Random models on regularity-integrability structures

I. Bailleul[1] & M. Hoshino[2]

Abstract

We prove a convergence result for a large class of random models that encompasses the case of the BPHZ models used in the study of singular stochastic PDEs. We introduce for that purpose a useful variation on the notion of regularity structure called a regularity-integrability structure. It allows to deal in a single elementary setting with models on a usual regularity structure and their first order Malliavin derivative.

1 – Introduction

The introduction by M. Hairer of the theory of regularity structures opened a new era in the domain of stochastic partial differential equations (PDEs). It provided in particular a robust solution theory for a number of equations, called 'singular', whose study is beyond the range of the methods based on stochastic calculus. The singular feature of these equations is related to the fact that their formulations involve some ill-defined products. The development of this theory was done in several steps. The analytic core was developed in Hairer' seminal work [17]. Its algebraic backbone was deepened in Bruned, Hairer & Zambotti's work [8]. The specific task of dealing with the ill-defined products of a singular stochastic PDE is called the renormalisation problem. This problem has a dynamics side and a probabilistic side. The dynamic meaning of the BPHZ renormalisation procedure of [8] was studied by Bruned, Chandra, Chevyrev & Hairer in [7], and lead to the identification of a solution to a singular stochastic PDE as the limit of solutions to renormalised versions of the initial equations, that is equations driven by a regularized noise with additional counterterms that typically diverge as the regularization parameter vanishes. The analytic machinery needs as an input an equation-dependent finite family of quantities built from a regularized noise. A systematic proof of probabilistic convergence of these quantities using the BPHZ renormalisation rule was given by Chandra & Hairer in [12]; this is the probabilistic side of the renormalisation problem.

Altogether the four works [17, 8, 7, 12] form an automated blackbox for the study of a well identified large class of equations, with prominent examples coming as scaling limits of some microscopic discrete systems of statistical mechanics. This is the case of the (KPZ) equation from continuous interface growth models, of the parabolic Anderson model equation giving the scale limit of branching particle systems, or of the Φ^4_3 equation from Euclidean quantum field theory. While the works [17, 8, 7] are now well understood by a growing community this is not the case of the work [12]. The latter uses ideas from the multiscale expansion method developed by Feldman, Magnen, Rivasseau & Sénéor in [15], for the study of divergent Feynman integrals, to analyse the convergence problem of an equation-dependent finite collection of iterated integrals. The sophistication of their analysis and the very general assumptions on the law of the noise adopted in [12] make their work very challenging.

Meanwhile, Otto developped with a number of co-authors a variant of the theory of regularity structures tailor made for the study of a certain class of singular quasilinear stochastic PDEs. Its analytic machinery was constructed in the works [28, 26] with Weber, and Sauer & Smith. The algebraic machinery was described in the work [22] with Linares & Tempelmayr. Importantly, they were able to identify in [26] a renormalisation procedure with a similar dynamic meaning as the BPHZ renormalisation process. Linares, Otto, Tempelmayr & Tsatsoulis proved in [23] the convergence result corresponding in their setting to the convergence result of [12]. Most interestingly, the authors of [23] used a set of assumptions and tools different from [12], trading assumptions on cumulants and questions on iterated integrals for a spectral gap assumption on the law of the random noise and an iterative control of the stochastic objects. Their approach bypasses in particular the intricate algebraic content of the BPHZ strategy. We note that the idea of differentiation with respect to the noise that is involved in the spectral gap assumption was used in a different form in the early 80s by Caswell & Kennedy [11] in their approach to perturbative

[1]Univ Brest, CNRS UMR 6205, Laboratoire de Mathématiques de Bretagne Atlantique, France. *E-mail*: ismael.bailleul@univ-brest.fr. Partial support from the ANR-11-LABX-0020-01 Labex CHL and the ANR project Smooth ANR-22-CE40-0017 is acknowledged.

[2]Graduate School of Engineering Science, Osaka University, Japan. *E-mail*: hoshino@sigmath.es.osaka-u.ac.jp. Support from JSPS KAKENHI Grant Number 23K12987 is acknowledged.

Keywords: Regularity structures, Renormalisation, Singular stochastic PDEs
AMS classification: 60L30, 35R60, 60H17

renormalisation of quantum field theories. The results [26, 22, 23] are not directly applicable to the study of semilinear subcritical singular stochastic PDEs. Hairer & Steele [20] gave very recently an improved and simplified version of the convergence result of [23], in the original regularity structure setting. Their general convergence result for the BPHZ renormalisation procedure provides an alternative to the result of [12] of similar scope for practical purposes. We provide in the present work an alternative proof of their convergence result that holds for a larger class of renormalisation procedures containing the BPHZ procedure of [20] as a particular example.

Like [23], the convergence proof of [20] is done by induction. The objects to control are renormalized models on a regularity structure. They are built from a regularized noise and come under the form of a family of distributions $\Pi_x^n\tau$ indexed by the points x of the state space and a finite, equation-dependent, family of symbols τ. The integer n accounts here for the regularization parameter. The stochastic convergence of these models as the regularization is removed is mainly controlled by the $L^p(\Omega)$ convergence of real-valued quantities of the form

$$\lambda^{-|\tau|}(\Pi_x^n\tau)(\varphi_x^\lambda) \tag{1.1}$$

where the smooth test functions φ_x^λ behave like Dirac masses at x as λ goes to 0 and $|\tau|$ is some real number. The spectral gap assumption on the law of the random noise allows to control the $L^p(\Omega)$ norm of (1.1) by its expectation and the quantity

$$\mathbb{E}\Big[\sup_{\|h\|_H\leq 1}\big|\big(d_\omega(\mathsf{\Pi}_x^n\tau)(h)\big)(\varphi_x^\lambda)\big|^p\Big], \tag{1.2}$$

where $d_\omega(\cdot)(h)$ stands for the Gâteaux derivative in the direction h, for $h\in H$ in some space H. It turns out that a good control on the expectation of (1.1) for $\lambda=1$ can be propagated by induction to any $0<\lambda\leq 1$ and all symbols. Building on the insight of [23] Hairer & Steele show that $d_\omega(\mathsf{\Pi}_x^n\tau)(h)$ can be represented as the reconstruction of a modelled distribution defined on an extended regularity structure that contains an extra noise symbol – a placeholder for a generic h. This representation comes with estimates that play a crucial role in the inductive procedure. To apply this strategy Hairer & Steele introduced a concept of *pointed modelled distribution* that allows to harvest the benefits associated with the improved regularity of the functions h involved in the spectral gap assumption, compared to the regularity of the noise, and get as a consequence a good scaling bound for (1.2). One then needs to extend the analytic core of the theory of regularity structures to the setting of pointed modelled distributions; a non-trivial task. Further, the construction of a pointed modelled distribution associated with the derivative $d_\omega(\mathsf{\Pi}_x^n\tau)(h)$ of the renormalized model is only done in [20] for BPHZ-like renormalisation procedures. We use a different strategy to prove the convergence of a larger family of renormalized models. These models are built from a class of maps that act on the linear space spanned by the symbols τ, called *preparation maps*. They were introduced by Bruned in [6] as a fundamental brick in the inductive construction of a large class of admissible models. BPHZ-like renormalisation procedures correspond to particular examples of preparation maps. The dynamic meaning of the renormalisation procedure associated with (strong) preparation maps was studied by Bailleul & Bruned in [1]. As in the BPHZ setting it involves renormalized equations that include additional counterterms.

In our setting we trade the testing operation (1.1) against a scaled centered function for a testing operation against some kernel $Q_t(x,\cdot)$ and we aim at getting some (probabilistic) bounds on quantities of the form $t^{-|\tau|/\ell}\mathcal{Q}_t(x,\mathsf{\Pi}_x^n\tau)$, where $\mathsf{\Pi}_x^n\tau$ is associated with a regularized noise and an n-dependent preparation map. At the informal level of this introduction, our main result, Theorem 6, reads as follows. See the latter half part of this section for the definitions of related notations.

1 – Theorem. *Assume the noise symbol is the only element of the regularity structure with degree less than or equal to $-|\mathfrak{s}|/2$. Assume that the law of the random noise has a spectral gap. Last, suppose that we have some preparation maps R_n for which the quantities $\mathbb{E}[\mathcal{Q}_1(0,\mathsf{\Pi}_0^n\tau)]$ converge for all the symbols τ with non-positive degree. Then the renormalized models associated with these preparation maps converge in $L^q(\mathbb{P})$ for any $1\leq q<\infty$.*

The spectral gap inequality is introduced in Section 2.1. We give in this work our convergence result in a situation where there is only one noise and one integration operator. The modifications needed to accommodate a situation with different noises and different integration operators, as in [20], are standard and left to the reader. To deal with models on a usual regularity structure and their first order Malliavin derivative in a single setting we introduce a useful variant of the notion of regularity structure that we call a *regularity-integrability structure*. Its symbol space is in particular graded by a subset of $\mathbb{R}\times[1,\infty]$, with the first component accounting for a regularity exponent and the second component accounting for an integrability exponent. An associated notion of modelled distribution, their reconstruction and Schauder estimates for some integration operator, can be developed in the regularity-integrability setting as in the classical setting. This is done in the companion work [21]. We only need to work with two integrability exponents $\infty, p \in [1,\infty]$ at a time in this work. The regularity degree for a symbols τ depends on the choice of p. The algebraic structure that defines the regularity-integrability structure then turns out to be p-dependent. On a technical level, in addition to the spectral gap assumption on the law of the random noise, our proof of Theorem 1 rests on the versions of the reconstruction theorem and Schauder estimates that hold in a regularity-integrability structure, and a useful comparison formula (Lemma 4) describing what happens to a renormalized model when we vary p, that is when we vary the regularity-integrability structure itself. The mechanics of the proof is detailed in Section 3.

The class of renormalized models from [8] is built from a subclass of preparation maps. Within that subclass there is a unique choice of preparation maps such that $\mathbb{E}[(\Pi^n\tau)(x)] = 0$ for all τ with negative degree, $x \in \mathbb{R}^d$, and n. The model $\mathsf{M}(\mathbb{P})$ associated to this preparation map is called *BPHZ model*. The next statement expresses a continuity property of the law of $\mathsf{M}(\mathbb{P})$ in the class of probability measures that have the same spectral gap. Tempelmayr [29] obtained recently a similar result in a different setting.

2 – Theorem. *Let $(\mathbb{P}_j)_{j\in\mathbb{N}}$ be a sequence of probability measures on Ω that converges weakly to a limit probability measure $\mathbb{P}$. If all the $\mathbb{P}_j$ satisfy a spectral gap inequality with the same constant then the law of $\mathsf{M}(\mathbb{P}_j)$ converges weakly to the law of $\mathsf{M}(\mathbb{P})$.*

We refer the reader to the reviews [13, 14] of Chandra & Weber and Corwin & Shen for some non-technical introductions to the domain of semilinear singular stochastic PDEs. One can refer to the books [16, 5] of Friz & Hairer and Berglund for mildly technical introductions to regularity structures, and to Bailleul & Hoshino's Tourist's Guide [3] for a thorough tour of the analytic and algebraic sides of the theory. Hairer's lecture notes [18, 19] are centered on the problems of renormalisation in the setting of Feynmann graphs and in the setting of singular stochastic PDEs, respectively. The lecture notes [27, 25] of Otto & co. give a gentle introduction to the tree-free approach [23] to the renormalisation of the random models that are involved in the analytic and algebraic settings of [28, 26, 22]. The present work is independent of any of these works.

Organisation of the work. Section 2 sets the scene for our main convergence result for random models. We specify the spectral gap assumption on the law of the random noise in Section 2.1. We introduce algebraic structures of decorated trees that depend on a parameter $p \in [2,\infty]$ in Section 2.2. A notion of differentiable sector is introduced in Section 2.3. It specifies a setting where one can talk of a sector that is stable by a natural noise-derivative operator. Section 2.4 introduces regularity-integrability structures. Since the degree of a tree depends on p so does a regularity-integrability structure, that is, the algebraic rules for making local expansions depend on p. Models on regularity-integrability structures are described in Section 2.5: The main point to get is that we use different Lebesgue spaces to measure some quantities indexed by trees depending on whether or not there is a derivative noise symbol in that tree. The fundamental results about modelled distributions in the setting of regularity-integrability structures are proved in the companion work [21] and stated in Appendix A. Preparation maps and their renormalized models are introduced in Section 2.6. Lemma 4, in this section, is important: It provides an explicit comparison for a fundamental quantity for two different values of p, that is, when the local expansion rules are possibly different. We state our main result, Theorem 6, in Section 2.7. The remaining two sections are dedicated to the proof of Theorem 6. We present the inductive core of the proof in Section 3 and defer to Section 4 the proof of a number of lemmas used in the induction. In a nutshell, we first introduce an order to construct inductively a limit model on an increasing finite sequence of spaces V_i, W_i. The trees of V_i have no derivative noise while the trees of W_i may have one derivative noise. The induction proceeds in three steps after proving the convergence result for the base case. In Step 1 we prove that the probabilistic convergence of the renormalized models on W_i implies its probabilistic convergence on V_i. We use for that purpose the reconstruction theorem and the spectral gap inequality. In Steps 2 and 3 we prove that the convergence on W_i and V_i implies the convergence on W_{i+1}. In Step 2 we use the multilevel Schauder estimate to control the g-part of the renormalized model on the elements of W_{i+1}. In Step 3 we use the comparison lemma, Lemma 4, to control the $\mathsf{\Pi}$-part of the model. Indeed one gets for free some analytic estimates when $p = 2$. The comparison lemma then allows to compare the $\mathsf{\Pi}$-part for an arbitrary p to its counterpart for $p = 2$. It turns out that the difference between the two quantities involves only some terms whose control is provided by the induction mechanics. We provide a sketch of the proof of Theorem 2 in Section 5.

Notations – *For a normed vector space X we will denote by $L^q(\Omega,\mathbb{P};X)$ the space of q-integrable X-valued random variables. Throughout this paper, we fix an integer $d\geq 1$, the* scaling $\mathfrak{s}=(\mathfrak{s}_1,\dots,\mathfrak{s}_d)\in[1,\infty)^d$, *and an exponent*

$$\ell>\max_{1\leq j\leq d}\mathfrak{s}_j.$$

Set

$$|\mathfrak{s}|:=\mathfrak{s}_1+\cdots+\mathfrak{s}_d.$$

For any multiindex $k=(k_j)_{j=1}^d\in\mathbb{N}^d$ we define

$$|k|_{\mathfrak{s}}:=\sum_{j=1}^d\mathfrak{s}_jk_j,\qquad k!:=\prod_{j=1}^d k_j!.$$

Also for every $x\in\mathbb{R}^d$ and $k=(k_j)_{j=1}^d\in\mathbb{N}^d$ we define

$$x^k:=\prod_{j=1}^d x_j^{k_j},\qquad \|x\|_{\mathfrak{s}}:=\sum_{j=1}^d|x_j|^{\frac{1}{\mathfrak{s}_j}}.$$

The functional setting within which we set our study is associated with a heat semigroup $(Q_t)_{t>0}$ of an (anisotropic) elliptic operator. Let

$$P(\lambda_1,\dots,\lambda_d)=\sum_{|k|_{\mathfrak{s}}\leq\ell}a_k\lambda^k$$

be a polynomial with real constant coefficients which satisfies

$$P(i\lambda_1,\dots,i\lambda_d)\leq-\delta\|\lambda\|_{\mathfrak{s}}^{\ell}$$

for some $\delta>0$ and for any $\lambda\in\mathbb{R}^d$. We denote by

$$Q_t(x)=e^{tP(\partial_1,\dots,\partial_d)}(x)\qquad(t>0,\ x\in\mathbb{R}^d)$$

the heat kernel of the differential operator $P(\partial_1,\dots,\partial_d)$. In Appendix A of [4], *it was proved that $(Q_t)_{t>0}$ satisfies the upper 'Gaussian' estimate*

$$|Q_t(x)|\lesssim G_t(x):=\frac{1}{t^{|\mathfrak{s}|/\ell}}\exp\left\{-c_1\sum_{j=1}^d\left(\frac{|x_j|}{t^{\mathfrak{s}_j/\ell}}\right)^{\frac{\ell}{\ell-\mathfrak{s}_j}}\right\}$$

for any $t>0$ and $x\in\mathbb{R}^d$, where $c_1>0$ is a fixed constant. We also define the family of weight functions $(w_c)_{c\geq0}$ on $\mathbb{R}^d$ by

$$w_c(x):=e^{-c\|x\|_{\mathfrak{s}}}.$$

It is elementary to show that for any $c\geq0$ and $a\geq0$, the inequality

$$\big\|G_t(x)w_c(x)^{-1}\|x\|_{\mathfrak{s}}^a\big\|_{L_x^p(\mathbb{R}^d)}\lesssim t^{-\frac{|\mathfrak{s}|}{\ell}(1-\frac{1}{p})+\frac{a}{\ell}}\tag{1.3}$$

holds uniformly over $p\in[1,\infty]$ and $t\in(0,1]$. This estimate will be used in Section 4. For $c\geq0$ and $p\in[1,\infty]$ we define the weighted L^p norm by

$$\|f\|_{L^p(w_c)}:=\|fw_c\|_{L^p(\mathbb{R}^d)}.$$

We define for each $t>0$ the operator $\mathcal{Q}_t$ on $C(\mathbb{R}^d)\cap L^p(w_c)$ by

$$\mathcal{Q}_t(x,f):=\int_{\mathbb{R}^d}Q_t(x-y)f(y)dy.$$

For every $\alpha\leq0$ and $p,q\in[1,\infty]$ we define the ***Besov space*** *$B_{p,q}^{\alpha,Q}(w_c)$ as the completion of $C(\mathbb{R}^d)\cap L^p(w_c)$ under the norm*

$$\|f\|_{B_{p,q}^{\alpha,Q}(w_c)}:=\|\mathcal{Q}_1f\|_{L^p(w_c)}+\big\|t^{-\alpha/\ell}\|\mathcal{Q}_tf\|_{L^p(w_c)}\big\|_{L^q((0,1];\frac{dt}{t})}.\tag{1.4}$$

See Section 2 of [21] *for detailed properties of Besov spaces associated with $(Q_t)_{t>0}$. Especially, the continuous embedding result*

$$B_{p,q}^{\alpha,Q}(w_c)\hookrightarrow B_{r,q}^{\alpha-|\mathfrak{s}|(\frac{1}{p}-\frac{1}{r}),Q}(w_c)\tag{1.5}$$

for any $\alpha\leq0$, $p,q,r\in[1,\infty]$ with $r\geq p$, and $c\geq0$, is important. Write

$$H^{\alpha,Q}(w_c):=B_{2,2}^{\alpha,Q}(w_c),\qquad C^{\alpha,Q}(w_c):=B_{\infty,\infty}^{\alpha,Q}(w_c).$$

Pick $0<\ell_1<\ell$ and some real constants $(b_l)_{|l|_\mathfrak{s}<\ell_1}$ and set

$$K_t(z) := \sum_{|l|_\mathfrak{s}\le \ell_1} b_l\, \partial_z^l Q_t(z).$$

Then $\{K_t\}_{t>0}$ is an $(\ell-\ell_1)$-regularizing kernel in the terminology of Section 5 of [21]*. For any $f\in C(\mathbb{R}^d)\cap L^p(w_c)$ and $k\in\mathbb{N}^d$ define*

$$\partial^k\mathcal{K}(x,f) := \int_0^1\int_{\mathbb{R}^d}\partial_x^k K_t(x-y)f(y)dydt. \tag{1.6}$$

As an example consider a two-dimensional situation and the elliptic polynomial $P(\lambda_1,\lambda_2)=\lambda_1^2-\lambda_2^4$. The factorization

$$\lambda_1^2-\lambda_2^4 = -(\lambda_1-\lambda_2^2)(\lambda_1+\lambda_2^2)$$

then gives a representation of the heat kernel

$$(\partial_1-\partial_2^2)^{-1}f = -\int_0^\infty(\partial_1+\partial_2^2)\mathcal{Q}_t(\cdot,f)dt \simeq \mathcal{K}(\cdot,f),$$

up to a smoothing operator, for the ad hoc choice of constants b_l. The operators $\partial^k\mathcal{K}$ has a continuous extension from $C^{\alpha,Q}(w_c)$ into $C^{\alpha+\ell-\ell_1-|k|_\mathfrak{s},Q}(w_c)$ when $\alpha+\ell-\ell_1<0$. We fix throughout this work a fixed number $\beta_0\in(0,\ell-\ell_1)$.

2 – A convergence result

2.1 Spectral gap. Let Ω be a separable Banach space and let H be a separable Hilbert space embedded continuously and densely into Ω. A function $F:\Omega\to\mathbb{R}$ is said to be ***(continuously) H-differentiable*** if there is a function $dF:\omega\in\Omega\mapsto d_\omega F\in H^*$ such that

$$\frac{d}{dt}F(\omega+th)\big|_{t=0} = (d_\omega F)(h).$$

Denote by $\|h^*\|_{H^*}=\sup_{\|h\|_H\le 1}|h^*(h)|$ the operator norm on H^*. A Borel probability measure $\mathbb{P}$ on Ω is said to satisfy the ***H-spectral gap inequality*** if there exists a constant $C>0$ such that

$$\mathbb{E}\big[(F-\mathbb{E}[F])^2\big]\le C\,\mathbb{E}\big[\|dF\|_{H^*}^2\big] = C\,\mathbb{E}\Big[\sup_{\|h\|_H\le 1}|dF(h)|^2\Big] \tag{2.1}$$

for any H-differentiable $F\in L^2(\Omega)$ such that $dF\in L^2(\Omega;H^*)$. Note that the supremum over h is indeed a random variable since $(d_\omega F)(h)$ is continuous in $h\in H$ and the supremum over h can be replaced by a countable supremum. By replacing F by F^2 and iterating, one further has

$$\mathbb{E}\big[F^{2^r}\big]\lesssim_r |\mathbb{E}[F]|^{2^r}+\mathbb{E}\big[\|dF\|_{H^*}^{2^r}\big] \tag{2.2}$$

for all $r\in\mathbb{N}$ (see Remark 2.21 of Hairer & Steele's work [20]). The H-spectral gap inequality holds if $(\Omega,H,\mathbb{P})$ is an abstract Wiener space (see e.g., Exercise 2.11.1 of [24]).

A typical example is the white noise measure on $\mathbb{R}^d$, which satisfies the H-spectral inequality with $H=L^2(w_0)$. (The weight w_0 is the constant function equation to 1.) In this paper, we consider an arbitrary $\kappa\ge 0$ and a wider Hilbert space

$$H := H^{-\kappa,Q}(w_0).$$

The $L^2(w_0)$-spectral gap inequality implies $H^{-\kappa,Q}(w_0)$-spectral gap inequality. By the Besov embedding (1.5) the space H is continuously embedded into the Banach space

$$\Omega := C^{\alpha_0,Q}(w_c)$$

for any $c>0$ and $\alpha_0<-|\mathfrak{s}|/2-\kappa$. We fix such exponents κ and α_0 in what follows.

2.2 Decorated trees. We introduce three node symbols $\mathbf{1},\bigcirc,\odot$, which will play in the sequel the role of the constant function 1, an element of Ω, and an element of H. Let $\mathbf{T}$ be the set of all rooted decorated trees τ, with vertex set N_τ and edge set E_τ, and equipped with two node decorations $\mathfrak{t}:N_\tau\to\{\mathbf{1},\bigcirc,\odot\}$, $\mathfrak{n}:N_\tau\to\mathbb{N}^d$, and one edge decoration $\mathfrak{e}:E_\tau\to\mathbb{N}^d$. For any parameters $\varepsilon\ge 0$ and $p\in[2,\infty]$, we define the degree map $|\cdot|_{\varepsilon,p}:\mathbf{T}\to\mathbb{R}$ by

$$|\mathbf{1}|_{\varepsilon,p}:=0,\qquad |\bigcirc|_{\varepsilon,p}:=\alpha_0-\varepsilon,\qquad |\odot|_{\varepsilon,p}:=\alpha_0-\varepsilon+\frac{|\mathfrak{s}|}{p},$$

$$|\tau_{\mathfrak{e}}^{\mathfrak{n}}|_{\varepsilon,p} := \sum_{v\in N_\tau} \big(|\mathfrak{t}(v)|_{\varepsilon,p} + |\mathfrak{n}(v)|_{\mathfrak{s}}\big) + \sum_{e\in E_\tau} \big(\beta_0 - |\mathfrak{e}(e)|_{\mathfrak{s}}\big).$$

The parameter ε will express an infinitesimal loss of regularity at each induction step (described in Section 3). The parameter p has more important role in this paper. We can see that the definition of $|\odot|_{\varepsilon,p}$ naturally comes from the embeddings

$$H = H^{-\kappa,Q}(w_0) \hookrightarrow B_{2,\infty}^{\alpha_0+\frac{|\mathfrak{s}|}{2},Q}(w_c) \hookrightarrow B_{p,\infty}^{\alpha_0+\frac{|\mathfrak{s}|}{p},Q}(w_c) \hookrightarrow C^{\alpha_0,Q}(w_c) = \Omega$$

between H and Ω. We denote by $\mathbf{T}^{(n)}$ be the set of all $\tau\in\mathbf{T}$ that have exactly n vertices v for which $\mathfrak{t}(v)=\odot$. As usual, we denote by $\tau\sigma$ the tree product of $\tau,\sigma\in\mathbf{T}$, and we write $\mathcal{I}_k(\tau)$ for the tree obtained from $\tau\in\mathbf{T}$ by grafting it to a new root with $\mathfrak{t}$-decoration $\mathbf{1}$ and $\mathfrak{n}$-decoration $0\in\mathbb{N}^d$, along an edge with $\mathfrak{e}$-decoration $k\in\mathbb{N}^d$. For technical reasons as in [17], we do not consider the trees of the form $\mathcal{I}_k(X^l)$. We denote by T the linear space spanned by $\mathbf{T}$ and define the linear map Δ from T to $T\widehat{\otimes}T$, analytic tensor product, by

$$\Delta(\bigcirc) = \bigcirc\otimes\mathbf{1},\qquad \Delta(\odot)=\odot\otimes\mathbf{1},\qquad \Delta(X^k) = \sum_{l+m=k}\frac{k!}{l!\,m!}X^l\otimes X^m,$$

$$\Delta(\tau\sigma) = (\Delta\tau)(\Delta\sigma),\qquad \Delta(\mathcal{I}_k(\tau)) = (\mathcal{I}_k\otimes\mathrm{id})\Delta\tau + \sum_{l\in\mathbb{N}^d}\frac{X^l}{l!}\otimes\mathcal{I}_{k+l}(\tau),$$

where we identify the single node decorated tree with node type $\mathbf{1}$ and node decoration k with the polynomial X^k. To avoid infinite linear spans we introduce the projection map $P_{\varepsilon,p}^+$ from T to the subalgebra $T_{\varepsilon,p}^+$ spanned by the symbols

$$X^k\prod_{i=1}^n \mathcal{I}_{k_i}(\tau_i) \tag{2.3}$$

with $n\in\mathbb{N}$, $k,k_i\in\mathbb{N}^d$, and $\tau_i\in\mathbf{T}$ such that $|\tau_i|_{\varepsilon,p}+\beta_0 > |k_i|_{\mathfrak{s}}$ for each i, and define

$$\Delta_{\varepsilon,p} := (\mathrm{id}\otimes P_{\varepsilon,p}^+)\Delta,\qquad \Delta_{\varepsilon,p}^+ := \big(P_{\varepsilon,p}^+\otimes P_{\varepsilon,p}^+\big)\Delta;$$

so $\Delta_{\varepsilon,p}$ and $\Delta_{\varepsilon,p}^+$ send T into the algebraic tensor products $T\otimes T_{\varepsilon,p}^+$ and $T_{\varepsilon,p}^+\otimes T_{\varepsilon,p}^+$, respectively. For any subset $\mathbf{C}\subset\mathbf{T}$, we denote by $\mathcal{P}(\mathbf{C})$ the set of all planted trees of the form $\mathcal{I}_k(\tau)$, with $\tau\in\mathbf{C}$ and $k\in\mathbb{N}^d$, and define $\mathrm{alg}_{\varepsilon,p}^+(\mathbf{C})$ as the subalgebra generated by the symbols (2.3) with τ_i chosen from $\mathbf{C}$. By a similar argument to [17, 8], we can see that $T_{\varepsilon,p}^+$ is a Hopf algebra with coproduct $\Delta_{\varepsilon,p}^+$ and T has a right comodule structure with coaction $\Delta_{\varepsilon,p}$. Denote by $S_{\varepsilon,p}^+$ the antipode of $(T_{\varepsilon,p}^+,\Delta_{\varepsilon,p}^+)$.

