Φ^4_3 measures on compact Riemannian 3-manifolds

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Abstract. We construct the Φ^4_3 measure on an arbitrary 3-dimensional compact Riemannian manifold without boundary as an invariant probability measure of a singular stochastic partial differential equation. Proving the nontriviality and the covariance under Riemannian isometries of that measure gives for the first time a non-perturbative, non-topological interacting Euclidean quantum field theory on curved spaces in dimension 3. This answers a longstanding open problem of constructive quantum field theory on curved 3 dimensional backgrounds. To control analytically several Feynman diagrams appearing in the construction of a number of random fields, we introduce a novel approach of renormalization using microlocal and harmonic analysis. This allows to obtain a renormalized equation which involves some universal constants independent of the manifold. We also define a new vectorial Cole-Hopf transform which allows to deal with the vectorial Φ^4_3 model where Φ is now a bundle valued random field. In a companion paper, we develop in a self-contained way all the tools from paradifferential and microlocal analysis that we use to build in our manifold setting a number of analytic and probabilistic objects.

Contents

1 – Introduction

In the setting of the discrete d-dimensional torus $\Lambda_d = \big(\frac{\mathbb{Z}}{N\mathbb{Z}}\big)^d$, for $d \geq 2$, a field is represented by a real-valued function σ on Λ_d. Write $i \sim j$ when two points i and j are neighbours in Λ_d. One can assign an energy

$$S(\sigma) = \frac{1}{2}\sum_{i\sim j}|\sigma_i - \sigma_j|^2 + \frac{1}{2}\sum_i |\sigma_i|^2 \tag{1.1}$$

to any field σ and define a Gibbs probability measure ν_{Λ_d} proportional to

$$e^{-S(\sigma)}\prod_{i\in\Lambda_d} d\sigma_i.$$

A random variable with values in the space $\mathcal{E}_d = \mathbb{R}^{\Lambda_d}$ of fields, with law ν_{Λ_d}, is called a discrete Gaussian free field. The continuum analogue of this random variable is the Gaussian free field on the torus $\mathbb{T}^d$, characterized by the fact that it is a random centered Gaussian field ζ with covariance $(1-\Delta)^{-1}$ on the torus $\mathbb{T}^d$. This means that for any smooth real-valued test function $f \in C^\infty(\mathbb{T}^d)$ the random variable $\zeta(f)$ is Gaussian with zero mean and covariance $\langle f, (1-\Delta)^{-1} f\rangle_{L^2(\mathbb{T}^d)}$. One can construct ζ as a random distribution that is almost surely of Besov-Hölder regularity $-\frac{d-2}{2} - \varepsilon$, for all $\varepsilon > 0$. Note the dependence of the regularity on the dimension. Going back to the discrete setting, we denote by $\Delta_{\mathcal{E}_d}$ the canonical Laplace operator on the field space $\mathcal{E}_d \simeq \mathbb{R}^{\Lambda_d}$ and by $\mathbf{1}$ the constant function on $\mathcal{E}_d$ equal to 1. A dynamical picture of the Gibbs measure ν_{Λ_d} can be obtained from the genuine identity

$$\nabla_{\mathcal{E}_d}\mathbf{1}_{\mathcal{E}_d} = 0$$

by rewriting it under the form

$$\nabla_{\mathcal{E}_d}\big(e^{-S}\nabla_{\mathcal{E}_d}e^{S}\big)(e^{-S}) = 0.$$

The density e^{-S} appears here as an element of the kernel of the dual of the conjugated operator

$$\big\{\nabla_{\mathcal{E}_d}\big(e^{-S}\nabla_{\mathcal{E}_d}e^{S}\big)\big\}^* = \Delta_{\mathcal{E}_d} - (\nabla_{\mathcal{E}_d}S)\cdot\nabla_{\mathcal{E}_d}.$$

This allows a construction of ν_{Λ_d} as the invariant measure of the Markov process on $\mathcal{E}_d$ with generator $\Delta_{\mathcal{E}_d} - (\nabla_{\mathcal{E}_d} S) \cdot \nabla_{\mathcal{E}_d}$ – provided this Markov process has indeed a unique invariant probability measure. The diffusion associated with the operator $\Delta_{\mathcal{E}_d} - (\nabla_{\mathcal{E}_d} S) \cdot \nabla_{\mathcal{E}_d}$ is the solution of the stochastic differential equation in $\mathcal{E}_d$

$$dz_t = \sqrt{2} dw_t - \nabla_{\mathcal{E}_d} S(z_t) dt,$$

for a Brownian motion w in $\mathcal{E}_d$.

