Gibbs measures for the 1D nonlinear Schrödinger equation with trapping potential

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Recontre ANR SMOOTH
22–24 mai 2024
Université de Bretagne-Occidentale

Outline

1d NLS with trapping potential

- We consider 1d nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u - Vu = \pm |u|^{p-2}u, \quad x \in \mathbb{R} \tag{NLS}$$

with $p > 2$ and the smooth trapping potential $V : \mathbb{R} \to \mathbb{R}_+$ satisfying

$$V(x) \sim |x|^s \quad (s > 0) \quad \text{as} \quad |x| \to \infty.$$

- Our goal is to construct the following, formally defined, Gibbs measures

$$d\mu(u) \text{ ``="} \frac{1}{Z} \exp\left(-\frac{1}{2} \int_{\mathbb{R}} |\partial_x u|^2 + V|u|^2 dx \mp \frac{1}{p} \int_{\mathbb{R}} |u|^p dx \right) du$$

and to prove their invariance under the flow of (NLS).

Motivation

- Physics: The many-body mean-field approximation of Bose-Einstein condensates:
  - the defocusing (+) measure (**Lewin, Nam, Rougerie '15, '18** ; **Fröhlich, Knowles, Schlein, Sohinger '17, '19**).
  - the focusing (-) measure (**Sohinger, Rout '22, '23**).

- Mathematics: Most work on Gibbs measures relies on the explicit knowledge of eigenfunctions of the linear operator $-\Delta + V(x)$, e.g.
  - torus (plan waves) **Lebowitz, Rose, Speer '88** ; **Bourgain '94** ; **Oh, Quastel '13** ; **Oh, Sosoe, Tolomeo '22** ,...
  - harmonic potential $V(x) = |x|^2$ (link with Hermite polynomials) **Burq, Thomann, Tzvetkov '13** ; **Deng '12** ; **Robert, Seong, Tolomeo, Wang '22** .
  - disk/sphere (link with Bessel functions) **Tzvetkov '06, '08** ; **Bourgain, Bulut '14** ; **Oh, Sosoe, Tolomeo '22** ; **Xian '23** .

  For $V(x) \sim |x|^s$ **at infinity**, such an **explicit knowledge on eigenfunctions is not available**.

1d NLS on torus

One can formally think it as (NLS) with $s = \infty$, $x \in [0,1]$ and a periodic boundary condition.

- defocusing: **McKean '95** .
- focusing:

$$d\mu(u) \text{ "=" } \frac{1}{Z}\exp\left(-\frac{1}{2}\int_{\mathbb{T}} |\partial_x u|^2 + |u|^2 dx + \frac{1}{p}\int_{\mathbb{T}} |u|^p dx\right) \mathbb{1}_{\{\int_{\mathbb{T}} |u|^2 dx \le m\}} du$$

- **Lebowitz, Rose, Speer '88** proved
  - normalizability ($Z < +\infty$) for $2 < p < 6$ and any $m > 0$;
  - non-normalizability ($Z = +\infty$)
    + for $p > 6$ and any $m > 0$;
    + for $p = 6$ and $m > \|Q\|^2_{L^2(\mathbb{R})}$, where Q is the unique (up to symmetries) optimizer of the Gagliardo-Nirenberg-Sobolev inequality

$$\|u\|^6_{L^6(\mathbb{R})} \le C_{\text{opt}} \|\partial_x u\|^2_{L^2(\mathbb{R})} \|u\|^4_{L^2(\mathbb{R})}.$$

- **Bourgain '94** proved
  - normalizability
    + for $2 < p < 6$ and any $m > 0$;
    + for $p = 6$ and $m > 0$ small;
  - invariance for $p > 2$ and $p \le 6$ for the focusing nonlinearity.
- **Oh, Sosoe, Tolomeo '22** proved the normalizability for $p = 6$ and $0 < m < \|Q\|^2_{L^2(\mathbb{R})}$ and $m = \|Q\|^2_{L^2(\mathbb{R})}$. This is remarkable since NLS admits blowup solutions with the minimal mass $\|Q\|^2_{L^2(\mathbb{R})}$ (**Ogawa, Tsutsumi '90**).