2.3 Differentiable sectors. For each $\tau\in\mathbf{T}$, denote by $\bigcirc_\tau$ the set of vertices $v\in N_\tau$ for which $\mathfrak{t}(v)=\bigcirc$, and write $|\tau|_\bigcirc$ for the number of such vertices in τ. For each $\tau\in\mathbf{T}$ and $v\in\bigcirc_\tau$, denote by $D_v\tau$ the same tree as τ except that the $\bigcirc$ symbol at vertex v has been replaced with the $\odot$ symbol. For any subset $\mathbf{C}\subset\mathbf{T}^{(0)}$ we define

$$\dot{\mathbf{C}} := \big\{D_v\tau\ ;\ \tau\in\mathbf{C},\ v\in\bigcirc_\tau\big\}.$$

Also we define the linear map $D:\mathrm{span}(\mathbf{T}^{(0)})\to\mathrm{span}(\mathbf{T}^{(1)})$ by setting for any $\tau\in\mathbf{T}^{(0)}$

$$D\tau := \sum_{v\in\bigcirc_\tau} D_v\tau \tag{2.4}$$

if $\bigcirc_\tau\neq\emptyset$, and $D\tau:=0$ otherwise. Note that D preserves the $|\cdot|_{\varepsilon,\infty}$-degree of trees because $|\odot|_{\varepsilon,\infty}=|\bigcirc|_{\varepsilon,\infty}$ but may change the $|\cdot|_{\varepsilon,p}$-degree when p is finite.

Definition – *Let $\mathbf{B}$ be a finite subset of $\mathbf{T}^{(0)}$. We call $V=\mathrm{span}(\mathbf{B})$ a **differentiable sector** if it satisfies the following properties.*

(a) *The vector space V is a sector (see Definition 2.5 of [17]), that is, setting*

$$V_{0,\infty}^+ := \mathrm{alg}_{0,\infty}^+(\mathbf{B})$$

one has $S_{0,\infty}^+(V_{0,\infty}^+)\subset V_{0,\infty}^+$ and

$$\Delta_{0,\infty}(V)\subset V\otimes V_{0,\infty}^+,\qquad \Delta_{0,\infty}^+(V_{0,\infty}^+)\subset V_{0,\infty}^+\otimes V_{0,\infty}^+.$$

(b) *Setting*

$$\dot V := \mathrm{span}(\dot{\mathbf{B}}),\qquad \dot V_{0,2}^+ := \mathrm{alg}_{0,2}^+(\mathbf{B}\cup\dot{\mathbf{B}})\cap\mathrm{span}(\mathbf{T}^{(1)}),$$

one has $S^+_{0,2}(\dot V^+_{0,2}) \subset \dot V^+_{0,2}$ *and*

$$\Delta_{0,2}(\dot V) \subset \Big(\dot V \otimes V^+_{0,\infty} + V \otimes \dot V^+_{0,2}\Big), \qquad \Delta^+_{0,2}(\dot V^+_{0,2}) \subset \Big(\dot V^+_{0,2} \otimes V^+_{0,\infty} + V^+_{0,\infty} \otimes \dot V^+_{0,2}\Big).$$

(c) *There exists a constant* $\varepsilon_0 = \varepsilon_0(\mathbf{B}) > 0$ *such that, for any* $\tau, \sigma \in \mathbf{B}$, $|\sigma|_{0,p} < |\tau|_{0,p}$ *implies* $|\sigma|_{\varepsilon,p} < |\tau|_{\varepsilon,p}$ *for any* $\varepsilon \in (0, \varepsilon_0)$ *and* $p \in \{2, \infty\}$.

We choose the letter $\mathbf{B}$ for 'basis'. Note that p has no influences on the trees in $\mathbf{B}$. The property (c) means that the coproduct on V, resp. $V^+_{0,\infty}$, is also independent of ε: $\Delta^{(+)}_{\varepsilon,p} = \Delta^{(+)}_{0,\infty}$ on V, resp. $V^+_{0,\infty}$, for any $(\varepsilon, p) \in [0, \varepsilon_0) \times [2, \infty]$. While the property that $|\sigma|_{0,p} < |\tau|_{0,p}$ implies $|\sigma|_{\varepsilon,p} < |\tau|_{\varepsilon,p}$ holds for generic $p \in [2, \infty]$ and any $\varepsilon \in (0, \varepsilon_0(p))$ with small $\varepsilon_0(p) > 0$, this $\varepsilon_0(p)$ cannot be uniform over all p. This is why we assume the property (c) only for $p \in \{2, \infty\}$.

We provide a more detailed description on the p-dependence of $\Delta_{\varepsilon,p}$. For any planted tree $\mu \in \mathcal{P}(\dot{\mathbf{B}})$ such that $0 < |\mu|_{\varepsilon,2} \le \frac{|\mathfrak{s}|}{2}$, we define the exponent

$$p_\varepsilon(\mu) := \frac{|\mathfrak{s}|}{\frac{|\mathfrak{s}|}{2} - |\mu|_{\varepsilon,2}} \in (2, \infty];$$

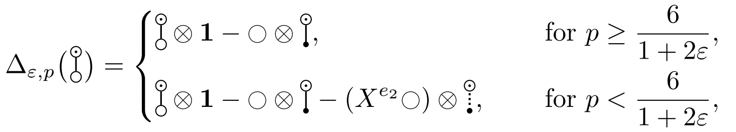
so $p_\varepsilon(\mu)$ is the unique p for which $|\mu|_{\varepsilon,p} = 0$. The set

$$I_\varepsilon := \Big\{ p_\varepsilon(\mu)\,;\ \mu \in \mathcal{P}(\dot{\mathbf{B}}),\ 0 < |\mu|_{\varepsilon,2} < \frac{|\mathfrak{s}|}{2} \Big\} \subset (2, \infty)$$

is finite and its associated 'floor function' is defined as

$$\lfloor p \rfloor_{I_\varepsilon} := \max \big\{ q \in \{2\} \cup I_\varepsilon\,;\ q < p \big\}$$

for any $p \in (2, \infty]$. The projection operator $P^+_{\varepsilon,p}$ is right continuous in p and constant in any interval which is disjoint with I_ε, so is the co-action operator $\Delta_{\varepsilon,p}$. Hence they behave like right continuous 'step functions'. The regularity structure of the one-dimensional multiplicative stochastic heat equation provides an elementary example. For instance, under the choices $\mathfrak{s} = (2, 1)$ and $\alpha_0 = -\frac{3}{2}$ we have


(2.5)

with a dotted line for the operator $\mathcal{I}_{e_2}$.

If $\mathbf{B}$ consists of all trees which strongly conform to a complete subcritical rule (see [8, Section 5]) and with degrees $|\cdot|_{0,\infty}$ less than some fixed number, then $V = \mathrm{span}(\mathbf{B})$ is a differentiable sector – See Section 4.1. In the sequel we fix a differentiable sector V and its basis $\mathbf{B}$. The defining properties of a differentiable sector ensure that the tuple

$$\mathscr{V}_\varepsilon := \Big((V, \Delta_{\varepsilon,\infty}), (V^+_{0,\infty}, \Delta^+_{\varepsilon,\infty}) \Big)$$

defines a concrete regularity structure in the sense of [3]. The structure of the (ε, p)-dependent tuple

$$\mathscr{W}_{\varepsilon,p} := \Big((W := V \oplus \dot V, \Delta_{\varepsilon,p}), (W^+_{\varepsilon,p} := V^+_{0,\infty} \oplus \dot V^+_{\varepsilon,p}, \Delta^+_{\varepsilon,p}) \Big),$$

where $\dot V^+_{\varepsilon,p} := \mathrm{alg}^+_{\varepsilon,p}(\mathbf{B} \cup \dot{\mathbf{B}}) \cap \mathrm{span}(\mathbf{T}^{(1)})$, is encapsulated in a useful variation on the notion of regularity structure that we call a *regularity-integrability structure.*

2.4 Regularity-integrability structures. We introduce this notion to deal with models on a usual regularity structure and their first order Malliavin derivative in a single setting. See the companion work [21] for more detailed descriptions.

Definition – *A* ***regularity-integrability structure*** (A, T, G) *consists of the following elements.*

(a) *The index set* A *is a subset of* $\mathbb{R} \times [1, \infty]$ *such that*

$$\big\{ (\gamma, r) \in A\,;\ \gamma < \beta,\ r \ge q \big\}$$

is finite for every $(\beta, q) \in \mathbb{R} \times [1, \infty]$, *where we define the strict partial order on* $\mathbb{R} \times [1, \infty]$ *by*

$$(\gamma, r) < (\beta, q) \quad \overset{\text{def}}{\iff} \quad \gamma < \beta \text{ and } r \ge q.$$

(b) *The vector space* $T = \bigoplus_{\mathbf{a} \in A} T_{\mathbf{a}}$ *is an algebraic sum of Banach spaces* $(T_{\mathbf{a}}, \|\cdot\|_{\mathbf{a}})$.

(c) *The structure group G is a group of continuous linear operators on T such that one has for all $\Gamma \in G$ and $\mathbf{a} \in A$*

$$(\Gamma - \mathrm{id})T_{\mathbf{a}} \subset \bigoplus_{\mathbf{a}' \in A,\, \mathbf{a}' < \mathbf{a}} T_{\mathbf{a}'}.$$

For $(\beta, q) \in A$ the number β is a regularity exponent and q is an integrability exponent. One says that the regularity-integrability structure has a *regularity* $\alpha_0 \in \mathbb{R}$ if $(\alpha_0, \infty) < \mathbf{a}$ for any $\mathbf{a} \in A$. Denoting by $P_{\mathbf{a}} : T \to T_{\mathbf{a}}$ the canonical projection, we set with a slight abuse of notations

$$\|\tau\|_{\mathbf{a}} := \|P_{\mathbf{a}}\tau\|_{\mathbf{a}}$$

for any $\tau \in T$ and $\mathbf{a} \in A$.

We return to the setting of Section 2.3 with fixed (ε, p). For each $\tau \in \mathbf{B} \cup \dot{\mathbf{B}}$ we define the integrability exponent

$$i_p(\tau) = \begin{cases} \infty & \text{if } \tau \in \mathbf{B}, \\ p & \text{if } \tau \in \dot{\mathbf{B}}. \end{cases} \tag{2.6}$$

Then we have the grading on $W = V \oplus \dot{V}$ by

$$A_{\varepsilon,p} := \big\{ \big(|\tau|_{\varepsilon,p}, i_p(\tau)\big) \,;\, \tau \in \mathbf{B} \cup \dot{\mathbf{B}} \big\} = \big\{ \big(|\tau|_{\varepsilon,\infty}, \infty\big) \,;\, \tau \in \mathbf{B} \big\} \cup \big\{ \big(|\dot{\tau}|_{\varepsilon,p}, p\big) \,;\, \dot{\tau} \in \dot{\mathbf{B}} \big\},$$

$$W = \bigoplus_{\mathbf{a} \in A_{\varepsilon,p}} W_{\mathbf{a}}, \qquad W_{\mathbf{a}} := \mathrm{span}\,\big\{ \tau \in \mathbf{B} \cup \dot{\mathbf{B}} \,;\, \big(|\tau|_{\varepsilon,p}, i_p(\tau)\big) = \mathbf{a} \big\}.$$

We further introduce the group $\mathsf{G}^+_{\varepsilon,p}$ of characters on the Hopf algebra

$$\big(W^+_{\varepsilon,p} := V^+_{0,\infty} \oplus \dot{V}^+_{\varepsilon,p}, \Delta^+_{\varepsilon,p}\big),$$

or equivalently, on the quotient Hopf algebra $\mathrm{alg}^+_{\varepsilon,p}(\mathbf{B} \cup \dot{\mathbf{B}})/\ddot{I}$, where $\ddot{I}$ the Hopf ideal generated by trees with more than one $\odot$ symbols. The group $\mathsf{G}^+_{\varepsilon,p}$ has a representation in $GL(W)$ where $\mathsf{g} \in \mathsf{G}^+_{\varepsilon,p}$ is mapped to $(\mathrm{id} \otimes \mathsf{g})\Delta_{\varepsilon,p}$. Denote by $G_{\varepsilon,p}$ the image group. Then the triple $(A_{\varepsilon,p}, W, G_{\varepsilon,p})$ is a regularity-integrability structure; it is said to be associated with the ***concrete regularity-integrability structure***

$$\mathscr{W}_{\varepsilon,p} := \big((W, \Delta_{\varepsilon,p}), (W^+_{\varepsilon,p}, \Delta^+_{\varepsilon,p})\big).$$

A ***substructure*** $(A'_{\varepsilon,p}, W', G'_{\varepsilon,p})$ of $(A_{\varepsilon,p}, W, G_{\varepsilon,p})$ is a regularity-integrability structure where $A'_{\varepsilon,p} \subset A_{\varepsilon,p}, W' \subset W$ and $G'_{\varepsilon,p} \subset G_{\varepsilon,p}$. We also have an analogous notion of concrete regularity-integrability substructure.

2.5 Models on regularity-integrability structures. Fix $(\varepsilon, p) \in [0, \varepsilon_0) \times [2, \infty]$ and $c > 0$. Assume we are given a pair of maps $\mathsf{M} = (\mathsf{\Pi}, \mathsf{g})$ such that

$$\mathsf{\Pi} : W \to C^{\alpha_0, Q}(w_c), \qquad \mathsf{g} : \mathbb{R}^d \mapsto \mathsf{G}^+_{\varepsilon,p}$$

and $\mathsf{\Pi}$ is continuous and linear. The map $\mathsf{\Pi}$ is called an ***interpretation map***. Set, for any $\tau \in W$ and $\mu \in W^+_{\varepsilon,p}$,

$$\mathsf{\Pi}^{\varepsilon,p}_x \tau := \big(\mathsf{\Pi} \otimes (\mathsf{g}_x \circ S^+_{\varepsilon,p})\big)\Delta_{\varepsilon,p}\tau,$$

$$\mathsf{g}^{\varepsilon,p}_{yx}(\mu) := \big(\mathsf{g}_y \otimes (\mathsf{g}_x \circ S^+_{\varepsilon,p})\big)\Delta^+_{\varepsilon,p}\mu.$$

We talk of the map $\mathsf{\Pi}^{\varepsilon,p}_x$ as the ***recentered interpretation map***.

Definition – *A pair of maps $\mathsf{M} = (\mathsf{\Pi}, \mathsf{g})$ as above is called a* ***model on $\mathscr{W}_{\varepsilon,p}$ (with weight w_c)*** *if*

$$\|\mathsf{\Pi}^{\varepsilon,p} : \tau\|_{w_c} := \sup_{0 < t \le 1} t^{-|\tau|_{\varepsilon,p}/\ell} \|\mathcal{Q}_t(x, \mathsf{\Pi}^{\varepsilon,p}_x \tau)\|_{L^{i_p(\tau)}_x(w_c)} < \infty \tag{2.7}$$

for any $\tau \in \mathbf{B} \cup \dot{\mathbf{B}}$ and

$$\|\mathsf{g}^{\varepsilon,p} : \mu\|_{w_c} := \|\mathsf{g}_x(\mu)\|_{L^{i_p(\mu)}_x(w_c)} + \sup_{y \in \mathbb{R}^d \setminus \{0\}} \left(w_c(y) \frac{\big\|\mathsf{g}^{\varepsilon,p}_{(x+y)x}(\mu)\big\|_{L^{i_p(\mu)}_x(w_c)}}{\|y\|^{|\mu|_{\varepsilon,p}}_{\mathfrak{s}}} \right) < \infty \tag{2.8}$$

for any $\mu \in \mathcal{P}(\mathbf{B} \cup \dot{\mathbf{B}}) \cap W^+_{\varepsilon,p}$, where $i_p(\mu) = p$ if $\mu \in \mathcal{P}(\dot{\mathbf{B}})$ and $i_p(\mu) = \infty$ otherwise. We denote by $\mathbf{M}(\mathscr{W}_{\varepsilon,p})_{w_c}$ the set of all models and define the quantity

$$\|\mathsf{M}\|_{\mathbf{M}(\mathscr{W}_{\varepsilon,p})_{w_c}} := \max_{\tau \in \mathbf{B} \cup \dot{\mathbf{B}}} \|\mathsf{\Pi}^{\varepsilon,p} : \tau\|_{w_c} + \max_{\mu \in \mathcal{P}(\mathbf{B} \cup \dot{\mathbf{B}}) \cap W^+_{\varepsilon,p}} \|\mathsf{g}^{\varepsilon,p} : \mu\|_{w_c}$$

for any $\mathsf{M} \in \mathbf{M}(\mathscr{W}_{\varepsilon,p})_{w_c}$. *We also define a metric* $\|\mathsf{M}_1 : \mathsf{M}_2\|_{\mathbf{M}(\mathscr{W}_{\varepsilon,p})_{w_c}}$ *on* $\mathbf{M}(\mathscr{W}_{\varepsilon,p})_{w_c}$ *setting for* $\mathsf{M}_1, \mathsf{M}_2 \in \mathbf{M}(\mathscr{W}_{\varepsilon,p})_{w_c}$

$$\|\mathsf{M}_1 : \mathsf{M}_2\|_{\mathbf{M}(\mathscr{W}_{\varepsilon,p})_{w_c}} := \max_{\tau \in \mathbf{B} \cup \dot{\mathbf{B}}} \|\Pi_1^{\varepsilon,p}, \Pi_2^{\varepsilon,p} : \tau\|_{w_c} + \max_{\mu \in \mathcal{P}(\mathbf{B} \cup \dot{\mathbf{B}}) \cap W_{\varepsilon,p}^+} \|\mathsf{g}_1^{\varepsilon,p}, \mathsf{g}_2^{\varepsilon,p} : \mu\|_{w_c},$$

where the quantities in the right hand side are defined in the same way as (2.7) *and* (2.8) *but with* $(\Pi_1)_x^{\varepsilon,p}\tau - (\Pi_2)_x^{\varepsilon,p}\tau$, $(\mathsf{g}_1)_x(\mu) - (\mathsf{g}_2)_x(\mu)$, *and* $(\mathsf{g}_1)_{yx}^{\varepsilon,p}(\mu) - (\mathsf{g}_2)_{yx}^{\varepsilon,p}(\mu)$ *in places of* $\Pi_x^{\varepsilon,p}\tau$, $\mathsf{g}_x(\mu)$, *and* $\mathsf{g}_{yx}^{\varepsilon,p}(\mu)$, *respectively.*

Since the exponents ε and p are involved not only in the definition of the recentered maps $\Pi_x^{\varepsilon,p}$ and $\mathsf{g}_{yx}^{\varepsilon,p}$ but also in the definition of the norms (2.7) and (2.8) via the degree map $|\cdot|_{\varepsilon,p}$, it would be more proper to write $\|(\cdot)^{\varepsilon,p} : \tau\|_{\varepsilon,p;w_c}$ for the norms. For the sake of readability, we use the above lightened notations.

In the setting of regularity-integrability structures one can prove analogues of the reconstruction theorem and the multilevel Schauder estimate. They are stated in Appendix A in Theorem 22 and Theorem 24. One can find the detailed self-contained proofs of them in [21].

In the proof of Theorem 6 we will consider restricted bounds on concrete regularity-integrability substructures $\mathscr{W}'_{\varepsilon,p} = \big((U, \Delta_{\varepsilon,p}), (U^+, \Delta^+_{\varepsilon,p})\big)$ of $\mathscr{W}_{\varepsilon,p}$ of the form

$$U = \mathrm{span}(\mathbf{A}), \qquad U^+ = \mathrm{alg}^+_{\varepsilon,p}(\mathbf{A}') \cap \mathrm{span}(\mathbf{T}^{(1)})$$

for some subsets $\mathbf{A}, \mathbf{A}' \subset \mathbf{B} \cup \dot{\mathbf{B}}$. For any $\mathsf{M} \in \mathbf{M}(\mathscr{W}_{\varepsilon,p})_{w_c}$ it is useful to define the restricted quantity

$$\begin{aligned} \|\mathsf{M}\|_{\mathbf{M}(\mathscr{W}'_{\varepsilon,p})_{w_c}} &:= \|\Pi^{\varepsilon,p} : U\|_{w_c} + \|\mathsf{g}^{\varepsilon,p} : U^+\|_{w_c} \\ &:= \max_{\tau \in \mathbf{A}} \|\Pi^{\varepsilon,p} : \tau\|_{w_c} + \max_{\mu \in \mathcal{P}(\mathbf{A}') \cap W^+_{\varepsilon,p}} \|\mathsf{g}^{\varepsilon,p} : \mu\|_{w_c}. \end{aligned} \tag{2.9}$$

In particular the restriction to the concrete regularity structure $\mathscr{V}_\varepsilon = \big((V, \Delta_{\varepsilon,\infty}), (V^+_{0,\infty}, \Delta^+_{\varepsilon,\infty})\big)$ is a model in the usual sense of [17]. Recall again that the parameter p is useless in $\mathscr{V}_\varepsilon$, but the parameter ε is involved in the norms via the degree map $|\cdot|_{\varepsilon,\infty}$.

Recall from (1.6) the definition of the operator $\mathcal{K}$ acting on functions over $\mathbb{R}^d$. An interpretation map Π is said to be $\mathcal{K}$-*admissible* if it satisfies

$$(\Pi X^k)(x) = x^k, \qquad \Pi(\mathcal{I}_k\tau) = \partial^k \mathcal{K}(\cdot, \Pi\tau),$$

for all k and τ. For any $\mathcal{K}$-admissible interpretation map Π such that

$$\Pi : W \to C^0_+(\mathbb{R}^d) := C(\mathbb{R}^d) \cap \bigcap_{c>0} L^\infty(w_c),$$

we can define a model $\mathsf{M}^{\varepsilon,p} = (\Pi, \mathsf{g}^{\varepsilon,p})$ on $\mathscr{W}_{\varepsilon,p}$ from the recursive definition of $\mathsf{g}^{\varepsilon,p}$ given by the formula

$$(\mathsf{g}_x^{\varepsilon,p})^{-1}(X^k) = (-x)^k, \qquad (\mathsf{g}_x^{\varepsilon,p})^{-1}(\mathcal{I}_k\tau) = -\sum_{l \in \mathbb{N}^d} \frac{(-x)^l}{l!} \mathbf{1}_{|\mathcal{I}_{k+l}\tau|_{\varepsilon,p} > 0}\, \partial^{k+l}\mathcal{K}(x, \Pi_x^{\varepsilon,p}\tau) \tag{2.10}$$

for the inverse $(\mathsf{g}_x^{\varepsilon,p})^{-1} := \mathsf{g}_x^{\varepsilon,p} \circ S^+_{\varepsilon,p}$. (See e.g. Section 3 of Bruned's work [6].) For any choice of $\xi = \Pi(\bigcirc)$ and $h = \Pi(\odot)$ both in $C^0_+(\mathbb{R}^d)$, we can define the unique multiplicative $\mathcal{K}$-admissible interpretation map $\Pi^{\xi,h}$; it is called the *naive interpretation map* associated to (ξ, h).

2.6 Renormalized models. Here is how to build a large family of $\mathcal{K}$-admissible interpretation maps from a naive one. Recall from [6] and Bruned & Nadeem's work [10] the following definition.

