# AN INTERACTIVE FIELD THEORY APPROACH TO THE STOCHASTIC SINE-GORDON MODEL

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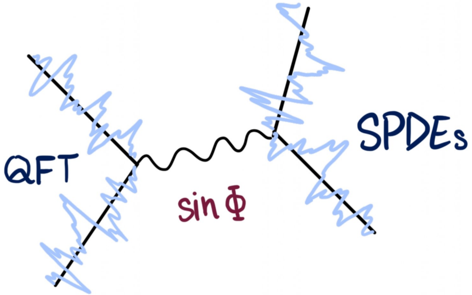
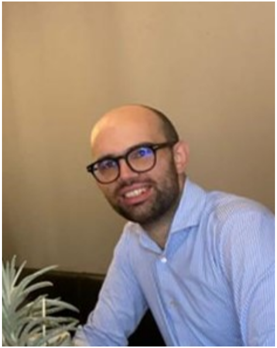



Rencontre ANR SMOOTH 2024

# Joint work with

(a) Claudio Dappiaggi
University of Pavia



(b) Paolo Rinaldi
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# Singular Stochastic PDEs

$$Lu = F(u) + H(u)\xi$$

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$\Rightarrow$ need for **renormalization**

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Present frameworks:

- Regularity structures
- Paracontrolled calculus
- Renormalization group techniques

| Pros | Cons |
|---|---|
| Well-posedness results | Few information on the solution |
| Widely applicable | |

# Sine-Gordon QFT

- **Geometry:** $2$-dim Minkowski $(\mathbb{R}^2, \eta)$
- **Field theory:** $a \in \mathbb{R}$, $g \in \mathcal{D}(\mathbb{R}^2)$

$$(\Box + m^2)\psi + \lambda g a \sin(a\psi) = 0$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \psi \partial^\mu \psi - \frac{1}{2} m^2 \psi^2 - \lambda g \cos(a\psi)$$

Finite ultraviolet regime: $a^2 < 4\pi/\hbar$

# pAQFT framework

2-step approach

1. construction of an **algebra of observables** $\mathcal{A}$
   - dynamics
   - causality
   - CCR/CAR ...

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1. construction of an **algebra of observables** $\mathcal{A}$
   - dynamics
   - causality
   - CCR/CAR ...

2. **state**: positive, normalized, linear functional $\omega : \mathcal{A} \to \mathbb{C}$
   $\to$ Expectations

Quantization? $\to$ Deformation of the algebraic structure

# Functional-based approach

$\mathcal{A}$ : functionals on the space of field configurations $\mathcal{E}(\mathbb{R}^2)$

$$\Phi_f(\varphi) = \int_{\mathbb{R}^2} f(t,x)\varphi(t,x)\,dtdx, \qquad \forall f \in \mathcal{D}(\mathbb{R}^2), \varphi \in \mathcal{E}(\mathbb{R}^2)$$
$$\Phi_f^2(\varphi) = \int_{\mathbb{R}^2} f(t,x)\varphi^2(t,x)\,dtdx$$

The theory goes **quantum** by switching to deformed products

$$F \star_{\hbar K} G := \mathcal{M} \circ e^{D_{\hbar K}}(F \otimes G)$$
$$D_{\hbar K} := \left\langle \hbar K, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \right\rangle = \int_{\mathbb{R}^2} dxdy\,\hbar K(x,y)\frac{\delta}{\delta\varphi(x)} \otimes \frac{\delta}{\delta\varphi(y)}$$

# Functional-based approach

Examples:

- $\Delta = \Delta_R - \Delta_A$ causal propagator $\to$ CCR
- $\omega = \frac{i}{2}\Delta + H$ Hadamard parametrix $\to$ Wick-ordered observables
- $\Delta_F$ Feynman propagator $\to$ time ordering

# Functional-based approach

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**Issues**:

1. Singular structure of the kernel = clash with the one of functional derivatives of the observables
2. $e^{D_{\hbar K}}$ yields a formal power series in $\hbar$

