Regularity structures for quasilinear singular SPDEs

Joint work with M. Hoshino & S. Kusuoka

► **Local in time well-posedness** – *Take* $u_0 \in C^{0^+}(\mathbf{T})$. *One can construct a regularity structure, containing infinitely many trees of any fixed degree, within which one can make sense of the equation*

$$\mathcal{L}^u u := \partial_t u - a(u)\partial_x^2 u = f(u)\xi + g(u)(\partial_x u)^2$$

and prove local in time well-posedness, in the full subcritical regime.

▶ **Local in time well-posedness** – *Take* $u_0 \in C^{0^+}(\mathbf{T})$*. One can construct a regularity structure, containing infinitely many trees of any fixed degree, within which one can make sense of the equation*

$$\mathcal{L}^u u := \partial_t u - a(u)\partial_x^2 u = f(u)\xi + g(u)(\partial_x u)^2$$

and prove local in time well-posedness, in the full subcritical regime.

Set $v := e^{\eta\partial_x^2}(u_0)$, with η small.

$$\mathcal{L}^v u = \partial_t u - a(v)\partial_x^2 u = f(u)\xi + g(u)(\partial u)^2 + \big(a(u) - a(v)\big)\partial_x^2 u \qquad (1)$$

Assumption 1 – *One can define the BPHZ model of the non-translation invariant (gKPZ) equation*

$$\partial_t z - a(v)\partial_x^2 z = f(z)\xi + g(z)(\partial z)^2.$$

▶ **Local in time well-posedness –** *Take $u_0 \in C^{0^+}(\mathbf{T})$. One can construct a regularity structure, containing infinitely many trees of any fixed degree, within which one can make sense of the equation*

$$\mathcal{L}^u u := \partial_t u - a(u)\partial_x^2 u = f(u)\xi + g(u)(\partial_x u)^2$$

and prove local in time well-posedness, in the full subcritical regime.

Set $v := e^{\eta\partial_x^2}(u_0)$, with η small.

$$\mathcal{L}^v u = \partial_t u - a(v)\partial_x^2 u = f(u)\xi + g(u)(\partial u)^2 + \big(a(u) - a(v)\big)\partial_x^2 u \qquad (1)$$

Assumption 1 – *One can define the BPHZ model of the non-translation invariant (gKPZ) equation*

$$\partial_t z - a(v)\partial_x^2 z = f(z)\xi + g(z)(\partial z)^2.$$

▶ **Renormalized equation –** *The solution u^ϵ to*

$$\mathcal{L}^{u^\epsilon} u^\epsilon = f(u^\epsilon)\xi^\epsilon + g(u^\epsilon)(\partial_x u^\epsilon)^2 - \sum_{\tau^{\mathbf{p}}} \frac{\ell^{a(v),\epsilon}(\tau^{\mathbf{p}},\cdot)}{S(\tau^{\mathbf{p}})} \big(a(u^\epsilon) - a(v)\big)^{|\mathbf{p}|} \chi^a(\tau)(u^\epsilon)\, \mathcal{F}(\tau)(u^\epsilon)$$

with initial condition $u_0 \in C^{0^+}(\mathbf{T})$ converges (in law) on a random time interval, as $\epsilon \downarrow 0$. Here $\tau^{\mathbf{p}}$ runs over an infinite number of rooted decorated trees, $\ell^{a(v),\epsilon}(\tau^{\mathbf{p}},\cdot)$ depends on v, $\chi^a(\tau)$ explicit polynomial of a and its derivatives.

For $\lambda > 0$ define the constant $I^{\lambda,\epsilon}_{\tau^{\mathbf{p}}}$ from the operator $(\partial_t - \lambda\partial_x^2)^{-1}$ and ξ^ϵ in the same way as we define the function $\ell^{a(v),\epsilon}(\tau^{\mathbf{p}}, \cdot)$ using the non-translation invariant operator $(\partial_t - a(v)\partial_x^2)^{-1}$ and ξ^ϵ.

Assumption 2 – *One has the ϵ-uniform*

$$\ell^{a(v),\epsilon}(\tau^{\mathbf{p}}, x) = I^{a(v(x)),\epsilon}(\tau^{\mathbf{p}}) + O_\tau\left(m^{|\mathbf{p}|}\right),$$

for some $m > 0$.

