Rough integrators on Banach manifolds

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Abstract. We introduce a notion of p-rough integrator on any Banach manifolds which plays the role of weak geometric Hölder p-rough paths in the usual Banach space setting, for any $p \geqslant 1$. The awaited results on rough differential equations driven by such objects are proved, and a canonical representation is given if the manifold is equipped with a connection.

1   Introduction

One of the most basic and illustrative example of a rough path is associated with the following 2-dimensional spinning signal

$$h_t^n = \frac{1}{n}\big(\cos(n^2 t), \sin(n^2 t)\big), \quad 0 \leqslant t \leqslant 1.$$

Although h^n converges uniformly to 0, this path sweeps an area t in any time interval $[0, t]$, independantly of n, suggesting that this sequence of paths may be really different from the zero signal. As a matter of fact, given any two vector fields V_1, V_2 on $\mathbb{R}^d$, of class C_b^3, it is remarkable that the solution path $x_\bullet^n$ to the ordinary controlled differential equation

$$\dot{x}_t^n = \big(\dot{h}_t^n\big)^i V_i\big(x_t^n\big)$$

converges uniformly to the solution path $x_\bullet$ to the ordinary differential equation

$$\dot{x}_t = \big[V_1, V_2\big](x_t).$$

This is due to the fact that the canonical lift of $h_\bullet^n$ as a weak geometric Hölder p-rough path, for any $2 < p < 3$, converges in a rough paths sense to the pure area rough path with area process $(t - s)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ – we refer the reader to the lecture notes [1, 2] and [3, 4] for basics on rough paths and rough differential equations.

This elementary example makes it clear that, when understood as controls, smooth paths may have a richer structure than expected at first sight, as the above constant path with non-trivial area process shows. Smooth paths have canonical lifts as rough paths; a smooth path in $\mathbb{R}^\ell$ whose rough path lift is non-canonical will be called a *spinning path*. While this may not be obvious, we shall see in this note that it actually makes perfect sense to talk of a spinning manifold-valued path, as an example of a manifold valued 'rough path'. One should keep in mind, however, that the classical notion of rough path only becomes really interesting when understood as a control in some differential equation. Does it make sense to think of a manifold-valued smooth path as a control? Yes. It is indeed common, in physics and differential geometry, to be given a bundle $\mathscr{B}$ over some manifold $\mathscr{M}$, together with a connection, given by a $T\mathscr{B}$-valued 1-form H on $\mathscr{M}$. Lifting a smooth $\mathscr{M}$-valued

path $\gamma_\bullet$ into a smooth horizontal $\mathscr{B}$-valued smooth path $e_\bullet$ is a basic procedure in which γ is used as a control in the ordinary differential equation

$$\dot{e}_t = \mathrm{H}(e_t)\dot{\gamma}_t$$

defining the path $e_\bullet$. What would happen if γ were a kind of spinning smooth $\mathscr{M}$-valued path? What about *spinning geodesics* in a Riemannian setting? The notion of p-rough integrator introduced below, and the results proved, provide a framework where one can answer such questions.

So far there has been only a few works dealing with rough paths in a geometrical setting, starting with the work [5] of Lyons and Qian. This seminal work investigated the well-posedness problem for the ordinary differential equation on the path space of a compact manifold generated by Itô vector fields, with an eye on probabilistic applications related to path space versions of the Cameron-Martin theorem and Driver's flow equation. It was enriched by the work [6] of Li and Lyons, showing that the Itô-Lyons solution map to a Young differential equation is Fréchet regular under appropriate conditions, when the driving signal has finite p-variation, with $1 < p < 2$, and by the work [7] of the author providing a general regularity result for the Itô-Lyons solution map, for any $p \geqslant 2$, in the setting of controlled paths. Another work [8] of Lyons and Qian addressed well-posedness issues for ordinary differential equations on path space associated with Itô vector fields obtained by varying the driving rough signal – see also [7] for similar results.

These works only use rough paths as an ingredient to construct some dynamics in a geometric configuration space. Cass, Litterer and Lyons made a step further in putting rough paths theory in a geometrical setting and proposed in [9] a notion of rough path on a manifold extending the classical notion defined in a linear setting. In the same way as a vector field on a manifold $\mathscr{M}$ can be understood in an analytic/algebraic setting as a differentiation in the ring of smooth functions on the manifold $\mathscr{M}$, a rough path is abstractly defined as a linear form on the space of sufficiently regular 1-forms on $\mathscr{M}$, which is required to have some continuity property; call it an *integrator*. This functional analytic definition rests on a basic chain rule which eventually enables to understand their notion of rough path on a manifold as an equivalence class of classical rough paths, related by some chain rule under change of coordinates. This situation is exactly similar to representing a tangent vector on a d-dimensional manifold as an equivalence class of vectors in $\mathbb{R}^d$, indexed by local diffeomorphisms of a neighbourhood of 0 (that is local changes of coordinates), and related by a change of coordinate rule which exactly balances the changes in the numerical representation of a given 1-form α on $\mathscr{M}$ associated with local coordinates, so the quantity $\alpha(u)$ is independent of any choice of coordinates used to compute it. Their approach rests however on a notion of Lip-γ manifold which prevents its easy use even with non-compact finite dimensional manifolds, not to speak about infinite dimensional manifolds.