# Interacting theory

- *S*-matrix

$$S(\lambda V) := \exp_{\star_{\hbar\Delta_F}}\left(\frac{i}{\hbar}\lambda V\right) := \sum_{n\geq 0}\frac{1}{n!}\left(\frac{i}{\hbar}\lambda\right)\underbrace{V \star_{\hbar\Delta_F} \ldots \star_{\hbar\Delta_F} V}_{n}$$

$$V_g := \frac{V_{a,g} + V_{-a,g}}{2}, \qquad V_{a,f}(\varphi) := \int_{\mathbb{R}^2} dx\, f(x)\, e^{ia\varphi(x)}$$

# Interacting theory

- $S$-matrix

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- Bogoliubov map: interacting version of $F \in \mathcal{F}((\mathbb{R}^2)^{\otimes m})$

$$R_{\lambda V}(F) = \sum_{n\geq 0}\frac{\lambda^n}{n!}R_{n,m}(V^{\otimes n}, F),$$

where $R_{n,m}(V^{\otimes n}, F)$ involves $\Delta_F$ and $\Delta_{AF}$

# Algebraic/microlocal approach to SPDEs

**Idea**: singular SPDEs as nonlinear QFTs

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**Idea**: singular SPDEs as nonlinear QFTs

state $\omega$ $\longleftrightarrow$ Covariance of the random field

$$\Delta u = \lambda u^3 + \xi$$

$\xi$ spacetime white noise: a Gaussian **random distribution** with

$$\mathbb{E}[\xi(f)] = 0, \qquad \mathbb{E}[\xi(f)\xi(h)] = \langle f, h\rangle_{L^2}, \qquad f, h \in \mathcal{D}(\mathbb{T}^d)$$

$\longrightarrow$ perturbation of the linear equation

$$\Delta u_0 = \xi$$

# Algebraic/microlocal approach to SPDEs

$G \in \mathcal{D}'(\mathbb{T}^d \times \mathbb{T}^d)$ fundamental solution

$$u_0 = G * \xi$$

$$\mathbb{E}[u_0(f)u_0(g)] = \int_{(\mathbb{T}^d)^2} \left( \int_{\mathbb{T}^d} G(\bar{z}, z) G(\bar{z}, z')\, d\bar{z} \right) f(z) g(z')\, dz dz' = Q(f, g)$$

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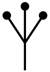
**Perturbative study** of the solution

$$u[\![\lambda]\!] = \sum_{n=0}^{\infty} \lambda^n u_n$$

$$u_0 = G * \xi =$$ 

$$u_1 = G * (G * \xi)^3 =$$ 

$$u_n = \sum_{k_1+k_2+k_3=n-1} G * (u_{k_1} u_{k_2} u_{k_3})$$

# Algebra of observables[1,2]

Inspired by Algebraic Quantum Field Theory

1. Promote the random field $u_0 = G * \xi$ to a functional-valued distribution $\Phi$ defined via

$$\Phi_f(\varphi) = \int_{\mathbb{R}^2} dtdx\, f(t,x)\varphi(t,x), \qquad \forall f \in \mathcal{D}(\mathbb{T}^d), \varphi \in \mathcal{E}(\mathbb{T}^d)$$

2. algebra of observables: polynomial (multi)local functionals + condition on the **wavefront set**

[1] C. Dappiaggi, N. Drago, P. Rinaldi, L. Zambotti, CCM (2022).
[2] A. B., C. Dappiaggi, P. Rinaldi, AHP (2023)

# Deformation of the algebra structure

Functionals don't know about the stochastic nature of the problem
$\Rightarrow$ **deformation** of the tensor product

$$D_Q(F)(f;\varphi) := \left\langle Q, \frac{\delta}{\delta\varphi_1} \otimes \frac{\delta}{\delta\varphi_2} F(f;\varphi) \right\rangle \qquad F \in \mathcal{T}(\mathcal{P}_{Loc})$$

$$\Gamma_Q(F) = e^{D_Q}(F)$$

$$\Gamma_Q(F_1 \otimes F_2) = \Gamma_Q(F_1) \star_Q \Gamma_Q(F_2)$$

**Algebra endomorphism**

Remark | These are formal expressions. One should either regularize $G$ or perform a renormalization procedure (Epstein-Glaser)

# Expectation values

The deformation map allows to **compute expectation values** of polynomial expressions in the random distribution $u_0$:

$$\mathbb{E}[P(u_0)(f)] = \Gamma_Q(P(\Phi))(f;0)$$

## Example

$$\Gamma_Q(\Phi^2)(f,0) = Q(f\delta_{Diag_2}) \equiv \mathbb{E}[u_0^2(f)]$$

$\Gamma_Q(\Phi^2) =$  $+$ 

# Renormalization

$Q \sim G^2$ ill-defined product of distributions

**Theorem**

The Fourier transform of a compactly supported smooth function is rapidly decreasing.