The assumption holds e.g. for the 2-dimensional quasilinear (gPAM) equation or the quasilinear (gKPZ) equation in the spacetime white noise regime. It should hold in much greater generality.

For $\lambda > 0$ define the constant $l^{\lambda,\epsilon}_{\tau^{\mathbf{p}}}$ from the operator $(\partial_t - \lambda\partial_x^2)^{-1}$ and ξ^ϵ in the same way as we define the function $\ell^{a(v),\epsilon}(\tau^{\mathbf{p}},\cdot)$ using the non-translation invariant operator $(\partial_t - a(v)\partial_x^2)^{-1}$ and ξ^ϵ.

Assumption 2 – *One has the ϵ-uniform*

$$\ell^{a(v),\epsilon}(\tau^{\mathbf{p}},x) = l^{a(v(x)),\epsilon}(\tau^{\mathbf{p}}) + O_\tau\big(m^{|\mathbf{p}|}\big),$$

for some $m > 0$.

The assumption holds e.g. for the 2-dimensional quasilinear (gPAM) equation or the quasilinear (gKPZ) equation in the spacetime white noise regime. It should hold in much greater generality.

- **Counterterm** – *Under the above assumption the counterterm*

$$\sum_{\tau^{\mathbf{p}}} \frac{\ell^{a(v),\epsilon}(\tau^{\mathbf{p}},\cdot)}{S(\tau^{\mathbf{p}})}\big(a(u^\epsilon)-a(v)\big)^{|\mathbf{p}|}\chi^a(\tau)(u^\epsilon)\,\mathcal{F}(\tau)(u^\epsilon)$$

$$= \sum_\tau \frac{l^{a(u^\epsilon(\cdot)),\epsilon}(\tau)}{S(\tau)}\chi^a(\tau)(u^\epsilon)\,\mathcal{F}(\tau)(u^\epsilon) + O(1)$$

where τ runs over a finite set of decorated trees and $O(1)$ is ϵ-uniform.

For $\lambda > 0$ define the constant $l^{\lambda,\epsilon}_{\tau^{\mathbf{p}}}$ from the operator $(\partial_t - \lambda\partial_x^2)^{-1}$ and ξ^ϵ in the same way as we define the function $\ell^{a(v),\epsilon}(\tau^{\mathbf{p}}, \cdot)$ using the non-translation invariant operator $(\partial_t - a(v)\partial_x^2)^{-1}$ and ξ^ϵ.

Assumption 2 – *One has the ϵ-uniform*

$$\ell^{a(v),\epsilon}(\tau^{\mathbf{p}}, x) = l^{a(v(x)),\epsilon}(\tau^{\mathbf{p}}) + O_\tau\left(m^{|\mathbf{p}|}\right),$$

for some $m > 0$.

The assumption holds e.g. for the 2-dimensional quasilinear (gPAM) equation or the quasilinear (gKPZ) equation in the spacetime white noise regime. It should hold in much greater generality.

▶ **Counterterm** – *Under the above assumption the counterterm*

$$\sum_{\tau^{\mathbf{p}}} \frac{\ell^{a(v),\epsilon}(\tau^{\mathbf{p}}, \cdot)}{S(\tau^{\mathbf{p}})} \big(a(u^\epsilon) - a(v)\big)^{|\mathbf{p}|} \chi^a(\tau)(u^\epsilon)\,\mathcal{F}(\tau)(u^\epsilon)$$

$$= \sum_{\tau} \frac{l^{a(u^\epsilon(\cdot)),\epsilon}(\tau)}{S(\tau)} \chi^a(\tau)(u^\epsilon)\,\mathcal{F}(\tau)(u^\epsilon) + O(1)$$

where τ runs over a finite set of decorated trees and $O(1)$ is ϵ-uniform.