The ideas of [9] have been reloaded in a different and more accessible form in the recent work [10] by Cass, Driver and Litterer, in which they define a weak geometric Hölder p-rough path on a finite dimensional compact embedded submanifold of $\mathbb{R}^d$ as an integrator, obtained by "projection" of a weak geometric Hölder p-rough path in the ambient Euclidean space, for $2 \leqslant p < 3$ only. Nothing is lost in working with submanifolds, and this notion is eventually shown to be independent of the embedding of the manifold in its environment, while these objects can be used to define and solve uniquely rough differential equations driven by weak geometric

Hölder p-rough paths on compact manifolds. They can construct parallel transport along manifold-valued rough paths in their sense, and use it to show a one-to-one correspondence between rough paths on a finite d-dimensional manifold and rough paths on d-dimensional Euclidean space. The recent work [11, 12] of Boutaib and Lyons follow the original intuition of [9] and clarify the geometric notion of Lipscthiz manifold needed to make it work. On the other hand, Driver and Semko adapt in [13] Gubinelli's notion of controlled path to a manifold setting, to deal with p-rough paths on manifolds with $2 \leqslant p < 3$, by using a formalism very close to the second order calculus developed by Schwartz in his analysis of Itô calculus on manifolds – see for instance Meyer expository article [14], or Emery's book [15].

We propose in this note an elementary and flexible notion of weak geometric Hölder p-rough path on Banach manifolds that goes beyond the previous works. Roughly speaking, a weak geometric Hölder p-rough path on a manifold $\mathscr{M}$ is a triple made up of a vector field valued 1-form F defined on some Banach space U, a weak geometric Hölder p-rough path $\mathbf{X}$ over U, and the maximal solution to the rough differential on $\mathscr{M}$ constructed from F and $\mathbf{X}$. The data of these three objects is sufficient to define and solve uniquely rough differential equations driven by "manifold-valued" rough paths, or better, by *p-rough integrators*. The above-mentionned spinning smooth paths on $\mathscr{M}$ are precisely those p-rough integrators whose associated $\mathscr{M}$-valued paths are smooth. Nothing else than the (smooth) manifold structure is needed to make sense of a p-rough integrator. One shows in section 3.2 that these objects have a canonical representation if the tangent bundle of $\mathscr{M}$ is equipped with a connection, in which case one can always choose for Banach space U in the preceeding description of a p-rough integrator the Banach space on which the manifold is modelled.

The interest of working with Banach manifolds comes from the fact that they naturally pop up in a number of geometric situations, as path or loop spaces over some finite dimensional manifold, as in the works [16, 17] of Brzezniak, Carroll and Elworthy, or the works [18, 19] of Inahama and Kawabi, or as manifolds of maps of a given finite dimensional manifold, as in the works [20, 21] of Elworthy and Brzezniak, to mention but a few works from the probability community. Note however that many interesting infinite dimensional manifolds are Fréchet manifolds, for which no theory of rough paths is presently available. As a matter of fact, working with Banach manifolds will not bring any additional difficulty along the way, as compared to working with finite dimensional manifolds. We refer the reader to the books [22] of Lang, and [23] of Abraham, Marsden and Ratiu, for the basics of differential geometry in an infinite dimensional setting. Let just recall that it makes perfect sense in a manifold setting to say that a (Banach space or real-valued) function f of class $\mathcal{C}^k$, defined on the domain of some local chart φ of $\mathscr{M}$, has bounded derivatives if the function $f \circ \varphi^{-1}$, defined on a neighbouurood of 0 in some Banach space, has bounded derivatives; the bounded character will indeed not depend on the precise chart used to define it, while the bounds themselves will depend on φ.

Notations. We gather here a few notations that will be used throughout the note.

- We shall denote by U a generic Banach space; the notation $T^{[p]}(\text{U})$ will be used for the truncated tensor product of order $[p]$, completed for some choice of tensor norms. The letter $\mathbf{X}$ will stand for a weak geometric Hölder p-rough path over U, for some $p \geqslant 2$, and for the above choice of tensor norm on $T^{[p]}(\text{U})$.

- We shall denote by $\mathscr{M}$ a Banach manifold modelled on some Banach space E. The set of continuous linear homomorphisms of E will be denoted by L(E), and the set of continuous linear isomorphisms of E will be denoted by GL(E).

2 Rough integrators on Banach manifolds

We define in this section what will play the role in a manifold setting of weak geometric Hölder p-rough paths. Rough integrators are basically defined as a triple, consisting of a traditional weak geometric Hölder p-rough path with values in some Banach space U, a $T\mathscr{M}$-valued vector field valued 1-form on U, and a solution to an associated rough differential equation on the manifold. This definition, given in section 2.2, requires that we recall some facts on rough differential equations with values in manifolds; this is done in section 2.1. We explain in section 2.3 how rough integrators can be used to define and solve rough differential equations.

