariance $(1+A)^{-1}$, i.e.

$$\mathbb{E}[X_\infty(x)X_\infty(y)] = e^{-|x-y|}, \qquad x,y\in\mathbb{R},$$

which is a Hölder continuous function for which the mild formulation (12) makes sense. The residual force is given by the well-defined potential $V_\infty(\varphi) = \int_{\mathbb{R}} \rho(x)\, v(\varphi(x))\mathrm{d}x$. By (9) we have

$$\mathbb{E}_{\mathbb{P}}[G(\psi_\infty)] = \mathbb{E}_{\mathbb{P}}[G(X_\infty)e^{-V_\infty(X_\infty)}] \tag{13}$$

and this shows that ψ_∞ is a (time-inhomogeneous) Markovian diffusion in the space parameter since X_∞ can be described also as a Markovian Ornstein–Uhlbenbeck process, solution of the stochastic differential equation

$$\mathrm{d}X_\infty(x) = -X_\infty(x)\mathrm{d}x + \mathrm{d}B_x,$$

where $(B_x)_{x\in\mathbb{R}}$ is a two-sided Brownian motion. Moreover if we take $\rho\to 1$, under suitable assumptions, the dynamics will converge to a time-homogeneous space-Markovian diffusion described by the WIDE

$$\mathrm{d}\psi_a = \dot{C}_a\mathbb{E}_a[-(-\Delta)\psi_\infty - v'(\psi_\infty)]\mathrm{d}a + \dot{C}_a^{1/2}\mathrm{d}W_a, \quad a\geqslant 0$$

and by the stochastic differential equation

$$\mathrm{d}\psi_\infty(x) = -[\psi_\infty(x) + v'(\psi_\infty(x))]\mathrm{d}x + \mathrm{d}B_x, \qquad x\in\mathbb{R}.$$

Note that, in absence of the IR cutoff ρ, the random field ψ_∞ is not absolutely continuous wrt. X_∞ and therefore the path-integral formulation (13) looses it meaning while the WIDE formulation remains valid.

This example is particularly instructive since it shows that a local gradient Wilson–Itô dynamics gives rise to a Markov process in the space variable. It is natural to conjecture that this is a general feature of (a wide class of) Wilson–Itô diffusions.

D. Variational formulation

Equation (11) can be interpreted as the Euler–Lagrange equations for a stochastic control problem. To derive this problem we test the equation with an adapted test field $(v_a)_a$ and integrate both in scale and in the probability space the process

$$u_a := -\dot{Q}_a\mathbb{E}_a[\mathrm{DV}_\infty(\psi_\infty)]$$

to get a weak formulation of the equation

$$\mathbb{E}\left[\int_0^\infty \langle v_a, u_a\rangle \mathrm{d}a + \left\langle \int_0^\infty \dot{Q}_a v_a \mathrm{d}a, \mathrm{DV}_\infty(\psi_\infty)\right\rangle\right] = 0 \tag{14}$$

Now note that this equation is the Euler–Lagrange equation for the problem of minimizing the functional

$$\Psi(u) := \mathbb{E}\left[V_\infty(\psi_\infty^u) + \frac{1}{2}\int_0^\infty \langle u_a, u_a\rangle \mathrm{d}a\right] \tag{15}$$

over all adapted controls $(u_a)_{a\geqslant 0}$, where

$$\psi_a^u := \int_0^a \dot{Q}_b u_b \mathrm{d}b + \int_0^a \dot{Q}_b^{1/2}\,\mathrm{d}W_b,$$

is the controlled process.