**Wavefront set**: singular points as well as singular directions
$\rightarrow$ products of distributions, composition...

**Scaling degree**: local behaviour
$\rightarrow$ extension over the singular support (renormalization)

# Back to stochastic sine-Gordon

Recent results[3] on convergence in the AQFT framework
$\Rightarrow$ we adapted them to our setting

$$(\Box + m^2)u + \lambda g a \sin(au) = \chi\xi$$

- Algebra of functionals $(\mathcal{F}^V \subset \mathcal{F}_{\mu c}, \star_{\hbar K}, *) \supset$ exponentials of the field

Standard approach: classical Möller map + action of $\Gamma_Q$ to get expectation values

[3] D. Bahns, N. Pinamonti, K. Rejzner, JMAA (2021)

## Strategy[4]

→ Cannot address the question on convergence of the perturbative series defining the expectation values.

We divide our approach in two steps:

1. **stochastic information within the quantum theory**
2. recovering expectations via **classical limit**

$$\Gamma_Q[r_{\lambda V_g}(\varphi)]|_{\varphi=0} = \lim_{\hbar\to 0^+} \Gamma_Q[R_{\lambda V_g}(\varphi)]|_{\varphi=0}$$

[4] A. B., C. Dappiaggi, P. Rinaldi, Arxiv (2023)

# Interplay between quantum and stochastic

$$F \in \mathcal{F}_{loc}(\mathbb{R}^2) \;\to\; R_{\lambda V_g}(F) = S(\lambda V_g)^{\star_{\hbar\omega}-1} \star_{\hbar\omega} (S(\lambda V_g) \star_{\hbar\Delta_F} F)$$

$$\Gamma_Q[R_{\lambda V_g}(F)] = \Gamma_Q[S(\lambda V_g)^{\star_{\hbar\omega}-1}] \star_{\hbar\omega+Q} \left[\Gamma_Q[S(\lambda V_g)] \star_{\hbar\Delta_F+Q} \Gamma_Q[F]\right]$$

We introduce the so called $Q-S$ **matrix**

$$\Gamma_Q[S(\lambda V_g)] = \sum_{n\geq 0} \frac{1}{n!}\left(\frac{i\lambda}{2\hbar}\right)^n \sum_{k=0}^{n} \binom{n}{k} \mathcal{T}^{\hbar\Delta_F+Q}\left[\Gamma_Q[V_{a,g}]^{\otimes k} \otimes \Gamma_Q[V_{-a,g}]^{\otimes n-k}\right]$$

# Interplay between quantum and stochastic

$$\Gamma_Q(V_{\pm a,g}) = \ldots = \int_{\mathbb{R}^2} dx\, g(x)\, e^{-\frac{a^2}{2} Q(x,x)}\, e^{\pm i a \varphi(x)} := V_{\pm a, g_Q}$$

$$g_Q(x) := g(x)\, e^{-\frac{a^2}{2} Q(x,x)} \in \mathcal{D}(\mathbb{R}^2)$$

$\Rightarrow \Gamma_Q$ simply modifies the localization

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$\Rightarrow \Gamma_Q$ simply modifies the localization

$$\left|[\Gamma_Q[S(\lambda V_g)]]_n\right| \le \frac{1}{n!}\left(\frac{\lambda}{\hbar}\right)^n \mathrm{ev}_0[\mathcal{T}_n^{\hbar H+Q}(V_g \otimes \ldots \otimes V_g)], \qquad H = \mathrm{Re}(\Delta_F)$$

Remark | $H$ and $Q$ are symmetric $\Rightarrow$ we switch to a commutative algebra

# Convergence of the Q-S matrix

- Conditioning and inverse conditioning (Euclidean QFT[5]): controlling the massive theory via the massless one