Definition – *A* ***preparation map*** *is a linear map*

$$R : W \to W$$

which leaves stable the subspace V *and has the following properties.*

- **(a)** *R fixes the polynomials and the noises: $R\tau = \tau$ for $\tau \in \{X^k, \bigcirc, \odot\}$.*
- **(b)** *For each $\tau \in \mathbf{B} \cup \dot{\mathbf{B}}$ there exist finitely many $\tau_i \in \mathbf{B} \cup \dot{\mathbf{B}}$ and constants λ_i such that*

$$R\tau = \tau + \sum_i \lambda_i \tau_i, \quad \textit{with} \quad |\tau_i|_{0,p} > |\tau|_{0,p} \textit{ for } p \in \{2, \infty\} \quad \textit{and} \quad |\tau_i|_{\bigcirc} < |\tau|_{\bigcirc}.$$

- **(c)** *R fixes planted trees: $R\mathcal{I}_k = \mathcal{I}_k$.*
- **(d)** *R commutes with the coproduct: $(R \otimes \mathrm{id})\Delta_{0,2} = \Delta_{0,2}R$.*
- **(e)** *R commutes with the derivative map* (2.4)*: $RD = DR$.*

It is easily checked that the triangular property (b) and the commutation (d) also hold for arbitrary exponents $(\varepsilon,p)\in[0,\varepsilon_0)\times[2,\infty]$. First, by the property (c) of differential sectors, $|\tau_i|_{0,p}>|\tau|_{0,p}$ implies $|\tau_i|_{\varepsilon,p}>|\tau|_{\varepsilon,p}$ for $p\in\{2,\infty\}$. Since $|\tau|_{\varepsilon,p}$ is affine for $\frac{1}{p}$, these inequalities extend to all $p\in[2,\infty]$. Next, we have the commutation $(R\otimes\mathrm{id})\Delta_{\varepsilon,2}=\Delta_{\varepsilon,2}R$ because $\Delta_{0,2}=\Delta_{\varepsilon,2}$ by the property (c) of differential sectors. Since $\Delta_{\varepsilon,p}=(\mathrm{id}\otimes P^+_{\varepsilon,p})\Delta_{\varepsilon,2}$ by definition, we also have

$$(R\otimes\mathrm{id})\Delta_{\varepsilon,p}=(\mathrm{id}\otimes P^+_{\varepsilon,p})(R\otimes\mathrm{id})\Delta_{\varepsilon,2}=(\mathrm{id}\otimes P^+_{\varepsilon,p})\Delta_{\varepsilon,2}R=\Delta_{\varepsilon,p}R.$$

A typical example of R is defined from the choice of constants $\{\ell(\sigma)\}_{\sigma\in\mathbf{B},\,|\sigma|_{\varepsilon,\infty}<0}$ by the extraction-contraction formula as in Corollary 4.5 of [6], or by its dual formula

$$R^*_\ell(\tau)=\sum_{\sigma\in\mathbf{B},\,|\sigma|_{\varepsilon,\infty}<0}\frac{\ell(\sigma)}{S(\sigma)}(\tau\star\sigma),\qquad(\tau\in\mathbf{B})\tag{2.11}$$

as in Bailleul & Bruned's work [1]. See [1] or Bruned & Manchon's work [9] for the precise definitions of all operators. These definitions are easily extended to all $\tau\in\mathbf{B}\cup\dot{\mathbf{B}}$.

For any given preparation map R, let

$$\widehat{M}^R:W\to W$$

be the linear map uniquely defined from R by requiring that $\widehat{M}^R$ is *multiplicative*, fixes polynomials and noises, and satisfies

$$\widehat{M}^R(\mathcal{I}_k\tau)=\mathcal{I}_k\big(\widehat{M}^R(R\tau)\big)$$

for all k and τ. Then the linear map

$$M^R:=\widehat{M}^RR.$$

is the ***renormalization map associated to the preparation map*** R. It follows from the property (c) of preparation maps that if Π is a $\mathcal{K}$-admissible interpretation map then so is the map

$$\Pi^R:=\Pi M^R.$$

Then we write

$$\mathsf{M}^{R;\varepsilon,p}:=(\Pi^R,\mathsf{g}^{R;\varepsilon,p})$$

for the model on $\mathscr{W}_{\varepsilon,p}$ constructed from the $\mathcal{K}$-admissible interpretation map Π^R by formula (2.10) with Π^R in place of Π. When $\Pi=\Pi^{\xi,h}$ is the naive interpretation map associated to $\xi,h\in C^0_+(\mathbb{R}^d)$, we denote the associated model by

$$\mathsf{M}^{\xi,h,R;\varepsilon,p}$$

In the setting of singular stochastic PDEs we insist on renormalizing models with preparation maps that are *interpretable*. Denote by $\mathbf{u}_R$ the solution, in a space of modelled distributions, to the regularity structure formulation of an equation $Lu=f(u,\xi)$ associated with a model built from a preparation map R and a smooth noise ξ. We say that R is interpretable if the reconstruction u_R of $\mathbf{u}_R$ is the solution to an equation of the form $Lu_R=f(u_R,\xi)+c(u_R)$, for some counterterms $c(u_R)$ that may depend on u_R and some of its derivatives. It was proved in Proposition 3.2 of [1] that the maps R of the form (2.11) are indeed interpretable preparation maps. The BPHZ renormalisation map from [8] corresponds to a particular choice of coefficients $\ell(\sigma)$ in the formula (2.11). Our proof of Theorem 6 works for any preparation map.

Before going to the main result we introduce two important algebraic identities. The proofs of them are given in Section 4.2. The following fact is also proved in Proposition 4.1 of [10].

3 – Lemma. *Let $\xi,h\in C^0_+(\mathbb{R}^d)$ and let R be a preparation map. For any $\tau\in\mathbf{B}$ one has*

$$d_\xi(\Pi^{\xi,R;\varepsilon,\infty}_x\tau)(h):=\frac{d}{dt}(\Pi^{\xi+th,R;\varepsilon,\infty}_x\tau)\big|_{t=0}=\Pi^{\xi,h,R;\varepsilon,\infty}_x(D\tau),\tag{2.12}$$

where the letter h is removed from $\Pi^{\xi,h,R;\varepsilon,\infty}_x\tau$ since this quantity is independent of h.

We remark that the integrability exponent in the right hand side of (2.12) is ∞, rather than any finite p. We further introduce some notations to make explicit the difference between $\Pi^{R;\varepsilon,p}_x$ and $\Pi^{R;\varepsilon,2}_x$. We define the linear map $\mathsf{h}^{R;\varepsilon,p}_x:W^+_{\varepsilon,2}\to\mathbb{R}$ by

$$\mathsf{h}^{R;\varepsilon,p}_x(\mathcal{I}_k\sigma):=\mathbf{1}_{|\mathcal{I}_k\sigma|_{\varepsilon,p}\le 0<|\mathcal{I}_k\sigma|_{\varepsilon,2}}\partial^k\mathcal{K}\Big(x,\Pi^{R;\varepsilon,\lfloor p_\varepsilon(\mathcal{I}_k\sigma)\rfloor_{I_\varepsilon}}_x\sigma\Big)\tag{2.13}$$

for any planted trees, and by $\mathsf{h}^{R;\varepsilon,p}_x(\tau):=0$ for non-planted trees τ. The strict inequality in the indicator implies that $\mathsf{h}^{R;\varepsilon,p}_x(\mu)=0$ if $\mu\in\mathcal{P}(\mathbf{B})$. For any arbitrary tree τ and $e=(e_+,e_-)\in E_\tau$ denote by $\{\tau^e_+,\tau^e_-\}$ the connected components of $\tau\setminus\{e\}$ such that τ^e_- contains the root and the node e_-. Last, given $k\in\mathbb{N}^d$

and $v \in N_\tau$ denote by $\uparrow_v^k \tau$ the decorated tree τ with the same decorations as τ except that the node v has now decoration $\mathfrak{n}(v) + k$. The following formula plays a crucial role in the proof of Theorem 6, especially in the proof of Lemma 15. Bruned & Nadeem proved in Proposition 3.7 of [10] a statement with a similar flavour.

4 – Lemma. *Let $\mathsf{M}^{R;\varepsilon,p}$ be arbitrary renormalized model constructed from the $\mathcal{K}$-admissible model and the preparation map R. For any $\tau \in \dot{\mathbf{B}}$ and any $p \in [2,\infty]$, one has*

$$\begin{aligned} \mathsf{\Pi}_x^{R;\varepsilon,p}\tau &= \mathsf{\Pi}_x^{R;\varepsilon,2}\tau + \sum_{e \in E_\tau} \sum_{k \in \mathbb{N}^d} \frac{1}{k!}\, \mathsf{h}_x^{R;\varepsilon,p}(\mathcal{I}_{\mathfrak{e}(e)+k}\tau_+^e)\, \mathsf{\Pi}_x^{R;\varepsilon,p}(\uparrow_{e_-}^k \tau_-^e) \\ &= \mathsf{\Pi}_x^{R;\varepsilon,2}\tau + \big(\mathsf{\Pi}_x^{R;\varepsilon,p} \otimes \mathsf{h}_x^{R;\varepsilon,p}\big)\Delta_{\varepsilon,2}\tau. \end{aligned} \tag{2.14}$$

Note that τ_-^e in (2.14) does not contain the $\odot$ symbol, so p in $\mathsf{\Pi}_x^{R;\varepsilon,p}(\uparrow_{e_-}^k \tau_-^e)$ can be arbitrary. In the example (2.5) with $\xi, h \in C_+^0(\mathbb{R}^d)$ and with an identity preparation map, one has

$$\mathsf{\Pi}_x^{\xi,h;\varepsilon,p}\Big(\overset{\odot}{\underset{\circ}{|}}\Big)(\cdot) = \begin{cases} \big(\mathcal{K}(\cdot,h) - \mathcal{K}(x,h)\big)\xi(\cdot), & \text{for } p \geq \dfrac{6}{1+2\varepsilon}, \\ \big(\mathcal{K}(\cdot,h) - \mathcal{K}(x,h) - \partial_2\mathcal{K}(x,h)(\cdot - x_2)\big)\xi(\cdot), & \text{for } p < \dfrac{6}{1+2\varepsilon}, \end{cases}$$

Then the formula (2.14) writes

$$\begin{aligned} \mathsf{\Pi}_x^{\xi,h;\varepsilon,p}\Big(\overset{\odot}{\underset{\circ}{|}}\Big)(\cdot) &= \mathsf{\Pi}_x^{\xi,h;\varepsilon,2}\Big(\overset{\odot}{\underset{\circ}{|}}\Big)(\cdot) + \partial_2\mathcal{K}(x,h)\,(\cdot - x_2)\xi(\cdot) \\ &= \mathsf{\Pi}_x^{\xi,h;\varepsilon,2}\Big(\overset{\odot}{\underset{\circ}{|}}\Big)(\cdot) + \mathsf{h}_x^{\xi,h;\varepsilon,p}\Big(\overset{\odot}{\underset{\bullet}{\vdots}}\Big)\mathsf{\Pi}_x^{\xi,h;\varepsilon,p}(X^{e_2}\bigcirc)(\cdot) \end{aligned}$$

for $p \geq \frac{6}{1+2\varepsilon}$.

2.7 A convergence result. Let $V = \mathrm{span}(\mathbf{B})$ be a differentiable sector. We fix the Hilbert space $H = H^{-\kappa,Q}(w_0)$ with $\kappa \geq 0$ and the Banach space $\Omega = C^{\alpha_0,Q}(w_c)$ for some $c > 0$ and $\alpha_0 < -|\mathfrak{s}|/2 - \kappa$. For a family of compactly supported smooth functions $\varrho_n \in C^\infty(\mathbb{R}^d)$ converge weakly to a Dirac mass at 0 as $n \in \mathbb{N}$ goes to ∞, we define a random variable ξ_n on Ω setting

$$\xi_n(\omega) := \varrho_n * \omega \in \Omega.$$

Set as well for $h \in H$

$$h_n := \varrho_n * h.$$

We denote by $\mathsf{\Pi}^{\xi_n,h_n}$ the naive interpretation map on W associated with (ξ_n, h_n) – it is a random variable $(\mathsf{\Pi}^{\xi_n,h_n}(\cdot))(\omega) := \mathsf{\Pi}^{\xi_n(\omega),h_n}(\cdot)$. If we are given a deterministic preparation map R_n on W we denote by $\mathsf{M}^{\xi_n,h_n,R_n;\varepsilon,p} = \big(\mathsf{\Pi}^{\xi_n,h_n,R_n}, \mathsf{g}^{\xi_n,h_n,R_n;\varepsilon,p}\big)$ the random ($\mathcal{K}$-admissible) model on $\mathscr{W}_{\varepsilon,p}$ associated to R_n and the random naive interpretation map $\mathsf{\Pi}^{\xi_n,h_n}$. The associated recentered interpretation map is denoted by $\mathsf{\Pi}_x^{\xi_n,h_n,R_n;\varepsilon,p}$. The following result is a direct consequence of Proposition 3.16 of [6].

5 – Lemma. *The renormalized model $\mathsf{M}^{\xi_n,h_n,R_n;\varepsilon,p}$ takes almost surely its values in $\mathbf{M}(\mathscr{W}_{\varepsilon,p})_{w_c}$.*

We can now state our main result.

6 – Theorem. *Pick $\kappa \geq 0$ and $\alpha_0 < -|\mathfrak{s}|/2 - \kappa$ and $c > 0$. Let $\mathbb{P}$ be a Borel probability measure on $\Omega = C^{\alpha_0,Q}(w_c)$ for some $c > 0$ that is stationary under the spacetime translations and satisfies the $H(= H^{-\kappa,Q}(w_0))$-spectral gap inequality. Let $V = \mathrm{span}(\mathbf{B})$ be a differentiable sector for which all $\tau \in \mathbf{B} \setminus \{\bigcirc\}$ satisfy*

$$|\tau|_{0,\infty} + \frac{|\mathfrak{s}|}{2} > 0.$$

Assume we are given a family $(R_n)_{n\geq 0}$ of deterministic preparation maps on W for which all the quantities

$$\mathbb{E}\big[\mathcal{Q}_1(0, \mathsf{\Pi}_0^{\xi_n,R_n;0,\infty}\tau)\big] \tag{2.15}$$

converge as n goes to ∞, for any $\tau \in \mathbf{B}$ with $|\tau|_{0,\infty} \leq 0$, where the useless letter h_n is removed. Then for any $c > 0$, $\varepsilon \in (0, \varepsilon_0)$, $p \in [2,\infty]$, and $q \in [1,\infty)$, one has

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{\|h\|_H \leq 1} \|\mathsf{M}^{\xi_n,h_n,R_n;\varepsilon,p}\|^q_{\mathbf{M}(\mathscr{W}_{\varepsilon,p})_{w_c}}\right] < \infty, \tag{2.16}$$

and that $\mathsf{M}^{\xi_n,h_n,R_n;\varepsilon,p}$ converges in $L^q\big(\Omega, \mathbb{P}; \mathbf{M}(\mathscr{W}_{\varepsilon,p})_{w_c}\big)$ as n goes to ∞.

As a direct consequence of Theorem 6, the (h_n-independent) restrictions to the concrete regularity structure $\mathscr{V}_\varepsilon = \big((V, \Delta_{\varepsilon,\infty}), (V^+_{0,\infty}, \Delta^+_{\varepsilon,\infty})\big)$ of the models $\mathsf{M}^{\xi_n, h_n, R_n;\varepsilon,\infty}$ converge in $L^q\big(\Omega, \mathbb{P}; \mathbf{M}(\mathscr{V}_\varepsilon)_{w_c}\big)$ as n goes to ∞ for any $q \in [1,\infty)$ and $c > 0$.

This result extends (the weighted version of) the main result of [20] into arbitrary preparation maps. Proceeding as in the proof of Theorem 6.18 of [8] one can see that there is a unique preparation map $R_n = R_n(\mathbb{P})$ of the form (2.11) such that the associated model (called BPHZ model) satisfies

$$\mathbb{E}\big[(\Pi^{\xi_n, R_n}\tau)(x)\big] = 0 \tag{2.17}$$

for any $x \in \mathbb{R}^d$ and $\tau \in \mathbf{B}$ of non-positive $|\cdot|_{0,\infty}$-degree. Moreover, we can construct an analogue here of $\overline{\text{BPHZ}}$ model of [20] by

$$\mathbb{E}\big[\mathcal{Q}_1(0, \Pi^{\xi_n, R_n;0,\infty}_x \tau)\big] = 0$$

for all $\tau \in \mathbf{B}$ of non-positive $|\cdot|_{0,\infty}$-degree. The convergence result of [20] is restricted to BPHZ and $\overline{\text{BPHZ}}$ models. $\overline{\text{BPHZ}}$ model obviously satisfies the assumption of our result. Also for BPHZ model, it is not difficult to show the convergence of (2.15) from the condition (2.17), following the induction steps in the proof of Theorem 6 described below.

We close this section by technical remarks on weights. The use of a weight w_c is a usual thing in the study of random fields on the full space $\mathbb{R}^d$. Such fields may grow to infinity as $|x|$ goes to ∞. On the analytic side, since we have modifications of Hölder inequality and Young inequality on weighted spaces (see Section 2.2 of [21]), these weights do not cause any serious problems in the proofs.

3 – The mechanics of convergence

We define a preorder $\preceq$ on $\mathbf{T}$ that we use to prove Theorem 6 by a finite induction. Set

$$\sigma \preceq \tau \quad \overset{\text{def}}{\iff} \quad \big(|\sigma|_\bigcirc, |E_\sigma|, |\sigma|_{0,\infty}\big) \leq \big(|\tau|_\bigcirc, |E_\tau|, |\tau|_{0,\infty}\big) \tag{3.1}$$

with the inequality $\leq$ in the right hand side standing here for the lexicographical order. Write

$$\mathbf{B} \setminus \{X^k\}_{k\in\mathbb{N}^d} = \big\{\tau_1 \preceq \tau_2 \preceq \cdots \preceq \tau_N\big\}.$$

Although there may be different choices τ_i for this representation of $\mathbf{B} \setminus \{X^k\}_{k\in\mathbb{N}^d}$, the choice has no consequence on Theorem 6 as it is only used as a tool in the proof of this result. Recall from (2.3) the definition of the algebra $\mathrm{alg}^+_{\varepsilon,p}(\mathbf{C})$, for an arbitrary subset $\mathbf{C}$ of $\mathbf{B}$. For $1 \leq i \leq N$ set

$$\mathbf{B}_i := \big\{\tau_1, \dots, \tau_i\big\}, \qquad V_i := \mathrm{span}\,\big(\mathbf{B}_i \cup \{X^k\}_{k\in\mathbb{N}^d}\big), \qquad V_i^+ := \mathrm{alg}^+_{0,\infty}(\mathbf{B}_i).$$

We also define

$$\dot{V}_i := \mathrm{span}(\dot{\mathbf{B}}_i), \qquad \dot{V}^+_{i,\varepsilon,p} := \mathrm{alg}^+_{\varepsilon,p}(\mathbf{B}_i \cup \dot{\mathbf{B}}_i) \cap \mathrm{span}(\mathbf{T}^{(1)}).$$

The proof of the following properties is given in Section 4.3.

7 – Lemma. *One has, for any $1 \leq i \leq N$ and $(\varepsilon, p) \in [0, \varepsilon_0) \times [2,\infty]$,*

$$\big(\Delta_{\varepsilon,p}(\tau) - \tau \otimes \mathbf{1}\big) \in V_{i-1} \otimes V^+_{i-1}, \qquad \tau \in \mathbf{B}_i$$

and

$$\big(\Delta_{\varepsilon,p}(\dot\tau) - \dot\tau \otimes \mathbf{1}\big) \in \Big(\dot{V}_{i-1} \otimes V^+_{i-1} + V_{i-1} \otimes \dot{V}^+_{i-1,\varepsilon,p}\Big), \qquad \dot\tau \in \dot{\mathbf{B}}_i.$$

For $1 \leq i \leq N$ set

$$V_0 := \mathrm{span}\,\big(\{X^k\}_{k\in\mathbb{N}^d}\big)$$

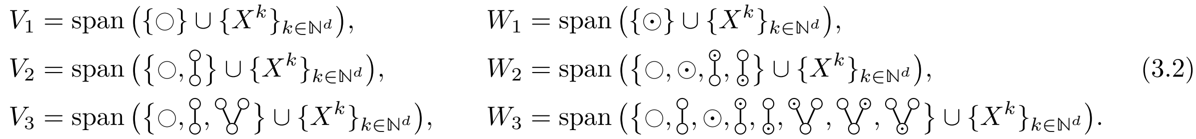
and

$$W_i := V_{i-1} \oplus \dot{V}_i, \qquad W^+_{i,\varepsilon,p} := V_i^+ \oplus \dot{V}^+_{i,\varepsilon,p},$$

It follows from Lemma 7 that

$$\mathscr{V}_{i,\varepsilon} := \Big((V_i, \Delta_{\varepsilon,\infty}), \big(V^+_{i-1}, \Delta^+_{\varepsilon,\infty}\big)\Big), \qquad \mathscr{W}_{i,\varepsilon,p} := \Big((W_i, \Delta_{\varepsilon,p}), \big(W^+_{i-1,\varepsilon,p}, \Delta^+_{\varepsilon,p}\big)\Big)$$

are concrete regularity-integrability substructures of $\mathscr{W}_{\varepsilon,p}$. As an example one could have



These spaces would be involved in the study of the three-dimensional parabolic Anderson model equation.

Let $\{\mathsf{M}^{\xi_n,h_n,R_n;\varepsilon,p}\}_n$ be the sequence of random models on $\mathscr{W}_{\varepsilon,p}$ under consideration. For any fixed $p \in [2,\infty]$, we write below

$$\mathbf{bd}(\mathscr{W}, i, p), \quad \text{resp.} \quad \mathbf{cv}(\mathscr{W}, i, p),$$

to mean the statements that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\sup_{\|h\|_H\le 1} \|\mathsf{M}^{\xi_n,h_n,R_n;\varepsilon,p}\|^q_{\mathbf{M}(\mathscr{W}_{i,\varepsilon,p})_{w_c}}\Big] < \infty$$

for any $\varepsilon \in (0,\varepsilon_0)$, $c > 0$, and $q \in [1,\infty)$, resp. the restriction of models $\mathsf{M}^{\xi_n,h_n,R_n;\varepsilon,p}$ on $\mathscr{W}_{i,\varepsilon,p}$ converges in $L^q\big(\Omega,\mathbb{P};\mathbf{M}(\mathscr{W}_{i,\varepsilon,p})_{w_c}\big)$ as n goes to ∞ for any ε, c, q, and $h \in H$ with $\|h\|_H \le 1$. We also write

$$\big\{\mathbf{bd}(\mathscr{W}, i, p)\big\}_p, \quad \text{resp.} \quad \big\{\mathbf{cv}(\mathscr{W}, i, p)\big\}_p,$$

to mean each statement holds for any $p \in [2,\infty]$. Similarly, we write

$$\mathbf{bd}(\mathscr{V}, i), \quad \text{resp.} \quad \mathbf{cv}(\mathscr{V}, i),$$

to mean the similar statements to $\mathbf{bd}(\mathscr{W}, i, p)$, resp. $\mathbf{cv}(\mathscr{W}, i, p)$ with $\mathscr{V}_{i,\varepsilon}$ in place of $\mathscr{W}_{i,\varepsilon,p}$. (It is independent of p and h.) Our induction is a three step process that can schematically be described as follows.

Step 1 (§3.2): $\mathbf{cv}(\mathscr{W}, i, \infty) \dashrightarrow \mathbf{cv}(\mathscr{V}, i)$

Step 2 (§3.3): $\mathbf{cv}(\mathscr{W}, i, p), \mathbf{cv}(\mathscr{V}, i) \longrightarrow \mathsf{g}\text{-part of } \mathbf{cv}(\mathscr{W}, i+1, p)$

Step 3 (§3.4): $\left.\begin{array}{r}\mathbf{cv}(\mathscr{W}, i, 2), \mathbf{cv}(\mathscr{V}, i) \longrightarrow \Pi\text{-part of } \mathbf{cv}(\mathscr{W}, i+1, 2)\\ \big\{\mathbf{cv}(\mathscr{W}, i, p)\big\}_p\end{array}\right\} \longrightarrow \Pi\text{-part of } \big\{\mathbf{cv}(\mathscr{W}, i+1, p)\big\}_p$

The *dashed line* in Step 1 is used to emphasize that this is a *probabilistic* step: We obtain stochastic estimates for the models on $\mathscr{V}_{i,\varepsilon}$ in terms of stochastic estimates for the models on $\mathscr{W}_{i,\delta,\infty}$ for some $\delta < \varepsilon$. On the other hand, we use *solid lines* in Step 2 and Step 3 to emphasize that they are *deterministic* steps: We obtain ω-wise estimates for the models on $\mathscr{W}_{i+1,\varepsilon,p}$ in terms of ω-wise estimates for the models on $\{\mathscr{W}_{i,\delta,q_a}\}_a$ and $\mathscr{V}_{i,\delta}$ for some finite set $\{q_a\} \subset [2,\infty]$ and $\delta < \varepsilon$. In the setting of example (3.2) we would first construct the deterministic model on $\odot$ (the initial case), then the random model on $\bigcirc$ (Step 1), and then successively on ,  (Steps 2 and 3), then  (Step 1), then , ,  (Steps 2 and 3), finally on  (Step 1).

We only describe in this section the flow of the proof of Theorem 6 and defer the proof of a number of lemmas to Section 4. To lighten the notations we will suppress from the notations the exponents ξ_n, h_n, R_n in the remainder of this section; so we simply write

$$\mathsf{M}^{n;\varepsilon,p} = (\Pi^n, \mathsf{g}^{n;\varepsilon,p})$$

rather than $\mathsf{M}^{\xi_n,h_n,R_n;\varepsilon,p}$. Occasionally we write $\mathsf{M}^{n,h;\varepsilon,p} = (\Pi^{n,h}, \mathsf{g}^{n,h;\varepsilon,p})$ to emphasize the dependence on h.