1d NLS with harmonic potential $V(x) = |x|^2$

► **Burq, Thomann, Tzvetkov '13** constructed the Gibbs measures for
- defocusing with $p > 2$;
- focusing

$$d\mu(u) \text{ “=” } \frac{1}{Z} \exp\left(-\frac{1}{2}\int_{\mathbb{R}} |\partial_x u|^2 + |x|^2|u|^2 dx + \frac{1}{p}\int_{\mathbb{R}} |u|^p dx\right) \mathbb{1}_{\{|\int_{\mathbb{R}} :|u|^2: dx| \leq m\}} du$$

only with $p = 4$ and any $m > 0$ (the renormalized mass cutoff).
They also proved the invariance for p **even integer**.

► **Robert, Seong, Tolomeo, Wang '22** considered the **focusing** measures and proved
- normalizability for $2 < p < 6$ and any $m > 0$;
- non-normalizability for $p \geq 6$ and any $m > 0$.

⚠ **No critical nonlinearity.**

Main results

Theorem 1 (Construction of Gibbs measures) D., Rougerie '23, D., Rougerie, Tolomeo, Wang '23+

- *Normalizability for* **defocusing** *nonlinearity with* $p > \max\{2, \frac{4}{s}\}$;
- **Focusing** *nonlinearity*
  - *Super-harmonic* ($s > 2$)

$$d\mu(u) \text{ “=” } \frac{1}{Z}\exp\left(-\frac{1}{2}\int_{\mathbb{R}} |\partial_x u|^2 + V|u|^2 dx + \frac{1}{p}\int_{\mathbb{R}} |u|^p dx\right)\mathbb{1}_{\{\int_{\mathbb{R}}|u|^2 dx \le m\}} du$$

    - *normalizability*
      + *for* $2 < p < 6$ *and any* $m > 0$;
      + *for* $p = 6$ *and* $0 < m < \|Q\|^2_{L^2(\mathbb{R})}$, *where* Q *is the unique (up to symmetries) optimizer of the Gagliardo-Nirenberg-Sobolev inequality*.
    - *non-normalizability*
      + *for* $p = 6$ *and* $m > \|Q\|^2_{L^2(\mathbb{R})}$;
      + *for* $p > 6$ *and any* $m > 0$.
  - *harmonic* ($s = 2$)

$$d\mu(u) \text{ “=” } \frac{1}{Z}\exp\left(-\frac{1}{2}\int_{\mathbb{R}} |\partial_x u|^2 + V|u|^2 dx + \frac{1}{p}\int_{\mathbb{R}} |u|^p dx\right)\mathbb{1}_{\{|\int_{\mathbb{R}}:|u|^2: dx| \le m\}} du$$

    - *normalizability for* $2 < p < 6$ *and any* $m > 0$.
    - *non-normalizability for* $p \ge 6$ *and any* $m > 0$.

Main results

Theorem 1 (Construction of Gibbs measures, continue)

- *sub-harmonic* ($1 < s < 2$)

$$d\mu(u) \text{ “=” } \frac{1}{Z}\exp\left(-\frac{1}{2}\int_{\mathbb{R}} |\partial_x u|^2 + V|u|^2 dx + \frac{\alpha}{p}\int_{\mathbb{R}} |u|^p dx\right)\mathbb{1}_{\{|\int_{\mathbb{R}} :|u|^2: dx| \le m\}} du$$

  - *normalizability*
    + *for* $\frac{4}{s} < p < 2+2s$, *any* $\alpha > 0$ *and any* $m > 0$;
    + *for* $p = 2+2s$, $m > 0$, *and* $0 < \alpha < \alpha_0$ *with* **some** $\alpha_0 = \alpha_0(m) > 0$.
  - *non-normalizability*
    + *for* $p = 2+2s$, $m > 0$, *and* $\alpha > \alpha_0$;
    + *for* $p > 2+2s$, *any* $\alpha > 0$ *and any* $m > 0$.

Theorem 2 (Invariance of Gibbs measures) D., Rougerie ’23

The above Gibbs measures are invariant under the flow of (NLS) provided

- *super-harmonic* ($s > 2$): $2 < p < 4+s$;
- *(sub)-harmonic* ($1 < s \le 2$): $\frac{4}{s} < p < 6$.

Consequently, (NLS) is globally well-posed almost surely on the support of Gibbs measures.

- **Open questions**:
  - ? normalizability for $m = \|Q\|^2_{L^2(\mathbb{R})}$ (for $s > 2$) and $\alpha = \alpha_0$ (for $1 < s < 2$);
  - ? invariance for $p \ge 4+s$ (for $s > 2$) and $p \ge 6$ (for $1 < s \le 2$).