Remark | to obtain positivity, one has to restrict to spacetime diamonds

$$D_\mu := \{(t,x) \in \mathbb{R}^2 \mid -\mu < t-x < \mu,\ -\mu < t+x < \mu\} \quad \Rightarrow \quad supp(g) \subseteq D_\mu$$

- Cauchy determinant: specific form of the propagators in $1+1$ dimensions

[5] J. Frölich, CMP (1976)

# Convergence of the Q-S matrix

## Theorem

Setting $\alpha := \frac{a^2\hbar}{4\pi}$ and for $0 < \alpha < 1$, there exist positive constants $\widetilde{C}$, $C_Q(\mu)$ and $K$ such that

$$\left|[\Gamma_Q[S(\lambda V_g)]]_n\right| \leq \frac{2(2\mu)^{n\alpha}(C_Q)^{n^2}}{(n!)^{1-1/p}}\left(\frac{2\lambda e^{2^{-1}a^2K}}{\hbar}\right)^n ||g||_{L^q}\widetilde{C}^{n/p},$$

for $p \in [1, \alpha^{-1})$, $\frac{1}{p} + \frac{1}{q} = 1$.

As a corollary, the series $\Gamma_Q[S(\lambda_{V_g})](\varphi) = \sum_{n\geq 0}[\Gamma_Q[S(\lambda V_g)]]_n$ is **absolutely convergent** for all $\varphi \in \mathcal{E}(\mathbb{R}^2)$

# Other convergence results

- Stochastic interacting field

$$\Gamma_Q[\Phi_{I,f}] := \Gamma_Q[R_{\lambda V_g}(\Phi_f)]$$

- $n$-point functions

$$\Gamma_Q[R_{\lambda V_g}(\Phi_{f_1} \ldots \Phi_{f_n})]$$

In both cases we get absolute convergence of the power series in $\lambda$ as a generalization of the $Q-S$ matrix case

# Classical limit $\hbar \to 0^+$

We must get rid of the quantum side

For $\lim_{\hbar\to 0^+} R_{\lambda V_g}(F)$ to exist, **the argument must not contain negative powers of $\hbar$**.

$\Rightarrow$ combining it with absolute convergence of the series ensures existence of the **non-perturbative momenta** of the solution

# Classical limit $\hbar \to 0^+$

Via combinatorial arguments we obtain that:

- all the non-vanishing contributions to $R_{n,m}(V_g^{\otimes n}, F)$ are such that any $V_g$ is connected with one of the entries of $F$.
- for any $n \geq 0$,

$$R_{n,m}(V_g^{\otimes n}, F) = \mathcal{O}(\hbar^0)$$

Non-trivial task:

$$\lim_{\hbar \to 0^+} R_{\lambda V_g}(F) \overset{?}{=} r_{\lambda V_g}(F)$$

$\rightarrow$ we are studying the actual solution to the stochastic sine-Gordon equation

# Stochastic bosonization ✪

**Bosonization**:

massless sine-Gordon model $\longleftrightarrow$ massive Thirring model

**Question**: does it survives at the stochastic level?

Convergence results for the SG model + algebraic approach to the stochastic Dirac equation[6]

- Convergence for Dirac?
- Similar results for SHE $\longleftrightarrow$ KPZ equation

[6] A. B., B. Costeri, C. Dappiaggi, P. Rinaldi, Arxiv (2023)

If you have any questions, I would be glad to (try to) answer them

# Algebras of functional-valued distributions

$\mathcal{D}'(M; Fun)$ space of polynomial functional-valued distributions

$$\tau : \mathcal{D}(M) \times \mathcal{E}(M) \to \mathbb{C}$$

linear in the first component and continuous in the locally convex topology of $\mathcal{D}(M) \times \mathcal{E}(M)$

- Functional derivatives $F^{(k)} \in \mathcal{E}'(\underbrace{M \times \ldots \times M}_{k}; \mathcal{D}'(M; Fun))$

$$F^{(k)}(f \otimes \eta_1 \otimes \ldots \otimes \eta_k; \varphi) := \frac{\partial^k}{\partial s_1 \ldots \partial s_k} F(f; \varphi + s_1\eta_1 + \ldots + s_k\eta_k)$$