We first prove the n-uniform bound (2.16) by proving the **bd**-versions of Steps 1-3 – this is the content of Sections 3.1-3.4. We use these uniform bounds together with some local Lipschitz estimates satisfied by the reconstruction and integration maps to prove in a second time the convergence result – Section 3.5.

Notation. *The following notations will be useful in the course of the proof. Set for all $x, y \in \mathbb{R}^d$*

$$\Gamma^{n;\varepsilon,p}_{yx} := \big(\mathrm{id} \otimes \mathsf{g}^{n;\varepsilon,p}_{yx}\big)\Delta_{\varepsilon,p}.$$

These operators leave each space V_i and W_i stable. (The pair of $(\Pi^n, \Gamma^{n;\varepsilon,p})$ is called a model in the original terminology of [17]*.) For any $\rho \in \mathbf{B} \cup \dot{\mathbf{B}}$ denote by $P_\rho : T \to \mathbb{R}$ the canonical projection to the ρ-coefficient. We define the quantities*

$$\|\Gamma^{n;\varepsilon,p} : V_i\|_{w_c} := \max_{\substack{\tau\in\mathbf{B}_i\\ \sigma\in\mathbf{B}_{i-1}}} \sup_{y\in\mathbb{R}^d\setminus\{0\}} \frac{\big\|P_\sigma \Gamma^{n;\varepsilon,p}_{(x+y)x}\tau\big\|_{L^\infty_x(w_c)}}{w_c(y)^{-1}\|y\|_{\mathfrak{s}}^{|\tau|_{\varepsilon,p}-|\sigma|_{\varepsilon,p}}}, \tag{3.3}$$

and

$$\|\Gamma^{n;\varepsilon,p} : W_i\|_{w_c} := \|\Gamma^{n;\varepsilon,p} : V_{i-1}\|_{w_c} + \max_{\tau\in\dot{\mathbf{B}}_i} \sup_{y\in\mathbb{R}^d\setminus\{0\}} \left(\max_{\sigma\in\dot{\mathbf{B}}_{i-1}} \frac{\big\|P_\sigma \Gamma^{n;\varepsilon,p}_{(x+y)x}\tau\big\|_{L^\infty_x(w_c)}}{w_c(y)^{-1}\|y\|_{\mathfrak{s}}^{|\tau|_{\varepsilon,p}-|\sigma|_{\varepsilon,p}}} + \max_{\eta\in\mathbf{B}_{i-1}} \frac{\big\|P_\eta \Gamma^{n;\varepsilon,p}_{(x+y)x}\tau\big\|_{L^p_x(w_c)}}{w_c(y)^{-1}\|y\|_{\mathfrak{s}}^{|\tau|_{\varepsilon,p}-|\eta|_{\varepsilon,p}}} \right).$$

In the second one, when we pick the $\sigma \in \dot{\mathbf{B}}_{i-1}$ component of $\Gamma^{n;\varepsilon,p}_{(x+y)x}\tau$ it only involves some trees in V^+_{i-1}, from Lemma 7. This is why we take an L^∞_x norm. A similar comment applies to the η terms, where we take an L^p_x norm.

Let $m_{\mathbf{B}}$ stand for the maximum of the number of branches at the roots of μ which appear in the right part of the tensor product $\Delta_{0,2}\tau = \sum \sigma \otimes \mu$ for arbitrary $\tau \in \mathbf{B}$. We obtain from Lemma 7 the estimates

$$\begin{aligned} \big\|\Gamma^{n;\varepsilon,p} : V_i\big\|_{w_{cm_{\mathbf{B}}}} &\lesssim \big(1 + \|\mathsf{g}^{n;\varepsilon,p} : V^+_{i-1}\|_{w_c}\big)^{m_{\mathbf{B}}}, \\ \big\|\Gamma^{n;\varepsilon,p} : \dot{V}_i\big\|_{w_{cm_{\mathbf{B}}}} &\lesssim \big(1 + \|\mathsf{g}^{n;\varepsilon,p} : V^+_{i-1}\|_{w_c}\big)^{m_{\mathbf{B}}-1}\big(1 + \|\mathsf{g}^{n;\varepsilon,p} : V^+_{i-1}\|_{w_c} + \|\mathsf{g}^{n;\varepsilon,p} : \dot{V}^+_{i-1,\varepsilon,p}\|_{w_c}\big), \qquad (3.4) \\ \big\|\Gamma^{n;\varepsilon,p} : W_i\big\|_{w_{cm_{\mathbf{B}}}} &\lesssim \big(1 + \|\mathsf{g}^{n;\varepsilon,p} : W^+_{i-1,\varepsilon,p}\|_{w_c}\big)^{m_{\mathbf{B}}}. \end{aligned}$$

3.1 Convergence for the initial case. For the initial case one has $W_1 = \mathrm{span}\big(\{\odot\} \cup \{X^k\}_{k\in\mathbb{N}^d}\big)$ and $W^+_{0,\varepsilon,p} = \mathrm{alg}^+_{\varepsilon,p}(\emptyset) = \mathrm{span}(\{X^k\}_{k\in\mathbb{N}^d})$, and it is sufficient to check the convergence of $\Pi^n_x(\odot) = h_n$ to h in $B^{|\odot|_{\varepsilon,p}}_{p,\infty}(w_0)$, with $|\odot|_{\varepsilon,p} = \alpha_0 - \varepsilon + \frac{|\mathfrak{s}|}{p}$. This entails the convergence in $H^{-\kappa,Q}(w_0)$, which is continuously embedded into $B^{|\odot|_{\varepsilon,p},Q}_{p,\infty}(w_0)$ for every $\varepsilon > 0$.

3.2 Step 1: From bd($\mathscr{W}, i, \infty$) to bd($\mathscr{V}, i$). Recall the definition of $\mathscr{V}_{i,\varepsilon} = (V_i, V^+_{i-1})$. Since $V_i = V_{i-1} \oplus \mathrm{span}\{\tau_i\}$, $V_{i-1} \subset W_i$ and $V^+_{i-1} \subset W^+_{i-1,\varepsilon,\infty}$, it is sufficient to prove the bound for

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\big\|\Pi^{n;\varepsilon,\infty} : \tau_i\big\|^q_{w_c}\Big]$$

in terms of the assumed boundedness results on $\mathscr{W}_{i,\varepsilon,\infty} = (W_i, W^+_{i-1,\varepsilon,\infty})$. We use different arguments depending on the sign of $|\tau_i|_{0,\infty}$. It turns out that if $|\tau_i|_{0,\infty} > 0$ we have an ω-wise bound of $\|\Pi^{n;\varepsilon,\infty} : \tau_i\|_{w_c}$ in terms of ω-wise bounds of $\mathsf{M}^{n;\varepsilon,\infty}$ on $\mathscr{W}_{i,\varepsilon,\infty}$. If $|\tau_i|_{0,\infty} \le 0$ we only have an $L^q(\Omega)$ control of $\|\Pi^{n;\varepsilon,\infty} : \tau_i\|_{w_c}$. Lemmas 8 to 10 for this step below are proved in Section 4.4.

If $|\tau_i|_{0,\infty} > 0$, then $|\tau_i|_{\varepsilon,\infty} > 0$ for any $\varepsilon \in (0, \varepsilon_0)$, and the proof reduces to an application of the reconstruction theorem (see Theorem 22). Indeed, the modelled distribution

$$f^{n;\varepsilon,\infty}_{\tau_i}(x) := \big(\mathrm{id} \otimes \mathsf{g}^{n;\varepsilon,\infty}_x\big)\Delta_{\varepsilon,\infty}\tau_i - \tau_i$$

belongs in that case to $D^{(|\tau_i|_{\varepsilon,\infty},\infty)}(V_{i-1}; \Gamma^{n;\varepsilon,\infty})_{w_{c_{\mathbf{B}}}}$, with a positive regularity exponent. Its reconstruction is uniquely determined and coincides with $\Pi^n\tau_i$. Moreover, the reconstruction theorem comes with a bound on

$$\Pi^{n;\varepsilon,\infty}_x \tau_i = \Pi^n\tau_i - \Pi^{n;\varepsilon,\infty}_x(f^n_{\tau_i}(x)).$$

8 – Lemma. *Let $|\tau_i|_{0,\infty} > 0$. For any $\varepsilon \in (0, \varepsilon_0)$ and $c > 0$, there exists a positive constant C which is independent of n and ω, one has*

$$\|\Pi^{n;\varepsilon,\infty} : \tau_i\|_{w_{c(m_{\mathbf{B}}+1)}} \le C\|\Pi^{n;\varepsilon,\infty} : V_{i-1}\|_{w_c}\, \|\Gamma^{n;\varepsilon,\infty} : V_i\|_{w_{cm_{\mathbf{B}}}}.$$

Recall from (3.4) that the bounds of $\|\Gamma^{n;\varepsilon,\infty} : V_i\|_{w_{cm_{\mathbf{B}}}}$ is obtained from the bounds of $\|\mathsf{g}^{n;\varepsilon,\infty} : V^+_{i-1}\|_{w_c}$, which is contained in the assumption **bd**($\mathscr{W}, i, \infty$). Consider next the case where $|\tau_i|_{0,\infty} \le 0$, so $|\tau_i|_{\varepsilon,\infty} < 0$ if $\varepsilon > 0$. We cannot use the above reconstruction argument as $f^{n;\varepsilon,\infty}_{\tau_i}$ no longer has a unique reconstruction. Instead, we use the H-spectral gap inequality and the algebraic identity

$$\begin{aligned} \big(d_\omega \Pi^{n;\varepsilon,\infty}_x \tau_i\big)(h) &= \big(d_{\xi_n} \Pi^{\xi_n,R_n;\varepsilon,\infty}_x \tau_i\big)\big((d_\omega \xi_n)(h)\big) \\ &= \Pi^{\xi_n,h_n,R_n;\varepsilon,\infty}_x(D\tau_i) = \Pi^{n,h;\varepsilon,\infty}_x(D\tau_i), \end{aligned}$$

which follows from Lemma 3 and the chain rule. Hence, for any finite exponent $q = 2^r$, one gets from the inequality (2.2) that

$$\mathbb{E}\big[\big|\mathcal{Q}_t(x, \Pi^{n;\varepsilon,\infty}_x \tau_i)\big|^q\big] \lesssim_q \big|\mathbb{E}\big[\mathcal{Q}_t(x, \Pi^{n;\varepsilon,\infty}_x \tau_i)\big]\big|^q + \mathbb{E}\Big[\sup_{\|h\|_H \le 1} \big|\mathcal{Q}_t\big(x, \Pi^{n,h;\varepsilon,\infty}_x(D\tau_i)\big)\big|^q\Big].$$

The following result holds for the expectation part, which is the only place in this work where we use the assumption that the law of the noise is invariant by the translations. Define the quantity

$$E^{n,R}_i := \mathbb{E}\big[\mathcal{Q}_1(0, \Pi^{n;\varepsilon,\infty}_0 \tau_i)\big] = \mathbb{E}\big[\mathcal{Q}_1(0, \Pi^{n;0,\infty}_0 \tau_i)\big],$$

which is uniformly bounded over n by assumption.

9 – Lemma. *Let $|\tau_i|_{0,\infty} \leq 0$. For any $\varepsilon \in (0,\varepsilon_0)$, there exists a positive constant C which is independent of n, one has*

$$\big\|\mathbb{E}\big[\mathcal{Q}_t(x, \Pi_x^{n;\varepsilon,\infty}\tau_i)\big]\big\|_{L_x^\infty(w_{c(m_{\mathbf{B}}+1)})} \leq C\Big(|E_i^{n,R}| + t^{\frac{|\tau_i|_{\varepsilon,\infty}}{\ell}}\,\mathbb{E}\big[\|\Pi^{n;\varepsilon,\infty} : V_{i-1}\|_{w_c}\,\|\Gamma^{n;\varepsilon,\infty} : V_i\|_{w_{cm_{\mathbf{B}}}}\big]\Big).$$

From that estimate and the H-spectral gap inequality, we obtain the bound

$$\begin{aligned}
&\mathbb{E}\big[\big|\mathcal{Q}_t(x, \Pi_x^{n;\varepsilon,\infty}\tau_i)\big|^q w_{c(m_{\mathbf{B}}+2)}(x)^q\big] \\
&\quad\lesssim w_c(x)^q\Big(\big|\mathbb{E}\big[\mathcal{Q}_t(x, \Pi_x^{n;\varepsilon,\infty}\tau_i)\big]\big|^q w_{c(m_{\mathbf{B}}+1)}(x)^q + \mathbb{E}\Big[\sup_{\|h\|_H\leq 1}\big|\mathcal{Q}_t\big(x, \Pi_x^{n,h;\varepsilon,\infty}(D\tau_i)\big)\big|^q w_c(x)^q\Big]\Big) \\
&\quad\lesssim w_c(x)^q\Big(|E_i^{n,R}|^q + t^{\frac{|\tau_i|_{\varepsilon,\infty}}{\ell}q}\,\mathbb{E}\big[\|\Pi^{n;\varepsilon,\infty} : V_{i-1}\|_{w_c}\,\|\Gamma^{n;\varepsilon,\infty} : V_i\|_{w_{cm_{\mathbf{B}}}}\big]^q \\
&\qquad\qquad + t^{\frac{|D\tau_i|_{\varepsilon,\infty}}{\ell}q}\,\mathbb{E}\Big[\sup_{\|h\|_H\leq 1}\|\Pi^{n,h;\varepsilon,\infty} : W_i\|_{w_c}^q\Big]\Big).
\end{aligned}$$

Recall that $D\tau_i \in \dot{V}_i \subset W_i$ and that D preserves the $|\cdot|_{\varepsilon,\infty}$ degree, so $|D\tau_i|_{\varepsilon,\infty} = |\tau_i|_{\varepsilon,\infty}$. By integrating the above estimate over x we get the bound

$$\begin{aligned}
&\mathbb{E}\big[\big\|\mathcal{Q}_t(x, \Pi_x^{n;\varepsilon,\infty}\tau_i)\big\|_{L_x^q(w_{c(m_{\mathbf{B}}+2)})}^q\big] \\
&\lesssim |E_i^{n,R}|^q + t^{\frac{|\tau_i|_{\varepsilon,\infty}}{\ell}q}\Big(\mathbb{E}\big[\|\Pi^{n;\varepsilon,\infty} : V_{i-1}\|_{w_c}\,\|\Gamma^{n;\varepsilon,\infty} : V_i\|_{w_{cm_{\mathbf{B}}}}\big]^q + \mathbb{E}\Big[\sup_{\|h\|_H\leq 1}\|\Pi^{n,h;\varepsilon,\infty} : W_i\|_{w_c}^q\Big]\Big)
\end{aligned} \tag{3.5}$$

To trade the $L_x^q(w_{c(m_{\mathbf{B}}+2)})$ norm for an $L_x^\infty(w_{c(m_{\mathbf{B}}+2)})$ norm in the above estimate we use an argument that is reminiscent of the proof of Besov embedding. Here we need a slight change of the parameter ε.

10 – Lemma. *Let $|\tau_i|_{0,\infty} \leq 0$. For any $\varepsilon \in (0, \frac{\varepsilon_0}{2})$ and $q \in [1,\infty)$, there exists a positive constant C which is independent of n, one has*

$$\begin{aligned}
&\mathbb{E}\big[\|\Pi^{n;2\varepsilon,\infty} : \tau_i\|_{w_{c(m_{\mathbf{B}}+2)}}^q\big] \\
&\qquad\leq C\Big(|E_i^{n,R}|^q + \mathbb{E}\big[\|\Pi^{n;\varepsilon,\infty} : V_{i-1}\|_{w_c}^q\,\|\Gamma^{n;\varepsilon,\infty} : V_i\|_{w_{cm_{\mathbf{B}}}}^q\big] + \mathbb{E}\Big[\sup_{\|h\|_H\leq 1}\|\Pi^{n,h;\varepsilon,\infty} : W_i\|_{w_c}^q\Big]\Big).
\end{aligned}$$

Lemma 8 and Lemma 10 provide an $L^q(\Omega)$ bound of $\|\Pi^{n;\varepsilon,\infty} : \tau_i\|_{w_{c(m_{\mathbf{B}}+2)}}$ in terms of the moment of $\|\mathsf{M}^{n;\varepsilon/2,p}\|_{\mathsf{M}(\mathscr{W}_{i,\varepsilon/2,\infty})_{w_c}}$, which is a part of $\mathbf{bd}(\mathscr{W}, i, \infty)$.

3.3 Step 2: From $\mathbf{bd}(\mathscr{W}, i, p)$ and $\mathbf{bd}(\mathscr{V}, i)$ to the g-part of $\mathbf{bd}(\mathscr{W}, i+1, p)$. We fix parameters $\varepsilon \in (0,\varepsilon_0)$ and $p \in [2,\infty]$ in this step. Recall that

$$\mathscr{W}_{i+1,\varepsilon,p} = \Big((W_{i+1}, \Delta_{\varepsilon,p}), (W_{i,\varepsilon,p}^+, \Delta_{\varepsilon,p}^+)\Big).$$

In this step, we show the ω-wise bounds of

$$\sup_{n\in\mathbb{N}}\|\mathsf{g}^{n;\varepsilon,p} : \mathcal{I}_k(\tau)\|_{w_c}$$

for any planted trees $\mathcal{I}_k(\tau)$ with $\tau \in \mathbf{B}_i \cup \dot{\mathbf{B}}_i$ and $k \in \mathbb{N}^d$ in terms of the assumed boundedness results on $\mathscr{W}_{i,\varepsilon,p}$ and $\mathscr{V}_{i,\varepsilon}$. Define for that purpose a modelled distribution $f_\tau^{n;\varepsilon,p}$ setting

$$f_\tau^{n;\varepsilon,p}(x) = \big(\mathrm{id}\otimes\mathsf{g}_x^{n;\varepsilon,p}\big)\Delta_{\varepsilon,p}\tau - \tau \in D^{(|\tau|_{\varepsilon,p}, i_p(\tau))}(W_i; \Gamma^{n;\varepsilon,p})_{w_{cm_{\mathbf{B}}}}.$$

We further define the linear map $\mathcal{I}_{\varepsilon,p}^+ : W_i \to W_{i,\varepsilon,p}^+$ by setting

$$\mathcal{I}_{\varepsilon,p}^+(\tau) := \begin{cases}\mathcal{I}(\tau), & \text{if } |\tau|_{\varepsilon,p} + \beta_0 > 0,\\ 0, & \text{if } |\tau|_{\varepsilon,p} + \beta_0 \leq 0,\end{cases}$$

which is an abstract integration map of order β_0 (see Definition 23). Then it turns out the model $\mathsf{M}^{n;\varepsilon,p}$ on $\mathscr{W}_{i,\varepsilon,p}$ and the model $\mathsf{M}^{n+;\varepsilon,p} = (\Pi^{n+;\varepsilon,p}, \Gamma^{n+;\varepsilon,p})$ on the concrete regularity-integrability structure $(W_{i,\varepsilon,p}^+, W_{i,\varepsilon,p}^+)$ given by

$$(\Pi^{n+;\varepsilon,p}\mu)(x) = \mathsf{g}_x^{n;\varepsilon,p}(\mu), \qquad \Gamma_{yx}^{n+;\varepsilon,p} := \big(\mathrm{id}\otimes\mathsf{g}_{yx}^{n;\varepsilon,p}\big)\Delta_{\varepsilon,p}^+$$

are compatible for $\mathcal{I}_{\varepsilon,p}^+$ – see the proof of Lemma 11. As in the usual setting of regularity structures, one needs some additional terms to turn $\mathcal{I}_{\varepsilon,p}^+(f_\tau^{n;\varepsilon,p})$ into a modelled distribution – here in a regularity-integrability setting. We define the modelled distribution $\mathcal{K}_{\varepsilon,p}^{n+}f_\tau^{n;\varepsilon,p}$ by the formula (A.1) with the model

$\mathsf{M}^{n;\varepsilon,p}$ and a reconstruction $\mathsf{\Pi}^n\tau$ of $f_\tau^{n;\varepsilon,p}$, which is certainly a reconstruction because the bound for $\mathsf{\Pi}_x^{n;\varepsilon,p}\tau = \mathsf{\Pi}^n\tau - \mathsf{\Pi}_x^{n;\varepsilon,p} f_\tau^{n;\varepsilon,p}$ is a part of $\mathbf{bd}(\mathscr{W},i,p)$ and $\mathbf{bd}(\mathscr{V},i)$.

11 – Lemma. *$\mathsf{g}_{yx}^{n;\varepsilon,p}(\mathcal{I}_k\tau)$ coincides with the X^k-coefficient of $(\mathcal{K}_{\varepsilon,p}^{n+} f_\tau^{n;\varepsilon,p})(y) - \mathsf{\Gamma}_{yx}^{n+;\varepsilon,p}\big(\mathcal{K}_{\varepsilon,p}^{n+} f_\tau^{n;\varepsilon,p}(x)\big)$.*

The multilevel Schauder estimate from Theorem 24 then gives an upper bound on $\mathsf{g}_{yx}^{n;\varepsilon,p}(\mathcal{I}_k\tau)$ in terms of the norm of $f_\tau^{n;\varepsilon,p}$. They take the following form, proved in Section 4.5.

12 – Lemma. *For any $\varepsilon \in (0,\varepsilon_0)$, $p \in [2,\infty]$, and $c > 0$, there exists a positive constant C which is independent of n, ω, and $h \in H$ with $\|h\|_H \le 1$, one has*

$$\|\mathsf{g}^{n;\varepsilon,p} : \mathcal{I}_k(\tau)\|_{w_{c(m_{\mathbf{B}}+2)}} \le \begin{cases} C\big(1 + \|\mathsf{M}^{n;\varepsilon,\infty}\|_{\mathsf{M}(\mathscr{V}_{i,\varepsilon})_{w_c}}\big)^{m_{\mathbf{B}}+2}, & \text{if } \tau \in \mathbf{B}_i, \\ C\big(1 + \|\mathsf{M}^{n;\varepsilon,p}\|_{\mathsf{M}(\mathscr{W}_{i,\varepsilon,p})_{w_c}}\big)^{m_{\mathbf{B}}+2}, & \text{if } \tau \in \dot{\mathbf{B}}_i. \end{cases}$$

Re-inserting the h in the notations $\mathsf{M}^{n,h}$ and $\mathsf{\Pi}^{n,h;\varepsilon,\infty}$ for M^n and $\mathsf{\Pi}^{n;\varepsilon,\infty}$ to emphasize the dependence on h of these objects, the n-uniform control of

$$\mathbb{E}\Big[\sup_{\|h\|_H\le 1} \|\mathsf{g}^{n,h;\varepsilon,p} : W_{i,\varepsilon,p}^+\|_{w_c}^q\Big]$$

is given by the joint assumptions $\mathbf{bd}(\mathscr{W},i,p)$ and $\mathbf{bd}(\mathscr{V},i)$.

3.4 Step 3: From $\big\{\mathbf{bd}(\mathscr{W},i,p)\big\}_p$ and $\mathbf{bd}(\mathscr{V},i)$ to the $\mathsf{\Pi}$-part of $\big\{\mathbf{bd}(\mathscr{W},i+1,p)\big\}_p$. It is sufficient to show the bound of

$$\sup_{n\in\mathbb{N}} \|\mathsf{\Pi}^{n;\varepsilon,p} : \tau\|_{w_c}$$

for any $\tau \in \dot{\mathbf{B}}_{i+1}$ in terms of the assumed boundedness results on $\mathscr{W}_{i,\varepsilon,p}$ and $\mathscr{V}_{i,\varepsilon}$. We first establish the result for $p = 2$ and next extend it into all $p \in [2,\infty]$ by using the formula (2.14).

Let τ be of the form $D_v\sigma$ with some $\sigma \in \mathbf{B}_{i+1}$ and $v \in \bigcirc_\sigma$. Since τ is not the initial tree $\odot$, the assumption of Theorem 6 gives here

$$|\tau|_{0,2} = |\tau|_{0,\infty} + \frac{|\mathfrak{s}|}{2} = |\sigma|_{0,\infty} + \frac{|\mathfrak{s}|}{2} > 0.$$

Thus, as in Lemma 8, one gets the following statement from an application of the reconstruction theorem to a modelled distribution

$$f_\tau^{n;\varepsilon,2}(x) = \big(\mathrm{id} \otimes \mathsf{g}_x^{n;\varepsilon,2}\big)\Delta_{\varepsilon,2}\tau - \tau \in D^{(|\tau|_{\varepsilon,2},2)}(V_i \oplus \dot{V}_i; \mathsf{\Gamma}^{n;\varepsilon,2})_{w_c}.$$

Note that $(V_i \oplus \dot{V}_i, W_{i-1,\varepsilon,2}^+)$ is a concrete regularity-integrability substructure and recall from (3.4) that the bound of $\|\mathsf{\Gamma}^{n;\varepsilon,2} : V_i \oplus \dot{V}_i\|_{w_{cm_{\mathbf{B}}}}$ follows from the assumption $\mathbf{bd}(\mathscr{W},i,2)$.

