Random models

Joint work with M. Hoshino

▶ **Theorem** (Convergence for general models) – *Assume the noise symbol is the only element of the regularity structure with degree less than or equal to* $-|\mathfrak{s}|/2$. *Assume that the law of the random noise has a spectral gap. Last, suppose we have some preparation maps* R_n *for which the quantities*

$$\mathbb{E}\big[\mathcal{Q}_1(0, \Pi_0^n \tau)\big]$$

converge for all the symbols τ *with non-positive degree. Then the renormalized models associated with these preparation maps converge in* $L^q(\mathbb{P})$ *for any* $1 \leq q < \infty$.

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($\mathcal{Q}_t$ convolution with a heat kernel.) Similar result proved first by **Hairer & Steele** for the **BPHZ model** $\mathsf{M}(\mathbb{P})$. Work below with Ω a certain Besov space.

▶ **Theorem** (Continuity in law for BPHZ) – *Let* $(\mathbb{P}_j)_{j\in\mathbb{N}}$ *be a sequence of probability measures on* Ω *that converges weakly to a limit probability measure* $\mathbb{P}$. *If all the* $\mathbb{P}_j$ *satisfy a spectral gap inequality with the same constant then the law of* $\mathsf{M}(\mathbb{P}_j)$ *converges weakly to the law of* $\mathsf{M}(\mathbb{P})$.

Result of a similar flavour proved first by **Tempelmayr** in a multi-index setting.

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1. a notion of (here parameter-dependent) regularity-integrability structure, the reconstruction and multilevel Schauder theorems for an associated notion of modelled distribution,
2. an inductive construction of regularized renormalised models based on the use of preparation maps,
3. a comparison result for models with different parameters.

Regularity-integrability structures (A, T, G)

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- $A \subset \mathbb{R} \times [1, \infty]$ *such that*

$$\{(\gamma, r) \in A\,;\, \gamma < \beta,\ r \geq q\}$$

is finite for every $(\beta, q) \in \mathbb{R} \times [1, \infty]$, *where*

$$(\gamma, r) < (\beta, q) \quad \overset{\text{def}}{\Longleftrightarrow} \quad \gamma < \beta \text{ and } r \geq q.$$

- $T = \bigoplus_{\mathbf{a} \in A} T_{\mathbf{a}}$
- G *a group of continuous linear operators on* T *such that*

$$(\Gamma - \mathrm{id})\, T_{\mathbf{a}} \subset \bigoplus_{\mathbf{a}' \in A,\, \mathbf{a}' < \mathbf{a}} T_{\mathbf{a}'}.$$

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Models on regularity-integrability structures **(RIS)** – For $\mathbf{a} \in A$

$$\|\Pi\|_{\mathbf{a}} = \max\Big\{ \sup_{0<t\leq 1} t^{-\alpha/\ell} \big\| \mathcal{Q}_t(x, \Pi_x \tau) \big\|_{L^p_x}\,;\, \tau \in T_{(\alpha,p)}, (\alpha, p) < \mathbf{a} \Big\},$$

$$\|\Gamma\|_{\mathbf{a}} = \max_{\substack{\tau \in T_{(\alpha,p)} \\ (\beta,q)<(\alpha,p)<\mathbf{a}}} \sup_{h \in \mathbb{R}^d \setminus \{0\}} \|h\|_{\mathfrak{s}}^{\beta-\alpha} \big\| \{\Gamma_{(x+h)x}\tau\}_{(\beta,q)} \big\|_{L^{p:q}_x}.$$

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Modelled distributions on **(RIS)** – For $\mathbf{c} = (\gamma, r)$ and $f \in \mathcal{D}^{\mathbf{c}}$

$$\|f\|_{\mathbf{c}} := \max_{(\alpha,p) < \mathbf{c}} \sup_{h \in \mathbb{R}^d \setminus \{0\}} \frac{\left\| \left\{ f(x+h) - \Gamma_{(x+h)x} f(x) \right\}_{(\alpha,p)} \right\|_{L_x^{r:p}}}{\|h\|_{\mathfrak{s}}^{\gamma - \alpha}}$$

There are versions of the reconstruction and multilevel Schauder theorems.