2.1 Rough differential equations

We adopt in this work the definition of a solution path to an $\mathscr{M}$-valued rough differential equation given in [25] in a Banach space setting, as it is perfectly suited for our needs. (This definition was also adopted in Cass and Weidner's work [26] on Hörmander's theorem for rough differential equations on manifolds.) It essentially amounts to requiring from a solution path that it satisfies some uniform Taylor-Euler expansion formulas, in the line of Davie' seminal work [24]. Let U be a Banach space, and F be a 1-form on U with values in the space of vector fields on a Banach manifold $\mathscr{M}$, of class $\mathcal{C}^{[p]+1}$. Given $u \in$ U, we identify the vector field $\mathrm{F}(\cdot\,; u)$ on $\mathscr{M}$ with its associated first order differential operator $\mathrm{F}^{\otimes}(u)$, and extend the definition of F to $T^{[p]}(\mathrm{U})$ setting $\mathrm{F}(1) = \mathrm{Id}$, and

$$\mathrm{F}(u_1 \otimes \cdots \otimes u_k) = \mathrm{F}(u_1)\cdots\mathrm{F}(u_k),$$

for $1 \leqslant k \leqslant [p]$, and $u_i \in$ U, and by linearity; so $\mathrm{F}(u_1 \otimes \cdots \otimes u_k)$ is a differential operator of order k. Given a weak geometric Hölder p-rough path $\mathbf{X}$ over U, recall that an $\mathscr{M}$-valued continuous path $(x_t)_{0 \leqslant t < \zeta}$ is said to solve the rough differential equation

$$dx_t = \mathrm{F}\big(x_t\,; \mathbf{X}(dt)\big) \tag{1}$$

if there exists a constant $a > 1$ such that, for any $0 \leqslant s < \zeta$, there exists an open neighbourhood V_s of x_s such that we have the Taylor-Euler expansion

$$f(x_t) = \big(\mathrm{F}(\mathbf{X}_{ts})f\big)(x_s) + O\big(|t-s|^a\big)$$

for all t close enough to s for x_t to belong to V_s, and any function f of class $\mathcal{C}^{[p]+1}$ defined on V_s, where it has bounded derivatives. The results of [25] show that such a rough differential equation has a unique maximal solution started from any given point, as awaited. It also follows from the results of [25], or other classical works, that the solution path $x_\bullet$ depends continuously on the driving signal $\mathbf{X}$ in the following sense. Fix $T < \zeta$ and cover the compact support of the path $(x_t)_{0 \leqslant t \leqslant T}$ by finitely many local chart domains $(O_i')_{1 \leqslant i \leqslant N}$ and $(O_i)_{1 \leqslant i \leqslant N}$, with $O_i' \subset O_i$ for all $1 \leqslant i \leqslant N$. Then, there exists a positive constant δ such that for $\mathbf{Y}$ δ-close to

X in the Hölder rough path distance, the solution path $y_\bullet$ to the rough differential equation

$$dy_t = F\big(y_t\,;\mathbf{Y}(dt)\big)$$

started from x_0, will be well-defined on the time interval $[0,T]$ and will remain in the open neighbourhood $\bigcup_{i=1}^N O_i$ of the support of $(x_t)_{0\leqslant t\leqslant T}$, with y_t in O_i whenever x_t is in O_i'.

Note here that the rough path setting allows to work with any Banach manifold, without any restriction on the Banach space on which it is modelled, unlike what needs to be done in a stochastic setting where a robust theory of stochastic integration is only available for some special types of Banach spaces, the so-called M-type 2 and UMD spaces, see [27, 28].

EXAMPLES. **1. Developping a spinning straight line.** *Pick $2 \leqslant p < 3$ and let $\mathscr{M}$ be a Riemannian surface; denote by H_1 and H_2 the canonical horizontal vector fields on the orthonormal frame bundle $\mathscr{OM}$ of $\mathscr{M}$, equipped with Levi-Civita connection. Denote by V_{12} the unique canonical vertical vector field and denote by Ω the (scalar-valued) curvature (tensor). Let (ϵ_1, ϵ_2) stand for the canonical basis of $\mathbb{R}^2$, and let $\mathbf{X}$ be the weak geometric Hölder p-rough path with first level $(t-s)\epsilon_1$ and area process $(t-s)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; this path was called above the spinning straight line. Denote by H the $T\mathscr{OM}$-valued 1-form $u \in \mathbb{R}^2 \mapsto u^1 H_1 + u^2 H_2$. Then the $\mathscr{OM}$-valued solution path to the rough differential equation*

$$de_t = H\big(e_t;\mathbf{X}(dt)\big) \qquad (2)$$

is actually the solution path to the ordinary differential equation

$$\dot{e}_t = \big(H_1 + \Omega\,V_{12}\big)(e_t).$$

2. Finding back a geodesic. *Assume for simplicity that Ω is constant, and take as a rough path in equation (2) the weak geometric Hölder q-rough path $\mathbf{Y}$, with $4 < q < 5$, whose logarithm is given by*

$$\log \mathbf{Y}_{ts} = (t-s)\Big(\epsilon_1 \oplus [\epsilon_1, \epsilon_2] \oplus \big[-\epsilon_1, \big[\epsilon_1, [\epsilon_1, \epsilon_2]\big]\big]\Big).$$

Then the solution path to the rough differential equation

$$de_t = H\big(e_t;\mathbf{Y}(dt)\big)$$

also solves the ordinary differential equation

$$\dot{e}_t = H_1(e_t),$$

that is, this path is the canonical lift to $\mathscr{OM}$ of a geodesic. This can be seen directly from the definition, or by appealing to the results proved in [29] by Friz and Oberhauser.