The energy S in (1.1) contains a kinetic term $\sum_{i\sim j}$ and a potential term $\sum_i$. One can add to the potential term an additional bit of the form

$$Q(\sigma) = \sum_i Q(\sigma_i).$$

for a real-valued function Q, typically a real polynomial bounded from below for the so–called Ginzburg–Landau models. The corresponding Gibbs probability measure μ_{Λ_d} can then be seen as a perturbation of the discrete Gaussian free field probability measure ν_{Λ_d}

$$\mu_{\Lambda_d}(d\sigma) \sim e^{Q(\sigma)} \nu_{\Lambda_d}(d\sigma).$$

The dynamics on $\mathcal{E}_d$ associated with this measure is given as above by the stochastic differential equation

$$dz_t = \sqrt{2} dw_t - \nabla_{\mathcal{E}_d} (S + Q)(z_t) dt. \tag{1.2}$$

There is a problem for taking the formal continuum limit of these measures as the Gaussian free field measure is supported by distributions of negative regularity, so an expression like $\int Q(\sigma)$ does not make sense for a nonlinear function of a distribution σ. It is the aim of the Euclidean quantum field theory of scalar fields to make sense of and construct such measures, for some particular examples of potentials Q. The Φ^4_3 measure corresponds to $Q(a) = a^4/4$ in a 3-dimensional setting.

Quantum field theory was developed as a theory describing interactions of elementary particles. It is arguably one of the most successful physical theory of the 20th century and has led to remarkable physical predictions with unprecedented numerical accuracy. However, in spite of its success in theoretical physics, a complete mathematical formulation and understanding of quantum field theory is still work in progress. A major difficulty in the subject comes from the divergences inherent to the formulation of the theory. In quantum field theory the perturbative calculation of any physical process involves a summation over an infinite number of virtual multi-particle states which is generically divergent, hence produces infinities. The divergences of perturbation theory in quantum field theory are directly linked to its short distance structure which is highly non-trivial because its description involves the infinity of multi-particle states. These divergences must be carefully subtracted in some organized way compatible with physical requirements such as locality, causality, unitarity. The methods developped to deal with these infinities were called *renormalization*. Constructive quantum field theory provides one of the mathematically rigorous approaches to quantum field theory. It was developped in the 70's with seminal contributions of Albeverio, Brydges, Feldman, Fröhlich, Gallavotti, Gawedzki, Glimm, Guerra, Jaffe, Kupiainen, Nelson, Rivasseau, Seiler, Sénéor, Spencer, Simon, Symanzik and Wightman to name but a few – see e.g. Glimm & Jaffe's book [40] and the references inside for an account of the early achievements in this domain. One of the first successes of constructive quantum field theory was the construction of the so called $P(\phi)_2$–model on $\mathbb{R}^2$ and of the ϕ^4_3 theory on $\mathbb{R}^3$ by Glimm-Jaffe [38, 39] in the 70's. The recent breakthroughs by Hairer [51] and Gubinelli–Perkowski–Imkeller [49] allowed several authors to recover the results of Glimm-Jaffe following the stochastic quantization program of Parisi & Wu [73], using only PDE and probabilistic techniques, without the intricate combinatorial methods from the constructive school. However, we would like to emphasize that the powerful cluster expansion techniques developed by the constructive school allowed, for both $P(\phi)_2, \phi^4_3$ theories, to show the Borel summability of the partition function in the couplings, to control the mass gap (exponential decay of the correlations), to study phase transitions (when there is no longer exponential decay of correlations), to investigate fine properties of the spectrum and scattering properties of these theories and finally, one of the deepest results of the constructive school was the stability proofs of Yang–Mills in 3 and 4 dimensions due to Balaban [12, 13]. Many of these results are currently out of reach of the stochastic methods, we refer to the book of Rivasseau [77] for more information on the topic. The dynamical construction of the Φ^4_3 measure was first done in finite volume on the flat 3-dimensional torus by Mourrat & Weber [68], Hairer & Mattingly [52] and Hairer & Schönbauer [53] and Albeverio & Kusuoka [3], then in the infinite volume 3-dimensional Euclidean space by different authors – Albeverio & Kusuoka [4], Moinat & Weber [66], Gubinelli & Hofmanová [45, 46], Barashkov & Gubinelli [14, 15], using different methods. A crucial integrability property of the ϕ_3^4 measure was proved in Hairer & Steele's work [54]. Gubinelli's lecture notes [43, 44] provide a remarkable source of inspiration and information on the stochastic quantization approach to the construction of the Φ_3^4 measure.