E. Rigorous results

Variants of Eq. (11), Eq. (14) or of the variational problem in Eq. (15) have been used to construct several Euclidean quantum fields including the Φ_2^4 and Φ_3^4 models [6, 5, 7], the Høegh-Krohn model [6], the Sine–Gordon model [4] and certain subcritical Euclidean fermionic field theories [15], both in finite and infinite volume. This shows that our approach is intrinsically non-perturbative and can be made rigorous. We invite the reader to compare this situation with the non-trivial mathematical difficulties of the path-integral formalism without cutoffs. As an example, take the Φ^4_3 Euclidan quantum field on a torus, constructed in [5] via a slightly different version of the variational formulation (15). It is known, and proven in [7], that the Φ^4_3 measure is not absolutely continuous with respect to the Gaussian free field, so there cannot be a rigorous path-integral for it. Similarly, in [6] it is shown that some variants of Eq. (14) provide effective tools to study the infinite volume limit of the Φ^4_2 and of the $\exp(\Phi)_2$ Euclidean fields.

III. PROPERTIES OF WILSON–ITÔ DIFFUSIONS

A. Coherent germs

Let $\mathring{O}$ be a germ for an observable O. For $0 \leqslant b \leqslant a < \infty$, one has by Itô formula

$$\begin{aligned} O_b &= \mathring{O}_b(\varphi_b) + R^O_b = \mathbb{E}_b\,[\mathring{O}_a(\varphi_a) + R^O_a] \\ &= \mathring{O}_b(\varphi_b) + \mathbb{E}_b\left[\int_b^a \mathscr{L}_c \mathring{O}_c(\varphi_c)\,\mathrm{d}c\right] + \mathbb{E}_b[R^O_a] \end{aligned}$$

where $\mathscr{L}_c$ is the generator (3) of the Wilso-Itô diffusion. The assumption that R^O goes to 0 in a strong enough sense gives

$$R^O_b = \mathbb{E}_b\left[\int_b^\infty \mathscr{L}_c \mathring{O}_c(\varphi_c)\,\mathrm{d}c\right]. \tag{16}$$

Therefore R^O, and hence O itself, is completely determined by the germ $\mathring{O}$ provided the integral

$$\int_b^\infty \|\mathbb{E}_b[\mathscr{L}_c \mathring{O}_c(\varphi_c)]\|\mathrm{d}c$$

converges absolutely. A germ which has this property is called a ***coherent germ***; its associated observable it is determined from it. Note that Equation (16) for the remainder is equivalent to the BSDE

$$\mathrm{d}R^O_a = -\mathscr{L}_a \mathring{O}_a(\varphi_a)\,\mathrm{d}a - Z^O_a \mathrm{d}W_a, \quad R^O_\infty = 0 \tag{17}$$

for the pair of adapted processes $(R^O_a, Z^O_a)_a$.

The effective force itself is an observable. For simplicity we assume that the diffusivity is taken constant $\sigma_a = 1$, similar considerations otherwise apply to it. Assume that the force has a germ $\mathring{f}_a(\varphi_a)$ and a remainder R^f which then, due to (16), satisfies

$$\begin{aligned} R^f_b &= \int_b^\infty \mathbb{E}_b\,[R^f_c \dot{C}_c \mathrm{D}\,\mathring{f}_c(\varphi_c)]\,\mathrm{d}c \\ &\quad + \int_b^\infty \mathbb{E}_b[\mathring{\mathscr{L}}_c\,\mathring{f}_c(\varphi_c)]\,\mathrm{d}c \end{aligned} \tag{18}$$

where we introduced the operator

$$\mathring{\mathscr{L}}_c := \partial_c + \mathring{f}_c \dot{C}_c \mathrm{D} + \frac{1}{2}\mathrm{Tr}\,\dot{C}_c \mathrm{D}^2.$$

Similarly to the general case of eq. (17), the eq. (18) give rise to an BSDEs for the pair $(R^f_a, Z^f_a)_a$ which reads

$$\begin{aligned} \mathrm{d}R^f_a &= -\mathring{\mathscr{L}}_a \mathring{f}_a(\varphi_a)\,\mathrm{d}a - R^f_a \dot{C}_a \mathrm{D}\,\mathring{f}_a(\varphi_a)\,\mathrm{d}a \\ &\quad - Z^f_a \mathrm{d}W_a, \\ R^O_\infty &= 0 \end{aligned} \tag{19}$$

A basic requirement for Eq. (18) is that the source term in (18) and (19) is convergent in the UV, i.e.

$$\int_{a_0}^\infty \left\| \mathbb{E}_{a_0}[\mathring{\mathscr{L}}_c\,\mathring{f}_c(\varphi_c)] \right\| \mathrm{d}c < \infty, \tag{20}$$

for some scale a_0. If equation (18) has indeed a unique solution then $\mathring{f}$ characterizes uniquely the force field f.