Sobolev spaces

- Denote

$$H = -\partial_x^2 + V(x).$$

Since V is trapping, the spectral theorem yields

$$H = \sum_{j\geq 1} \lambda_j |u_j\rangle\langle u_j|,$$

where

$$0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_j \to +\infty$$

are eigenvalues of H and $\{u_j\}_{j\geq 1}$ are the corresponding normalized eigenfunctions which form an orthonormal basis of $L^2(\mathbb{R})$.

- Let $1 \leq p \leq \infty$ and $\beta \in \mathbb{R}$. Sobolev spaces associated to H are defined by

$$\mathscr{W}^{\beta,p}(\mathbb{R}) = \left\{ u \in \mathscr{S}'(\mathbb{R}) : H^{\beta/2} u \in L^p(\mathbb{R}) \right\}.$$

When $p = 2$, we write $\mathscr{W}^{\theta,2}(\mathbb{R}) = \mathscr{H}^{\theta}(\mathbb{R})$.

- **Equivalence norms (Yajima, Zhang '01)** :

For $\beta > 0$ and $1 < p < \infty$,

$$\|H^{\beta/2} u\|_{L^p(\mathbb{R})} \sim \|\langle D\rangle^{\beta} u\|_{L^p(\mathbb{R})} + \|\langle x\rangle^{\beta s/2} u\|_{L^p(\mathbb{R})},$$

where $\langle D\rangle = \sqrt{1 - \partial_x^2}$.

Gaussian measure

- By writing

$$d\mu(u) = \frac{1}{Z}\exp\left(\mp\frac{1}{p}\int_{\mathbb{R}}|u|^p dx\right)\exp\left(-\frac{1}{2}\int_{\mathbb{R}}|\partial_x u|^2 + V|u|^2 dx\right)du,$$

we aim at defining μ as an absolutely continuous probability measure with respect to the **Gaussian measure** formally given by

$$d\rho(u) = \frac{1}{C}\exp\left(-\frac{1}{2}\int_{\mathbb{R}}|\partial_x u|^2 + V|u|^2 dx\right)du.$$

- For $u \in \mathscr{S}'(\mathbb{R})$, we decompose

$$u = \sum_{j\geq 1}\alpha_j u_j, \quad \alpha_j = \langle u_j, u\rangle_{L^2} \in \mathbb{C}.$$

We thus write

$$d\rho(u) = \frac{1}{C}\exp\left(-\frac{1}{2}\langle u, Hu\rangle_{L^2}\right)du = \frac{1}{C}\exp\left(-\frac{1}{2}\sum_{j\geq 1}\lambda_j|\alpha_j|^2\right)\prod_{j\geq 1}d\alpha_j.$$

- We can think of defining ρ as

$$d\rho(u) = \prod_{j\geq 1}\frac{\lambda_j}{2\pi}e^{-\frac{1}{2}\lambda_j|\alpha_j|^2}d\alpha_j \tag{GM}$$

with

$$C = \prod_{j\geq 1}\frac{2\pi}{\lambda_j}.$$

Gaussian measure

- To define rigorously the Gaussian measure ρ, we will take the limit $\Lambda \to \infty$ of the finite dimensional Gaussian measure

$$d\rho_\Lambda(u) = \prod_{\lambda_j \leq \Lambda} \frac{\lambda_j}{2\pi} e^{-\frac{1}{2}\lambda_j |\alpha_j|^2} d\alpha_j$$

defined on

$$E_\Lambda = \text{span}\{u_j : \lambda_j \leq \Lambda\}.$$

- **But, taking the limit on which topology?**

- to view ρ_Λ as an induced probability measure under the randomization

$$u_\Lambda^\omega = \sum_{\lambda_j \leq \Lambda} \frac{g_j(\omega)}{\sqrt{\lambda_j}} u_j,$$

where $\{g_j\}_{j\geq 1}$ are i.i.d. complex-valued **standard** Gaussian random variables ($\mathscr{N}_\mathbb{C}(0,1)$) on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

- to show

$$\mathbb{E}_\mathbb{P}[\| u_\Theta^\omega - u_\Lambda^\omega \|^2_{\mathscr{H}^\theta}] = \sum_{\Lambda < \lambda_j \leq \Theta} \lambda_j^{\theta - 1} \to 0 \text{ as } \Theta, \Lambda \to \infty$$

for a suitable θ. For that, we need

$$\sum_{j\geq 1} \lambda_j^{\theta-1} < \infty.$$

Regularity of Gaussian measure

- **Under which condition on θ, we have $\sum_{j\geq 1} \lambda_j^{\theta-1} < \infty$?**
  - For 1d torus, $\lambda_j \sim j^2$, so we need $\theta < \frac{1}{2}$.
  - For 1d harmonic potential, $\lambda_j \sim j$, hence we require $\theta < 0$.
- For general trapping potential, we use the **Lieb–Thirring inequality** (**Dolbeault, Felmer, Loss, Paturel '06**): for $p > \frac{1}{2}$,

$$\mathrm{Tr}[H^{-p}] \leq C(p) \iint_{\mathbb{R}\times\mathbb{R}} \frac{dxd\xi}{(|\xi|^2 + V(x))^p}.$$