Yet most results in constructive quantum field theory are proved in the geometric settings of either $\mathbb{R}^3$ or the flat torus $\mathbb{T}^3$. On the other hand, quantum field theory on curved spacetimes has been studied since the 70s. Recent breakthroughs by Brunetti & Fredenhagen [20], Hollands & Wald [55, 56] ad Rejzner [76] on Lorentzian manifolds, see [27] for a detailed mathematical exposition of part of this approach, and Kopper & Müller [61] in a Riemannian setting, lead to complete proofs of *perturbative renormalization* on (pseudo)-Riemannian manifolds of dimension less than or equal to 4. We also mention the approach of Costello [24] that is designed to work on Riemannian manifolds, possibly with a boundary after Albert's work [2]. However all these results are perturbative and construct quantum field theory objects as formal power series. They do not provide any probability measures, Hilbert spaces or operators in a straightforward way. Constructive quantum fields on manifolds have been earlier addressed only on *compact surfaces* in [74] by Pickrell and [30] by Dimock for the $P(\varphi)_2$ theories, in [62] by Lévy for the $2d$ Yang–Mills theory and in [48] by Guillarmou–Kupiainen–Rhodes–Vargas for the Liouville field theory on Riemann surfaces. In Lorentz signature, we would also like to mention the work [16] of Barata-Jaekel-Mund who managed to define the $P(\varphi)_2$ theory on 2-dimensional de Sitter space, extending previous work [36] of Figari-Høegh-Krohn-Nappi. From the PDE side, the constructions of Gibbs measures on Riemannian surfaces that we are aware of, come from [22] by Burq-Thomann-Tzvetkov for dynamical $P(\Phi)_2$ and from [72] by Oh-Robert-Tzvetkov-Wang for the dynamical Liouville model. It is therefore a longstanding open problem in both constructive quantum field theory and field theory on curved spaces to construct the ϕ_3^4 measure on an arbitrary closed, compact 3-dimensional Riemannian manifold. As emphasized by Witten in the recent work [80]

"*If a theory exists perturbatively in curved spacetime, and non-perturbatively in flat spacetime, one would expect that it works non-perturbatively in curved spacetime. Unfortunately, not much is available in terms of rigorous theorems, except for special models like two-dimensional conformal field theories. That reflects the general mathematical difficulty of understanding quantum field theory rigorously. One would think that rigorous results for a superrenormalizable theory in curved spacetime might be relatively accessible, but such results are not available.*"

Let us mention that both the $2d$ Yang–Mills and Liouville theories are *integrable* as mentioned above by Witten. Our work seems to give the first construction of a nonintegrable, interacting quantum field theory on 3–manifolds.

Let M stand for a closed 3-dimensional Riemannian manifold. This work is dedicated to constructing the Φ_3^4 measure over M, formally the ill-defined functional integral measure

$$\frac{e^{-\int_M |\nabla u|^2 - \int_M \frac{u^4}{4}}}{\int e^{-\int_M |\nabla u|^2 - \int_M \frac{u^4}{4}}[\mathcal{D}u]},$$

as an invariant probability measure of the dynamics

$$\partial_t u = \xi + (\Delta - 1)u - u^3 \tag{1.3}$$

where ξ stands for a spacetime white noise. This noise plays in a continuum setting the role of the Brownian motion w in (1.2) in a discrete setting while the terms $(\Delta - 1)u$ and u^3 come from the gradient terms of the energy S and the quartic potential Q respectively. The construction of the Φ_3^4 measure as the hopefully unique invariant probability measure of this dynamics was first put forward by Parisi & Wu in a famous work of the early 80s; it comes under the name of *stochastic quantization*. Note we define the Φ_3^4 measure as an invariant measure from equation (1.3). This is a priori not equivalent to obtaining the measure as a scaling limit of lattice models which is a highly non-trivial issue on manifolds since there is no canonical way of discretizing quantum field theories on manifolds. We do not try to relate here our construction of the Φ_3^4 measure with any such limiting procedure.