3. Rough differential equations with values in Banach Lie groups. *Let G be a Banach Lie group, with Lie algebra $\mathrm{LIE}(\mathsf{G})$. Think for instance to the loop groups, made up of $\mathcal{C}^k$ maps from some finite dimensional manifold M_0 to some finite dimensional Lie group G_0, with pointwise multiplication and inversion operations. Their Lie algebra is the set $\mathcal{C}^k\big(M_0, \mathfrak{g}_0\big)$ of $\mathcal{C}^k$ maps from M_0 to the Lie algebra $\mathfrak{g}_0$ of G_0, with bracket defined pointwise by the relation $\big[\mathsf{u}, \mathsf{v}\big](x) := \big[\mathsf{u}(x), \mathsf{v}(x)\big]$, for any u, v in $\mathcal{C}^k\big(M_0, \mathfrak{g}_0\big)$ and $x \in M_0$. These groups are extensively used in gauge theory or quantum field theory.*

Write L_g for the left translation by g in G. One defines a continuous linear map from $\mathrm{LIE}(\mathsf{G})$ to the space of smooth vector fields on G setting

$$\mathrm{F}(g\,;u) := \Big(D_{\mathrm{Id}}L_g\Big)(u),$$

for any $g \in \mathsf{G}$ and $u \in \mathrm{LIE}(\mathsf{G})$; so $\mathrm{F}(\cdot\,;u)$ is for any $u \in \mathrm{LIE}(\mathsf{G})$ a smooth left invariant vector field on G. It is elementary to proceed as in the proof of theorem 4.20 in [10] and see that for any weak geometric Hölder p-rough path $\mathbf{X}$ over $\mathrm{LIE}(\mathsf{G})$, defined on some interval $[0,T)$, for some $0 \leqslant T \leqslant \infty$, the solution path to equation (3) cannot explode before time T.

2.2 Rough integrators on Banach manifolds

The definition of a p-rough integrator given below is best understood in the light of the following example. Given a 1-form α on a d-dimensional manifold $\mathscr{M}$ equipped with a connection, how can one define the integral $\int_0^1 \alpha(\circ dx_t)$, of α along an $\mathscr{M}$-valued continuous semimartingale $(x_t)_{0\leqslant t\leqslant 1}$? The classical answer consists in showing that $x_\bullet$ can always be constructed as the projection in $\mathscr{M}$ of a $\mathrm{GL}(\mathscr{M})$-valued path $(e_t)_{0\leqslant t\leqslant 1}$ obtained by solving a stochastic differential equation in $\mathrm{GL}(\mathscr{M})$ involving the horizontal vector fields associated with the connection and some $\mathbb{R}^d$-valued semimartingale $(w_t)_{0\leqslant t\leqslant 1}$ – see for instance Chapter 2 in the book [30] of Hsu. One then defines $\int_0^1 \alpha(\circ dx_t)$ in terms of w only, setting

$$\int_0^1 \alpha(\circ dx_t) = \int_0^1 \big(\alpha \circ e_t^{-1}\big)\circ dw_t.$$

So the datum of w and the horizontal vector fields are all we need to define $x_\bullet$ and use it as a control. The next definition adopts a similar point of view in the present setting.

1. **Definition** $-$ *Let $p \geqslant 2$ be given. A* **weak geometric Hölder p-rough path on $\mathscr{M}$** *is a a triple $\Theta = \big((x_t)_{0\leqslant t<\zeta}, \mathrm{F}, \mathbf{X}\big)$, where*

   - $\mathbf{X}$ *a weak geometric Hölder p-rough path over some Banach space U, defined on the time interval $[0,\zeta)$,*
   - F *is a continuous linear map from U to the space of vector fields on $\mathscr{M}$ of class $\mathcal{C}^{[p]+1}$,*
   - *the path $(x_t)_{0\leqslant t<\zeta}$ solves the rough differential equation*

   $$dx_t = \mathrm{F}\big(x_t\,;\mathbf{X}(dt)\big). \tag{3}$$

   We also call Θ a **basic p-rough integrator***.*

   Given a p-rough integrator Θ as above, the triple $\big((x_t)_{0\leqslant t\leqslant T}, \mathrm{F}, \mathbf{X}\big)$, with $T < \zeta$, is also called a basic p-rough integrator. Let us insist on the fact that the Banach space U in the above definition is not fixed a priori, and depends on the basic p-rough integrator Θ on $\mathscr{M}$. So any weak geometric Hölder p-rough path $\mathbf{Y}$ over some Banach space V can be canonically seen as a weak geometric Hölder p-rough path in the above sense by choosing $\mathrm{U} = \mathrm{V}$ and $\mathbf{X} = \mathbf{Y}$, the identity map for F, and the first level of the rough path $\mathbf{X}_t$ for x_t. Note that we do not endow the set of basic p-rough integrators with a topology or a distance as this is somewhat subtle and we do not need it in the sequel.

The definition of a solution to the rough differential equation (3) makes it clear that if g is any sufficiently regular diffeomorphism between $\mathscr{M}$ and another manifold $\mathscr{M}'$, one defines a weak geometric Hölder p-rough path on $\mathscr{M}'$ setting

$$g^*\Theta := \Big(\big(g(x_t)\big)_{0 \leqslant t < \zeta}, g^*\mathrm{F}, \mathbf{X}\Big),$$

where $g^*\mathrm{F}(y\,;u) := \big(D_{g^{-1}(y)}\big)g\Big(\mathrm{F}\big(g^{-1}(y)\,;u\big)\Big)$, is the push forward on $\mathscr{M}'$ by g of the vector field $\mathrm{F}(\cdot\,;u)$ on $\mathscr{M}$.