  Applying this, we have

$$\sum_{j\geq 1} \lambda_j^{\theta-1} = \mathrm{Tr}[H^{\theta-1}] \leq 2^{1-\theta}\mathrm{Tr}[(H+\lambda_1)^{\theta-1}] \leq C(\theta) \iint_{\mathbb{R}\times\mathbb{R}} \frac{dxd\xi}{(|\xi|^2 + V(x) + \lambda_1)^{1-\theta}}.$$

  If $V(x) \geq C|x|^s$, then $\sum_{j\geq 1} \lambda_j^{\theta-1} < \infty$ provided $\theta < \frac{1}{2} - \frac{1}{s}$.

Lemma 1 (Regularity of ρ)

ρ is supported on $\mathscr{H}^\theta(\mathbb{R})$ for any $\theta < \frac{1}{2} - \frac{1}{s}$.

$\rightarrow$ When $s \leq 2$, a mass renormalization is needed, i.e.,

$$\int_{\mathbb{R}} :|u|^2: dx = \int_{\mathbb{R}} (|u|^2 - \mathbb{E}_\rho[|u|^2])dx = \int_{\mathbb{R}} |u|^2 dx - \mathbb{E}_\rho\left[\int_{\mathbb{R}} |u|^2 dx\right].$$

Integrability of Gaussian measure

- For $\beta > 0$ and $1 < p < \infty$, we can use Khintchine's inequality

$$\mathbb{E}_{\mathbb{P}}\left[\left|\sum_j a_j g_j(\omega)\right|^p\right] \leq C(p)\|a_j\|_{\ell^2}^p$$

to estimate $\mathbb{E}_\rho[\|u\|^p_{\mathscr{W}^{\beta,p}}]$. Indeed,

$$\begin{aligned}
\mathbb{E}_\rho[\|u\|^p_{\mathscr{W}^{\beta,p}}] &= \int_{\mathbb{R}} \mathbb{E}_\rho[|H^{\beta/2}u(x)|^p]dx \\
&= \int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}}\left[\left|H^{\beta/2}\left(\sum_j \frac{g_j(\omega)}{\sqrt{\lambda_j}}u_j(x)\right)\right|^p\right]dx \\
&= \int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}}\left[\left|\sum_j \lambda_j^{\frac{\beta-1}{2}} u_j(x) g_j(\omega)\right|^p\right]dx \\
&\leq C(p)\int_{\mathbb{R}} \Big(\underbrace{\sum_j \lambda_j^{\beta-1}|u_j(x)|^2}_{=H^{\beta-1}(x,x)}\Big)^{p/2} dx,
\end{aligned}$$

where $H^{\beta-1}(x,y) = \sum_{j\geq 1} \lambda_j^{\beta-1} u_j(x)\overline{u_j}(y)$ is the integral kernel of $H^{\beta-1}$.
$\Rightarrow$ ρ is supported in $\mathscr{W}^{\beta,p}(\mathbb{R})$ as long as

$$H^{\beta-1}(.,.) \in L^{p/2}(\mathbb{R}).$$

Integrability of Gaussian measure

For which values of β and q, we have $H^{\beta-1}(.,.) \in L^q$?

- For 1d torus, $|u_j(x)|^2 = 1$, so for $q \geq 1$,

$$\|H^{\beta-1}(.,.)\|_{L^q(\mathbb{T})} \leq \sum_{j\geq 1} \lambda_j^{\beta-1} < \infty$$

  provided $\beta < \frac{1}{2}$. So $H^{\beta-1}(.,.) \in L^q(\mathbb{T})$ for all $0 \leq \beta < \frac{1}{2}$ and any $1 \leq q \leq \infty$.
- For 1d harmonic potential, we have (**Koch, Tataru '05**)

$$\|u_j\|_{L^p(\mathbb{R})} \lesssim \begin{cases} \lambda_j^{-\frac{1}{6}+\frac{1}{3p}} & \text{if } 2 \leq p < 4, \\ \lambda_j^{-\frac{1}{12}} & \text{if } p \geq 4. \end{cases}$$