13 – Lemma. *For any $\varepsilon \in (0,\varepsilon_0)$ and $c > 0$, there exists a positive constant C which is independent of n, ω, and $h \in H$ with $\|h\|_H \le 1$ one has*

$$\|\mathsf{\Pi}^{n;\varepsilon,2} : \tau\|_{w_{c(m_{\mathbf{B}}+1)}} \le C\|\mathsf{\Pi}^{n;\varepsilon,2} : V_i \oplus \dot{V}_i\|_{w_c} \|\mathsf{\Gamma}^{n;\varepsilon,2} : V_i \oplus \dot{V}_i\|_{w_{cm_{\mathbf{B}}}}.$$

The proof is almost the same as that of Lemma 8 and left to the reader. Next we recall the algebraic formula (2.14)

$$\mathsf{\Pi}_x^{n;\varepsilon,p}\tau = \mathsf{\Pi}_x^{n;\varepsilon,2}\tau + \big(\mathsf{\Pi}_x^{n;\varepsilon,p} \otimes \mathsf{h}_x^{n;\varepsilon,p}\big)\Delta_{\varepsilon,2}\tau$$

to infer from the bound of Lemma 13 a similar bound on $\mathsf{\Pi}_x^{n;\varepsilon,p}\tau$. We can estimate as follows the size of the $\mathsf{h}_x^{n;\varepsilon,p}$ terms. Recall from Section 2.3 the definitions of the exponent $p_\varepsilon(\mu)$ and the floor function $\lfloor p\rfloor_{I_\varepsilon}$.

14 – Lemma. *For any $\mu \in \mathcal{P}(\dot{\mathbf{B}}_i)$, $p \in [2,\infty]$, and $p_\varepsilon(\mu) > r \ge \lfloor p_\varepsilon(\mu)\rfloor_{I_\varepsilon}$ one has*

$$\|\mathsf{h}_x^{n;\varepsilon,p}(\mu)\|_{L_x^r(w_{c(m_{\mathbf{B}}+1)})} \lesssim \|\mathsf{\Pi}^{n;\varepsilon,r} : W_i\|_{w_c}\big(1 + \|\mathsf{\Gamma}^{n;\varepsilon,r} : W_i\|_{w_c}\big).$$

The proof is given in Section 4.6. Formula (2.14) then leads to the following estimate.

15 – Lemma. *For any $\varepsilon \in (0,\varepsilon_0)$, $p \in [2,\infty]$, and $c > 0$, there exist a finite subset $R_{\varepsilon,p}$ of $[2,\infty]$, an exponent $\varepsilon' \in (0,\varepsilon)$, and a positive constant C, which are independent of n, ω, and $h \in H$ with $\|h\|_H \le 1$, one has*

$$\|\mathsf{\Pi}^{n;\varepsilon,p} : \tau\|_{w_{c(m_{\mathbf{B}}+2)}} \le C\Bigg(1 + \sum_{q\in R_{\varepsilon,p}} \|\mathsf{M}^{n;\varepsilon',q}\|_{\mathsf{M}(\mathscr{W}_{i,\varepsilon',q})_{w_c}} + \|\mathsf{M}^{n;\varepsilon',\infty}\|_{\mathsf{M}(\mathscr{V}_{i,\varepsilon'})_{w_c}}\Bigg)^{m_{\mathbf{B}}+2}.$$

Thus we see from Lemma 15 that the n-uniform control of

$$\mathbb{E}\Big[\sup_{\|h\|_H\le 1}\|\Pi^{n,h;\varepsilon,p}:\tau\|^q_{w_{c(m_{\mathbf{B}}+2)}}\Big]$$

is given by the joint assumptions $\{\mathbf{bd}(\mathscr{W},i,p)\}_p$ and $\mathbf{bd}(\mathscr{V},i)$.

A finite number of iterations of Steps 1-3 starting from the initial case of Section 3.1 proves the n-uniform bounds of M^n in $L^q\big(\Omega,\mathbb{P};\mathbf{M}(\mathscr{W}_{\varepsilon,p})_{w_c}\big)$ for any $p\in[2,\infty]$, $q\in[1,\infty)$, $\varepsilon\in(0,\varepsilon_0)$ and $c>0$.

3.5 From uniform boundedness to convergence results. Since the models M^n stay in a bounded set of $L^q\big(\Omega,\mathbb{P};\mathbf{M}(\mathscr{W}_{\varepsilon,p})_{w_c}\big)$ for any p,q,ε,c, we can use the local Lipschitz estimates satisfied by the reconstruction operator and the $\mathcal{K}^{n+}_{\varepsilon,p}$ maps – see Theorem 22 and Theorem 24 in Appendix A, to prove the Cauchy property

$$\lim_{n,m\to\infty}\mathbb{E}\Big[\sup_{\|h\|_H\le 1}\|\mathsf{M}^{n;\varepsilon,p}:\mathsf{M}^{m;\varepsilon,p}\|^q_{\mathbf{M}(\mathscr{W}_{\varepsilon,p})_{w_c}}\Big]=0 \tag{3.6}$$

for any p,q,ε,c. Since $\mathbf{M}(\mathscr{W}_{\varepsilon,p})_{w_c}$ is complete, this implies the convergence in $L^q\big(\Omega,\mathbb{P};\mathbf{M}(\mathscr{W}_{\varepsilon,p})_{w_c}\big)$ of $\{\mathsf{M}^{n;\varepsilon,p}\}_n$. We can prove the Cauchy property (3.6) starting from the convergence result of the initial case (Section 3.1) and following the same induction steps as above. We only collect below the statements corresponding to the different lemmas from sections 3.2, 3.3 and 3.4. A statement corresponding to Lemma k in one of the previous sections is numbered here Lemma k'. We denote by

$$Q_n$$

the arbitrary nonnegative random variable which polynomially depends on $\sup_{\|h\|_H\le 1}\|\mathsf{M}^{n;\varepsilon,p}\|_{\mathbf{M}(\mathscr{W}_{\varepsilon,p})_{w_c}}$ where the parameters ε,p,c run over an n-independent finite set. We know that

$$\sup_{n\in\mathbb{N}}\mathbb{E}\big[Q_n^q\big]<\infty$$

for any $q\in[1,\infty)$.

3.5.1 Step 1: From $\mathbf{cv}(\mathscr{W},i,\infty)$ to $\mathbf{cv}(\mathscr{V},i)$.

8' – Lemma. *For $|\tau_i|_{0,\infty}>0$, one has the ω-wise estimate*

$$\|\Pi^{n;\varepsilon,\infty},\Pi^{n;\varepsilon,\infty}:\tau_i\|_{w_{c(m_{\mathbf{B}}+2)}}\le (Q_n+Q_m)\|\mathsf{M}^{n;\varepsilon,\infty}:\mathsf{M}^{m;\varepsilon,\infty}\|_{\mathbf{M}(\mathscr{V}_{i-1,\varepsilon,p})_{w_c}}$$

for any $n,m\in\mathbb{N}$.

10' – Lemma. *For $|\tau_i|_{0,\infty}\le 0$, there exists a positive constant C which is independent of $n,m\in\mathbb{N}$ and one has the moment estimate*

$$\begin{aligned}&\mathbb{E}\big[\|\Pi^{n;2\varepsilon,\infty},\Pi^{n;2\varepsilon,\infty}:\tau_i\|^q_{w_{c(m_{\mathbf{B}}+2)}}\big]\\&\le C\Big(|E_i^{n,R}-E_i^{m,R}|^q+\mathbb{E}\big[\|\mathsf{M}^{n;\varepsilon,\infty}:\mathsf{M}^{m;\varepsilon,\infty}\|^{2q}_{\mathbf{M}(\mathscr{V}_{i-1,\varepsilon,p})_{w_c}}\big]+\mathbb{E}\Big[\sup_{\|h\|_H\le 1}\|\Pi^{n,h;\varepsilon,\infty},\Pi^{m,h;\varepsilon,\infty}:W_i\|^q_{w_c}\Big].\Big)\end{aligned}$$

for any $q\in[1,\infty)$.

3.5.2 Step 2: From $\mathbf{cv}(\mathscr{W},i,p)$ and $\mathbf{cv}(\mathscr{V},i)$ to the g-part of $\mathbf{cv}(\mathscr{W},i+1,p)$. Fix $\tau\in\mathbf{B}_i\cup\dot{\mathbf{B}}_i$ and $k\in\mathbb{N}^d$.

12' – Lemma. *One has*

$$\|\mathsf{g}^{n;\varepsilon,p},\mathsf{g}^{m;\varepsilon,p}:\mathcal{I}_k(\tau)\|_{w_c}\le\begin{cases}(Q_n+Q_m)\|\mathsf{M}^{n;\varepsilon,\infty}:\mathsf{M}^{m;\varepsilon,\infty}\|_{\mathbf{M}(\mathscr{V}_{i,\varepsilon})_{w_c}}, & \textit{if } \tau\in\mathbf{B}_i,\\(Q_n+Q_m)\|\mathsf{M}^{n;\varepsilon,p}:\mathsf{M}^{m;\varepsilon,p}\|_{\mathbf{M}(\mathscr{W}_{i,\varepsilon,p})_{w_c}}, & \textit{if } \tau\in\dot{\mathbf{B}}_i.\end{cases}$$

3.5.3 Step 3: From $\mathbf{cv}(\mathscr{W},i,p)$ and $\mathbf{cv}(\mathscr{V},i)$ to the Π-part of $\mathbf{cv}(\mathscr{W},i+1,p)$. Fix $\tau\in\dot{\mathbf{B}}_{i+1}$.

15' – Lemma. *With the same choice of $R_{\varepsilon,p}\subset[2,\infty]$ and $\varepsilon'\in(0,\varepsilon)$ as Lemma 15, one has*

$$\|\Pi^{n;\varepsilon,p},\Pi^{m;\varepsilon,p}:\tau\|_{w_c}\le(Q_n+Q_m)\Bigg(\sum_{q\in R_{\varepsilon,p}}\|\mathsf{M}^{n;\varepsilon',q}:\mathsf{M}^{m;\varepsilon',q}\|_{\mathbf{M}(\mathscr{W}_{i,\varepsilon',q})_{w_c}}+\|\mathsf{M}^{n;\varepsilon',\infty}:\mathsf{M}^{m;\varepsilon',\infty}\|_{\mathbf{M}(\mathscr{V}_{i,\varepsilon'})_{w_c}}\Bigg)$$

4 – Proofs of the lemmas

We give in this section the proofs of the lemmas used in our proof of Theorem 6.

4.1 Examples of differentiable sectors. Let $\mathbf{B}$ be a set of all trees which strongly conform to a complete subcritical rule and with degrees $|\cdot|_{0,\infty}$ less than some fixed number. We show that $V = \mathrm{span}(\mathbf{B})$ is a differentiable sector focusing on only (b). It is obvious for the initial case $\odot \in \mathbf{B}$. Pick $\tau \in \dot{\mathbf{B}}$ of the form $\tau = D_v\sigma$ with $\sigma \in \mathbf{B}$ and $v \in \bigcirc_\sigma$. If v is a root of τ, then τ is of the form $\odot\eta$ and the root of $\eta \in \mathbf{B}$ has a type $\mathbf{1}$. Then $\Delta_{0,\infty}\eta = \Delta_{0,2}\eta$ is of the form $\sum \eta_1 \otimes \eta_2$, where η_1 strongly confirms the rule and has the root with type $\mathbf{1}$, and $\eta_2 \in V^+_{0,\infty}$, and $\bigcirc\eta_1$ also strongly confirms the rule. Hence $\odot\eta_1 = D_v(\bigcirc\eta_1) \in \dot{\mathbf{B}}$ and one has

$$\Delta_{0,2}\tau = \Delta_{0,2}(\odot\eta) = (\Delta_{0,2}\odot)(\Delta_{0,2}\eta) = \sum(\odot\eta_1) \otimes \eta_2 \in \dot{V} \otimes V^+_{0,\infty}.$$

The proof for the case that v is not the root is similar. The proof of the property on $\Delta^+_{0,2}$ is a simple modification.

4.2 Proofs of algebraic identities. We prove two algebraic identities stated in Section 2.6. In the proofs, we use the fact proved by Bruned in Proposition 3.15 of [6] that one can factorize the renormalized interpretation map by

$$\Pi^{R;\varepsilon,p}_x = \widehat{\Pi}^{R;\varepsilon,p}_x R, \tag{4.1}$$

where $\widehat{\Pi}^{R;\varepsilon,p}_x$ is a linear and some *multiplicative* map defined by

$$\widehat{\Pi}^{R;\varepsilon,p}_x(X^k) = (\cdot - x)^k, \quad \widehat{\Pi}^{R;\varepsilon,p}_x(\mathcal{I}_k\tau) = \partial^k\mathcal{K}(\cdot, \Pi^{R;\varepsilon,p}_x\tau) - \sum_{l\in\mathbb{N}^d,\, |l|_\mathfrak{s}<|\mathcal{I}_k\tau|_{\varepsilon,p}} \frac{(\cdot - x)^l}{l!}\partial^{k+l}\mathcal{K}(x, \Pi^{R;\varepsilon,p}_x\tau).$$

Proof of Lemma 3 – In addition to (2.12), we also prove the similar identity

$$d_\xi(\widehat{\Pi}^{\xi,R;\varepsilon,\infty}_x\tau)(h) = \widehat{\Pi}^{\xi,h,R;\varepsilon,\infty}_x(D\tau) \tag{4.2}$$

simultaneously. The proof is an induction on the preorder $\preceq$ defined by (3.1). Both (2.12) and (4.2) are obvious for the initial cases $\tau \in \{\bigcirc, X^k\}$. Let τ be a planted tree of the form $\mathcal{I}_k(\sigma)$ with $\sigma \in \mathbf{B}$. If σ satisfies (2.12), then by the definition of $\widehat{\Pi}^{\xi,R;\varepsilon,\infty}_x$ operator,

$$\begin{aligned} d_\xi(\widehat{\Pi}^{\xi,R;\varepsilon,\infty}_x\tau)(h) &= d_\xi\Big(\partial^k\mathcal{K}(\cdot, \Pi^{\xi,R;\varepsilon,\infty}_x\sigma) - \sum_{|l|_\mathfrak{s}<|\mathcal{I}_k\sigma|_{\varepsilon,\infty}} \frac{(\cdot - x)^l}{l!}\partial^{k+l}\mathcal{K}(x, \Pi^{\xi,R;\varepsilon,\infty}_x\sigma)\Big)(h) \\ &= \partial^k\mathcal{K}(\cdot, \Pi^{\xi,h,R;\varepsilon,\infty}_x D\sigma) - \sum_{|l|_\mathfrak{s}<|\mathcal{I}_k\sigma|_{\varepsilon,\infty}} \frac{(\cdot - x)^l}{l!}\partial^{k+l}\mathcal{K}(x, \Pi^{\xi,h,R;\varepsilon,\infty}_x D\sigma) \\ &= \widehat{\Pi}^{\xi,h,R;\varepsilon,\infty}_x(\mathcal{I}_k D\sigma) = \widehat{\Pi}^{\xi,h,R;\varepsilon,\infty}_x(D\tau). \end{aligned}$$

In the third equality, we use the fact that D preserves the $|\cdot|_{\varepsilon,\infty}$-degree: $|\mathcal{I}_k(D\sigma)|_{\varepsilon,\infty} = |\mathcal{I}_k\sigma|_{\varepsilon,\infty}$. Thus τ satisfies (4.2). Next we consider a non-planted tree τ factorized by $\tau = \prod_{i=0}^N \eta_i$ with $\eta_0 = X^k$ and planted trees η_i $(1 \le i \le N)$. If each η_i satisfies (4.2), then we use the multiplicativity of $\widehat{\Pi}^{\xi,R;\varepsilon,\infty}_x$ and Leibniz rules for d_ξ and D to derive

$$\begin{aligned} d_\xi(\widehat{\Pi}^{\xi,R;\varepsilon,\infty}_x\tau)(h) &= d_\xi\Big(\prod_i(\widehat{\Pi}^{\xi,R;\varepsilon,\infty}_x\eta_i)\Big)(h) = \sum_i d_\xi(\widehat{\Pi}^{\xi,R;\varepsilon,\infty}_x\eta_i)(h)\prod_{j\neq i}(\widehat{\Pi}^{\xi,R;\varepsilon,\infty}_x\eta_j) \\ &= \sum_i(\widehat{\Pi}^{\xi,h,R;\varepsilon,\infty}_x D\eta_i)(h)\prod_{j\neq i}(\widehat{\Pi}^{\xi,R;\varepsilon,\infty}_x\eta_j) = \widehat{\Pi}^{\xi,h,R;\varepsilon,\infty}_x\Big(\sum_i(D\eta_i)\prod_{j\neq i}\eta_j\Big) = \widehat{\Pi}^{\xi,h,R;\varepsilon,\infty}_x D\tau. \end{aligned}$$

Thus τ satisfies (4.2). Finally, by using the commutation (e) between R and D, we have

$$\begin{aligned} d_\xi(\Pi^{\xi,R;\varepsilon,\infty}_x\tau)(h) &= d_\xi(\widehat{\Pi}^{\xi,R;\varepsilon,\infty}_x R\tau)(h) = \widehat{\Pi}^{\xi,h,R;\varepsilon,\infty}_x D(R\tau) \\ &= \widehat{\Pi}^{\xi,h,R;\varepsilon,\infty}_x R(D\tau) = \Pi^{\xi,h,R;\varepsilon,\infty}_x(D\tau). \end{aligned}$$

In the third equality, we use the triangular property (b) of R and the inductive assumption that all trees σ with $|\sigma|_\bigcirc < |\tau|_\bigcirc$ satisfy (4.2). Thus τ satisfies (2.12).

Proof of Lemma 4 – To simplify the notations we suppress the R, ε dependence in the notation and we write $\Pi^p_x, \mathsf{h}^p_x, \Delta_p, |\tau|_p, p(\mu)$ for $\Pi^{R;\varepsilon,p}_x, \mathsf{h}^{R;\varepsilon,p}_x, \Delta_{\varepsilon,p}, |\tau|_{\varepsilon,p}, p_\varepsilon(\mu)$ respectively.

The proof proceeds by a similar induction to the proof of Lemma 3. In addition to (2.14), we prove

$$\widehat{\Pi}^p_x\tau = \widehat{\Pi}^2_x\tau + \sum_{e\in E_\tau}\sum_{k\in\mathbb{N}^d} \frac{1}{k!}\,\mathsf{h}^p_x(\mathcal{I}_{\mathfrak{e}(e)+k}\tau^e_+)\,\widehat{\Pi}^p_x(\uparrow^k_{e_-}\tau^e_-) \tag{4.3}$$

simultaneously. It is obvious for the initial case $\tau = \odot$. Let τ be a planted tree $\mathcal{I}_k(\sigma)$ with $\sigma \in \dot{\mathbf{B}}$. By the definition of $\widehat{\Pi}^p_x$ and the inductive assumption that (2.14) holds for $\sigma \in \dot{\mathbf{B}}$,

$$\begin{aligned}\widehat{\Pi}^p_x\tau &= \partial^k\mathcal{K}(\cdot,\Pi^p_x\sigma) - \sum_{|l|_\mathfrak{s}<|\tau|_p} \frac{(\cdot-x)^l}{l!}\,\partial^{k+l}\mathcal{K}(x,\Pi^p_x\sigma)\\ &= \Big(\partial^k\mathcal{K}(\cdot,\Pi^2_x\sigma) - \sum_{|l|_\mathfrak{s}<|\tau|_p} \frac{(\cdot-x)^l}{l!}\,\partial^{k+l}\mathcal{K}(x,\Pi^2_x\sigma)\Big)\\ &\quad+\sum_{e,m}\frac{1}{m!}\,\mathsf{h}^p_x(\sigma^{e,m}_+)\Big(\partial^k\mathcal{K}(\cdot,\Pi^p_x\sigma^{e,m}_-) - \sum_{|l|_\mathfrak{s}<|\tau|_p}\frac{(\cdot-x)^l}{l!}\,\partial^{k+l}\mathcal{K}(x,\Pi^p_x\sigma^{e,m}_-)\Big),\end{aligned}$$

where we write $\sigma^{e,m}_+ = \mathcal{I}_{m+\mathfrak{e}(e)}\sigma^e_+$ and $\sigma^{e,m}_- = \uparrow^m_{e_-}\sigma^e_-$ for simplicity. Here we can compare the two quantities in the big parentheses with $\widehat{\Pi}^2_x\tau$ and $\widehat{\Pi}^p_x\mathcal{I}_k(\sigma^{e,m}_-)$ except different domains for l in both cases. Recall that $|\tau|_p \le |\tau|_2$. Also, since $|\sigma^{e,m}_+|_p \le 0$ for m such that $\mathsf{h}^p_x(\sigma^{e,m}_+) \neq 0$, we have

$$|\tau|_p = |\mathcal{I}_k\sigma|_p = |\sigma^{e,m}_+|_p + |\mathcal{I}_k(\sigma^{e,m}_-)|_p \le |\mathcal{I}_k(\sigma^{e,m}_-)|_p.$$

One therefore has

$$\begin{aligned}\widehat{\Pi}^p_x\tau &= \widehat{\Pi}^2_x\tau + \sum_{|\tau|_p\le|l|_\mathfrak{s}<|\tau|_2}\frac{(\cdot-x)^l}{l!}\,\partial^{k+l}\,\mathcal{K}(x,\Pi^2_x\sigma)\\ &\quad+\sum_{e,m}\frac{1}{m!}\,\mathsf{h}^p_x(\sigma^{e,m}_+)\Big(\widehat{\Pi}^p_x\mathcal{I}_k(\sigma^{e,m}_-) + \sum_{|\tau|_p\le|l|_\mathfrak{s}<|\mathcal{I}_k(\sigma^{e,m}_-)|_p}\frac{(\cdot-x)^l}{l!}\,\partial^{k+l}\mathcal{K}(x,\Pi^p_x\sigma^{e,m}_-)\Big).\end{aligned}\tag{4.4}$$

By using the assumption (2.14) on σ with the integrability exponent $p(\mathcal{I}_{k+l}\sigma) - r$ for sufficiently small $r > 0$ (fixed later), we have that the second term in the right hand side of (4.4) is equal to

$$\begin{aligned}&\sum_{|\tau|_p\le|l|_\mathfrak{s}<|\tau|_2}\frac{(\cdot-x)^l}{l!}\,\partial^{k+l}_x\mathcal{K}(x,\Pi^{p(\mathcal{I}_{k+l}\sigma)-r}_x\sigma)\\ &\qquad-\sum_{|\tau|_p\le|l|_\mathfrak{s}<|\tau|_2}\frac{(\cdot-x)^l}{l!}\sum_{e,m}\frac{1}{m!}\,\mathsf{h}^{p(\mathcal{I}_{k+l}\sigma)-r}_x(\sigma^{e,m}_+)\,\partial^{k+l}K(x,\Pi^{p(\mathcal{I}_{k+l}\sigma)-r}_x\sigma^{e,m}_-).\end{aligned}$$

The first term is of the form $\sum_l \frac{1}{l!}\mathsf{h}^p_x(\mathcal{I}_{k+l}\sigma)\widehat{\Pi}^p_x X^l$, which appears in (4.3) in the case that e is the unique edge connected to the root of $\tau = \mathcal{I}_k\sigma$. The other terms in (4.3) appear in the third term of the right hand side of (4.4) as the form $\sum_{e,m}\frac{1}{m!}\,\mathsf{h}^p_x(\sigma^{e,m}_+)\widehat{\Pi}^p_x\mathcal{I}_k(\sigma^{e,m}_-)$. Therefore to show that (4.3) holds for τ it is sufficient to show the identity

$$\begin{aligned}&\sum_{e,m}\frac{1}{m!}\,\mathsf{h}^p_x(\sigma^{e,m}_+)\sum_{|\tau|_p\le|l|_\mathfrak{s}<|\mathcal{I}_k(\sigma^{e,m}_-)|_p}\frac{(\cdot-x)^l}{l!}\,\partial^{k+l}\mathcal{K}(x,\Pi^p_x\sigma^{e,m}_-)\\ &\quad=\sum_{|\tau|_p\le|l|_\mathfrak{s}<|\tau|_2}\frac{(\cdot-x)^l}{l!}\sum_{e,m}\frac{1}{m!}\,\mathsf{h}^{p(\mathcal{I}_{k+l}\sigma)-r}_x(\sigma^{e,m}_+)\,\partial^{k+l}K(x,\Pi^{p(\mathcal{I}_{k+l}\sigma)-r}_x\sigma^{e,m}_-),\end{aligned}$$

that is, to show that the following two conditions are equivalent when we choose any small $r > 0$.

(a) $|\sigma^{e,m}_+|_p \le 0 < |\sigma^{e,m}_+|_2$ and $|\tau|_p \le |l|_\mathfrak{s} < |\mathcal{I}_k(\sigma^{e,m}_-)|_p$,

(b) $|\tau|_p \le |l|_\mathfrak{s} < |\tau|_2$ and $|\sigma^{e,m}_+|_{p(\mathcal{I}_{k+l}\sigma)-r} \le 0 < |\sigma^{e,m}_+|_2$.

First we assume (a). Since $|\mathcal{I}_k(\sigma^{e,m}_-)|_2 = |\mathcal{I}_k(\sigma^{e,m}_-)|_p > |l|_\mathfrak{s}$,

$$|\tau|_2 = |\mathcal{I}_k\sigma|_2 = |\sigma^{e,m}_+|_2 + |\mathcal{I}_k(\sigma^{e,m}_-)|_2 > |l|_\mathfrak{s}.$$

Also, since $|\mathcal{I}_{k+l}\sigma|_{p(\mathcal{I}_{k+l}\sigma)-r} \to 0$ as $r \to 0$ and $|\mathcal{I}_{k+l}(\sigma^{e,m}_-)|_p$ is a p-independent positive number, if $r > 0$ is small enough then

$$|\sigma^{e,m}_+|_{p(\mathcal{I}_{k+l}\sigma)-r} = |\mathcal{I}_{k+l}\sigma|_{p(\mathcal{I}_{k+l}\sigma)-r} - |\mathcal{I}_{k+l}(\sigma^{e,m}_-)|_{p(\mathcal{I}_{k+l}\sigma)-r} \le 0.$$

Thus (b) holds. Next we assume (b). Note that $|\cdot|_p$-degree is monotonically decreasing in p. Since $|\mathcal{I}_{k+l}\sigma|_p = |\tau|_p - |l|_\mathfrak{s} \le 0$ implies $p \ge p(\mathcal{I}_{k+l}\sigma)$, we have

$$|\sigma^{e,m}_+|_p \le |\sigma^{e,m}_+|_{p(\mathcal{I}_{k+l}\sigma)-r} \le 0.$$

Moreover, since $|\mathcal{I}_{k+l}\sigma|_{p(\mathcal{I}_{k+l}\sigma)-r} > 0$ for any $r > 0$, one also has

$$|\mathcal{I}_{k+l}(\sigma_-^{e,m})|_p = |\mathcal{I}_{k+l}(\sigma_-^{e,m})|_{p(\mathcal{I}_{k+l}\sigma)-r} = |\mathcal{I}_{k+l}\sigma|_{p(\mathcal{I}_{k+l}\sigma)-r} - |\sigma_+^{e,m}|_{p(\mathcal{I}_{k+l}\sigma)-r} > 0$$

Thus (a) holds. Since (a) and (b) are equivalent, (4.3) holds for τ.