If further ξ stationary Gaussian, and white in time if time-dependent and additive noise, then counterterm of the form

$$l^{a(u^\epsilon(x)),\epsilon}(\tau) = \frac{\ell^\epsilon(\tau)}{a(u^\epsilon(x))^{\theta_\tau}}.$$

A regularity structure

- Index set $\{\tau^{\mathbf{p}}\}$ where τ trees of the RS of (gKPZ) and $\mathbf{p}$ integer decorations on each edge of τ.
- The $\mathbf{p}$ decoration has no effect of the algebraic structure Δ, Δ^+, homogeneity.
- Fix parameter m. For fixed homogeneity β consider *series*

$$\Big\| \sum_{|\tau^{\mathbf{p}}|=\beta} c_{\tau^{\mathbf{p}}} \tau^{\mathbf{p}} \Big\|^2 := \sum_{|\tau^{\mathbf{p}}|=\beta} |c_{\tau^{\mathbf{p}}}|^2 m^{2|\mathbf{p}|}.$$

A regularity structure

- Index set $\{\tau^{\mathbf{p}}\}$ where τ trees of the RS of (gKPZ) and $\mathbf{p}$ integer decorations on each edge of τ.
- The $\mathbf{p}$ decoration has no effect of the algebraic structure Δ, Δ^+, homogeneity.
- Fix parameter m. For fixed homogeneity β consider *series*

$$\Big\| \sum_{|\tau^{\mathbf{p}}|=\beta} c_{\tau^{\mathbf{p}}} \tau^{\mathbf{p}} \Big\|^2 := \sum_{|\tau^{\mathbf{p}}|=\beta} |c_{\tau^{\mathbf{p}}}|^2 m^{2|\mathbf{p}|}.$$

Models

$$\big| g_{z'z}(\tau^{\mathbf{p}}) \big| \lesssim m^{|\mathbf{p}|} \|z'-z\|_{\mathfrak{s}}^{|\tau|}, \qquad \big| \mathcal{Q}_\theta \big(\Pi_z^g \sigma^{\mathbf{p}} \big)(z) \big| \lesssim m^{|\mathbf{p}|} \theta^{|\sigma|/4},$$

for a heat type operator $\mathcal{Q}_\theta$.

Admissibility with respect to $\mathrm{K}^{a(v)} \simeq \big(\partial_t - a(v)\partial_x^2 \big)^{-1}$

$$\Pi(\mathcal{I}_{\mathbf{n}}^p \tau) = \partial^{\mathbf{n}} \Big(\mathrm{K}^{a(v)} \circ (\partial_x^2 \mathrm{K}^{a(v)})^{\circ p} \Big) (\Pi\tau)$$

Modelled distributions – As usual, with a reconstruction map $\mathcal{R}^M$ and multilevel Schauder estimates.

- **Local in time well-posedness**
  - Series instead of finite sums does not cause any problem: In Picard iteration

$$u \simeq K^{a(v),M}\Big(F(u)\zeta + G(u)(Du)^2 + \big(A(u) - A(v)\big)D^2 u\Big), \qquad (2)$$

each time you go inside D^2 you get an *a priori small factor* $A(u) - A(v)$ $\rightsquigarrow$ you produce converging series.

- **Local in time well-posedness**
  - Series instead of finite sums does not cause any problem: In Picard iteration
$$u \simeq K^{a(v),M}\Big(F(u)\zeta + G(u)(Du)^2 + \big(A(u) - A(v)\big)D^2 u\Big), \tag{2}$$
each time you go inside D^2 you get an *a priori small factor* $A(u) - A(v)$ $\rightsquigarrow$ you produce converging series.
  - As in semilinear setting, small factor for contraction comes from the gain in time explosion in multilevel Schauder estimates for $K^{a(v),M}$.

▶ **Renormalized equation** – An automated approach in a (second order) semilinear setting:

- The coefficients u_τ of the modelled distribution $\boldsymbol{u} = \sum u_\tau \tau$ solution of equation satisfy a *coherence property*

$$u_\tau = \mathcal{F}(\tau)\big(u_{\mathbf{1}}, u_{X^{(0,1)}}\big)$$

  for some explicit functions $\mathcal{F}(\tau)$ defined inductively.

- The function $\tau \mapsto \mathcal{F}(\tau)$ satisfies a morphism property.

▶ **Renormalized equation** – An automated approach in a (second order) semilinear setting:

- The coefficients u_τ of the modelled distribution $\boldsymbol{u} = \sum u_\tau \tau$ solution of equation satisfy a *coherence property*

$$u_\tau = \mathcal{F}(\tau)(u_{\mathbf{1}}, u_{X^{(0,1)}})$$

for some explicit functions $\mathcal{F}(\tau)$ defined inductively.