  Thus

$$\|H^{\beta-1}(.,.)\|_{L^q(\mathbb{R})} \leq \sum_{j\geq 1} \lambda_j^{\beta-1} \|u_j\|^2_{L^{2q}(\mathbb{R})} \lesssim \begin{cases} \sum_{j\geq 1} \lambda_j^{\beta-1-\frac{1}{3}+\frac{1}{3q}} & \text{if } 1 \leq q < 2, \\ \sum_{j\geq 1} \lambda_j^{\beta-1-\frac{1}{6}} & \text{if } q \geq 2. \end{cases}$$

  Since $\lambda_j \sim j$, we have $H^{\beta-1}(.,.) \in L^q(\mathbb{R})$ for

$$\begin{cases} 0 \leq \beta < \frac{1}{3} - \frac{1}{3q} & \text{if } 1 \leq q < 2, \\ 0 \leq \beta < \frac{1}{6} & \text{if } q \geq 2. \end{cases}$$

Integrability of Gaussian measure

Lemma 2 (Integrability of diagonal integral kernel)

Let $0 \leq \beta < \frac{1}{2}$. Then $H^{\beta-1}(.,.) \in L^q(\mathbb{R})$ for any

$$\max\left\{1, \frac{2}{s(1-2\beta)}\right\} < q \leq \infty.$$

Proof:

- $q = \infty$: $H \geq C(1-\partial_x^2) \to H^{\beta-1} \leq C(\beta)(1-\partial_x^2)^{\beta-1}$ (operator monotonicity) $\to H^{\beta-1}(x,x) \leq C(\beta)G(x,x)$, where

$$G(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{i(x-y)\xi}}{(1+|\xi|^2)^{1-\beta}} d\xi$$

  is the Green function of $(1-\partial_x^2)^{1-\beta}$. Note that $G(x,x) < \infty$ as long as $\beta < \frac{1}{2}$.
- For $1 < q < \infty$, we want to show

$$\sup_{\substack{\chi \in L^{q'}(\mathbb{R}) \\ \chi \geq 0}} \int_{\mathbb{R}} H^{\beta-1}(x,x)\chi(x) dx \leq C\|\chi\|_{L^{q'}(\mathbb{R})}.$$

Integrability of Gaussian measure

We write

$$\int_{\mathbb{R}} H^{\beta-1}(x,x)\chi(x)dx = \mathrm{Tr}[\chi^{1/2}H^{\beta-1}\chi^{1/2}] = \|H^{(\beta-1)/2}\chi^{1/2}\|^2_{\mathfrak{S}^2},$$

where

$$\mathfrak{S}^p = \left\{A : \|A\|_{\mathfrak{S}^p} = \left(\mathrm{Tr}[(A^*A)^{p/2}]\right)^{1/p} < \infty\right\}$$

is the p-th Schatten space. For $\alpha > 0$, we write

$$H^{(\beta-1)/2}\chi^{1/2} = \underbrace{H^{\alpha+(\beta-1)/2}}_{\mathfrak{S}^{2q}}\left(\underbrace{H^{-\alpha}(1-\partial_x^2)^{\alpha}}_{\mathfrak{S}^{\infty}}\right)\underbrace{(1-\partial_x^2)^{-\alpha}\chi^{1/2}}_{\mathfrak{S}^{2q'}}.$$

- $H^{\alpha+(\beta-1)/2} \in \mathfrak{S}^{2q} \to \left(\frac{1-\beta}{2} - \alpha\right)2q > \frac{1}{2} + \frac{1}{s}$.
- $H \geq C(1-\partial_x^2) \to H^{-2\alpha} \leq C^{-2\alpha}(1-\partial_x^2)^{-2\alpha} \to (1-\partial_x^2)^{\alpha}H^{-2\alpha}(1-\partial_x^2)^{\alpha} \leq C^{-2\alpha}$, hence $H^{-\alpha}(1-\partial_x^2)^{\alpha} \in \mathfrak{S}^{\infty}$.
- $(1-\partial_x^2)^{-\alpha}\chi^{1/2} \in \mathfrak{S}^{2q'}$. We use the **Kato–Seiler–Simon** inequality: for $1 \leq p < \infty$,

$$\|f(-i\nabla)g(x)\|_{\mathfrak{S}^p} \leq \|f\|_{L^p(\mathbb{R})}\|g\|_{L^p(\mathbb{R})}.$$

It requires $4\alpha q' > 1$ or $\alpha > \frac{1}{4} - \frac{1}{4q}$.