We now consider non-planted trees of the form $\tau = \sigma\mu$ with $\sigma \in \mathbf{B}$ and $\mu \in \{\odot\} \cup \mathcal{P}(\dot{\mathbf{B}})$. Note that the cut at e in (4.3) makes sense only if $e \in E_\mu$ since h_x^p vanishes on trees without $\odot$ symbol. Assuming that (4.3) holds for μ, and recalling the multiplicativity of $\widehat{\Pi}_x^p$, we have

$$\begin{aligned}
\widehat{\Pi}_x^p\tau &= (\widehat{\Pi}_x^p\sigma)(\widehat{\Pi}_x^p\mu) \\
&= (\widehat{\Pi}_x^p\sigma)\Big(\widehat{\Pi}_x^2\mu + \sum_{e\in E_\mu}\sum_{k\in\mathbb{N}^d}\frac{1}{k!}\,\mathsf{h}_x^p(\mathcal{I}_{\mathfrak{e}(e)+k}\mu_+^e)\,\widehat{\Pi}_x^p(\uparrow_{e_-}^k\ \mu_-^e)\Big) \\
&= \widehat{\Pi}_x^2(\sigma\mu) + \sum_{e\in E_\mu}\sum_{k\in\mathbb{N}^d}\frac{1}{k!}\,\mathsf{h}_x^p(\mathcal{I}_{\mathfrak{e}(e)+k}\mu_+^e)\,\widehat{\Pi}_x^p\big(\sigma(\uparrow_{e_-}^k\ \tau_-^e)\big) \\
&= \widehat{\Pi}_x^2\tau + \sum_{e\in E_\tau}\sum_{k\in\mathbb{N}^d}\frac{1}{k!}\,\mathsf{h}_x^p(\mathcal{I}_{\mathfrak{e}(e)+k}\tau_+^e)\,\widehat{\Pi}_x^p(\uparrow_{e_-}^k\ \tau_-^e).
\end{aligned}$$

In the third equality, we use the fact that $\widehat{\Pi}_x^p\sigma$ is independent of p for $\sigma \in \mathbf{B}$. Thus (4.3) holds for τ.

Finally, by using the commutation (d) between R and Δ_2 and the triangular property (b) of R with respect to $|\cdot|_\bigcirc$, we have

$$\begin{aligned}
(\Pi_x^p - \Pi_x^2)\tau = (\widehat{\Pi}_x^p - \widehat{\Pi}_x^2)R\tau &= (\widehat{\Pi}_x^p \otimes \mathsf{h}_x^p)\Delta_2 R\tau \\
&= (\widehat{\Pi}_x^p R \otimes \mathsf{h}_x^p)\Delta_2\tau = (\Pi_x^p \otimes \mathsf{h}_x^p)\Delta_2\tau.
\end{aligned}$$

Thus (2.14) holds for τ.

4.3 Proof of Lemma 7.

4.3 Proof of Lemma 7. The triangular structure of coproducts as in Lemma 7 is well known in the literature, so experts can skip this part. Recall from (3.1) the definition of the preorder $\preceq$ and write $\sigma \prec \tau$ if $(|\sigma|_\bigcirc, |E_\sigma|, |\sigma|_{0,\infty}) < (|\tau|_\bigcirc, |E_\tau|, |\tau|_{0,\infty})$ in the lexicographical order. For any subset $\mathbf{C} \subset \mathbf{T}$ and $\tau \in \mathbf{T}$ we define

$$\mathbf{C}_{\prec\tau} := \big\{\sigma \in \mathbf{C}\ ;\ \sigma \prec \tau\big\}.$$

The following statement is used in the proof of Lemma 7.

16 – Lemma. *For any $\tau \in \mathbf{T}$,*

$$\big(\Delta_{0,2}\tau - \tau\otimes\mathbf{1}\big) \in \mathrm{span}(\mathbf{T}_{\prec\tau}) \otimes \mathrm{alg}_{0,2}^+(\mathbf{T}_{\prec\tau}). \tag{4.5}$$

Furthermore, if $\mathbf{B}$ is a differentiable sector, then for any $\tau \in \mathbf{B}$,

$$\big(\Delta_{0,2}\tau - \tau\otimes\mathbf{1}\big) \in \mathrm{span}(\mathbf{B}_{\prec\tau}) \otimes \mathrm{alg}_{0,2}^+(\mathbf{B}_{\prec\tau}),$$

and for any $\tau \in \dot{\mathbf{B}}$,

$$\big(\Delta_{0,2}\tau - \tau\otimes\mathbf{1}\big) \in \Big(\ \mathrm{span}(\dot{\mathbf{B}}_{\prec\tau}) \otimes \mathrm{alg}_{0,2}^+(\mathbf{B}_{\prec\tau}) + \mathrm{span}(\mathbf{B}_{\prec\tau}) \otimes \big(\ \mathrm{alg}_{0,2}^+(\mathbf{B}_{\prec\tau} \cup \dot{\mathbf{B}}_{\prec\tau}) \cap \mathrm{span}(\mathbf{T}^{(1)})\big)\Big).$$

Proof – We can show the first assertion by the induction. It is trivial for the initial cases $\tau \in \{\bigcirc, \odot, X^k\}$. If $\tau, \sigma \in \mathbf{T}$ satisfy (4.5) we have that $\Delta_{0,2}(\tau\sigma) = (\Delta_{0,2}\tau)(\Delta_{0,2}\sigma)$ belongs to

$$\begin{aligned}
&(\tau\sigma)\otimes\mathbf{1} + \big(\tau\,\mathrm{span}(\mathbf{T}_{\prec\sigma})\big) \otimes \mathrm{alg}_{0,2}^+(\mathbf{T}_{\prec\sigma}) + \big(\sigma\,\mathrm{span}(\mathbf{T}_{\prec\tau})\big) \otimes \mathrm{alg}_{0,2}^+(\mathbf{T}_{\prec\tau}) \\
&\qquad + \big(\,\mathrm{span}(\mathbf{T}_{\prec\tau})\,\mathrm{span}(\mathbf{T}_{\prec\sigma})\big) \otimes \mathrm{alg}_{0,2}^+(\mathbf{T}_{\prec\tau} \cup \mathbf{T}_{\prec\sigma}) \\
&\subset \Big((\tau\sigma)\otimes\mathbf{1} + \mathrm{span}(\mathbf{T}_{\prec\tau\sigma}) \otimes \mathrm{alg}_{0,2}^+(\mathbf{T}_{\prec\tau\sigma})\Big);
\end{aligned}$$

hence $\tau\sigma$ satisfies (4.5). Moreover if $\tau \in \mathbf{T}$ satisfies (4.5) one has $\tau \prec \mathcal{I}_k\tau$ and

$$\begin{aligned}
\big(\Delta_{0,2}(\mathcal{I}_k\tau) - \mathcal{I}_k\tau\otimes\mathbf{1}\big) &\in \Big(\mathcal{I}_k(\mathrm{span}(\mathbf{T}_{\prec\tau})) \otimes \mathrm{alg}_{0,2}^+(\mathbf{T}_{\prec\tau}) + \mathrm{span}\{X^l\}_l \otimes \mathrm{alg}_{0,2}^+(\{\tau\})\Big) \\
&\in \mathrm{span}(\mathbf{T}_{\prec\mathcal{I}_k\tau}) \otimes \mathrm{alg}_{0,2}^+(\mathbf{T}_{\prec\mathcal{I}_k\tau});
\end{aligned}$$

so $\mathcal{I}_k\tau$ satisfies (4.5). The remaining assertions follow from the definition of a differential sector, so we are done. ▷

Proof of Lemma 7 – It is sufficient to show that

$$\tau\in\mathbf{B}_i \quad\Rightarrow\quad \mathbf{B}_{\prec\tau}\subset\mathbf{B}_{i-1},$$
$$\mu\in\dot{\mathbf{B}}_i \quad\Rightarrow\quad \mathbf{B}_{\prec\mu}\subset\mathbf{B}_{i-1},\quad \dot{\mathbf{B}}_{\prec\mu}\subset\dot{\mathbf{B}}_{i-1}.$$

The first statement is left to the reader. To show the second statement, let $\mu=D_v\tau$ for some $\tau\in\mathbf{B}_i$. If $\sigma\in\mathbf{B}_{\prec\mu}$, then since

$$|\sigma|_\bigcirc\le|\mu|_\bigcirc<|\tau|_\bigcirc,$$

we have $\sigma\prec\tau$ and $\sigma\in\mathbf{B}_{i-1}$. If $\rho=D_w\eta\in\dot{\mathbf{B}}_{\prec\mu}$ with $\eta\in\mathbf{B}$, then since

$$\big(|\rho|_\bigcirc,|E_\rho|,|\rho|_{0,\infty}\big)=\big(|\eta|_\bigcirc-1,|E_\eta|,|\eta|_{0,\infty}\big),$$

$\rho\prec\mu$ implies $\eta\prec\tau$. Thus $\eta\in\mathbf{B}_{i-1}$ and $\rho\in\dot{\mathbf{B}}_{i-1}$.

4.4 Proof of the lemmas of Section 3.2. We have three statements to prove.

Proof of Lemma 8 – Recall from Definition 20 the definition of the quantities $(\!|f|\!)_{\mathbf{c};w_b}$ and $\|f\|^{\mathsf{\Gamma}}_{\mathbf{c};w_b}$ involved in the definition of modelled distribution. Also recall that we can write

$$f^{n;\varepsilon,\infty}_{\tau_i}(x)=\mathsf{\Gamma}^{n;\varepsilon,\infty}_x\tau_i-\tau_i$$

with $\mathsf{\Gamma}^{n;\varepsilon,\infty}_x=\big(\mathrm{id}\otimes\mathsf{g}^{n;\varepsilon,\infty}_x\big)\Delta_{\varepsilon,\infty}$ which satisfies $\mathsf{\Gamma}^{n;\varepsilon,\infty}_{yx}\mathsf{\Gamma}^{n;\varepsilon,\infty}_x=\mathsf{\Gamma}^{n;\varepsilon,\infty}_y$. Then the relation

$$\begin{aligned}&\mathsf{\Gamma}^{n;\varepsilon,\infty}_{yx}f^{n;\varepsilon,\infty}_{\tau_i}(x)-f^{n;\varepsilon,\infty}_{\tau_i}(y)\\&=\mathsf{\Gamma}^{n;\varepsilon,\infty}_{yx}(\mathsf{\Gamma}^{n;\varepsilon,\infty}_x-\mathrm{id})\tau_i-(\mathsf{\Gamma}^{n;\varepsilon,\infty}_y-\mathrm{id})\tau_i=(\mathrm{id}-\mathsf{\Gamma}^{n;\varepsilon,\infty}_{yx})\tau_i.\end{aligned}$$

yields that $f^{n;\varepsilon,\infty}_{\tau_i}$ belongs to the space $D^{(|\tau_i|_{\varepsilon,\infty},\infty)}(V_{i-1};\mathsf{\Gamma}^{n;\varepsilon,\infty})_{w_{cm_{\mathbf{B}}}}$ with the norms

$$(\!|f^{n;\varepsilon,\infty}_{\tau_i}|\!)_{(|\tau_i|_{\varepsilon,\infty},\infty);w_{cm_{\mathbf{B}}}}\lesssim\big(1+\|\mathsf{g}^{n;\varepsilon,\infty}:V^+_{i-1}\|_{w_c}\big)^{m_{\mathbf{B}}},\qquad \|f^{n;\varepsilon,\infty}_{\tau_i}\|^{\mathsf{\Gamma}^{n;\varepsilon,\infty}}_{(|\tau_i|_{\varepsilon,\infty},\infty);w_{cm_{\mathbf{B}}}}\le\|\mathsf{\Gamma}^{n;\varepsilon,\infty}:V_i\|_{w_{cm_{\mathbf{B}}}}.$$

Since $|\tau_i|_{\varepsilon,\infty}>0$ there exists a unique reconstruction of $f^{n;\varepsilon,\infty}_{\tau_i}$ for the restriction of $\mathsf{M}^{n;\varepsilon,\infty}$ to the concrete regularity structure (V_{i-1},V^+_{i-1}). Moreover, since the relation $\mathsf{\Pi}^{n;\varepsilon,\infty}_x\mathsf{\Gamma}^{n;\varepsilon,\infty}_x=\mathsf{\Pi}^n$ yields

$$\mathsf{\Pi}^{n;\varepsilon,\infty}_x\tau_i=\mathsf{\Pi}^n\tau_i-\mathsf{\Pi}^{n;\varepsilon,\infty}_x\big(f^{n;\varepsilon,\infty}_{\tau_i}(x)\big),$$

the reconstruction of $f^{n;\varepsilon,\infty}_{\tau_i}$ is noting but $\mathsf{\Pi}^n\tau_i$. Hence the reconstruction theorem implies the inequality

$$\begin{aligned}\|\mathsf{\Pi}^{n;\varepsilon,\infty}:\tau_i\|_{w_{c(m_{\mathbf{B}}+1)}}&=\sup_{0<t\le1}t^{-\frac{|\tau|_{\varepsilon,\infty}}{\ell}}\|\mathcal{Q}_t(x,\mathsf{\Pi}^{n;\varepsilon,\infty}_x\tau_i)\|_{L^\infty_x(w_{c(m_{\mathbf{B}}+1)})}\\&\lesssim\|\mathsf{\Pi}^{n;\varepsilon,\infty}:V_{i-1}\|_{w_c}\|f^{n;\varepsilon,\infty}_{\tau_i}\|^{\mathsf{\Gamma}^{n;\varepsilon,\infty}}_{(|\tau_i|_{\varepsilon,\infty},\infty);w_{cm_{\mathbf{B}}}},\end{aligned}$$

which gives the result.

Before going to the proof of Lemma 9, we prove some useful technical results.

17 – Lemma. *There exists a finite positive constant C_c depending only on $(Q_t)_{t>0}$ and $c>0$ such that, for any $q\ge p\in[1,\infty]$, $t\in(0,1]$, and $f\in L^p(w_c)$, one has*

$$\|G_t*f\|_{L^q(w_c)}\le C_c\,t^{\frac{|\mathfrak{s}|}{\ell}(\frac1q-\frac1p)}\|f\|_{L^p(w_c)}.$$

Proof – Since $|(G_t*f)w_c|\le(G_tw_c^{-1})*(|f|w_c)$, the result follows from usual Young inequality. The proportional constant is $\|G_tw_c^{-1}\|_{L^r(\mathbb{R}^d)}$, where $\frac1r=1+\frac1q-\frac1p$. By (1.3) we have the bound

$$\|G_tw_c^{-1}\|_{L^r(\mathbb{R}^d)}\lesssim t^{-\frac{|\mathfrak{s}|}{\ell}(1-\frac1r)}=t^{\frac{|\mathfrak{s}|}{\ell}(\frac1q-\frac1p)}.$$

▷

18 – Lemma. *For any $\varepsilon\in(0,\varepsilon_0)$, $p\in[2,\infty]$, and $c,d>0$, there exists a positive constant C, which is independent of n, ω, and $h\in H$ with $\|h\|_H\le1$, one has*

$$\Big\|\int_{\mathbb{R}^d}G_t(x-y)\big|\mathcal{Q}_t\big(y,\mathsf{\Pi}^{n;\varepsilon,p}_x\tau-\mathsf{\Pi}^{n;\varepsilon,p}_y\tau\big)\big|dy\Big\|_{L^{i_p(\tau)}_x(w_{c+d})}\le Ct^{\frac{|\tau|_{\varepsilon,p}}{\ell}}\|\mathsf{\Gamma}^{n;\varepsilon,p}:W_{\preceq\tau}\|_{w_d}\|\mathsf{\Pi}^{n;\varepsilon,p}:W_{\prec\tau}\|_{w_c}$$

for any $\tau\in\mathbf{B}\cup\dot{\mathbf{B}}$, where $W_{\prec\tau}$ is a linear space spanned by all trees $\sigma\prec\tau$ and $W_{\preceq\tau}:=W_{\prec\tau}\oplus\mathrm{span}\{\tau\}$.

Proof – To simplify the notations we omit the useless symbols n,ε,p from the model. By the change of variable $y\to x-z$ and the expansion of $\mathsf{\Gamma}_{yx}$, we can write the quantity inside $L^{i_p(\tau)}_x$ norm as

$$\int_{\mathbb{R}^d}G_t(z)\big|\mathcal{Q}_t\big(x-z,\mathsf{\Pi}_{x-z}(\mathsf{\Gamma}_{(x-z)x}-\mathrm{id})\tau\big)\big|dz=\sum_{\sigma\prec\tau}\int_{\mathbb{R}^d}G_t(z)|P_\sigma\mathsf{\Gamma}_{(x-z)x}\tau||\mathcal{Q}_t(x-z,\mathsf{\Pi}_{x-z}\sigma)|dz.$$

By Hölder inequality and the elementary inequality $\|f(x-z)\|_{L_x^p(\mathbb{R}^d)} \le w_c(z)^{-1}\|f\|_{L_x^p(\mathbb{R}^d)}$, we can bound the $L_x^{i_p(\tau)}$ norm of above quantity from above by

$$
\begin{aligned}
&\sum_{\sigma\prec\tau}\int_{\mathbb{R}^d} G_t(z)\|P_\sigma\Gamma_{(x-z)x}\tau\|_{L_x^{i_p(\tau):i_p(\sigma)}}\|\mathcal{Q}_t(x-z,\Pi_{x-z}\sigma)\|_{L_x^{i_p(\sigma)}}dz\\
&\le \sum_{\sigma\prec\tau}\int_{\mathbb{R}^d} G_t(z)w_{c+d}(z)^{-1}\|z\|_{\mathfrak{s}}^{|\tau|_{\varepsilon,p}-|\sigma|_{\varepsilon,p}}t^{\frac{|\sigma|_{\varepsilon,p}}{\ell}}\|\Gamma:W_{\preceq\tau}\|_{w_d}\|\Pi:W_{\prec\tau}\|_{w_c}dz\\
&\lesssim \sum_{\sigma\prec\tau} t^{\frac{|\tau|_{\varepsilon,p}-|\sigma|_{\varepsilon,p}}{\ell}}t^{\frac{|\sigma|_{\varepsilon,p}}{\ell}}\|\Gamma:W_{\preceq\tau}\|_{w_d}\|\Pi:W_{\prec\tau}\|_{w_c} = t^{\frac{|\tau|_{\varepsilon,p}}{\ell}}\|\Gamma:W_{\preceq\tau}\|_{w_d}\|\Pi:W_{\prec\tau}\|_{w_c}.
\end{aligned}
$$

Here the exponent $p:q$ for $1\le p\le q\le\infty$ is defined at the beginning of Appendix A; $i_p(\tau):i_p(\sigma)=p$ if $\tau\in\dot{\mathbf{B}}$ and $\sigma\in\mathbf{B}$, otherwise $i_p(\tau):i_p(\sigma)=\infty$. $\rhd$

Proof of Lemma 9 – A similar statement was proved by Linares, Otto, Tempelmayr & Tsatsoulis in Proposition 4.6 of [23]. We follow here their reasoning. To simplify the notations we write Π_x,Γ,E_i for $\Pi_x^{n;\varepsilon,\infty},\Gamma^{n;\varepsilon,\infty},E_i^{n,R}$, respectively.

Note that $\mathbb{E}\big[\mathcal{Q}_t(x,\Pi_x\tau_i)\big]$ is independent of x by stationarity of the law of the random noise under the translations of $\mathbb{R}^d$. One therefore has for any $0<s<t$

$$
\begin{aligned}
\partial_t\mathbb{E}\big[\mathcal{Q}_t(x,\Pi_x\tau_i)\big] &= \int_{\mathbb{R}^d}\partial_t Q_{t-s}(x,y)\mathbb{E}\big[\mathcal{Q}_s(y,\Pi_x\tau_i)\big]dy\\
&= \int_{\mathbb{R}^d}\partial_t Q_{t-s}(x,y)\mathbb{E}\big[\mathcal{Q}_s\big(y,\Pi_x\tau_i-\Pi_y\tau_i\big)\big]dy.
\end{aligned}
$$

Picking $s=\frac{t}{2}$ we apply Lemma 18 and have

$$
\big\|\partial_t\mathbb{E}\big[\mathcal{Q}_t(x,\Pi_x\tau_i)\big]\big\|_{L_x^\infty(w_{c(m_{\mathbf{B}}+1)})} \lesssim t^{\frac{|\tau|_{\varepsilon,\infty}}{\ell}-1}\mathbb{E}\big[\|\Gamma:V_i\|_{w_{cm_{\mathbf{B}}}}\|\Pi:V_{i-1}\|_{w_c}\big].
$$

Since $|\tau_i|_{\varepsilon,\infty}<0$ and

$$
\mathbb{E}\big[\mathcal{Q}_1(x,\Pi_x\tau_i)\big] = \mathbb{E}\big[\mathcal{Q}_1(0,\Pi_0\tau)\big] = E_i
$$

by the stationarity of the law of the noise, we obtain the result by writing

$$
\begin{aligned}
\big\|\mathbb{E}\big[\mathcal{Q}_t(x,\Pi_x\tau_i)\big]\big\|_{L_x^\infty(w_{c(m_{\mathbf{B}}+1)})} &\le \big\|\mathbb{E}\big[\mathcal{Q}_1(x,\Pi_x\tau_i)\big]\big\|_{L_x^\infty(w_{c(m_{\mathbf{B}}+1)})} + \int_t^1 \big\|\partial_s\mathbb{E}\big[\mathcal{Q}_s(x,\Pi_x\tau_i)\big]\big\|_{L_x^\infty(w_{c(m_{\mathbf{B}}+1)})}ds\\
&\lesssim |E_i| + t^{\frac{|\tau_i|_{\varepsilon,\infty}}{\ell}}\,\mathbb{E}\big[\|\Gamma:V_i\|_{w_{cm_{\mathbf{B}}}}\|\Pi:V_{i-1}\|_{w_c}\big].
\end{aligned}
$$

Proof of Lemma 10 – Here we omit the parameters n,∞,R similarly to the proof of Lemma 9, but we leave the parameter ε to make explicit the ε-dependence of the quantities $\|(\cdot)^{n;\varepsilon,\infty}:\tau_i\|_{w_c}$. Note that $\Pi_x^\varepsilon\tau_i=\Pi_x^{2\varepsilon}\tau_i$ by the property (c) of differentiable sectors. By using the integral formula

$$
\mathcal{Q}_t(x,\Pi_x^\varepsilon\tau_i) = \mathcal{Q}_1(x,\Pi_x^\varepsilon\tau_i) - \int_t^1\partial_s\mathcal{Q}_s(x,\Pi_x^\varepsilon\tau_i)ds,
$$

which is also used in the proof of Lemma 9, we have

$$
\begin{aligned}
\mathbb{E}\big[\|\Pi^{2\varepsilon}:\ \tau_i\|_{w_{c(m_{\mathbf{B}}+2)}}^q\big] &= \mathbb{E}\Big[\sup_{0<t\le1} t^{-\frac{|\tau_i|_{2\varepsilon,\infty}}{\ell}q}\|\mathcal{Q}_t(x,\Pi_x^\varepsilon\tau_i)\|_{L_x^\infty(w_{c(m_{\mathbf{B}}+2)})}^q\Big]\\
&\le \mathbb{E}\Big[\|\mathcal{Q}_1(x,\Pi_x^\varepsilon\tau_i)\|_{L_x^\infty(w_{c(m_{\mathbf{B}}+2)})}^q\Big] + \mathbb{E}\left[\sup_{0<t\le1}\left|\int_t^1 s^{-\frac{|\tau_i|_{2\varepsilon,\infty}}{\ell}}\|\partial_s\mathcal{Q}_s(x,\Pi_x^\varepsilon\tau_i)\|_{L_x^\infty(w_{c(m_{\mathbf{B}}+2)})}ds\right|^q\right]\\
&=: A_1+A_2.
\end{aligned}
$$

We focus on A_2. By the semigroup property of Q_s we have

$$
\begin{aligned}
\partial_s\mathcal{Q}_s(x,\Pi_x^\varepsilon\tau_i) &= \int_{\mathbb{R}^d}(\partial_s Q)_{\frac{s}{2}}(x,y)\mathcal{Q}_{\frac{s}{2}}(y,\Pi_x^\varepsilon\tau_i)dy\\
&= \int_{\mathbb{R}^d}(\partial_s Q)_{\frac{s}{2}}(x,y)\mathcal{Q}_{\frac{s}{2}}(y,\Pi_y^\varepsilon\tau_i)dy + \int_{\mathbb{R}^d}(\partial_s Q)_{\frac{s}{2}}(x,y)\mathcal{Q}_{\frac{s}{2}}(y,\Pi_x^\varepsilon\tau_i-\Pi_y^\varepsilon\tau_i)dy.
\end{aligned}
$$

In the last line, the $L_x^\infty(w_{c(m_{\mathbf{B}}+1)})$ norm of the second term is bound above by

$$
s^{\frac{|\tau_i|_{\varepsilon,\infty}}{\ell}-1}(\star)_\varepsilon, \qquad (\star)_\varepsilon := \|\Gamma^\varepsilon:V_i\|_{w_{cm_{\mathbf{B}}}}\|\Pi^\varepsilon:V_{i-1}\|_{w_c}
$$

by Lemma 18. For the remaining term we can use Lemma 17 and the bound $|\partial_t Q_t(x)| \lesssim t^{-1}G_t(x)$ to get

$$\Big\| \int_{\mathbb{R}^d} (\partial_s Q)_{\frac{s}{2}}(x,y)\mathcal{Q}_{\frac{s}{2}}(y,\Pi_y^\varepsilon \tau_i)dy\Big\|_{L_x^\infty(w_{c(m_{\mathbf{B}}+2)})} \lesssim s^{-\frac{|\mathfrak{s}|}{\ell}\frac{1}{q}-1}\|\mathcal{Q}_{\frac{s}{2}}(y,\Pi_y^\varepsilon\tau_i)\|_{L_y^q(w_{c(m_{\mathbf{B}}+2)})}.$$

Write q' for the conjugate exponent of q. If we choose sufficiently large q such that $q > \frac{\ell+|\mathfrak{s}|}{|\tau_i|_{\varepsilon,\infty}-|\tau_i|_{2\varepsilon,\infty}}$, then since $\big(\frac{|\tau_i|_{\varepsilon,\infty}-|\tau_i|_{2\varepsilon,\infty}}{\ell} - \frac{|\mathfrak{s}|}{\ell}\frac{1}{q}-1\big)q' > -1$, we get from Hölder inequality in t the estimate

$$\begin{aligned}
&\int_0^1 s^{-\frac{|\tau_i|_{2\varepsilon,\infty}}{\ell}}\|\partial_s \mathcal{Q}_s(x,\Pi_x^\varepsilon\tau_i)\|_{L_x^\infty(w_{c(m_{\mathbf{B}}+2)})}ds\\
&\lesssim \int_0^1 s^{\frac{|\tau_i|_{\varepsilon,\infty}-|\tau_i|_{2\varepsilon,\infty}}{\ell}-\frac{|\mathfrak{s}|}{\ell}\frac{1}{q}-1} s^{-\frac{|\tau_i|_{\varepsilon,\infty}}{\ell}}\|\mathcal{Q}_{\frac{s}{2}}(y,\Pi_y^\varepsilon\tau_i)\|_{L_y^q(w_{c(m_{\mathbf{B}}+2)})}ds + (\star)_\varepsilon \int_0^1 s^{\frac{|\tau_i|_{\varepsilon,\infty}-|\tau_i|_{2\varepsilon,\infty}}{\ell}-1}ds\\
&\lesssim \Big(\int_0^1 s^{-\frac{|\tau_i|_{\varepsilon,\infty}}{\ell}q}\|\mathcal{Q}_{\frac{s}{2}}(y,\Pi_y^\varepsilon\tau_i)\|^q_{L_y^q(w_{c(m_{\mathbf{B}}+2)})}ds\Big)^{\frac{1}{q}} + (\star)_\varepsilon.
\end{aligned}$$

We conclude from the estimate (3.5) that

$$\begin{aligned}
A_2 &\lesssim \int_0^1 s^{-\frac{|\tau_i|_{\varepsilon,\infty}}{\ell}q}\,\mathbb{E}\big[\|\mathcal{Q}_{\frac{s}{2}}(y,\Pi_y^\varepsilon\tau_i)\|^q_{L_y^q(w_{c(m_{\mathbf{B}}+2)})}\big]ds + \mathbb{E}[(\star)^q]\\
&\lesssim |E_i|^q + \mathbb{E}\Big[\sup_{\|h\|_H\le 1}\|\Pi^{n,h;\varepsilon,\infty}:W_i\|^q_{w_c}\Big] + \mathbb{E}[(\star)^q].
\end{aligned}$$

The proof of the estimate of A_1 is similar and left to the reader.