Equation (1.3) involves the fundamental problem of considering a nonlinear function of a distribution. Spacetime white noise on a 3-dimensional Riemannian manifold has indeed almost surely a parabolic [1] Besov-Hölder regularity $-5/2-\varepsilon$, for all $\varepsilon > 0$, so one does not expect from a possible solution u to equation (1.3) that it has parabolic regularity better than $-1/2-\varepsilon$, as a consequence of Schauder estimate. The term u^3 in (1.3) is thus ill-defined. This kind of problem in a stochastic partial differential equation (PDE) is characteristic of the class of singular stochastic PDEs, a field that was opened around 2014 by the groundbreaking works of M. Hairer on regularity structures [51] and Gubinelli, Imkeller & Perkowski [49] on paracontrolled calculus. The tools of regularity structures and paracontrolled calculus were used to run the stochastic quantization approach to the construction of the Φ^4_3 measure over a 3-dimensional torus and Euclidean space in a series of works. Local well-posedness of equation (1.3) was proved first by Hairer in [51] – see also Catellier & Chouk's work [23] for a proof of that result with the tools of paracontrolled calculus. Mourrat & Weber proved in [68] an a priori estimate that gives the long time existence (and well-posedness) of the solution to (1.3) and the existence of an invariant probability measure. The uniqueness of this invariant probability measure comes from the works of Hairer & Mattingly [52], on the strong Markov property of transition semigroups associated to singular stochastic PDEs, and Hairer & Schönbauer [53] on the support of the laws of solutions to singular stochastic PDEs. See Hairer & Steele's work [54] for more references.

None of the previous works are readily available in a manifold setting, and so far the only works on singular stochastic PDEs in a manifold setting are the works [7, 8, 9] of Bailleul & Bernicot, and the works [29], [70] and [11] of Dahlqvist, Diehl & Driver, Mouzard and Bailleul, Dang & Mouzard on the Anderson operator on a 2-dimensional Riemannian manifold. We refer to [71] for a lucid exposition of some of the above results. We would also like to mention the forthcoming work of Hairer–Singh [42] which develops a generalisation of the original Theory of Regularity Structures which is able to treat SPDEs on manifolds with values in vector bundles in full generality. This could possibly lead to similar results as ours. The aim of the present work together with our companion work [10] is to develop in a self-contained way all the tools needed to run the analysis in a 3 dimensional closed Riemannian manifold. On the purely analytical side

- We follow Jagannath & Perkowski' simple approach [57] of equation (1.3) to prove that this equation is locally well-posed. Their formulation of the problem avoids the use of regularity structures or paracontrolled calculus.
- We study the vectorial Φ^4_3 model (sometimes called $O(N)$ model in the physics litterature) where Φ is now E-valued where (E,h) is some Hermitian vector bundle over M. In this case, the SPDE reads
$$\partial_t\Phi + (1-\Delta)\,\Phi = -\Phi\,\langle\Phi,\Phi\rangle_h + \xi_E$$
where Δ is a generalized laplacian acting on E-valued sections, ξ_E is some E-valued white noise and the Φ^4_3 measure is still invariant under the Markovian dynamics. The crucial ingredient for this part is a novel vectorial Cole-Hopf transform that we introduce in definition 23.
- We give a simple and short proof of an L^p "*coming down from infinity*" property satisfied by the solution to equation (1.3) using energy methods. The longtime existence of a unique solution to equation (1.3) follows as a consequence.
- As usual in the study of singular stochastic PDEs we need to feed the analytic machinery with a number of random distributions whose formal definitions involve some ill-defined products, and whose actual definitions involve some probabilistic constructions based on regularization and renormalization. Our approach to the renormalization problem is a far reaching generalization of the Epstein-Glaser point of view where we benefit from the many improvements contained in [75, 20, 55, 56]. We reduce the problem of renormalization to an extension problem for distributions on a configuration space defined outside all the diagonals, for which we develop a general machinery. We note that there is also a new approach to SPDE's relying on the Epstein-Glaser renormalization in the works [31] by Dappiaggi-Drago-Rinaldi-Zambotti and [19] by Bonicelli-Dappiaggi-Rinaldi. However it seems that these authors work only at a perturbative level whereas our results are nonperturbative.

