Segal's Axioms & Probability.

I. Quantum Mechanics of a Single (free) particle moving in $\mathbb{R}$

* (possible) position of the particle described by $x \in \mathbb{R}$.

Message from QM : DO NOT know precise position/momentum of the particle, but $\exists$ "wave function" $f \in L^2(\mathbb{R}, d_{\mathbb{R}})$

possible positions   Lebesgue measure

s.t. $|f(x)|^2$ = prob. density of finding particle at x. ($\Rightarrow \|f\|_{L^2} = 1$).

Schrödinger's picture of time evolution

particle starts at time $t=0$ with w.f. f_0

$\Rightarrow$ w.f. at time $t>0$ is $f_t = e^{itH} f_0$, $H = -\partial_x^2$ (free particle)

i.e. $f_t(x) = \int_{\mathbb{R}} e^{itH}(x,y) f_0(y)\, dy$, $\quad e^{itH}(x,y)$ = integral kernel

$\approx \langle \delta_x, e^{itH} \delta_y \rangle_{L^2}$,

= "amplitude" for particle to travel from y to x.

$\boxed{\text{Want more details of what happens during the process "} e^{itH} \text{".}}$

Recall: heat kernel $e^{-tH}(x,y) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{1}{4}\cdot\frac{|x-y|^2}{t}}$

$t \mapsto -it$ (Wick rotation) $\Rightarrow e^{itH}(x,y) = \sqrt{\frac{i}{4\pi t}}\, e^{\frac{i}{4}\cdot\frac{|x-y|^2}{t}}$

$$\boxed{\underline{\underline{\text{magic}}} \quad \left(\frac{i}{4\pi t}\right)^{\frac{1}{2}} e^{\frac{i}{4}\cdot t\frac{|x-y|^2}{t^2}}}$$

$$\boxed{\begin{array}{l} q(t_j) = y_j = \text{position of particle at } t_j = j(\delta t). \\ t_0 = 0,\ t_N = t.\quad y_0 = y,\ y_N = x. \end{array}}$$

Now pick $N \gg 1$. $\delta t \overset{\text{def}}{=} t/N$.

$$e^{itH}(x,y) = \left(e^{i\frac{t}{N}H}\right)^N (x,y)$$

$$= \underbrace{\int\cdots\int}_{N-1} e^{i\frac{t}{N}H}(x, y_{N-1})\, e^{i\frac{t}{N}H}(y_{N-1}, y_{N-2}) \cdots e^{i\frac{t}{N}H}(y_1, y)\, dy_1 \cdots dy_{N-1}$$

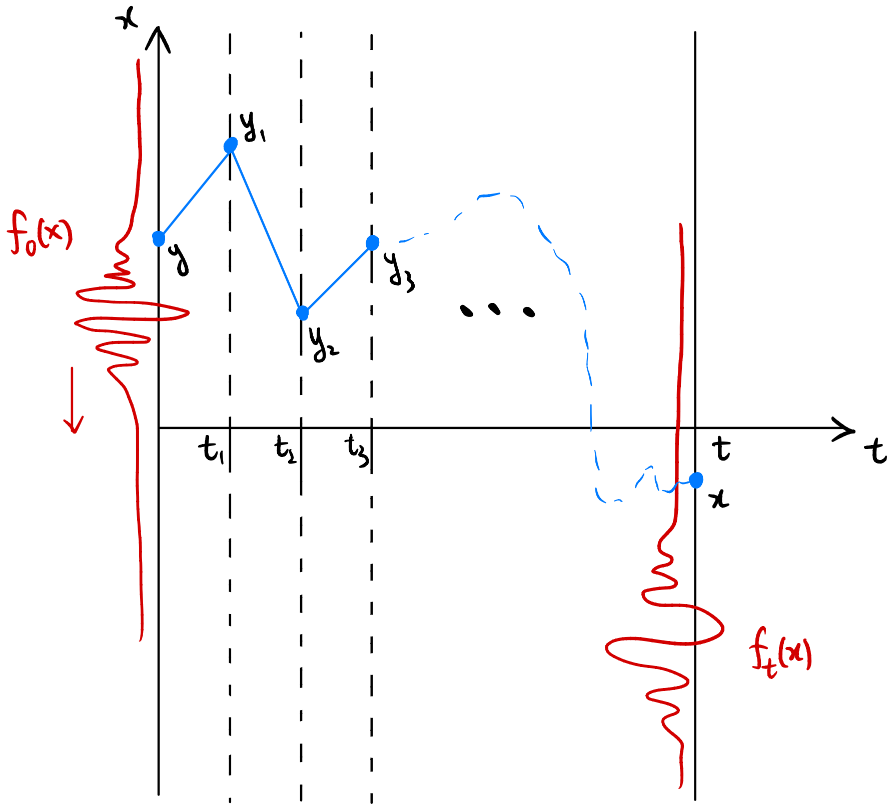
$$= \int\cdots\int \prod_{j=0}^{N-1} e^{\frac{i}{4}(\delta t)\frac{|q(t_{j+1})-q(t_j)|^2}{(\delta t)^2}} \left(\frac{i}{4\pi\delta t}\right)^{\frac{N}{2}} \prod_{j=1}^{N-1} dq(t_j).$$