- The function $\tau \mapsto \mathcal{F}(\tau)$ satisfies a morphism property.

Bruned, Chandra, Chevyrev & Hairer (Renormalising SPDEs in regularity structures) use *multi-pre-Lie structures*. We use a *different algebraic structure* introduced by **Bruned & Manchon** (Algebraic deformations for (S)PDEs), used in **Bailleul & Bruned** (Locality for singular stochastic PDEs), well adapted to an inductive construction of admissible models based on the use of strong *preparation maps*.

▶ **Renormalized equation** – An automated approach in a (second order) semilinear setting:

- The coefficients u_τ of the modelled distribution $u = \sum u_\tau \tau$ solution of equation satisfy a *coherence property*

$$u_\tau = \mathcal{F}(\tau)(u_{\mathbf{1}}, u_{X^{(0,1)}})$$

for some explicit functions $\mathcal{F}(\tau)$ defined inductively.

- The function $\tau \mapsto \mathcal{F}(\tau)$ satisfies a morphism property.

Bruned, Chandra, Chevyrev & Hairer (Renormalising SPDEs in regularity structures) use *multi-pre-Lie structures*. We use a *different algebraic structure* introduced by **Bruned & Manchon** (Algebraic deformations for (S)PDEs), used in **Bailleul & Bruned** (Locality for singular stochastic PDEs), well adapted to an inductive construction of admissible models based on the use of strong *preparation maps*.

- The star map

$$(\sigma, \tau) \in T_\bullet \times T \mapsto \sigma \star \tau$$

is a generalisation of the grafting map, with $T_\bullet = \{X^k \prod \mathcal{I}_{\mathbf{n}_i}(\tau_i)\}$.

- The morphism property reads here

$$\mathcal{F}^a\left(\left\{X^{\mathbf{k}} \prod_{i=1}^{j} \mathcal{I}_{\mathbf{n}_i}(\sigma_i)\right\} \star \tau\right) = \left\{\partial^{\mathbf{k}} D_{\mathbf{n}_1} \cdots D_{\mathbf{n}_j} \mathcal{F}^a(\tau)\right\} \prod_{i=1}^{j} \mathcal{F}^a(\sigma_i).$$

From **Bailleul & Bruned**, for a model M built from a continuous noise ξ and a strong preparation map R, and u^M the solution to the RS equation (2) with model M, the function

$$u = \mathcal{R}^M(u^M)$$

is a solution to

$$\partial u - a(u)\partial_x^2 u = f(u)\xi + g(u)(\partial_x u)^2 + \sum_m \mathcal{F}^a\big((R - \mathrm{id})^* \zeta_m\big)(u, \partial_x u, v, \partial_x v).$$

From **Bailleul & Bruned**, for a model M built from a continuous noise ξ and a strong preparation map R, and u^M the solution to the RS equation (2) with model M, the function

$$u = \mathcal{R}^M(u^M)$$

is a solution to

$$\partial u - a(u)\partial_x^2 u = f(u)\xi + g(u)(\partial_x u)^2 + \sum_m \mathcal{F}^a\big((R - \mathrm{id})^* \zeta_m\big)(u, \partial_x u, v, \partial_x v).$$

For BPHZ-type strong preparation maps

$$\mathcal{F}^a(\cdots)(u, \partial_x u, v, \partial_x v) = \mathcal{F}^a(\cdots)(u, \partial_x u, v).$$

From **Bailleul & Bruned**, for a model M built from a continuous noise ξ and a strong preparation map R, and u^M the solution to the RS equation (2) with model M, the function

$$u = \mathcal{R}^M(u^M)$$

is a solution to

$$\partial u - a(u)\partial_x^2 u = f(u)\xi + g(u)(\partial_x u)^2 + \sum_m \mathcal{F}^a\big((R - \mathrm{id})^* \zeta_m\big)(u, \partial_x u, v, \partial_x v).$$

For BPHZ-type strong preparation maps

$$\mathcal{F}^a(\cdots)(u, \partial_x u, v, \partial_x v) = \mathcal{F}^a(\cdots)(u, \partial_x u, v).$$