Regularity-integrability structures

- **An example** – For $\alpha_0 < -|\mathfrak{s}|/2 - \kappa$ think of

$$H^{-\kappa} \hookrightarrow B_{p,\infty}^{\alpha_0 + \frac{|\mathfrak{s}|}{p}} \hookrightarrow C^{\alpha_0}.$$

For a parameter $p \in [1, \infty]$ set

$$|\bigcirc|_p = \alpha_0, \qquad |\odot|_p = \alpha_0 + \frac{|\mathfrak{s}|}{p}$$

$$T = \bigoplus_{\alpha} T_{(\alpha,\infty)} \oplus \bigoplus_{\beta} T_{(\beta,p)} = \{\text{no } \odot\} \oplus \{\text{exactly one } \odot\}$$

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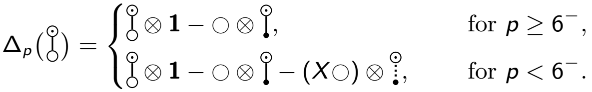
$P_p^+ : T \to T^+$ projection on forests of trees with positive $|\cdot|_p$ degree. The expansion map

$$\Delta_p = (\mathrm{id} \otimes P_p^+)\Delta$$

e.g., for one-dimensional multiplicative stochastic heat equation



Renormalised models

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▶ **Definition of preparation maps** – *Linear maps* R *that fix the polynomials, noises* $\bigcirc, \odot$ *and planted trees, with*

$$(R \otimes id)\Delta_2 = \Delta_2 R, \qquad RD = DR,$$

and

$$R\tau = \tau + \sum_i \lambda_i \tau_i,$$

with $|\tau_i|_r > |\tau|_r$ *for* $r \in \{2, \infty\}$ *and* $|\tau_i|_{\bigcirc} < |\tau|_{\bigcirc}$.

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These maps somehow renormalise only what happens at the root of a tree. We propagate this definition by defining inductively $M^{\times R}$ as a *multiplicative* map such that

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Renormalised interpretation map

$$\Pi^R = \Pi M^{\times R} R$$

and a unique associated admissible model on T.

Renormalised models

Our **(RIS)** depends on a *parameter* p. The recentered quantities depend on p, write $\Pi_x^{R;p}\tau$ or $\Pi_x^{\xi,h,R;p}\tau$, where $\Pi(\bigcirc) = \xi$ and $\Pi(\odot) = h$.

- **Derivative lemma** – *For a 'smooth' noise ξ and for τ with no derivative noise one has*

$$d_\xi(\Pi_x^{\xi,R;\infty}\tau)(h) := \frac{d}{dt}(\Pi_x^{\xi+th,R;\infty}\tau)\Big|_{t=0} = \Pi_x^{\xi,h,R;\infty}(D\tau).$$

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Extension to $p \in (2,\infty)$ of a statement with a similar flavour in **Bruned & Nadeem**. *Analogy with* **Linares, Otto, Tempelmayr & Tsatsoulis**: $\Pi_x^{R,\infty}(D\beta) \simeq \delta\Pi_{x\beta}$, $\Pi_x^{R,2}(D\beta)(y) \simeq \left(\delta\Pi_{x\beta} - (d\Gamma^*_{xy})\Pi_{y\beta}\right)(y)$ – *comes with good estimates*, $\left((\Pi_x^{R;\infty} \otimes \mathsf{k}_x^{R;p\infty})\Delta_2\beta\right)(y) \simeq \left((d\Gamma^*_{xy})\Pi_{y\beta}\right)(y)$.

The inductive mechanics

- A (pre)order for the induction

$$\sigma \preceq \tau \quad \overset{\text{def}}{\Longleftrightarrow} \quad \left(|\sigma|_{\bigcirc}, |E_\sigma|, |\sigma|_\infty\right) \leq \left(|\tau|_{\bigcirc}, |E_\tau|, |\tau|_\infty\right)$$

with $\leq$ the lexicographical order. Write
$\mathbf{B} \setminus \{X^k\}_{k \in \mathbb{N}^d} = \left\{\tau_1 \preceq \tau_2 \preceq \cdots \preceq \tau_N\right\}$.