$$\underset{\delta t \to 0}{\overset{N\to\infty}{=\!=}} \int_{\left\{\substack{\text{paths } q:[0,t]\to\mathbb{R} \\ \text{with } q(0)=x,\ q(t)=y}\right\}} \underbrace{e^{\frac{i}{4}\int_0^t |\dot q(s)|^2 ds}\, [\mathcal{D}q]}_{A}$$

$\rightarrow$ "Lebesgue measure" on path space.

$$[\mathcal{D}q] \approx \lim_{N\to\infty} \left(\frac{i}{4\pi\delta t}\right)^{\frac{N}{2}} \prod_{j=1}^{N-1} dq(t_j).$$

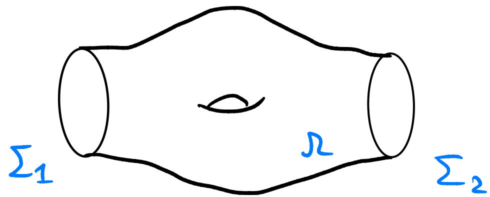
$\boxed{\text{Rmk. Without } i,\ A \text{ (normalized)} = \text{Wiener measure for Brownian motion (Bridge), rigorous.}}$



Remark: this work focus on the **probabilistic/Euclidean side** (without i)
rather than the unitary/quantum side (with i)

80-90s

$\boxed{\text{Atiyah-Segal Axioms}}$ for an abstract 2D QFT.



A 2D QFT consists of the data (of association).

① Circle $\Sigma \rightsquigarrow$ (real) Hilbert Space $\mathcal{H}_\Sigma$.

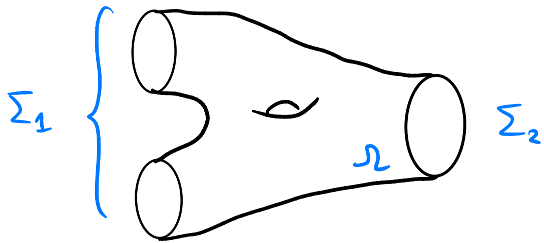
$\Sigma_1 \sqcup \Sigma_2 \rightsquigarrow \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$.

(Σ is Riemannian with radius $= R$)

② Cobordism $\Omega \rightsquigarrow U_\Omega : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2}$

Riemannian surface with metric g connecting Σ_1 and Σ_2.

+ Rules (next page).

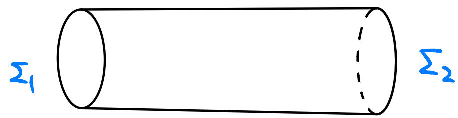


☆ 1D analogue $=$ QM.

In the model mentioned above:

possible positions at time$=t$

① A point (denoted $t \in \mathbb{R}$) $\rightsquigarrow \mathcal{H}_t \equiv L^2(\mathbb{R}, d_{\mathbb{R}})$



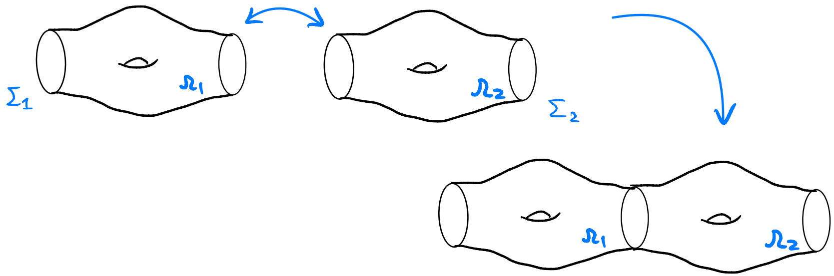
② An interval $[t_1, t_2] \rightsquigarrow$ the evolution

$$e^{i(t_2-t_1)H} : \mathcal{H}_{t_1} \to \mathcal{H}_{t_2}.$$

Atiyah-Segal Axioms

Rules:

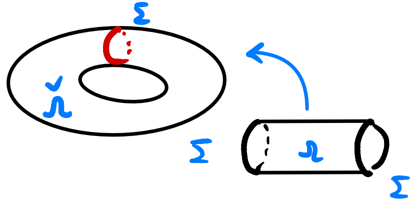
④ $\boxed{U_{\Omega_2 \cup_{\Sigma'} \Omega_1} = U_{\Omega_2} \circ U_{\Omega_1}}$



③ $Z_{\check{\Omega}} = \mathrm{tr}(U_\Omega)$.

⑤ $U_{\Omega^*} = U_\Omega^\dagger$. ← adjoint.

↳ co-orient. of boundaries reversed.



Problem: Construct a concrete model satisfying ① – ⑤ (with interaction).

Recipe: do in analogue with QM (path integral).

Model: a (rough) closed string moving in $\mathbb{R}$

* (possible) configuration of string described by a real distribution $\varphi \in \mathcal{D}'(\mathbb{S}^1)$

Alternatively, $\varphi \in \mathcal{D}'(\mathbb{S}^1)$ is config. of real scalar field over $\mathbb{S}^1$.

$$\Rightarrow \mathcal{H}_{\mathbb{S}^1} \overset{\text{def}}{=\!=} L^2(\mathcal{D}'(\mathbb{S}^1), \mu_{???})$$

needs to be a proba measure ($\neq$ Lebesgue).

$F \in \mathcal{H}_{\mathbb{S}^1}$ are "wave functions".

s.t. $|F(\varphi)|^2 =$ prob. density of finding string at config. $\varphi \in \mathcal{D}'(\mathbb{S}^1)$.

RN density WRT $\mu_{???}$

Rmk. $\mu_{???} =$ Gaussian measure with covariance $(\Delta + m^2)^{-\frac{1}{2}}$.

$\boxed{\text{Time evolution.}}$ Suppose $\partial\Omega = \Sigma_{in} \sqcup \Sigma_{out}$. Ω with metric g.

$\cancel{e^{itH}}$

$\Rightarrow$ define directly integral kernel using integration on path space.

i.e. $(U_t F)(\varphi_{out}) = \int A_\Omega(\varphi_{in}, \varphi_{out}) F(\varphi_{in})\, d\mu_{???}(\varphi_{in})$.

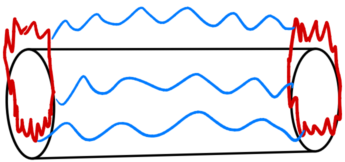
"$P(\phi)$ model".

with
$$A_\Omega(\varphi_{\text{in}}, \varphi_{\text{out}}) = \int_{\{\phi|\partial\Omega=(\varphi_{\text{in}},\varphi_{\text{out}})\}} e^{-\int_\Omega \frac{1}{2}(|\nabla\phi|_g^2 + m^2\phi^2) + \underbrace{P(\phi(x))}_{\text{interaction potential}} dV_\Omega(x)} [\mathcal{L}\phi],$$

$\subseteq \mathcal{D}'(\Omega)$

mass >0

interaction potential
"
polynomial bounded below



Σ_{in} Σ_{out}

(Ω, g).

Example: $P(\phi) = \phi^4$.

In case M = closed Riemannian surface,

$$Z_M = \int_{\mathcal{D}'(M)} e^{-\int_M P(\phi(x))dV_M(x)} e^{-\frac{1}{2}\int_M (|\nabla\phi|_g^2 + m^2\phi^2)dV_M} [\mathcal{L}\phi]$$

Problem.

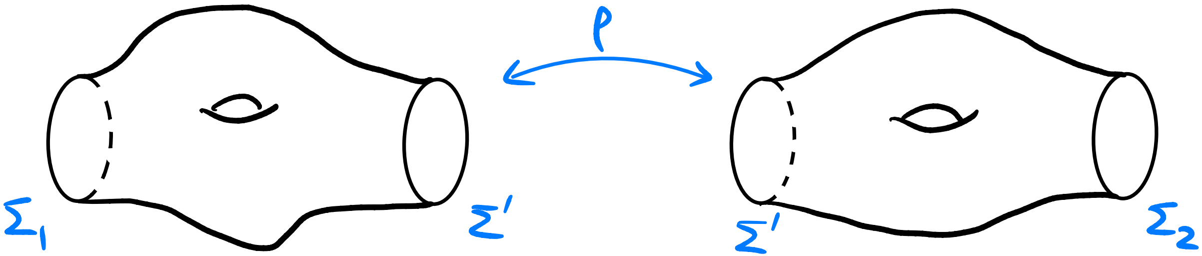
☆ a) define $\mathcal{A}_\Omega(\varphi_{in}, \varphi_{out})$ in a rigorous and natural manner.