Normalizability

Lemma 3 (Integrability of Gaussian measure)

ρ is supported on $\mathcal{W}^{\beta,p}(\mathbb{R})$ for any $0 \le \beta < \frac{1}{2}$ and

$$\max\left\{2, \frac{4}{s(1-2\beta)}\right\} < p \le \infty.$$

- **Defocusing** : We want to show

$$Z = \mathbb{E}_\rho\left[\exp\left(-\frac{1}{p}\int_{\mathbb{R}} |u|^p dx\right)\right] \in (0,\infty).$$

Jensen's inequality gives

$$Z \ge \exp\left(-\frac{1}{p}\mathbb{E}_\rho\left[\int_{\mathbb{R}} |u|^p dx\right]\right) > 0$$

provided $p > \max\{2, \frac{4}{s}\}$.

Normalizability

▶ **Focusing** : The idea is to use the Boué-Dupuis variational formula as follows. Denote a centered Gaussian process $Y(t)$ by

$$Y(t) = \sum_j \frac{B_j(t)}{\sqrt{\lambda_j}} e_j,$$

where $\{B_j\}_{j\geq 1}$ is a sequence of independent complex-valued Brownian motions, i.e., $B_j(t) \sim \mathcal{N}_{\mathbb{C}}(0,t)$.

Lemma 4 (Boué-Dupuis variational formula)

Fix $\Lambda > 0$ and denote P_Λ the projection on E_Λ, i.e., $P_\Lambda u = \sum_{\lambda_j \leq \Lambda} \alpha_j u_j$. Let $F : C^\infty(\mathbb{R}) \to \mathbb{R}$ be a "nice" measurable function. Then

$$-\log \mathbb{E}_{\mathbb{P}}[\exp(-F(P_\Lambda Y(1)))] = \inf_{\theta \in \mathbb{H}_a} \mathbb{E}_{\mathbb{P}} \left[F(P_\Lambda Y(1) + P_\Lambda I(\theta)(1)) + \frac{1}{2} \int_0^1 \|\theta(t)\|^2_{L^2(\mathbb{R})} dt \right],$$

where $\mathbb{H}_a$ is the space of drifts (progressively measurable processes belonging to $L^2([0,1], L^2(\mathbb{R}))$) and

$$I(\theta)(t) = \int_0^t H^{-\frac{1}{2}} \theta(\tau) d\tau.$$

Normalizability

It suffices to prove that

$$\sup_{\Lambda} \mathbb{E}_{\rho} \left[\exp\left(\frac{1}{p} \int_{\mathbb{R}} |P_\Lambda u|^p dx \cdot \mathbb{1}_{\{|\int_{\mathbb{R}} |P_\Lambda u|^2 dx| \le m\}} \right) \right] < \infty.$$

Applying the Boué-Dupuis variational formula to

$$F(P_\Lambda Y(1)) = -\frac{1}{p} \int_{\mathbb{R}} |P_\Lambda Y(1)|^p dx \cdot \mathbb{1}_{\{|\int_{\mathbb{R}} |P_\Lambda Y(1)|^2 dx| \le m\}},$$

it suffices to **bound from below**

$$\mathbb{E}_{\mathbb{P}} \left[-\frac{1}{p} \int_{\mathbb{R}} |P_\Lambda Y(1) + P_\Lambda I(\theta)(1)|^p dx \cdot \mathbb{1}_{\{|\int_{\mathbb{R}} |P_\Lambda Y(1) + P_\Lambda I(\theta)(1)|^2 dx| \le m\}} + \frac{1}{2} \int_0^1 \|\theta(t)\|^2_{L^2(\mathbb{R})} dt \right]$$

independent of $\theta \in \mathbb{H}_a$ and independent of Λ.