Definition 1 encompasses the characterization of a weak geometric Hölder p-rough path on a submanifold of $\mathbb{R}^d$ given in [10], only for $2 \leqslant p < 3$, in terms of a priori extrinsic considerations. Roughly speaking, they define their class of weak geometric p-rough paths on a compact submanifold $\mathscr{M}$ of $\mathbb{R}^d$, for $2 \leqslant p < 3$, as the class of all weak geometric rough paths $\mathbf{Y}$ over $\mathbb{R}^d$ with the property that $\mathbf{Y}_0 = (1, y_0, 0, \dots)$ with $y_0 \in \mathscr{M}$, and

$$d\mathbf{Y}_t = Q(y_t)\,d\mathbf{Y}_t \tag{4}$$

for some projector-valued sufficiently regular map Q such that $Q(y)$ has range included in $T_y\mathscr{M}$ if $y \in \mathscr{M}$; this equation ensures that $y_t \in \mathscr{M}$. If $\mathscr{M}' \subset \mathbb{R}^{d'}$ is any other embedded compact manifold diffeomorphic to $\mathscr{M}$ through $\varphi : \mathscr{M} \to \mathscr{M}'$, equation (4) and the chain rule for rough integrals shows that one defines an image rough path $\mathbf{Z}$ over $\mathbb{R}^{d'}$ setting

$$d\mathbf{Z}_t = d\varphi(d\mathbf{Y}_t)$$

and that it satisfies the identity

$$\int_0^1 (\varphi^*\alpha)(d\mathbf{Z}_t) = \int_0^1 \alpha(d\mathbf{Y}_t),$$

for any sufficiently regular 1-form α on $T\mathscr{M}$. (The expression $\varphi^*\alpha$ stands for the push-forward of the 1-form α by φ, and both integrals make sense because of condition (4).) So the class of weak geometric p-rough paths is intrinsically defined as a class of rough integrators, independently of any particular embedding of $\mathscr{M}$, seen as an abstract manifold. Note that they use general controls $\omega(s,t)$ to measure the size of their rough paths, while we simply use $\omega(s,t) = t - s$ and Hölder scales. A weak geometric p-rough path in the sense of [10] is simply described in our terms as a special kind of weak geometric p-rough path in the ambient linear space via equation (4). The interest of working with the intrinsic definition 1 is clear in an infinite dimensional setting where embeddings of Banach manifolds into larger Banach spaces are less natural and rarely happen, unless $\mathscr{M}$ is parallelizable [31]. Note however that the frame bundle of $\mathscr{M}$ is always parallelizable if $T\mathscr{M}$ is equipped with a connection. See section 3.

One of the nice features of the notion of rough integrator on a manifold proposed in [9] and [10] is the possibility to push forward a rough integrator by a sufficiently regular map g from $\mathscr{M}$ to another manifold $\mathscr{N}$, giving as a result a rough integrator on $\mathscr{N}$. The rough integral of a 1-form α on $\mathscr{N}$ against the image rough integrator is simply defined as the rough integral of $\alpha \circ dg$ against the rough integrator on $\mathscr{M}$. A similar transport operation can be done in our framework, by seeing the push forward of a p-rough integrator Θ on $\mathscr{M}$ by $g : \mathscr{M} \to \mathscr{N}$ as the p-rough integrator $g\Theta$ on $\mathscr{M} \times \mathscr{N}$ associated with $(\mathrm{F}, dg \circ \mathrm{F})$ and $\mathbf{X}$.

It may happen that the vector fields $\mathrm{F}(\cdot\,;u)$ and $\mathrm{F}(\cdot\,;v)$ commute for any $u, v \in \mathrm{U}$, and $\mathbf{X}$ has null first level, so the path $x_\bullet$ is constant. We say in that case that Θ is

a **pure rough path**. It can still give rise to some dynamics, when seen as a rough integrator in a rough differential equation, as shown in the next section.

2.3 Weak geometric Hölder p-rough paths as integrators

Recall the construction of a line integral along a continuous semimartingale on a manifold described at the beginning of section 2.2. The next proposition shows that a weak geometric Hölder p-rough path on $\mathscr{M}$ can be used as an integrator in a rough differential equation, which justifies calling it a *p-rough integrator*.