# Algebras of functional-valued distributions

- Microcausal functionals: condition on the WF
  $\longrightarrow$ Analogy with the problem of Wick renormalization

$$\mathcal{D}'_C(M^k; Pol) := \left\{ F \in \mathcal{D}'(M^k; Pol) \mid WF(F^{(n)}) \subseteq C_{k+n} \, \forall n \geq 0 \right\}$$

$C_n$ sort of Cartesian product of diagonals

$$\mathcal{A}_0 := \mathcal{E}\{\mathbf{1}, \Phi\}, \qquad \mathcal{A}_j := \mathcal{E}\{\mathcal{A}_{j-1} \cup G * \mathcal{A}_{j-1}\} \qquad \mathcal{A} := \varinjlim \mathcal{A}_j$$

Algebraic structure: **pointwise product** $[\tau_1 \tau_2](f; \eta) := \tau_1 \otimes \tau_2(f \delta_{Diag_2}; \eta)$

## Deformations

### Theorem

Let $\Gamma_Q : \mathcal{A} \to \mathcal{D}'_C(M; Pol)$ be the deformation map constructed before. Then $(\mathcal{A}_{\cdot_Q} := \Gamma_Q(\mathcal{A}), \cdot_Q)$ is a unital, commutative and associative algebra w.r.t. the product

$$\tau_1 \cdot_Q \tau_2 := \Gamma_Q[\Gamma_Q^{-1}\tau_1 \Gamma_Q^{-1}\tau_2]$$

**Corollary**: $\Gamma_Q$ is an algebra homomorphism

# Nonlocal algebra - correlations

$$\mathcal{T}(\mathcal{A}_{\cdot_Q}) := \mathcal{E}(M) \oplus \bigoplus_{n\geq 1} \mathcal{A}_{\cdot_Q}^{\otimes n}, \qquad \mathcal{T}'_C(M;Pol) := \mathbb{C} \oplus \bigoplus_{n\geq 1} \mathcal{D}'_C(M^n, Pol)$$

We look for a map $\Gamma_{\bullet_Q} : \mathcal{T}(\mathcal{A}_{\cdot_Q}) \to \mathcal{T}'_C(M;Pol)$ implementing the covariance of the noise

$\Rightarrow$ we just remove the pullback on the diagonal from $\cdot_Q$.

## Theorem

Given $\Gamma_{\bullet_Q} : \mathcal{T}(\mathcal{A}_{\cdot_Q}) \to \mathcal{T}'_C(M;Pol)$, $(\mathcal{A}_{\bullet_Q} := \Gamma_{\bullet_Q}(\mathcal{T}(\mathcal{A}_{\cdot_Q})), \cdot_Q)$ is a unital, commutative and associative algebra w.r.t. the product

$$\tau_1 \bullet_Q \tau_2 := \Gamma_{\bullet_Q}[\Gamma^{-1}_{\bullet_Q}\tau_1 \otimes \Gamma^{-1}_{\bullet_Q}\tau_2], \qquad \tau_1, \tau_2 \in \mathcal{A}_{\bullet_Q}$$

## Renormalized $\varphi^4_n$ equation

Applying $\Gamma_Q$ to the mild equation we get

$$\Psi_{\cdot_Q} = \Phi - \lambda G * (\Psi_{\cdot_Q} \cdot_Q \Psi_{\cdot_Q} \cdot_Q \Psi_{\cdot_Q})$$

To express it in terms of pointwise products, there is a price to pay

**Theorem**

There exists a sequence of functional-valued linear operators $\{M_n\}_{n\in\mathbb{N}}$ such that

- $M_n(\varphi) : \mathcal{E}(M) \to \mathcal{E}(M)$ for all $\varphi \in \mathcal{E}(M)$, $\forall n \in \mathbb{N}$,
- $M_n(\varphi)$ has an even polynomial dependence on $\varphi$,
- defining $M := \sum_{n\geq 0} \lambda^n M_n$, $\Psi_{\cdot_Q}$ solves the equation

$$\Psi_{\cdot_Q} = \Phi - \lambda G * \Psi^3_{\cdot_Q} - G * M\Psi_{\cdot_Q}$$