Normalizability

Main ingredients :

- Gagliardo-Nirenberg-Sobolev inequality;
- regularity and integrability of Gaussian measure;
- the pathwise regularity bound

$$\| I(\theta)(1)\|^2_{\mathscr{H}^1(\mathbb{R})} \leq \int_0^1 \|\theta(t)\|^2_{L^2(\mathbb{R})} dt.$$

Invariance of Gibbs measures

The invariance of Gibbs measures follows from **Bourgain's argument '94, '96**:

- **Step 1. Local theory** :
  - When $s > 2$: deterministic LWP in the support of Gibbs measure, i.e., *for $u_0 \in \mathcal{H}^\theta(\mathbb{R}) \supset supp(\mu)$, there exists $\delta \sim \|u_0\|_{\mathcal{H}^\theta}^{-\sigma}$ with $\sigma > 0$ such that solution to (NLS) exists on $[-\delta, \delta]$*.
    Tool: Strichartz estimates with a loss of derivatives due to **Yajima, Zhang '04** .
  - When $s \leq 2$: probabilistic LWP in the support of the Gibbs measure, i.e., *there exists $\Sigma \subset \mathcal{H}^\theta(\mathbb{R}) \supset supp(\mu)$ with $\mu(\Sigma) = 1$ such that for $u_0 \in \Sigma$, there exists $\delta > 0$ so that solution to (NLS) exists on $[-\delta, \delta]$.*
    Idea: to use the integrability of Gaussian measureto derive **probabilistic Strichartz estimates**: for all $T > 0$, $q \geq 1$ and all $\lambda > 0$,

$$\rho(\|e^{-itH} f\|_{L^q([-T,T],\mathcal{W}^{\beta,p})} > \lambda) \leq Ce^{-c\frac{\lambda^2}{T^{2/q}}}.$$

Invariance of Gibbs measures

- **Step 2. Approximate NLS** : Consider

$$i\partial_t u_\Lambda - Hu_\Lambda = \pm Q_\Lambda(|Q_\Lambda u_\Lambda|^2 Q_\Lambda u_\Lambda),$$

where $Q_\Lambda = \chi(H/\Lambda)$ for a suitable cutoff χ. Decomposition

$$u_\Lambda = u_\Lambda^{\text{low}} + u_\Lambda^{\text{high}}, \quad u_\Lambda^{\text{low}} = P_\Lambda u_\Lambda, \quad u_\Lambda^{\text{high}} = P_\Lambda^\perp u_\Lambda.$$

→ u_Λ exists globally in time.
The approximate measure

$$d\eta_\Lambda(u) = d\mu_\Lambda(u) \otimes d\rho_\Lambda^\perp(u)$$

where

$$d\mu_\Lambda(u) = \frac{1}{Z_\Lambda} \exp\left(\mp\frac{1}{2}\int_{\mathbb{R}} |Q_\Lambda u|^4 dx\right) d\rho_\Lambda(u)$$

is invariant under the flow of the approximate NLS.
→ to use it as a **substitution for the conservation law** to derive a **uniform** (in Λ) estimate for the approximate solutions.

Invariance of Gibbs measure

▸ **Step 3. Estimation of the difference** : to use a PDE approximation argument to estimate the difference between the approximate and the exact solutions.
→ the same uniform estimate holds for the exact solution. Moreover, *for T and $\varepsilon > 0$, there exists $\Sigma_{T,\varepsilon}$ such that*

- $\mu(\Sigma_{T,\varepsilon}^c) < \varepsilon$;
- solution to (NLS) exists on $[-T, T]$ for $u_0 \in \Sigma_{T,\varepsilon}$.

▸ **Step 4. Almost sure GWP and measure invariance** :

- Fix $\varepsilon > 0$ and let $T_n = 2^n$ and $\varepsilon_n = 2^{-n}\varepsilon$. We have the set $\Sigma_n = \Sigma_{T_n,\varepsilon_n}$ as above.
- Let $\Sigma_\varepsilon = \cap_{n=1}^\infty \Sigma_n$. Then solution to (NLS) exists globally in time for data in Σ_ε and

$$\mu(\Sigma_\varepsilon^c) = \mu\left(\bigcup_{n=1}^\infty \Sigma_n^c\right) \le \sum_{n=1}^\infty \mu(\Sigma_n^c) < \sum_{n=1}^\infty 2^{-n}\varepsilon = \varepsilon.$$

- Let $\Sigma = \cup_{j=1}^\infty \Sigma_{1/j}$. Then solution to (NLS) exists globally in time for data in Σ and

$$\mu(\Sigma^c) = \mu\left(\bigcap_{j=1}^\infty \Sigma_{1/j}^c\right) \le \inf_j \mu(\Sigma_{1/j}^c) = 0.$$

From almost sure GWP → measure invariance.