2. Definition − *Let Θ be a weak geometric Hölder p-rough path on $\mathscr{M}$ in the sense of definition 1. Let also $\pi : \mathscr{B} \mapsto \mathscr{M}$ be a smooth fiber bundle over $\mathscr{M}$, equipped with a connection, of class $\mathcal{C}^{[p]+1}$, given under the form of a smooth $T\mathscr{B}$-valued 1-form H on $\mathscr{M}$ such that $\pi_* : T_e\mathscr{B} \to T_{\pi(e)}\mathscr{M}$, is an isomorphism for every $e \in \mathscr{B}$. Given a rough integrator $\Theta = \big((x_t)_{0\leqslant t<\zeta}, \mathrm{F}, \mathbf{X}\big)$ in $\mathscr{M}$, and $e_0 \in \mathscr{B}$ with $\pi(e_0) = x_0$, a path $(e_t)_{0\leqslant t<\zeta'}$ in $\mathscr{B}$ is said to be **a lift of** $(x_t)_{0\leqslant t<\zeta'}$, if $\zeta' \leqslant \zeta$ and it is the solution path to the $\mathscr{B}$-valued rough differential equation*

$$de_t = \mathrm{H}\Big(\mathrm{F}\big(\pi(e_t); \mathbf{X}(dt)\big)\Big) \tag{5}$$

*started from e_0. This equation ensures in particular that $\pi(e_t) = x_t$. The p-rough integrator $\big((e_t)_{0\leqslant t<\zeta'}, \mathrm{H}\circ\mathrm{F}\circ\pi, \mathbf{X}\big)$ is then said to be the **lift of Θ to** $\mathscr{B}$ started from e_0.*

A commonly encountered situation is to have $\mathscr{B} = \mathscr{M}\times\mathscr{N}$, for some other Banach manifold $\mathscr{N}$, and a connection form defined for any $p \in T_x\mathscr{M}$ and $y \in \mathscr{N}$ by the formula

$$\mathrm{H}(x,y)p = \big(p, \mathrm{G}(y, x\,; p)\big),$$

with $T(\mathscr{M}\times\mathscr{N})$ canonically identified with $T\mathscr{M}\times T\mathscr{N}$, and some function $\mathrm{G} : \mathscr{N}\times T\mathscr{M} \to T\mathscr{N}$ of class $\mathcal{C}^{[P]+1}$, with $\mathrm{G}(y,x;p) \in T_y\mathscr{N}$, for each $p \in T_x\mathscr{M}$, and $x \in \mathscr{M}$. This definition is reminiscent of the approach chosen by Cass, Litterer and Lyons, in section 5 of [9], to define in their setting rough differential equations driven by manifold-valued rough paths. Note however that their theory requires the use of the rigid and non-trivial notion of $\mathrm{Lip}-\gamma$ manifold; no such constraint holds in our setting.

The fact that the two vector fields $\big(\mathrm{H}\circ\mathrm{F}\circ\pi\big)(\cdot,\cdot\,; u)$ and $\big(\mathrm{H}\circ\mathrm{F}\circ\pi\big)(\cdot,\cdot\,; v)$ may not commute while $\mathrm{F}(\cdot\,; u)$ and $\mathrm{F}(\cdot\,; v)$ may commute, for some $u, v \in \mathrm{U}$, explains why pure rough paths can generate dynamics. The results on rough differential equations recalled in the introduction of this section apply and show that

3. Proposition − *The rough differential equation* (5) *has a unique maximal solution started from any given point.*

Solving successively some rough differential equations of the form (5) with some basic p-rough integrators $\Theta^{(1)} = \big((x_t^{(1)})_{0\leqslant t\leqslant t_1}, \mathrm{F}^{(1)}, \mathbf{X}^{(1)}\big), \ldots, \Theta^{(k)}$, with $x_0^{(j)} = x_{t_{j-1}}^{(j-1)}$, $y_0^{(j)} = y_{t_{j-1}}^{(j-1)}$ and $t_{j-1} < \zeta'_{j-1} \leqslant \zeta_{j-1}$, for $2 \leqslant j \leqslant k$, defines, whenever this makes sense, the *concatenation* $\Theta^{(k)}\cdots\Theta^{(1)}$ *of the basic p-rough integrators* $\Theta^{(i)}$. We call such an object a p-**rough integrator**.

3 Canonical representation of rough integrators

We show in this section that p-rough integrators have a canonical representation when the tangent bundle of $\mathscr{M}$ is equipped with a connection. This representation is the analogue of the representation of a regular path γ on $\mathscr{M}$ by a regular path in $T_{\gamma_0}\mathscr{M}$ using Cartan's development map.

3.1 Cartan's moving frame method

Let $\mathscr{M}$ be a manifold of finite dimension d. One owes to E. Cartan the introduction in differential geometry of the *moving frame method*, which provides a chart for the set of $\mathscr{M}$-valued paths of class $\mathcal{C}^1$ based at some fixed starting point, in terms of $\mathcal{C}^1$ paths in $\mathbb{R}^d$ started from 0. Its construction requires the use of a connection on $T\mathscr{M}$ and of the frame bundle of $\mathscr{M}$; it can basically be described as follows. Given a $\mathcal{C}^1$ path $\gamma_\bullet = (\gamma_t)_{0 \leqslant t \leqslant 1}$ on $\mathscr{M}$ and a frame e_0 at γ_0, its parallel transport along the path $\gamma_\bullet$ defines a section $(e_t)_{0 \leqslant t \leqslant 1}$ of the frame bundle above $\gamma_\bullet$. One can use these frames to describe $\dot{\gamma}_t$ at any time in terms of its coordinates $\left(\dot{u}_t^i\right)_{1 \leqslant i \leqslant d}$ in e_t. The $\mathbb{R}^d$-valued path $u_\bullet$ defined by the formula