Assume now we are given an observable O which has a germ $\mathring{O}$. Under proper assumptions on $\mathring{O}$, and for a choice of $\mathring{f}$ that ensure an appropriate strong decay of R^f_a as $a \to \infty$, the coherence relation

$$\int_b^\infty \|\mathbb{E}_b[\mathscr{L}_c \mathring{O}_c(\varphi_c)]\|\,\mathrm{d}c < \infty$$

is a consequence of the relation

$$\int_b^\infty \left\| \mathbb{E}_b[\mathring{\mathscr{L}}_c\,\mathring{O}_c(\varphi_c)] \right\| \mathrm{d}c < \infty. \tag{21}$$

A family $\mathring{O}$ that satisfies the estimate (21) is called an ***approximately coherent germ***. In those terms, Condition (20) states that $\mathring{f}$ is an approximately coherent germ for the force field f.

These considerations lead to the following strategy for *constructing* the law of a random field φ. Associate to each force germ $(\mathring{f}_c)_c$ and each scale parameter $a_0 > 0$ the solution to the coupled forward-backward stochastic differential equations

$$\begin{aligned} \mathrm{d}\varphi^{a_0}_b &= \dot{C}_b(\mathring{f}_b(\varphi^{a_0}_b) + R^{f,a_0}_b)\mathrm{d}b + \dot{C}_b^{1/2}\mathrm{d}W_b, \\ R^{f,a_0}_b &= \int_b^{a_0} \mathbb{E}_b\,[R^{f,a_0}_c \dot{C}_c \mathrm{D}\,\mathring{f}_c(\varphi^{a_0}_c)]\mathrm{d}c \\ &\quad + \int_b^{a_0} \mathbb{E}_b[(\mathring{\mathscr{L}}_c\,\mathring{f}_c)(\varphi^{a_0}_c)]\,\mathrm{d}c \end{aligned} \tag{22}$$

for $0 \leqslant b \leqslant a_0$ with mixed initial/final conditions

$$\varphi^{a_0}_0 = 0, \qquad R^{f,a_0}_{a_0} = 0. \tag{23}$$

Now we need to find $(\mathring{f}_b)_{b \geqslant 0}$ such that the coupled system (22) has a unique solution for all a_0 and a_0-uniform estimates that entail sufficient compactness to pass to the limit in (22). We should ask that $(\mathring{f}_a)_a$ contains a part that accounts for our elementary description of the physics involved. We say in that case that $(\mathring{f}_a)_a$ has the ***correct physical content***.

B. Link with factorization algebras

Definition 1 gives rise only to a vector space structure on the set of observables. Since the product of two martingales is, generally speaking, not a martingale, we do not have a natural way of multiplying observables. However for two observables O^1, O^2 that have some coherent germs $\mathring{O}^1, \mathring{O}^2$ with disjoint supports, say they are at distance strictly larger than $1/a_0$, we can set for $a \geqslant a_0$

$$\mathring{O}_a^{(12)} := \mathring{O}_a^{(1)} \mathring{O}_a^{(2)}.$$

Since $\operatorname{Tr} \dot{C}_a^{1/2} \sigma_a^2 \dot{C}_a^{1/2} \mathrm{D} \mathring{O}_a^{(1)} \mathrm{D} \mathring{O}_a^{(2)} = 0$, for $a \geqslant a_0$, because of the support condition, one has