Fixed mass Gibbs measures

Consider

$$d\mu^m(u) \text{ “=” } \frac{1}{Z^m} \exp\left(-\frac{1}{2}\int_{\mathbb{R}} |\partial_x u|^2 + V|u|^2 dx \mp \frac{1}{p}\int_{\mathbb{R}} |u|^p dx\right) \mathbb{1}_{\{\int_{\mathbb{R}} |u|^2 dx = m\}} du.$$

▶ **Oh and Quastel '13** proved:

- normalizability:
  - (defocusing) for $p > 2$ and any $m > 0$;
  - (focusing) for $2 < p < 6$ and any $m > 0$; for $p = 6$ and $m > 0$ small.
- invariance: (defocusing) for $p > 2$; (focusing) for $2 < p \leq 6$.

Theorem 3 (Fixed mass Gibbs measures) D., Rougerie '23

Let $p = 4$ and $m > 0$. Then μ^m is well-defined for **defocusing** *nonlinearity with $s > 1$ and* **focusing** *nonlinearity with $s > \frac{8}{5}$. In addition, μ^m is invariant under the flow of (NLS).*

Fixed mass Gibbs measures

- Write

$$d\mu^m(u) = \frac{1}{Z^m} \exp\left(\mp\frac{1}{4}\int_{\mathbb{R}} |u|^4 dx\right) d\rho^m(u)$$

where

$$d\rho^m(u) \text{ “=” } \frac{1}{C^m} \exp\left(-\frac{1}{2}\int_{\mathbb{R}} |\partial_x u|^2 + V|u|^2 dx\right) \mathbb{1}_{\{\int_{\mathbb{R}} |u|^2 dx = m\}} du$$

is the fixed mass Gaussian measure.

- Take the limit $\epsilon \to 0$ of

$$\begin{aligned} d\rho^{m,\epsilon}(u) \text{ “=” } & \frac{1}{A^{m,\epsilon}} \exp\left(-\frac{1}{2}\int_{\mathbb{R}} |\partial_x u|^2 + V|u|^2 dx - \frac{1}{\epsilon}\left(\int_{\mathbb{R}} |u|^2 dx - m\right)^2\right) du \\ = & \frac{1}{C^{m,\epsilon}} \exp\left(-\frac{1}{\epsilon}\left(\int_{\mathbb{R}} |u|^2 dx - m\right)^2\right) d\rho(u). \end{aligned}$$

⚠ **The problem is $C^{m,\epsilon} \to 0$ as $\epsilon \to 0$.**

Fixed mass Gibbs measures

- Take a Borelian set B of $E_\Lambda = \mathrm{span}\{u_j : \lambda_j \le \Lambda\}$, we write

$$\begin{aligned}\rho^{m,\epsilon}(B) &= \frac{1}{C^{m,\epsilon}} \int_B \exp\left(-\frac{1}{\epsilon}\left(\|u\|^2 - m\right)^2\right) d\rho(u) \\ &= \frac{1}{C^{m,\epsilon}} \int_B \left(\exp\left(-\frac{1}{\epsilon}\left(\|P_\Lambda u\|^2 + \|P_\Lambda^\perp u\|^2 - m\right)^2\right) d\rho_\Lambda^\perp(u)\right) d\rho_\Lambda(u) \\ &= \int_B \left(\int_0^\infty \exp\left(-\frac{1}{\epsilon}(\eta + \|P_\Lambda u\|^2 - m)^2\right) f_\Lambda(\eta) d\eta\right) \div \left(\int_0^\infty \exp\left(-\frac{1}{\epsilon}(\eta - m)^2\right) f_0(\eta) d\eta\right) d\rho_\Lambda(u),\end{aligned}$$

where f_Λ is the density function of $\|P_\Lambda^\perp u\|^2$ with respect to $\rho_\Lambda^\perp$.

Lemma (Density function)

For each $\Lambda \ge 0$, f_Λ is bounded and uniformly continuous on $(0,\infty)$. In addition, $f_0(m) > 0$ for all $m > 0$.

As a result, we have

$$\frac{1}{\sqrt{\epsilon}} \int_0^\infty \exp\left(-\frac{1}{\epsilon}\left(\eta + \|P_\Lambda u\|^2 - m\right)^2\right) f_\Lambda(\eta) d\eta \to f_\Lambda\left(m - \|P_\Lambda u\|^2\right)\sqrt{\pi}$$

and

$$\frac{1}{\sqrt{\epsilon}} \int_0^\infty \exp\left(-\frac{1}{\epsilon}(\eta - m)^2\right) f_0(\eta) d\eta \to f_0(m)\sqrt{\pi}.$$

In particular, $\rho^{m,\epsilon}(B)$ has a limit as $\epsilon \to 0$.

Thank You!