4.5 Proofs of the Lemmas of Section 3.3. We prove the following two statements. We suppress the useless exponents n,ε,p from the objects for more readability.

Proof of Lemma 11 – The proof is classical and recalled here for completeness. Similar arguments can be found at Section 4 of [2].

We first prove the compatibility between $\mathsf{M} = \mathsf{M}^{n;\varepsilon,p}$ and $\mathsf{M}^+ = \mathsf{M}^{n+;\varepsilon,p}$. Recall that $\mathsf{g}_x = \mathsf{g}_x^{n;\varepsilon,p}$ is defined by the formulas (2.10). We define another character $\mathsf{f}_x = \mathsf{f}_x^{n;\varepsilon,p}$ by the formula

$$\mathsf{f}_x(X^k) = x^k, \qquad \mathsf{f}_x(\mathcal{I}_k\tau) = \mathbf{1}_{|\mathcal{I}_k\tau|>0}\,\partial^k\mathcal{K}(x,\Pi_x\tau). \tag{4.6}$$

Then the characters g_x and f_x are related to each other by the equivalent identities

$$\mathsf{g}_x^{-1}(\mathcal{I}_k\tau) = -\sum_{l\in\mathbb{N}^d}\frac{(-x)^l}{l!}\mathsf{f}_x(\mathcal{I}_{k+l}\tau), \qquad \mathsf{f}_x(\mathcal{I}_k\tau) = -\sum_{m\in\mathbb{N}^d}\frac{x^m}{m!}\mathsf{g}_x^{-1}(\mathcal{I}_{k+m}\tau),$$

and we have the explicit representation of g_x

$$\mathsf{g}_x(\mathcal{I}_k^+\tau) = (\mathsf{f}_x\otimes\mathsf{g}_x)(\mathcal{I}_k^+\otimes\mathrm{id})\Delta\tau. \tag{4.7}$$

We can obtain the last one by applying the operator $\mathsf{g}_x\otimes\mathsf{g}_x^{-1}\otimes\mathsf{g}_x$ to the identity

$$\sum_{m\in\mathbb{N}^d}\frac{X^m}{m!}\otimes\Delta^+\mathcal{I}_{k+m}^+\tau = \sum_{m\in\mathbb{N}^d}\frac{X^m}{m!}\otimes(\mathcal{I}_{k+m}^+\otimes\mathrm{id})\Delta\tau + \sum_{m,l\in\mathbb{N}^d}\frac{X^m}{m!}\otimes\frac{X^l}{l!}\otimes\mathcal{I}_{k+l+m}^+\tau$$

and using the fact $(\mathsf{g}_x^{-1}\otimes\mathsf{g}_x)\Delta^+\mathcal{I}_{k+m}^+\tau = 0$ and the binomial theorem. The identity (4.7) yields the formula

$$\mathsf{g}_{yx}\big((\mathcal{I}^+ + \mathcal{J}^{\mathsf{M}}(x))\tau\big) = \mathsf{f}_y(\mathcal{I}^+\Gamma_{yx}\tau),$$

as follows.

$$\begin{aligned}
\mathsf{f}_y(\mathcal{I}^+\Gamma_{yx}\tau) &= (\mathsf{f}_y\otimes\mathsf{g}_y\otimes\mathsf{g}_x^{-1})(\mathcal{I}^+\otimes\mathrm{id}\otimes\mathrm{id})(\mathrm{id}\otimes\Delta^+)\Delta\tau\\
&= (\mathsf{f}_y\otimes\mathsf{g}_y\otimes\mathsf{g}_x^{-1})(\mathcal{I}^+\otimes\mathrm{id}\otimes\mathrm{id})(\Delta\otimes\mathrm{id})\Delta\tau\\
&= (\mathsf{g}_y\otimes\mathsf{g}_x^{-1})(\mathcal{I}^+\otimes\mathrm{id})\Delta\tau\\
&= (\mathsf{g}_y\otimes\mathsf{g}_x^{-1})\Big(\Delta^+\mathcal{I}^+\tau - \sum_{k\in\mathbb{N}^d}\frac{X^k}{k!}\otimes\mathcal{I}_k^+\tau\Big)\\
&= \mathsf{g}_{yx}(\mathcal{I}^+\tau) + \sum_{k,l\in\mathbb{N}^d}\frac{y^k}{k!}\frac{(-x)^l}{l!}\mathsf{f}_x(\mathcal{I}_{k+l}\tau)
\end{aligned}$$

$$= \mathsf{g}_{yx}(\mathcal{I}^+\tau) + \sum_{m\in\mathbb{N}^d} \frac{(y-x)^m}{m!}\mathsf{f}_x(\mathcal{I}_m\tau) = \mathsf{g}_{yx}\big((\mathcal{I}^+ + \mathcal{J}^{\mathsf{M}}(x))\tau\big).$$

Thus for any $x,y,z\in\mathbb{R}^d$ we have

$$\begin{aligned}\mathsf{g}_{zy}\Big\{\Gamma^+_{yx}(\mathcal{I}^+ + \mathcal{J}^{\mathsf{M}}(x))\tau - \big(\mathcal{I}^+ + \mathcal{J}^{\mathsf{M}}(y)\big)\Gamma_{yx}\tau\Big\} &= \mathsf{g}_{zx}(\mathcal{I}^+ + \mathcal{J}^{\mathsf{M}}(x))\tau - \mathsf{f}_z(\mathcal{I}^+\Gamma_{zy}\Gamma_{yx}\tau)\\ &= \mathsf{f}_z(\mathcal{I}^+\Gamma_{zx}\tau) - \mathsf{f}_z(\mathcal{I}^+\Gamma_{zx}\tau) = 0.\end{aligned}$$

Since the quantity inside g_{zy} is in the subspace $\mathrm{span}\{X^k\}_k$, on which g_{zy} is injective, we have the compatibility

$$\Gamma^+_{yx}\big(\mathcal{I}^+ + \mathcal{J}^{\mathsf{M}}(x)\big)\tau = \big(\mathcal{I}^+ + \mathcal{J}^{\mathsf{M}}(y)\big)\Gamma_{yx}\tau.$$

Next we prove Lemma 11. It is straightforward from the definition of $\mathcal{K}^+$ to show the explicit representation

$$(\mathcal{K}^+ f_\tau)(x) = \mathcal{I}^+\big(f_\tau(x)\big) + \sum_{k\in\mathbb{N}^d} \mathsf{g}_x(\mathcal{I}_k\tau)\,\frac{X^k}{k!}.$$

for $f_\tau = f_\tau^{n;\varepsilon,p}$. This formula further transformed into

$$(\mathcal{K}^+ f_\tau)(x) = (\mathrm{id}\otimes\mathsf{g}_x)\Delta^+\mathcal{I}^+\tau - \mathcal{I}^+\tau = f_{\mathcal{I}^+\tau}.$$

Thus we have the result in a similar argument to the proof of Lemma 8.

Proof of Lemma 12 – We consider the case where $\tau\in\dot{\mathbf{B}}_i$ and let the reader treat the case where $\tau\in\mathbf{B}_i$. We saw in a similar argument to the proof of Lemma 8 that f_τ belongs to the space

$$D^{(|\tau|_{\varepsilon,p},p)}(W_i;\Gamma)_{w_{cm_{\mathbf{B}}}},$$

with norms

$$\|\!| f_\tau |\!\|^{\Gamma}_{(|\tau|_{\varepsilon,p},p);w_{cm_{\mathbf{B}}}} \lesssim \big(1 + \|\mathsf{g}: W^+_{i-1,\varepsilon,p}\|_{w_c}\big)^{m_{\mathbf{B}}}.$$

Therefore we have from Theorem 24 the estimate

$$\begin{aligned}\|\!|\mathcal{K}^+ f_\tau|\!\|_{(|\tau|_{\varepsilon,p}+\beta_0,p);w_{c(m_{\mathbf{B}}+2)}} &\lesssim \|\Pi: W_i\|_{w_c}\big(1 + \|\Gamma: W_i\|_{w_c}\big)\|\!| f_\tau|\!\|^{\Gamma}_{(|\tau|_{\varepsilon,p},p);w_{cm_{\mathbf{B}}}} + [\![\Pi\tau]\!]^{\Pi,f_\tau}_{(|\tau|_{\varepsilon,p},p);w_{c(m_{\mathbf{B}}+2)}}\\ &\lesssim \big(1 + \|\!|\mathsf{M}|\!\|_{\mathbf{M}(\mathscr{W}_{i,\varepsilon,p})_{w_c}}\big)^{m_{\mathbf{B}}+2},\end{aligned}$$

on which one reads the conclusion.

4.6 Proofs of the lemmas of Section 3.4.

Recall $\tau\in\dot{\mathbf{B}}_{i+1}$. We have two statements to prove.

Proof of Lemma 14 – Recall from (2.13) the definition of $\mathsf{h}_x^{n;\varepsilon,p}$. The proof leads to the estimate of the character f_x defined by (4.6). This is a content of Lemma 5.6 of [21], but the proof is recalled here for completeness. Pick $\sigma\in\dot{\mathbf{B}}_i$ with $\mu=\mathcal{I}_k(\sigma)$. Note that, by the behavior of $\Pi^{n;\varepsilon,p}$ as a 'step function' of p, we can write

$$\mathsf{h}_x^{n;\varepsilon,p}(\mu) := \mathbf{1}_{|\mu|_{\varepsilon,p}\le 0<|\mu|_{\varepsilon,2}}\mathsf{f}_x^{n;\varepsilon,r}(\mu), \qquad \mathsf{f}_x^{n;\varepsilon,r}(\mu) := \partial^k\mathcal{K}(x,\Pi_x^{n;\varepsilon,r}\sigma).$$

We can write the $\mathsf{f}_x^{n;\varepsilon,r}(\mu)$ by the form

$$\int_0^1 \partial^k\mathcal{K}_t(x,\Pi_x^{n;\varepsilon,r}\sigma)dt = \int_0^1\int_{\mathbb{R}^d}\partial_x^k K_{\frac{t}{2}}(x,y)\mathcal{Q}_{\frac{t}{2}}(y,\Pi_x^{n;\varepsilon,r}\sigma)dydt.$$

By Lemma 18, we can replace $\Pi_x^{n;\varepsilon,r}\sigma$ by $\Pi_y^{n;\varepsilon,r}\sigma$ with an error $\|\Gamma^{n;\varepsilon,r}: W_i\|_{w_{cm_{\mathbf{B}}}}\|\Pi^{n;\varepsilon,r}: W_i\|_{w_c}$, where note that $|\mu|_{\varepsilon,r} > |\mu|_{\varepsilon,p_\varepsilon(\mu)} = 0$ and the t-dependent factor $t^{\frac{|\mu|_{\varepsilon,r}}{\ell}-1}$ is integrable. After the replacement, we obtain the statement as a direct consequence of the size estimate (2.7) satisfied by $\mathcal{Q}_{\frac{t}{2}}(y,\Pi_y^{n;\varepsilon,r}\sigma)$.

Proof of Lemma 15 – To lighten the notations we suppress in this proof the exponent n from the models and their associated quantities. By the semigroup property of Q_t we have

$$\begin{aligned}\mathcal{Q}_t(x,\Pi_x^{\varepsilon,p}\tau) &= \int_{\mathbb{R}^d} Q_{\frac{t}{2}}(x,y)\mathcal{Q}_{\frac{t}{2}}(y,\Pi_x^{\varepsilon,p}\tau)dy\\ &= \int_{\mathbb{R}^d} Q_{\frac{t}{2}}(x,y)\mathcal{Q}_{\frac{t}{2}}(y,\Pi_y^{\varepsilon,p}\tau)dy + \int_{\mathbb{R}^d} Q_{\frac{t}{2}}(x,y)\mathcal{Q}_{\frac{t}{2}}(y,\Pi_x^{\varepsilon,p}\tau - \Pi_y^{\varepsilon,p}\tau)dy\\ &=: B_1 + B_2.\end{aligned}$$

We can apply Lemma 18 to bound above the $L^p_x(w_{c(m_{\mathbf{B}}+1)})$ norm of B_2 by

$$t^{\frac{|\tau|_{\varepsilon,p}}{\ell}}\|\Gamma^{\varepsilon,p}: W_i\|_{w_{cm_{\mathbf{B}}}}\|\Pi^{\varepsilon,p}: W_i\|_{w_c},$$

For B_1 write (2.14) in synthetic form

$$\mathsf{\Pi}_y^{\varepsilon,p}\tau = \mathsf{\Pi}_y^{\varepsilon,2}\tau + \sum_{\tau_1,\tau_2} \mathsf{h}_y^{\varepsilon,p}(\tau_1)\, \mathsf{\Pi}_y^{\varepsilon,p}\tau_2 \tag{4.8}$$

with $\tau_1 \in \dot{\mathbf{B}}_i$, $\tau_2 \in \mathbf{B}_i$, $|\tau_1|_{\varepsilon,p} + |\tau_2|_{\varepsilon,p} = |\tau|_{\varepsilon,p}$, and $p \geq p_\varepsilon(\tau_1)$. Note the elementary yet fundamental identities

$$|\tau|_{\varepsilon,2} + |\mathfrak{s}|\Big(\frac{1}{p} - \frac{1}{2}\Big) = |\tau|_{\varepsilon,p}, \qquad |\tau_2|_{\varepsilon,p} + |\mathfrak{s}|\Big(\frac{1}{p} - \frac{1}{q_1}\Big) = |\tau|_{\varepsilon,p} - |\tau_1|_{\varepsilon,q_1}.$$

For any small $\zeta > 0$ (fixed later), we can choose $q_1 = q_1(\varepsilon,\zeta)$ such that $\lfloor p_\varepsilon(\tau_1)\rfloor_{I_\varepsilon} \leq q_1 < p_\varepsilon(\tau_1)$ and

$$|\tau|_{\varepsilon,p} - |\tau|_{\varepsilon+\zeta,p} > |\tau_1|_{\varepsilon,q_1} > 0 = |\tau_1|_{\varepsilon,p_\varepsilon(\tau_1)}.$$

We then have from Lemmas 13 and 14 an upper bound for $\|B_1\|_{L_x^p(w_{c(m_{\mathbf{B}}+2)})}$ of the form

$$\begin{aligned}
&\Big\| \int_{\mathbb{R}^d} G_t(x-y)\big|\mathcal{Q}_{\frac{t}{2}}(y, \mathsf{\Pi}_y^{\varepsilon,2}\tau)\big|dy + \sum_{\tau_1,\tau_2}\int_{\mathbb{R}^d} G_t(x-y)|\mathsf{h}_y^{\varepsilon,p}(\tau_1)|\big|\mathcal{Q}_{\frac{t}{2}}(y, \mathsf{\Pi}_y^{\varepsilon,p}\tau_2)\big|dy\Big\|_{L_x^p(w_{c(m_{\mathbf{B}}+2)})} \\
&\lesssim t^{\frac{|\mathfrak{s}|}{\ell}(\frac{1}{p}-\frac{1}{2})}\|\mathcal{Q}_{\frac{t}{2}}(y, \mathsf{\Pi}_y^{\varepsilon,2}\tau)\|_{L_y^2(w_c)} + \sum_{\tau_1,\tau_2} t^{\frac{|\mathfrak{s}|}{\ell}(\frac{1}{p}-\frac{1}{q_1})}\|\mathsf{h}_y^{\varepsilon,p}(\tau_1)\|_{L_y^{q_1}(w_{c(m_{\mathbf{B}}+1)})}\big\|\mathcal{Q}_{\frac{t}{2}}(y, \mathsf{\Pi}_y^{\varepsilon,p}\tau_2)\big\|_{L_y^\infty(w_c)} \\
&\lesssim t^{\frac{|\mathfrak{s}|}{\ell}(\frac{1}{p}-\frac{1}{2})} t^{\frac{|\tau|_{\varepsilon,2}}{\ell}}\|\mathsf{\Pi}^{\varepsilon,2} : \tau\|_{w_c} + \sum_{\tau_1,\tau_2} t^{\frac{|\mathfrak{s}|}{\ell}(\frac{1}{p}-\frac{1}{q_1})} t^{\frac{|\tau_2|_{\varepsilon,\infty}}{\ell}}\big(1 + \|\mathsf{M}^{\varepsilon,q_1}\|_{\mathbf{M}(\mathscr{W}_{i,\varepsilon,q_1})_{w_c}}\big)^{m_{\mathbf{B}}+1}\|\mathsf{\Pi}^{\varepsilon,\infty} : V_i\|_{w_c} \\
&= t^{\frac{|\tau|_{\varepsilon,p}}{\ell}}\|\mathsf{\Pi}^{\varepsilon,2} : \tau\|_{w_c} + \sum_{\tau_1,\tau_2} t^{\frac{|\tau|_{\varepsilon,p} - |\tau_1|_{\varepsilon,q_1}}{\ell}}\big(1 + \|\mathsf{M}^{\varepsilon,q_1}\|_{\mathbf{M}(\mathscr{W}_{i,\varepsilon,q_1})_{w_c}}\big)^{m_{\mathbf{B}}+1}\|\mathsf{\Pi}^{\varepsilon,\infty} : V_i\|_{w_c} \\
&\lesssim t^{\frac{|\tau|_{\varepsilon+\zeta,p}}{\ell}}\left(1 + \sum_{q_1}\|\mathsf{M}^{\varepsilon,q_1}\|_{\mathbf{M}(\mathscr{W}_{i,\varepsilon,q_1})_{w_c}} + \|\mathsf{M}^{\varepsilon,\infty}\|_{\mathbf{M}(\mathscr{V}_{i,\varepsilon})_{w_c}}\right)^{m_{\mathbf{B}}+2}.
\end{aligned}$$

To obtain the expected bound we use the left continuity of $\mathsf{\Pi}_x^{\varepsilon,p}\tau$ with respect to ε. Choosing small $\zeta > 0$ such that $\mathsf{\Pi}_x^{\varepsilon,p}\tau = \mathsf{\Pi}_x^{\varepsilon-\zeta,p}\tau$ (it depends on ε, p) and using the above estimates we get the bound

$$\begin{aligned}
\|\mathcal{Q}_t(x, \mathsf{\Pi}_x^{\varepsilon,p}\tau)\|_{L_x^p(w_{c(m_{\mathbf{B}}+2)})} &= \|\mathcal{Q}_t(x, \mathsf{\Pi}_x^{\varepsilon-\zeta,p}\tau)\|_{L_x^p(w_{c(m_{\mathbf{B}}+2)})} \\
&\lesssim t^{\frac{|\tau|_{\varepsilon,p}}{\ell}}\left(1 + \sum_{q_1(\varepsilon-\zeta,\zeta)}\|\mathsf{M}^{\varepsilon-\zeta,q_1}\|_{\mathbf{M}(\mathscr{W}_{i,\varepsilon-\zeta,q_1})_{w_c}} + \|\mathsf{M}^{\varepsilon-\zeta,\infty}\|_{\mathbf{M}(\mathscr{V}_{i,\varepsilon-\zeta})_{w_c}}\right)^{m_{\mathbf{B}}+2},
\end{aligned}$$

which implies the result.

5 – Proof of Theorem 2

Our proof of Theorem 2 follows the pattern of proof of the corresponding statement in Tempelmayr's recent work [29]. Since this part is independent of the parameters ε, p, c, we suppress them from the corresponding objects. For instance, we write $\mathbf{M}(\mathscr{W})$ for $\mathbf{M}(\mathscr{W}_{\varepsilon,p})_{w_c}$.

Assume now that we are given a sequence $(\mathbb{P}_j)_{j\geq 0}$ of stationary Borel probability measures on $\Omega = C^{\alpha_0,Q}(w_c)$ that all satisfy the spectral gap inequality (2.1) with the same constant C. Also, assume that $\mathbb{P}_j$ tend weakly to a probability $\mathbb{P}_\infty$. Then $\mathbb{P}_\infty$ also satisfies the spectral gap inequality with the same C. It is easily checked for any bounded cylindrical functions $F : \Omega \to \mathbb{R}$ of the form

$$F(\omega) = f\big(\varphi_1(\omega), \ldots, \varphi_m(\omega)\big),$$

where $\varphi_1, \ldots, \varphi_m \in \Omega^*$ and $f \in C^\infty(\mathbb{R}^n)$ and extended to all F by a density argument – see e.g. Lemma 2.23 in Hairer & Steele's work [20] for the density argument. For each $j \in \mathbb{N} \cup \{\infty\}$ and $n \in \mathbb{N}$, let $\mathsf{M}_j^n(\omega) = \mathsf{M}^{\xi_n(\omega), R_j^n}$ be the BPHZ model on $\mathscr{W}$ associated with the random variable $\xi_n(\omega) = \varrho_n * \omega$ and the unique BPHZ-type preparation map R_j^n defined by (2.17). We also denote by $\mathsf{M}_j = \lim_{n\to\infty}\mathsf{M}_j^n$ the $L^q(\mathbb{P}_j)$-limit for any $q < \infty$ ensured by Theorem 6. From its proof, it is obvious that the quantities

$$\sup_{n\in\mathbb{N}} \mathbb{E}_j\big[\|\mathsf{M}_j^n\|_{\mathbf{M}(\mathscr{W})}^q\big]$$

have the j-uniform upper bound and the convergence

$$\lim_{n\to\infty} \mathbb{E}_j\big[\|\mathsf{M}_j^n : \mathsf{M}_j\|_{\mathbf{M}(\mathscr{W})}^q\big] = 0$$

is j-uniform.