$$\mathscr{L}_a \mathring{O}_a^{(12)} = (\mathscr{L}_a \mathring{O}_a^{(1)}) \mathring{O}_a^{(2)} + \mathring{O}_a^{(1)} (\mathscr{L}_a \mathring{O}_a^{(2)})$$

therefore provided the germs are coherent and we impose sufficient decay on $\mathscr{L}_a \mathring{O}_a^{(i)}$ and moderate growth conditions on $\mathring{O}_a^{(i)}$, it is possible to guarantee that $\mathscr{L}_a \mathring{O}_a^{(12)}$ is integrable and thefore $\mathring{O}_a^{(12)}$ is an approximately coherent germ which defines a unique observable $O^{(12)} =: O^{(1)} * O^{(2)}$. This gives on a subspace of observables a natural pre-factorization algebra structure, as defined in Costello–Gwilliam [12, 13].

IV. GAUGE THEORIES

In this section we assume that the field φ is a connection on a principal bundle over $\mathbb{R}^d$, with finite dimensional compact structure group $\mathfrak{G}$ and Lie algebra $\mathfrak{g}$. Recall that the space of connections is affine with underlying vector space the space of $\mathfrak{g}$-valued 1-forms. The gauge group consists of $\mathfrak{G}$-valued functions and acts on connections by $g \cdot \varphi = \mathrm{Ad}_g \varphi - (\mathrm{d} g)\, g^{-1}$ and forms by $g \cdot f = \mathrm{Ad}_g f$. We assume moreover that the force field is a function of the underlying field $f_a = f_a(\varphi_a)$ (taking values in the space of 1-forms) that is gauge covariant:

$$g \cdot f_a(\psi) = f_a\, (g \cdot \psi),$$

for every $g \in \mathfrak{G}$ and connection ψ. Given a connection φ, denote by $h^{xy}(\varphi)$ the φ-holonomy along the geodesic from x to y. Recall that

$$h^{xy}\, (g \cdot \varphi) = g(x)\, h^{xy}(\varphi)\, g(y)^{-1}. \tag{24}$$

Let $\chi_a(x, y)$ be a symmetric function of (x, y). We define a map $\dot{C}_a^{1/2}(\varphi)$ acting on 1-forms by

$$\big(\dot{C}_a^{1/2}(\varphi)\, \omega\big)(x) := \frac{1}{a^{1/2}} \int \chi_a(x, y) \mathrm{Ad}_{h^{xy}(\varphi)} \omega(y) \mathrm{d} y.$$

The operator $\dot{C}_a^{1/2}$ is symmetric and $\dot{C}_a(\varphi) := \big(\dot{C}_a^{1/2}(\varphi)\big)^2$ is symmetric and non-negative. By (24), it is also gauge covariant:

$$g \cdot (\dot{C}_a(\varphi)\, \omega) = \dot{C}_a\, (g \cdot \varphi)\, (g \cdot \omega).$$

The tangent space at a given connection φ of the gauge orbit in the connection space is spanned by the elements of the form $\mathrm{d}_\varphi h$, where d_φ is the φ-covariant derivative and h an arbitrary (smooth enough) function ($\mathfrak{g}$-valued 0-form), $\mathrm{d}_\varphi h = (\partial_i h + [\varphi_i, h]) \mathrm{d} x_i$. Two scale-dependent families of connections $(\varphi_a)_{a \geqslant 0}$ and $(\varphi'_a)_{a \geqslant 0}$ are said to be *gauge equivalent* if there exists a scale-dependent family of gauge transforms $(g_a)_{a \geqslant 0}$ such that $\varphi'_a = g_a \cdot \varphi_a$ for all $a \geqslant 0$. We note that if $(\varphi_a)_{a \geqslant 0}$ is a solution of the Wilson–Itô equation (2) with $\sigma_a \equiv 1$, and $(g_a)_{a \geqslant 0}$ is an adapted process that is differentiable in a and takes values in $C^1(M, \mathfrak{G})$ then $\varphi_a^g := g_a \cdot \varphi_a$ satisfies the equation