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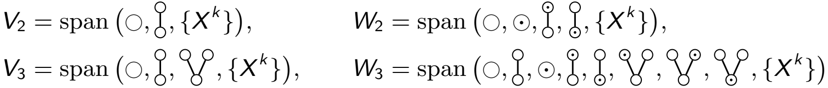
- Example

$$V_1 = \mathrm{span}\left(\bigcirc, \{X^k\}\right), \qquad W_1 = \mathrm{span}\left(\odot, \{X^k\}\right),$$



Associated regularity structures $\mathscr{V}_{i,p}$ and $\mathscr{W}_{i,p}$ and spaces of modelled distributions $\mathbf{M}(\mathscr{V}_{i,p})$ and $\mathbf{M}(\mathscr{W}_{i,p})$.

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- *H-spectral gap assumption* – For $H = H^{-\kappa}$ and $\Omega = C^{\alpha_0}$
$$\mathbb{E}\big[(F - \mathbb{E}[F])^2\big] \lesssim \mathbb{E}\Big[\sup_{\|h\|_H \le 1} |dF(h)|^2\Big].$$

Set
$$\xi_n(\omega) := \varrho_n * \omega \in \Omega, \quad h_n := \varrho_n * h, \quad (h \in H).$$

Given any preparation map R_n write $M^{n;p} = \mathsf{M}^{\xi_n, h_n, R_n; p}$ for the random admissible model associated with ξ_n, h_n, R_n or its restriction to $\mathscr{V}_{i,p}$ or $\mathscr{W}_{i,p}$.

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for any $q\in[1,\infty)$, *and the restrictions of the models* $\mathsf{M}^{n;p}$ *on* $\mathscr{W}_{i,p}$ *converge in* $L^q\big(\Omega,\mathbb{P};\mathbf{M}(\mathscr{W}_{i,p})\big)$, *for any* $1\leq q<\infty$ *and any* $h\in H$ *with* $\|h\|_H \leq 1$.

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Similar definition of $\mathbf{cv}(\mathscr{V}, i)$ (Does not depend on p.). Write $\{\mathbf{cv}(\cdot)\}_p$ to mean $\mathbf{cv}(\cdot, p)$ for all $1 \leq p \leq \infty$. Induction hypothesis: $\{\mathbf{cv}(\mathscr{W}, i, p)\}_p$.

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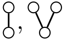
- **Example** (○, ) and above spaces V_1, V_2, V_3. Initial case ⊙.
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Step 1 is *probabilistic*, uses spectral gap assumption, the centering condition and reconstruction theorem.

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Step 3 **(a)** $\Big(\mathbf{cv}(\mathscr{W}, i, 2) + \mathbf{cv}(\mathscr{V}, i)\Big) \Longrightarrow \Pi\text{-part of } \mathbf{cv}(\mathscr{W}, i+1, 2)$ – Builds $\Pi_x^{R,2}(D\sigma)$

(b) $\Big(\Pi\text{-part of } \mathbf{cv}(\mathscr{W}, i+1, 2) + \big\{\mathbf{cv}(\mathscr{W}, i, p)\big\}_p\Big)$
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Step 1 is *probabilistic*, uses spectral gap assumption, the centering condition and reconstruction theorem.
Steps 2 and **3** are *deterministic*. **Step 2** uses multilevel Schauder estimates.

The inductive mechanics: structure of the induction

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Step 3 (a): When $p = 2$ *all non-trivial trees have positive* $|.|_2$*-degree* and one can use the reconstruction theorem for free on some well-chosen modelled distributions to get estimates on $\Pi_x^{R,2}(D\sigma)$.
(b): The Comparison Lemma shows that $\Pi_x^{n;p}(D\sigma) - \Pi_x^{n;2}(D\sigma)$ is a sum of terms which can be controlled by the induction hypothesis.

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Comparison with Hairer & Steele's work –

- We use a recursive construction of models based on *preparation maps*. (No need to work with trees with the extended $\mathfrak{o}$-decoration.)
- We trade the use of pointed modelled distributions for a notion of regularity-integrability structure. Straightforward commutation of the noise derivative operator and $p = \infty$ renormalized model.
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Details:

- A functional setting based on semigroups vs scaled centered functions.
- Our result stated for one noise and one integration operator. Can be generalized to multiple noises and systems.

Thank you for your attention!