$$u_t := \int_0^t \dot{u}_s^i \, ds$$

is called the **anti-development of the path** γ. One finds back the $\mathscr{M}$-valued path $\gamma_\bullet$ from the $\mathbb{R}^d$-valued $u_\bullet$ and (x_0, e_0) as the projection on $\mathscr{M}$ of the unique solution of the ordinary differential equation in the frame bundle

$$\nabla_{\dot{\gamma}_t} e_t = 0, \quad \dot{\gamma}_t := e_t(\dot{u}_t).$$

The interest of Cartan's moving frame method is that it somehow provides an optimal coding of all paths in $\mathscr{M}$ in terms of vector space-valued paths living in a space of minimal dimension. Here is a trivial illustration of this fact. Let $u_\bullet$ be an $\mathbb{R}^{10}$-valued path and $\gamma_t = \sum_{i=1}^{10} u_t^i$ be an $\mathbb{R}$-valued path. The coding of $\gamma_\bullet$ by the 10 components of $u_\bullet$ is optimized by coding it with its real value at each time. Replacing $\mathbb{R}^{10}$ and $\mathbb{R}$ by infinite dimensional Banach spaces emphasizes the importance of such parcimonious representations.

Get back to the general setting and recall that $\mathscr{M}$ is modelled on some Banach space E. Under some additional mild condition, one can give a parcimonious description of weak geometric Hölder p-rough paths on $\mathscr{M}$ similar to the above one, providing a description of these objects in terms of weak geometric Hölder p-rough paths in the tensor space $T^{[p]}(\mathrm{E})$, as opposed to the a priori unrelated Banach space U involved in the primary definition of a weak geometric Hölder p-rough path, see definition 1. As in Cartan's moving frame method, this requires the tangent bundle $T\mathscr{M} \to \mathscr{M}$ to be equipped with a connection; this is a non-trivial assumption in an infinite dimensional setting, linked to the fact that there exists some Banach spaces that do not even admit a smooth partition of unity. No such pathology happens in finite dimension, and finite dimensional manifolds can always be endowed with an arbitrary connection; so the results of the forthcoming section 3.2 always hold for rough integrators on finite dimensional manifolds.

The frame bundle $\mathrm{GL}(\mathscr{M})$ of $\mathscr{M}$ will have a crucial role in that play. This is the collection of all bi-continuous isomorphisms from E to $T_m\mathscr{M}$, for $m \in \mathscr{M}$; it has a natural manifold structure modelled on $\mathrm{GL}(\mathrm{E}) \times \mathrm{E}$. We shall denote by e a

generic element of $\mathrm{GL}(\mathscr{M})$, and by $\mathcal{V}\mathrm{GL}(\mathscr{M})$ the vertical sub-bundle of $T\mathrm{GL}(\mathscr{M})$, canonically identified with the Lie algebra $\mathrm{gl}(\mathrm{E}) = \mathrm{L}(\mathrm{E})$ of $\mathrm{GL}(\mathrm{E})$. The connection ∇ on the bundle $T\mathscr{M} \to \mathscr{M}$ is naturally lifted into a connection on the bundle $\pi : \mathrm{GL}(\mathscr{M}) \to \mathscr{M}$, still denoted by the same symbol. Remark that π_* is a bi-continuous isomorphism between the horizontal sub-bundle $\mathcal{H}^\nabla\mathrm{GL}(\mathscr{M})$ in $T\mathrm{GL}(\mathscr{M})$ and $T\mathscr{M}$. The horizontal destribution $\mathcal{H}^\nabla\mathrm{GL}(\mathscr{M})$ in $T\mathrm{GL}(\mathscr{M})$ can be used to define a continuous linear map H^∇ from E to the space of horizontal vector fields on $T\mathrm{GL}(\mathscr{M})$, defined by the requirement that $\mathrm{H}^\nabla(e\,;a) \in T_e\mathrm{GL}(\mathscr{M})$ is horizontal and corresponds to $e(a) \in T\mathscr{M}$, for any $a \in \mathrm{E}$ and $e \in \mathrm{GL}(\mathscr{M})$. This vector field valued 1-form H^∇ on E is called the **canonical horizontal 1-form**.

Once again, we refer the reader to the nice books [22] and [23] for the basics of differential geometry on Banach manifolds.

3.2 Canonical representation of rough integrators

Theorem 4 below gives a canonical and parcimonious representation of a given weak geometric Hölder p-rough path, seen as a rough integrator. We assume for that purpose that the manifold $\mathscr{M}$ is endowed with a connection ∇. Given a 1-form F on U with values in the space of vector fields on $\mathscr{M}$ as in the above definition of a weak geometric Hölder p-rough path on $\mathscr{M}$, we denote by F^∇ its lift to a 1-form with values in the space of ∇-horizontal vector fields on $\mathrm{GL}(\mathscr{M})$. **Parallel translation along a weak geometric Hölder p-rough path** $\Theta = \big((x_t)_{0\leqslant t<\zeta}, \mathrm{F}, \mathbf{X}\big)$ is defined as the solution path to the rough differential equation in $\mathrm{GL}(\mathscr{M})$

$$de_t = \mathrm{F}^\nabla\big(e_t, \mathbf{X}(dt)\big)$$

started from any given frame $e_0 \in \mathrm{GL}(\mathscr{M})$ above x_0. It is elementary to see that the paths $e_\bullet$ and $x_\bullet$ are defined on the same maximal interval.