$$\begin{aligned} \mathrm{d} \varphi_a^g &= g_a \cdot (\mathrm{d} \varphi_a) - \mathrm{d}_{\varphi_t^a} (\dot{g}_t g_t^{-1} \mathrm{d} a) \\ &= (\dot{C}_a\, f_a(\varphi_a^g) - \mathrm{d}_{\varphi_a^g} (\dot{g}_a\, g_a^{-1}))\, \mathrm{d} a \\ &\quad + \dot{C}_a^{1/2} (\varphi_a^g) \mathrm{d} W_a. \end{aligned} \tag{25}$$

This is a particular case of a more general situation.

Proposition 3. *For any adapted process $(h_a)_{a \geqslant 0}$ with values in $C^1(M, \mathfrak{g})$ the solution $(\varphi_a^{(h)})_{a \geqslant 0}$ to the equation*

$$\mathrm{d} \varphi_a^{(h)} = \big((\dot{C}_a\, f_a)(\varphi_a^{(h)}) + \mathrm{d}_{\varphi_a^{(h)}} h_a\big)\, \mathrm{d} a + \dot{C}_a^{1/2} (\varphi_a^{(h)})\, \mathrm{d} W_a$$

is gauge equivalent to the solution of a Wilson–Itô equation

$$\mathrm{d} \varphi_a^{[h]} = (\dot{C}_a\, f_a)(\varphi_a^{[h]})\, \mathrm{d} a + \dot{C}_a^{1/2} (\varphi_a^{[h]}) \mathrm{d} W_a^h$$

driven by another Brownian motion W^h. So the law of the gauge orbit of $(\varphi_a^{(0)})_{a \geqslant 0}$ is well-defined.

The proof proceeds by solving the ordinary differential equation $\dot{g}_a\, g_a^{-1} = h_a$ and remarking that $\varphi_a^{[h]} := g_a \cdot \varphi_a^{(h)}$ solves the claimed equation with $\mathrm{d} W_a^h := g_a \cdot \mathrm{d} W_a$, which is a Brownian motion by Itô isometry.

As Parisi–Wu's stochastic quantization scheme [18] this approach allows to quantize gauge theories without path integrals and their associated ghosts or BRS symmetries. Adding terms of the form $\mathrm{d}_{\varphi_a} h_a$ in the drift allows to perform gauge-fixing in analogy to the Zwanziger–DeTurck–Sadun trick [14]. Moreover it points to a covariant formulation of the flow equation for the effective force f_a

$$\partial_a f_a + f_a \dot{C}_a \mathrm{D} f_a + \frac{1}{2} \operatorname{Tr} \dot{C}_a \mathrm{D}^2 f_a = 0 \tag{26}$$

with terminal condition f_∞. This is similar to Polchinski equation (8) with the difference that the cutoff propagator has been replaced by a covariant averaging operator.

V. CONCLUSIONS

We introduced Wilson-Itô fields, a novel class of random fields described by "local interactions". Our point of view is inscribed in the general idea of ***stochastic quantization*** [14] initiated by Parisi-Wu [18], which replaces the use of functional integrals with stochastic partial differential equations for the Euclidean fields. This topic has recently witnessed a growth of interest from the mathematical community since SPDEs can be used to study rigorously Euclidean fields: see e.g. [9, 10] for non-Abelian gauge theories and [3, 2] for Φ^4_3 on Riemmanian manifolds and also [16] for a partial verification of the Osterwalder–Schrader axioms in Φ^4_3. However, at variance with more classical stochastic quantization methods, our approach does not require to introduce fictious additional parameters and depends on a physically relevant scale of observation, motivated by the Kadanoff–Wilson picture of renormalization and by recent mathematical frameworks for the definition of Euclidean QFT via effective theories, see e.g. [11]. Wilson-Itô fields have an intrinsic natural definition, independent of cutoff procedures and it seems interesting to pursue further their study, e.g. determine conditions under which they possess spatial Markov properties or satisfy reflection postivity, uniqueness/non-uniqueness of solutions, numerical simulations, description of scale-invariant and conformal invariant fields, etc…