Main difficulty: define interaction with boundary condition as RV.

b) Check axioms ③ — ⑤.

(Markov property & Conditioning under μ_{GFF}).

Main difficulty is present in showing the axioms for free field.
(i.e. $P=0$).



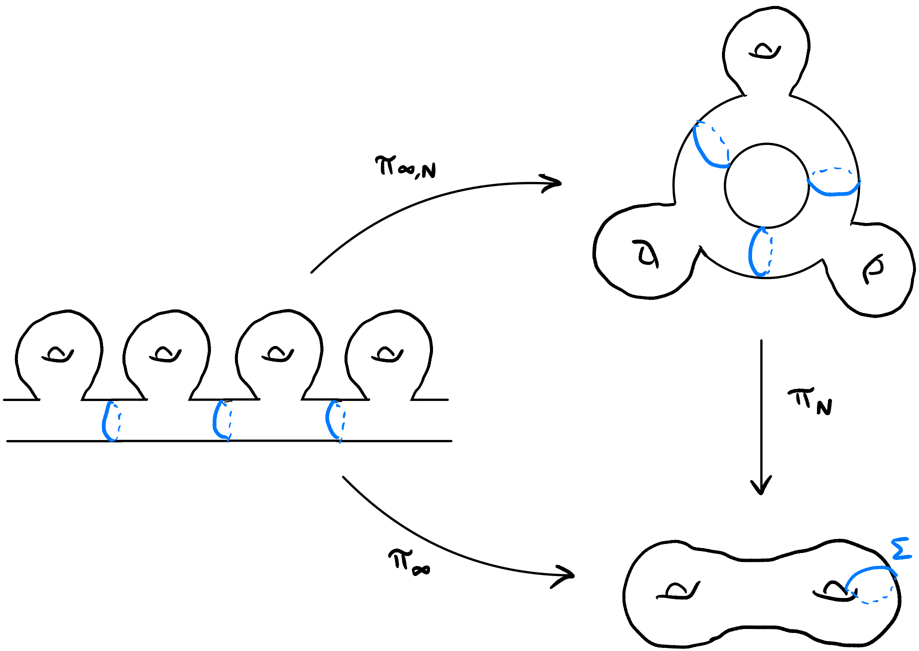
Composition Axiom

$$\Updownarrow$$

$$\mathcal{A}_{\Omega_1 \cup_\rho \Omega_2}(\varphi_2, \varphi_1) = \int \mathcal{A}_{\Omega_2}(\varphi_2, \varphi') \, \mathcal{A}_{\Omega_1}(\varphi', \varphi_1) \, [d\varphi']$$

A nice application.

Start with a fundamental surface M and manufacture periodic surfaces M_N for $N \in \mathbb{N}$.



Result:

$$\exists \lim_{N\to\infty} \frac{\log(Z_{M_N})}{N}.$$

Related work:

[1] Naud, Frédéric. Determinants of Laplacians on random hyperbolic surfaces. Journal d'Analyse Mathématique 151.1 (2023)

[2] Upcoming work of Dang-L-Naud-Shen.

Previous work on Segal Problem.

[1] Pickrell, Doug, P(φ) 2 Quantum Field Theories and Segal's Axioms, Commun. Math. Phys. 280, 403–425, 2008.

[2] Guillarmou, C., Kupiainen, A., Rhodes, R., and Vargas, V. (2021). Segal's axioms and bootstrap for Liouville Theory. arXiv:2112.14859.

[3] S. Kandel, P. Mnev and K. Wernli, Two-dimensional perturbative scalar QFT and Atiyah-Segal gluing, Adv. Theor. Math. Phys. 25 (2021) no.7, 1847-1952.

For M = closed Riemannian Surface.

partition function

$$Z_M = \int_{\mathcal{D}'(M)} e^{-\int_M P(\phi(x))\mathrm{d}V_M(x)} \underbrace{e^{-\frac{1}{2}\int_M (|\nabla\phi|_g^2 + m^2\phi^2)\mathrm{d}V_M}[\mathcal{L}\phi]}_{\det_\zeta(\Delta+m^2)\cdot \mu_{GFF}^M}$$

$P \neq 0$. $\dim M = 2$. $\rightarrow$ define as R.V. in $L^1(\mu_{GFF})$.

$$\boxed{\int \boxed{e^{-\int_M \phi(x)^4 dx}} d\mu_{GFF}^M(\phi) < \infty}$$

Nelson's argument. '60s.