Given another Banach manifold $\mathscr{N}$, and a 1-form $\mathrm{G} : \mathscr{N} \times T\mathscr{M} \to T\mathscr{N}$, as used above to write down a rough differential equation in definition 2, define $\mathfrak{G}$ on $\mathrm{GL}(\mathscr{M}) \times \mathscr{N} \times \mathrm{U}$ by the formula

$$\mathfrak{G}(e,y\,;u) := \Big(\mathrm{F}^\nabla(e\,;u),\, \mathrm{G}\big(y, \pi(e)\,; \pi_*\mathrm{F}^\nabla(e\,;u)\big)\Big),$$

and $\mathfrak{G}^\nabla$ on $\mathrm{GL}(\mathscr{M}) \times \mathscr{N} \times \mathrm{E}$ by the formula

$$\mathfrak{G}^\nabla(e,y\,;a) := \Big(\mathrm{H}^\nabla(e\,;a),\, \mathrm{G}\big(y, \pi(e)\,; \pi_*\mathrm{H}^\nabla(e\,;a)\big)\Big).$$

(Insist on the fact that the one-form H^∇ depends only on the connection ∇ on $T\mathscr{M}$.)

4. Theorem $-$ *Let $\mathscr{M}$ be a Banach manifold modeled on the Banach space E, and endowed with a connection ∇. Let $\Theta = \big((x_t)_{0\leqslant t<\zeta}, \mathrm{F}, \mathbf{X}\big)$ be a basic p-rough integrator on $\mathscr{M}$.*

(a) *Given $e_0 \in \mathrm{GL}(\mathscr{M})$, one defines a weak geometric Hölder p-rough path $\mathbf{Z}$ over E, on the time interval $[0,\zeta)$, by solving the rough differential equation*

$$\begin{aligned} de_t &= \mathrm{F}\big(e_t\,; \mathbf{X}(dt)\big), \\ d\mathbf{Z}_t &= \mathbf{Z}_t \otimes e_t^{-1}\mathrm{F}\big(\pi(e_t)\,; \mathbf{X}(dt)\big). \end{aligned} \tag{6}$$

in $\mathrm{GL}(\mathscr{M}) \times T^{[p]}(\mathrm{E})$, started from (e_0, Id).

(b) *The solution paths (e_t, y_t) and (f_t, z_t) in $\mathrm{GL}(\mathcal{M}) \times \mathcal{N}$ to the rough differential equations*

$$d(e_t, y_t) = \mathfrak{G}\big(e_t, y_t \, ; \mathbf{X}(dt)\big),$$
$$d(f_t, z_t) = \mathfrak{G}^\nabla\big(f_t, z_t \, ; \mathbf{Z}(dt)\big), \tag{7}$$

coincide if they start from the same initial condition.

It is elementary to state an analogue of this theorem when $\mathcal{M} \times \mathcal{N}$ is replaced by a fiber bundle $\mathcal{B}$ over $\mathcal{M}$, equipped with a regular connection G. There is no more difficulties in stating a similar result for any p-rough integrator $\Theta = \Theta^{(k)} \cdots \Theta^{(1)}$ instead of basic p-rough integrators. The interesting point is that although the rough paths in each $\Theta^{(i)}$ may be defined on *different* Banach spaces, the canonical representation of the concatenation of the $\Theta^{(i)}$ only involves a rough path over E.

Proof – The first claim follows from general principles. For the second point, pick $p < p' < [p] + 1$. Using the continuity result for the Itô map recalled in section 2.1, in the topology of Hölder p'-rough path, together with the continuous embedding of the space $WG_p(\mathrm{U})$ of weak geometric Hölder p-rough paths in U into the space $G_{p'}(\mathrm{U})$ of geometric Hölder p'-rough paths in U, and the continuous embedding of $G_{p'}(\mathrm{U})$ into $WG_{p'}(\mathrm{U})$, we see that it suffices to prove the claim when $\mathbf{X}$ is the weak geometric Hölder (p'-) rough path associated with a smooth U-valued control. The result is clear in that case as it appears as a rephrasing of Cartan's moving frame method, described above in section 3.1. $\qquad\qquad \triangleright$

So one can always understand the solution of a rough differential equation driven by a p-rough integrator Θ as the solution to another rough differential equation driven by a p-rough integrator involving a weak geometric Hölder p-rough path over the model space E, and the canonical horizontal 1-form H^∇. The dependence on F in the original p-rough integrator is hidden in the definition of $\mathbf{Z}$ in this reformulation, equation (6). Given $e_0 \in \mathrm{GL}(\mathcal{M})$ above x_0, the "$\mathrm{GL}(\mathcal{M})$-valued" p-rough integrator $\big((e_t)_{0 \leqslant t < \zeta}, \mathrm{H}^\nabla, \mathbf{Z}\big)$ is said to be the **canonical representation of Θ**. This result echoes one of the main results of [10], corollary 6.19, where a one-to-one correspondence between the classes of weak geometric rough path on a given finite d-dimensional manifold, weak geometric horizontal rough paths on its frame bundle, and classical weak geometric rough paths on $\mathbb{R}^d$, is proved in their setting, using Cartan's development map as an essential ingredient; Theorem 4 has wider scope; applications of our setting will be developed in a forthcoming work.

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