REFERENCES

[1] S. Albeverio, L. Borasi and F. C. De Vecchi, and M. Gubinelli. Grassmannian stochastic analysis and the stochastic quantization of Euclidean fermions. *Probability Theory and Related Fields*, 183(3):909–995, 2022.

[2] I. Bailleul. Uniqueness of the Φ^4_3 measures on closed Riemannian 3-manifolds, arXiv:2306.07616, 2023.

[3] I. Bailleul, N. V. Dang, L. Ferdinand, and T. D. Tô. Φ^4_3 measures on compact Riemannian 3-manifolds, arXiv:2304.10185, 2023.

[4] N. Barashkov. A stochastic control approach to Sine Gordon EQFT. *arXiv:2203.06626 [math-ph]*, 2022.

[5] Ni. Barashkov and M. Gubinelli. A variational method for ϕ^4_3. *Duke Mathematical Journal*, 169(17):3339–3415, 2020.

[6] N. Barashkov and M. Gubinelli. On the variational method for euclidean quantum fields in infinite volume. *ArXiv*, arXiv:2112.05562, 2021.

[7] N. Barashkov and M. Gubinelli. The ϕ^4_3 measure via Girsanov's theorem. *Electronic Journal of Probability*, 26:1–29, 2021.

[8] R. Bauerschmidt and Th. Bodineau. Log-Sobolev Inequality for the Continuum Sine-Gordon Model. *Communications on Pure and Applied Mathematics*, 74(10):2064–2113, 2021.

[9] A. Chandra, I. Chevyrev, M. Hairer and H. Shen. Langevin dynamic for the 2D Yang-Mills measure. arXiv:2006.04987, 2020.

[10] A. Chandra, I. Chevyrev, M. Hairer and H. Shen. Stochastic quantisation of Yang-Mills-Higgs in 3D. arXiv:2201.03487, 2022.

[11] K. Costello. *Renormalization and Effective Field Theory*, volume 170 of *Math. Surv. Monogr.* American Mathematical Society (AMS), Providence, RI, 2011.

[12] K. Costello and O. Gwilliam. *Factorization Algebras in Quantum Field Theory. Vol. 1*, volume 31 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2017.

[13] K. Costello and O. Gwilliam. *Factorization Algebras in Quantum Field Theory. Vol. 2*, volume 41 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2021.

[14] P. H. Damgaard and H. Hüffel. *Stochastic Quantization*. World Scientific, 1988.

[15] F. C. De Vecchi, L. Fresta, and M. Gubinelli. A stochastic analysis of subcritical Euclidean fermionic field theories. *arXiv:2210.15047 [math-ph]*, 2022.

[16] M. Gubinelli and M. Hofmanová. A PDE construction of the Euclidean Φ^4_3 quantum field theory. *Communications in Mathematical Physics*, 384(1):1–75, 2021.

[17] J. Ma and J. Yong. *Forward-Backward Stochastic Differential Equations and Their Applications*, volume 1702 of *Lecture Notes in Mathematics*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2007.

[18] G. Parisi and Y.-S. Wu. Perturbation theory without gauge fixing. *Scientia Sinica. Zhongguo Kexue*, 24(4):483–496, 1981.

[19] J. Polchinski. Renormalization and effective lagrangians. *Nuclear Physics B*, 231(2):269–295, jan 1984.

[20] L. C. G. Rogers and D. Williams. *Diffusions, Markov processes, and martingales. Vol. 2*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000. Itô calculus, Reprint of the second (1994) edition.

[21] D.W. Stroock. *Markov Processes from K. Itô's Perspective*, volume 155 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2003.