The solution of the coherence relation has here a particular structure

$$\mathcal{F}^a\left(\tau^{\mathbf{p}}\right)\left(\mathrm{c}_0, \mathrm{c}_{(0,1)}, \mathrm{c}_0'\right) = \chi^a(\tau)(\mathrm{c}_0)\left(a(\mathrm{c}_0) - a(\mathrm{c}_0')\right)^{|\mathbf{p}|}\mathcal{F}(\tau)(\mathrm{c}_0)$$

for some functions $\chi^a(\tau)$ and $\mathcal{F}(\tau)$ defined inductively, with $\chi^a(\tau)$ a polynomial function of a and its derivatives.

▶ **Counterterm –** Still with continuous noise ξ, for $\lambda > 0$ denote by $\tau \mapsto l^{\lambda}(\tau^{\mathbf{p}})$ the BPHZ character built from the operator $\partial_t - \lambda\partial_x^2$. **Assumption 2** trades BPHZ character built from $\partial_t - a(v)\partial_x^2$ for $l^{a(v)}$.

▶ **Counterterm** – Still with continuous noise ξ, for $\lambda > 0$ denote by $\tau \mapsto l^\lambda(\tau^{\mathbf{p}})$ the BPHZ character built from the operator $\partial_t - \lambda\partial_x^2$. **Assumption 2** trades BPHZ character built from $\partial_t - a(v)\partial_x^2$ for $l^{a(v)}$.

○ Lemma – *For any τ with null* **p***-decoration the function*

$$\lambda \mapsto l^\lambda(\tau)$$

is analytic in any given bounded interval $(a, b) \subset (0, +\infty)$ with $a > 0$ and

$$\frac{1}{n!}\,\partial_\lambda^n l^\lambda(\tau) = \sum_{\mathbf{p}\in\mathbb{N}^{E_\tau},\, |\mathbf{p}|=n} l^\lambda(\tau^{\mathbf{p}}).$$

▶ **Counterterm** – Still with continuous noise ξ, for $\lambda > 0$ denote by $\tau \mapsto l^\lambda(\tau^{\mathbf{p}})$ the BPHZ character built from the operator $\partial_t - \lambda\partial_x^2$. **Assumption 2** trades BPHZ character built from $\partial_t - a(v)\partial_x^2$ for $l^{a(v)}$.

○ Lemma – *For any τ with null $\mathbf{p}$-decoration the function*

$$\lambda \mapsto l^\lambda(\tau)$$

is analytic in any given bounded interval $(a,b) \subset (0,+\infty)$ with $a > 0$ and

$$\frac{1}{n!}\,\partial_\lambda^n l^\lambda(\tau) = \sum_{\mathbf{p}\in\mathbb{N}^{E_\tau},\,|\mathbf{p}|=n} l^\lambda(\tau^{\mathbf{p}}).$$

Then one has

$$\begin{aligned}
\sum_{\tau^{\mathbf{p}}} \frac{l^{a(v(\cdot))}(\tau^{\mathbf{p}})}{S(\tau^{\mathbf{p}})}\,\chi^a(\tau)(u)\,\big(a(u)-a(v)\big)^{|\mathbf{p}|}\,\mathcal{F}(\tau)(u) \\
&= \sum_\tau \frac{\chi^a(\tau)(u)\,\mathcal{F}(\tau)(u)}{S(\tau)} \sum_{\mathbf{p}\in\mathbb{N}^{E_\tau}} l^{a(v(\cdot))}(\tau^{\mathbf{p}})\,\big(a(u)-a(v)\big)^{|\mathbf{p}|} \\
&= \sum_\tau \frac{\chi^a(\tau)(u)\,\mathcal{F}(\tau)(u)}{S(\tau)} \sum_{n=0}^{\infty}\big(a(u)-a(v)\big)^n \sum_{|\mathbf{p}|=n} l^{a(v(\cdot))}(\tau^{\mathbf{p}}) \\
&\overset{\text{Lemma}}{=} \sum_\tau \frac{\chi^a(\tau)(u)\,\mathcal{F}(\tau)(u)}{S(\tau)}\, l^{a(u(\cdot))}(\tau).
\end{aligned}$$