__Rmk.__ does NOT work in 3D. (target measure is mutually singular WRT μ_{GFF}^M) $\rightsquigarrow$ SPDE method (regularity str. / paracontrol)

Problem ϕ = dist. low regularity, ϕ^2, ϕ^3, etc. not defined.

$\rightsquigarrow$ need renormalization.

Related work involving Nelson’s argument / regularisation-renormalisation:

[1] Tadahiro Oh, Tristan Robert, Nikolay Tzvetkov, Yuzhao Wang, Stochastic quantization of Liouville conformal field theory, arXiv:2004.04194;

[2] Nicolas Burq; Laurent Thomann; Nikolay Tzvetkov. Remarks on the Gibbs measures for nonlinear dispersive equations. Annales de la Faculté des sciences de Toulouse : Mathématiques, Série 6, Tome 27 (2018) no. 3, pp. 527-597.

In our case, ① first regularize $\phi_\varepsilon := K_\varepsilon \phi$, K_ε = Smoothing Op.

② replace $\phi_\varepsilon(x)^4 \rightsquigarrow :\phi_\varepsilon(x)^4: = \phi_\varepsilon(x)^4 - 6\mathbb{E}[\phi_\varepsilon(x)^2]\phi_\varepsilon(x)^2 + 3\mathbb{E}[\phi_\varepsilon(x)^2]^2$

$\rightsquigarrow$ Corresponds to proj. of R.V. $\phi_\varepsilon(x)^4$ onto $\underline{\mathrm{Sym}^4(H^{-1}(M))}$ in $L^2(\mu_{GFF})$.

③ discover that $\int_M :\phi_\varepsilon(x)^4: dx \xrightarrow{\varepsilon \to 0}$ well-defined R.V. $\in L^2(\mu_{GFF})$

$$\text{well-defined R.V.} = S_M(\phi)$$

☆ $\varepsilon, \varepsilon' > 0$, $G_{\varepsilon,\varepsilon'}(x,y) = \mathrm{Kernel}\left(K_\varepsilon^* (\Delta + m^2)^{-1} K_\varepsilon\right)$

$$= \mathbb{E}[\phi_\varepsilon(x)\phi_{\varepsilon'}(y)]$$

$$|G_{\varepsilon,\varepsilon'}(x,y)| \leq \text{integrable func. uniformly in } \varepsilon, \varepsilon'.$$

$$|G_{\varepsilon,\varepsilon}(x,y) - G_{\varepsilon,\varepsilon'}(x,y)| \leq \text{integrable func.} \cdot o(|\varepsilon - \varepsilon'|)$$

Example. $K_\varepsilon = e^{-\varepsilon(\Delta + m^2)}$. Remember $\dim M = 2$

$$\Rightarrow \Big\{ \underbrace{\int_M :\phi_\varepsilon(x)^4 : dx}_{= S_{M,\varepsilon}} \Big\} \text{ Cauchy in } L^2(\mu_{GFF}).$$

④ Hypercontractivity.

$X =$ deg $\leq n$ poly. of Gaussian R.V.s

$$\Rightarrow \quad \mathbb{E}[|X|^p]^{\frac{1}{p}} \leqslant (p-1)^{\frac{n}{2}} \mathbb{E}[X^2]^{\frac{1}{2}}$$

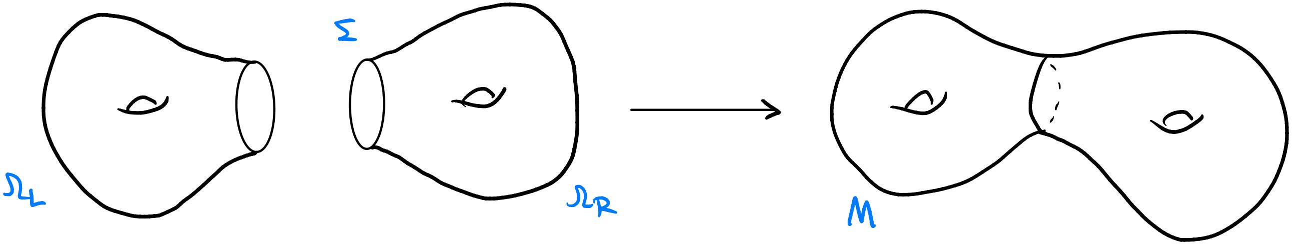
$\Rightarrow$ Cauchy in L^p, $\forall\, 1 \leq p < \infty$