By Skhorohod representation theorem there is a probability space $(E,\mathcal{E},\mathbb{Q})$ and random variables

$$\widetilde{\xi}_j : E \to \Omega$$

for each $j \in \mathbb{N}\cup\{\infty\}$ such that the $\mathbb{Q}$-law of $\widetilde{\xi}_j$ is equal to $\mathbb{P}_j$ and $\widetilde{\xi}_j$ converges $\mathbb{Q}$-almost surely to $\widetilde{\xi}_\infty$ as $j\to\infty$. Once we define the random variables

$$\widetilde{\mathsf{M}}_j^n := \mathsf{M}_j^n(\widetilde{\xi}_j) : E \to \mathbf{M}(\mathscr{W}), \qquad \widetilde{\mathsf{M}}_j := \mathsf{M}_j(\widetilde{\xi}_j) : E \to \mathbf{M}(\mathscr{W}),$$

we have that

(1) for each $j \in \mathbb{N}\cup\{\infty\}$, the $\mathbb{Q}$-law of $\widetilde{\mathsf{M}}_j^n$ (resp. $\widetilde{\mathsf{M}}_j$) is equal to the $\mathbb{P}_j$-law of M_j^n (resp. M_j), and $\widetilde{\mathsf{M}}_j^n$ converges to $\widetilde{\mathsf{M}}_j$ in $L^q(E,\mathbb{Q};\mathbf{M}(\mathscr{W}))$ for any $q<\infty$ as n goes to ∞,

(2) for each $n\in\mathbb{N}$, the random variables $\widetilde{\mathsf{M}}_j^n$ converges $\mathbb{Q}$-almost surely to $\widetilde{\mathsf{M}}_\infty^n$ in $\mathbf{M}(\mathscr{W})$ as $j\to\infty$.

Point (2) follows from the facts that $\widetilde{\mathsf{M}}_j^n$ is a constinuous function of the smooth function $\widetilde{\xi}_j^n = \varrho_n * \widetilde{\xi}_j$ and the $\mathbb{Q}$-almost sure convergence of $\widetilde{\xi}_j^n$ to $\widetilde{\xi}_\infty^n$ in the space of smooth functions as $j\to\infty$. Denoting by $\mathsf{E}_{\mathbb{Q}}[\cdot]$ the expectation operator associated to $\mathbb{Q}$, one has for every $j,k\in\mathbb{N}$ and $n\in\mathbb{N}$,

$$\mathsf{E}_{\mathbb{Q}}\big[\|\widetilde{\mathsf{M}}_j : \widetilde{\mathsf{M}}_\infty\|_{\mathbf{M}(\mathscr{W})}^q\big] \lesssim \mathsf{E}_{\mathbb{Q}}\big[\|\widetilde{\mathsf{M}}_j : \widetilde{\mathsf{M}}_j^n\|_{\mathbf{M}(\mathscr{W})}^q\big] + \mathsf{E}_{\mathbb{Q}}\big[\|\widetilde{\mathsf{M}}_j^n : \widetilde{\mathsf{M}}_\infty^n\|_{\mathbf{M}(\mathscr{W})}^q\big] + \mathsf{E}_{\mathbb{Q}}\big[\|\widetilde{\mathsf{M}}_\infty^n : \widetilde{\mathsf{M}}_\infty\|_{\mathbf{M}(\mathscr{W})}^q\big].$$

By point (1), the first and third terms in the right hand side are bounded above by (j,k)-independent constant C_n which vanishes as n goes to ∞. By point (2) and the j-uniformly integrability of $\|\widetilde{\mathsf{M}}_j^n\|_{\mathbf{M}(\mathscr{W})}$, the second term vanishes as $j\to\infty$ for each n by Vitali convergence theorem. Therefore we have

$$\lim_{j\to\infty} \mathsf{E}_{\mathbb{Q}}\big[\|\widetilde{\mathsf{M}}_j : \widetilde{\mathsf{M}}_\infty\|_{\mathbf{M}(\mathscr{W})}^q\big] = 0,$$

which implies that the $\mathbb{P}_j$-law of M_j converges to the $\mathbb{P}_\infty$-law of M_∞ as $j\to\infty$.

A – Reconstruction and multilevel Schauder estimates in regularity-integrability structures

We give in this section a summary of the results on modelled distributions over a regularity-integrability structure proved in the companion work [21]. Given $1\le p\le q\le\infty$ we define the exponent $p:q\in[p,\infty]$ from the relation

$$\frac{1}{p:q} + \frac{1}{q} = \frac{1}{p}.$$

19 – Definition. *Let $\mathscr{T}=(A,T,G)$ be a regularity-integrability structure of regularity $\alpha_0\le 0$. For any $c>0$, we define $\mathbf{M}(\mathscr{T})_{w_c}$ as the set of pairs $\big(\{\Pi_x\}_{x\in\mathbb{R}^d},\{\Gamma_{xy}\}_{x,y\in\mathbb{R}^d}\big)$ with the following properties.*

(1) *(Algebraic conditions) The maps $\Pi_x : T\to C^{\alpha_0,Q}(w_c)$ are linear continuous, $\Gamma_{xy}\in G$, and $\Pi_x\Gamma_{xy}=\Pi_y$, $\Gamma_{xx}=\mathrm{Id}$, and $\Gamma_{xy}\Gamma_{yz}=\Gamma_{xz}$ for any $x,y,z\in\mathbb{R}^d$.*

(2) *(Analytic conditions) For any $\mathbf{a}\in\mathbb{R}\times[1,\infty]$, one has*

$$\|\Pi\|_{\mathbf{a};w_c} := \max_{\substack{(\alpha,p)\in A\\(\alpha,p)<\mathbf{a}}} \sup_{0<t\le 1} t^{-\alpha/\ell}\left\| \sup_{\tau\in T_{(\alpha,p)}\setminus\{0\}} \frac{|\mathcal{Q}_t(x,\Pi_x\tau)|}{\|\tau\|_{(\alpha,p)}}\right\|_{L_x^p(w_c)} < \infty$$

and

$$\|\Gamma\|_{\mathbf{a};w_c} := \max_{\substack{(\alpha,p),(\beta,q)\in A\\(\beta,q)<(\alpha,p)<\mathbf{a}}} \sup_{y\in\mathbb{R}^d\setminus\{0\}} \left\{ \frac{w_c(y)}{\|y\|_{\mathfrak{s}}^{\alpha-\beta}} \left\| \sup_{\tau\in T_{(\alpha,p)}\setminus\{0\}} \frac{\|\Gamma_{(x+y)x}\tau\|_{(\beta,q)}}{\|\tau\|_{(\alpha,p)}}\right\|_{L_x^{p:q}(w_c)}\right\} < \infty.$$

We write

$$\|\mathsf{M}\|_{\mathbf{a};w_c} := \|\Pi\|_{\mathbf{a};w_c} + \|\Gamma\|_{\mathbf{a};w_c}.$$

Furthermore, for any two models $\mathsf{M}_i=(\Pi_i,\Gamma_i)\in\mathbf{M}(\mathscr{T})_{w_c}$ with $i\in\{1,2\}$, define the pseudo-metric

$$\|\mathsf{M}_1;\mathsf{M}_2\|_{\mathbf{a};w_c} := \|\Pi_1-\Pi_2\|_{\mathbf{a};w_c} + \|\Gamma_1-\Gamma_2\|_{\mathbf{a};w_c}$$

by replacing Π and Γ above with $\Pi_1-\Pi_2$ and $\Gamma_1-\Gamma_2$ respectively.

The topological space $\mathbf{M}(\mathscr{T})_{w_c}$ is complete with respect to the pseudo-metrics $\|\mathsf{M}_1;\mathsf{M}_2\|_{\mathbf{a};w_c}$.

20 – Definition. *Let $\mathsf{M}=(\Pi,\Gamma)\in\mathbf{M}(\mathscr{T})_{w_c}$. For every $\mathbf{c}=(\gamma,r)\in\mathbb{R}\times[1,\infty]$ and $b>0$, we define $D^{\mathbf{c}}(\Gamma)_{w_b}$ as the space of all functions $f:\mathbb{R}^d\to T_{<\mathbf{c}} := \bigoplus_{\mathbf{a}\in A,\,\mathbf{a}<\mathbf{c}} T_{\mathbf{a}}$ such that*

$$(\!|f|\!)_{\mathbf{c};w_b} := \max_{(\alpha,p)<\mathbf{c}} \big\|\|f(x)\|_{(\alpha,p)}\big\|_{L_x^{r:p}(w_b)} < \infty,$$

$$\|f\|^{\Gamma}_{\mathbf{c};w_b} := \max_{(\alpha,p)<\mathbf{c}} \sup_{h\in\mathbb{R}^d\setminus\{0\}} w_b(y) \frac{\big\| \|\Delta^{\Gamma}_{x;h} f\|_{(\alpha,p)} \big\|_{L^{r;p}_x(w_b)}}{\|h\|_{\mathfrak{s}}^{\gamma-\alpha}} < \infty,$$

where

$$\Delta^{\Gamma}_{x;h} f := f(x+h) - \Gamma_{(x+h)x} f(x).$$

We call each element of $D^{\mathbf{c}}(\Gamma)_{w_b}$ a modelled distribution. In addition, we write

$$\|\!| f |\!\|^{\Gamma}_{\mathbf{c};w_b} := (\!| f |\!)_{\mathbf{c};w_b} + \|f\|^{\Gamma}_{\mathbf{c};w_b}.$$

Furthermore, for any models $\mathsf{M}_i = (\Pi_i, \Gamma_i) \in \mathbf{M}(\mathscr{T})_{w_c}$ and functions $f_i \in D^{\mathbf{c}}(\Gamma_i)_{w_b}$ with $i \in \{1,2\}$, we define

$$\|\!| f_1; f_2 |\!\|^{\Gamma_1;\Gamma_2}_{\mathbf{c};w_b} := (\!| f_1 - f_2 |\!)_{\mathbf{c};w_b} + \|f_1; f_2\|^{\Gamma_1;\Gamma_2}_{\mathbf{c};w_b}$$

by

$$(\!| f_1 - f_2 |\!)_{\mathbf{c};w_b} := \max_{(\alpha,p)<\mathbf{c}} \big\| \|f_1(x) - f_2(x)\|_{(\alpha,p)} \big\|_{L^{r;p}_x(w_b)},$$

$$\|f_1; f_2\|^{\Gamma_1;\Gamma_2}_{\mathbf{c};w_b} := \max_{(\alpha,p)<\mathbf{c}} \sup_{h\in\mathbb{R}^d\setminus\{0\}} w_b(y) \frac{\big\| \|\Delta^{\Gamma_1}_{x;h} f_1 - \Delta^{\Gamma_2}_{x;h} f_2\|_{(\alpha,p)} \big\|_{L^{r;p}_x(w_b)}}{\|h\|_{\mathfrak{s}}^{\gamma-\alpha}}.$$

For any subspace V of T which is stable under G, we denote by $D^{\mathbf{c}}(V;\Gamma)_{w_b}$ the space of all modelled distributions $f \in D^{\mathbf{c}}(\Gamma)_{w_b}$ which takes values in V

Recall from (1.4) the definition of the weighted Besov spaces $B^{\alpha,Q}_{pq}(w)$ associated to Q.

21 – Definition. *Let $\mathsf{M} = (\Pi, \Gamma) \in \mathbf{M}(\mathscr{T})_{w_c}$ and $\mathbf{c} = (\gamma, r) \in \mathbb{R} \times [1,\infty]$. Then for any $f \in D^{\mathbf{c}}(\Gamma)_{w_b}$, any distribution $\Lambda \in B^{\alpha_0,Q}_{r,\infty}(w_{b+c})$ satisfying*

$$[\![\Lambda]\!]^{\Pi,f}_{\mathbf{c};w_{b+c}} := \sup_{0<t\le 1} t^{-\gamma/\ell} \big\| \mathcal{Q}_t(x, \Lambda^{\Pi,f}_x) \big\|_{L^r_x(w_{b+c})} < \infty, \qquad \big(\Lambda^{\Pi,f}_x := \Lambda - \Pi_x f(x)\big)$$

is called a ***reconstruction*** *of f for M. Furthermore, for any models $\mathsf{M}_i = (\Pi_i, \Gamma_i) \in \mathbf{M}(\mathscr{T})_{w_c}$ and functions $f_i \in D^{\mathbf{c}}(\Gamma_i)_{w_b}$ with $i \in \{1,2\}$, if there is a reconstruction Λ_i for each i, we define*

$$[\![\Lambda_1; \Lambda_2]\!]_{\mathbf{c};w_{b+c}} := \sup_{0<t\le 1} t^{-\gamma/\ell} \big\| \mathcal{Q}_t\big(x, (\Lambda_1)^{\Pi_1,f_1}_x - (\Lambda_2)^{\Pi_2,f_2}_x\big) \big\|_{L^r_x(w_{b+c})}.$$

The next statement is the regularity-integrability counterpart of the reconstruction theorem.

22 – Theorem. *Pick $\mathsf{M} = (\Pi, \Gamma) \in \mathbf{M}(\mathscr{T})_{w_c}$ and $\mathbf{c} = (\gamma, r) \in \mathbb{R} \times [1,\infty]$ with $\gamma > 0$. There exists a unique reconstruction $\mathcal{R}^{\mathsf{M}} f$ of $f \in D^{\mathbf{c}}(\Gamma)_{w_b}$ for M. Moreover, it holds that*

$$\|\mathcal{R}^{\mathsf{M}} f\|_{B^{\alpha_0,Q}_{r,\infty}(w_{b+c})} \lesssim \|\Pi\|_{\mathbf{c};w_c} \|\!| f |\!\|^{\Gamma}_{\mathbf{c};w_b},$$

$$[\![\mathcal{R}^{\mathsf{M}} f]\!]^{\Pi,f}_{\mathbf{c};w_{b+c}} \lesssim \|\Pi\|_{\mathbf{c};w_c} \|f\|^{\Gamma}_{\mathbf{c};w_b}.$$

Moreover there is an affine function $C_\lambda > 0$ of $\lambda > 0$ such that

$$\big\| \mathcal{R}^{\mathsf{M}_1} f_1 - \mathcal{R}^{\mathsf{M}_2} f_2 \big\|_{B^{\alpha_0,Q}_{r,\infty}(w_{b+c})} \le C_\lambda \Big(\|\Pi_1 - \Pi_2\|_{\mathbf{c};w_c} + \|\!| f_1; f_2 |\!\|^{\Gamma_1;\Gamma_2}_{\mathbf{c};w_b} \Big),$$

$$\big[\!\big[\mathcal{R}^{\mathsf{M}_1} f_1; \mathcal{R}^{\mathsf{M}_2} f_2 \big]\!\big]_{\mathbf{c};w_{b+c}} \le C_\lambda \Big(\|\Pi_1 - \Pi_2\|_{\mathbf{c};w_c} + \|f_1; f_2\|^{\Gamma_1;\Gamma_2}_{\mathbf{c};w_b} \Big),$$

for any models $\mathsf{M}_i = (\Pi_i, \Gamma_i) \in \mathbf{M}(\mathscr{T})_{w_c}$ and functions $f_i \in D^{\mathbf{c}}(\Gamma_i)_{w_b}$ with $i \in \{1,2\}$ such that $\|\!| \mathsf{M}_i |\!\|_{\mathbf{c};w_c} \le \lambda$ and $\|\!| f_i |\!\|_{\mathbf{c};w_b} \le \lambda$.

We now lift the operator $\mathcal{K}$ into an operator that maps modelled distributions on modelled distributions.

23 – Definition. *Let $\mathscr{T} = (A, T, G)$ be a regularity-integrability structure. In addition, let $\overline{\mathscr{T}} = (\overline{A}, \overline{T}, \overline{G})$ be a regularity-integrability structure satisfying the following properties.*

1. *$\mathbb{N}[\mathfrak{s}] \times \{\infty\} \subset \overline{A}$, where $\mathbb{N}[\mathfrak{s}] := \big\{ |k|_{\mathfrak{s}} \,;\, k \in \mathbb{N}^d \big\}$.*
2. *For each $\alpha \in \mathbb{N}[\mathfrak{s}]$, the vector space $\overline{T}_{(\alpha,\infty)}$ contains all X^k with $|k|_{\mathfrak{s}} = \alpha$.*
3. *The linear space $\mathrm{span}\{X^k\}_{k\in\mathbb{N}^d}$ is closed under the action of the group $\overline{G}$.*

For $\mathsf{M} \in \mathbf{M}(\mathscr{T})_{w_c}$ we define the linear map $\mathcal{J}^{\mathsf{M}}(x) : T \to \mathrm{span}\{X^k\} \subset \overline{T}$ by setting

$$\mathcal{J}^{\mathsf{M}}(x)\tau := \sum_{|k|_{\mathfrak{s}} < \alpha + \beta_0} \frac{X^k}{k!} \, \partial^k \mathcal{K}(x, \Pi_x \tau)$$

for any $(\alpha,p)\in A$ *and* $\tau\in T_{(\alpha,p)}$. *A continuous linear map* $\mathcal{I}:T\to\overline{T}$ *is called an* ***abstract integration map of order*** $\beta_0\in(0,\ell-\ell_1)$ *if*

$$\mathcal{I}:T_{(\alpha,p)}\to\overline{T}_{(\alpha+\beta_0,p)}$$

for any $(\alpha,p)\in A$. *The models* $\mathsf{M}=(\Pi,\Gamma)\in\mathbf{M}(\mathscr{T})_{w_c}$ *and* $\overline{\mathsf{M}}=(\overline{\Pi},\overline{\Gamma})\in\mathbf{M}(\overline{\mathscr{T}})_{w_c}$ *are said to be* ***compatible for*** $\mathcal{I}$ *if they satisfy the following properties.*

(i) For any $k\in\mathbb{N}^d$ *one has*

$$(\overline{\Pi}_x X^k)(\cdot)=(\cdot-x)^k,\qquad \overline{\Gamma}_{yx}X^k=\sum_{l\le k}\binom{k}{l}(y-x)^l X^{k-l}.$$

(ii) For any $x,y\in\mathbb{R}^d$ *one has*

$$\overline{\Gamma}_{yx}\circ\big(\mathcal{I}+\mathcal{J}^{\mathsf{M}}(x)\big)=\big(\mathcal{I}+\mathcal{J}^{\mathsf{M}}(y)\big)\circ\Gamma_{yx}.$$

For any model $\mathsf{M}=(\Pi,\Gamma)\in\mathbf{M}(\mathscr{T})_{w_c}$, $f\in D^{\mathbf{c}}(\Gamma)_{w_b}$ with $\mathbf{c}=(\gamma,r)\in\mathbb{R}\times[1,\infty]$, and its reconstruction Λ, we set

$$\mathcal{N}^{\mathsf{M}}(x;f,\Lambda):=\sum_{|k|_s<\gamma+\beta_0}\frac{X^k}{k!}\,\partial^k\mathcal{K}\big(x,\Lambda_x^{\Pi,f}\big).$$

Finally, we define

$$\mathcal{K}^{\mathsf{M}}f(x):=\mathcal{I}(f(x))+\mathcal{J}^{\mathsf{M}}(x)(f(x))+\mathcal{N}^{\mathsf{M}}(x;f,\Lambda). \tag{A.1}$$

24 – Theorem. *Let* $\beta_0\in(0,\ell-\ell_1)$ *and* $\mathbf{c}=(\gamma,r)\in\mathbb{R}\times[1,\infty]$. *Then for any* $\mathsf{M}=(\Pi,\Gamma)\in\mathbf{M}(\mathscr{T})_{w_c}$, $f\in D^{\mathbf{c}}(\Gamma)_{w_b}$, *and any reconstruction* Λ *of* f *for* M, *the function* $\mathcal{K}^{\mathsf{M}}f$ *belongs to* $D^{(\gamma+\beta_0,r)}(\overline{\Gamma})_{w_{2c+b}}$, *and*

$$(\!|\mathcal{K}^{\mathsf{M}}f|\!)_{(\gamma+\beta_0,r);w_{2c+b}}\lesssim(1+\|\Pi\|_{\mathbf{c};w_c})(1+\|\Gamma\|_{\mathbf{c};w_c})(\!|f|\!)_{\mathbf{c};w_b}+[\![\Lambda]\!]^{\Pi,f}_{\mathbf{c};w_{c+b}},$$
$$\|\mathcal{K}^{\mathsf{M}}f\|^{\overline{\Gamma}}_{(\gamma+\beta_0,r);w_{2c+b}}\lesssim(1+\|\Pi\|_{\mathbf{c};w_c})(1+\|\Gamma\|_{\mathbf{c};w_c})\|f\|^{\Gamma}_{\mathbf{c};w_b}+[\![\Lambda]\!]^{\Pi,f}_{\mathbf{c};w_{c+b}}.$$

Moreover there is a quadratic function $C_\lambda>0$ *of* $\lambda>0$ *such that*

$$|\!|\!|\mathcal{K}^{\mathsf{M}_1}f_1;\mathcal{K}^{\mathsf{M}_2}f_2|\!|\!|_{(\gamma+\beta_0,r);w_{2c+b}}\le C_\lambda\Big(|\!|\!|\mathsf{M}_1;\mathsf{M}_2|\!|\!|_{\mathbf{c};w_c}+|\!|\!|f_1;f_2|\!|\!|_{\mathbf{c};w_b}+\|\Lambda_1;\Lambda_2\|_{\mathbf{c};w_{c+b}}\Big),$$

for any models $\mathsf{M}_i=(\Pi_i,\Gamma_i)\in\mathbf{M}(\mathscr{T})_{w_c}$ *and* $\overline{\mathsf{M}}_i=(\overline{\Pi}_i,\overline{\Gamma}_i)\in\mathbf{M}(\overline{\mathscr{T}})_{w_c}$ *such that* M_i *and* $\overline{\mathsf{M}}_i$ *are compatible, and functions* $f_i\in D^{\mathbf{a}}(\Gamma_i)_{w_b}$ *with* $i\in\{1,2\}$ *such that* $|\!|\!|\mathsf{M}_i|\!|\!|_{\mathbf{c};w_c}\le\lambda$ *and* $|\!|\!|f_i|\!|\!|_{\mathbf{c};w_b}\le\lambda$.

References

[1] I. Bailleul and Y. Bruned, *Renormalised singular stochastic PDEs.* arXiv:2101.11949, (2021).
[2] I. Bailleul and M. Hoshino, *Paracontrolled calculus and regularity structures (1).* J. Math. Soc. Japan, **73**:553–595, (2021).
[3] I. Bailleul and M. Hoshino, *A tourist's guide to regularity structures and singular stochastic PDEs.* arXiv:2006.03524, (2020).
[4] I. Bailleul, M. Hoshino and S. Kusuoka, *Regularity structures for quasilinear singular SPDEs.* arXiv:2209.05025, (2022).
[5] N. Berglund, *An Introduction to Singular Stochastic PDEs. Allen-Cahn Equations, Metastability, and Regularity Structures.* EMS Series of Lectures in Mathematics, (2022).
[6] Y. Bruned, *Recursive formulae in regularity strutures.* Stoch. PDE: Anal. Comp., **6**:525–564, (2018).
[7] Y. Bruned, A. Chandra, I. Chevyrev and M. Hairer, *Renormalising SPDEs in regularity structures.* J. Europ. Math. Soc., **23**(3):869–947, (2021).
[8] Y. Bruned, M. Hairer and L. Zambotti, *Algebraic renormalisation of regularity structures.* Invent. Math., **215**(3):1039–1156, (2019).
[9] Y. bruned and D. Manchon, *Algebraic deformation for (S)PDEs.* J. Math. Soc. Japan, **75**(2):485–526, (2023).
[10] Y. Bruned and U. Nadeem, *Diagram-free approach for convergence of trees based model in Regularity Structures.* arXiv:2211.11428, (2022).
[11] W.E. Caswell and A.D. Kennedy, *Simple approach to renormalization theory.* Physical Review D, **25**(2):392–408, (1982).
[12] A. Chandra and M. Hairer, *An analytic BPHZ theorem for regularity structures.* arXiv:1612.08138, (2016).
[13] A. Chandra and H. Weber, *Stochastic PDEs, regularity structures and interacting particle systems.* Ann. Fac. Sci. Toulouse, **26**(4):847–909, (2017).
[14] I. Corwin and H. Shen, *Some recent progress in singular stochastic PDEs.* Bull. Amer. Math. Soc., **57**(3):409–454, (2020).
[15] J. Feldman, J. Magnen, V. Rivasseau and R. Sénéor, *Bounds on renormalized Feynman graphs.* Comm. Math. Phys., **100**:23–55, (1985).
[16] P. Friz and M. Hairer, *A Course on Rough Paths, With an introduction to regularity structures.* Universitext, Springer, (2020).

[17] M. Hairer, *A theory of regularity structures.* Invent. Math., **198**(2):269–504, (2014).
[18] M. Hairer, *An analyst's take on the BPHZ theorem.* Proceedings of the Abel Symposium, (2016).
[19] M. Hairer, *Renormalisation of parabolic stochastic PDEs.* Japanese J. Math., **13**:187–233, (2018).
[20] M. Hairer and R. Steele, *The BPHZ Theorem for Regularity Structures via the Spectral Gap Inequality.* arXiv:2301.10081, (2023).
[21] M. Hoshino, *A semigroup approach to reconstruction theorems and multilevel Schauder estimates.* arXiv:2310.07396, (2023).
[22] P. Linares, F. Otto and M. Tempelmayr, *The structure group for quasi-linear equations via universal enveloping algebras.* Comm. Amer. Math. Soc., **3**:1–64, (2022).
[23] P. Linares, F. Otto, M. Tempelmayr and P. Tsatsoulis, *A diagram-free approach to the stochastic estimates in regularity structures.* arXiv:2112.10739, (2021).
[24] I. Nourdin and G. Peccati, *Normal approximations with Malliavin calculus. From Stein's method to universality.* Cambridge University Press, Cambridge, (2012).
[25] F. Otto, *Lectures at the 51th Probability Summer School, St Flour*, https://www.mis.mpg.de/applan/lectures/summer-term-2023.html, (2023).
[26] F. Otto, J. Sauer, S. Smith and H. Weber, *A priori bounds for quasi-linear SPDEs in the full sub-critical regime.* arXiv:2103.11039, (2021).
[27] F. Otto, K. Seong and M. Tempelmayr, *Lecture notes on tree-free regularity structures.* arXiv:2301.00778, (2023).
[28] F. Otto and H. Weber, *Quasilinear SPDEs via Rough Paths*, Arch. Rat. Mech. Anal., **232**(2):873–950, (2019).
[29] M. Tempelmayr, *Characterizing models in regularity structures: A quasilinear case.* arXiv:2303.18192, (2023).