⑤
$$\begin{aligned}
\mathbb{P}\big(e^{-S_{M,\chi}} \geqslant e^{b_2|\log(2\varepsilon)|^n+1}\big) &= \mathbb{P}\left(S_{M,\chi} \leqslant -b_2|\log(2\varepsilon)|^n - 1\right) \\
&\leqslant \mathbb{P}\left(|S_{M,\chi} - S_{M,\chi,\varepsilon}| \geqslant 1\right) \\
&\leqslant \|S_{M,\chi} - S_{M,\chi,\varepsilon}\|^p_{L^p(\mu_{\mathrm{GFF}})} && \leftarrow \text{Chebychev} \\
&\leqslant (p-1)^{\frac{np}{2}} C_1^p \varepsilon^{\frac{p}{2}} \|\chi\|^p_{L^4} && \leftarrow \text{③ \& ④}, \ \forall p! \\
&\lesssim \|\chi\|^p_{L^4} p^{\frac{n}{2}p} (C_1 \varepsilon^{\frac{1}{2}})^p, \\
&\lesssim \exp\left(-C_2 (\varepsilon^{\frac{1}{2}} \|\chi\|_{L^4})^{-\frac{1}{n}}\right). && \leftarrow \text{minimize over } 1 \leq p < \infty
\end{aligned}$$

(Goal: $e^{-S_M} \in L^1$)

⑥
$$\mathbb{E}\left[e^{-S_{M,\chi}}\right] = \underbrace{\int_0^\infty \mathbb{P}\left(e^{-S_{M,\chi}} \geqslant t\right) dt}_{< \infty} \Rightarrow 🏆$$

$\boxed{\text{The case for boundary.}}$ Best illustrated by the following simple case:



Unpacking definition, the composition property boils down to

$$C_{det}\int e^{-S_M(\phi)}d\mu_{GFF}^{M} = C'_{det}\iiint e^{-S_L(\phi_L|\varphi)}e^{-S_R(\phi_R|\varphi)}\underbrace{d\mu_{GFF}^{\Omega_L,D}(\phi_L)\otimes d\mu_{GFF}^{\Omega_R,D}(\phi_R)\otimes d\mu_{DN}^{\Sigma}(\varphi)}_{\text{"Markov" decomposition of GFF.}}$$

boundary condition $\phi_L|_\Sigma = \varphi$.

Heuristically, $S_L(\phi_L|\varphi) \overset{\text{def}}{=\!=} \int :P(\phi_L^D(x)+PI\varphi(x)): dx$, $\phi_L^D \sim$ GFF on Ω_L with (zero) Dirichlet condition on Σ

- ⊛ How to define $S_L(\phi_L|\varphi)$ on Ω_L independent of any $\Omega_L \hookrightarrow M$. for ϕ_L^D, φ both random?
- ⊛ $S_M(\phi_L^D+\phi_R^D+PI\varphi) = S_L(\phi_L|\varphi)+S_R(\phi_R|\varphi)$ as RV?

Technical Requirement.

⊛ Need a freedom in choosing the regulators $(K_\varepsilon)_{\varepsilon>0}$.
More precisely, we show the Nelson procedure produce the same RV. $S_M(\Phi)$, as long as $K_\varepsilon \xrightarrow{\varepsilon\to 0^+} \mathbb{1}$ in $\Psi^{0+}(M)$.

⊛ Existence of a local regulator ($\approx$ conv. with $\mathcal{C}^\infty$-bump, supp $\to 0$).
Rmk. $e^{-\varepsilon\Delta} \neq$ local

⊛ Define $S_L(\phi_L|\varphi) \overset{\text{def}}{=} \int_M 1_{\Omega_L} :P(\phi(x)): dx$ for any embedding $\Omega_L \hookrightarrow M$.

Show $S_L(\phi_L|\varphi)$ measurable WRT σ-algebra generated by
$\{\phi(f) \mid \operatorname{supp} f \subseteq \Omega_L\}$, $\phi \sim$ GFF.

$\rightsquigarrow$ it's an RV well-defined under $\mu_{GFF}^{\Omega_L, D} \otimes \mu_{DN}^{\Sigma}$

Thank you!