[1]Meaning the time variable has weight 2 whereas space variables have weight 1

One remarkable feature of our approach is that we are able to renormalize equation (1.3) using universal counterterms – they do not depend on the metric on M. We emphasize that fact in the following statement where $\varepsilon \in (0, 1/8)$ is a positive constant and $\xi_r := e^{r(\Delta-1)}\xi$ stands for a *space regularization* of ξ by heat operator– so ξ_r is still white in time. Set

$$a_r := \frac{r^{-1/2}}{8\sqrt{2}\,\pi^{3/2}}, \qquad b_r := \frac{|\log r|}{128\,\pi^2}. \tag{1.4}$$

Theorem 1 – *Pick $\phi \in C^{-1/2-\varepsilon}(M)$. The equation*

$$(\partial_t - \Delta + 1)u_r = \xi_r - u_r^3 + 3(a_r - b_r)u_r \tag{1.5}$$

with initial condition ϕ, has a unique solution over $[0,\infty) \times M$ in some appropriate function space. For any $0 < T < \infty$ this random variable converges in probability in $C\big([0,T], C^{-1/2-\varepsilon}(M)\big)$ as $r > 0$ goes to 0 to a limit u.

The function u is what we define as the solution to equation (1.3) and it turns out to be a Markov process. The a priori estimate encoded in the coming down from infinity property provides a compactness statement from which the existence of an invariant probability measure for the Markovian dynamics follows.

Theorem 2 – *The dynamics of u is Markovian and its associated semigroup on $C^{-1/2-\varepsilon}(M)$ has an invariant non-Gaussian probability measure.*

A Φ^4_3 measure over M is that invariant measure. Note that the constants a_r, b_r in (1.5) are universal in the sense that they do not depend on the Riemannian metric on M; they depend however on the regularization scheme we use, here the heat regularization in space which is *fully covariant* w.r.t the Riemannian structure.

The study of equation (1.3) on an arbitrary fixed time interval $[0,T]$ is the object of Section 2. We follow Jagannath & Perkowski's robust formulation of equation (1.3) which avoids the use of regularity structures or paracontrolled calculus. The local in time well-posedness of (1.3) is proved in Section 2.1. We get the longtime existence from the r-uniform L^p '*coming down from infinity*' property satisfied by u_r, proved in Section 2.2. This property is proved in Jagannath & Perkowski's formulation of equation (1.3). The results of Section 2 show that u_r depends continuously on a finite family $\widehat{\xi_r}$ of multilinear functionals of ξ_r. The functional setting needed to prove their convergence in some appropriate space as r goes to 0 is detailed in Section 3. A crucial role is played here by a set of distributions with given wavefront sets and a certain scaling property with respect to some submanifolds. The notion of scaling field is introduced in Section 3.1 and the preceding set of distributions is introduced in Section 3.2. We prove our main workhorse in Section 3.3, Theorem 11. It provides a numerical criterion for a distribution defined outside a submanifold, with some wavefront set bound and some scaling property with respect to that submanifold, to have a possibly unique extension to the whole manifold. We draw consequences of this general statement for the particular case of a configuration space in Section 3.4. The relevance of Theorem 11 to the convergence problem for $\widehat{\xi_r}$ stems from the fact that we can formulate the latter as a quantitative extension problem. This point of view is inspired by the Epstein-Glaser approach of renormalization. The convergence of $\widehat{\xi_r}$ in an appropriate space is the object of Section 4. At that point of the analysis of equation (1.3) one can make sense of its $r = 0$ version as a Markovian Feller dynamics in the state space $C^{-1/2-\varepsilon}(M)$, for any fixed sufficiently small $\varepsilon > 0$. The r-uniform L^p coming down property from Section 2.2 is used to get the existence of an invariant probability measure for this dynamics. The non-triviality of this invariant probability measure is proved in Section 5.2. Last, Section 6 is dedicated to the construction of a Φ^4_3 measure corresponding to a space-dependent coupling constant and also we discuss the bundle case. The detailed proofs of a number of basic tools from microlocal and harmonic analysis are given in our companion paper [10]. They are mainly put to work in our analysis of the convergence of the enhanced noise $\widehat{\xi_r}$ in Section 4.

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Notations – *Given* $0 < T < \infty$ *and a Banach space* E *we write* $C_T E$ *for* $C([0,T],E)$. *The parabolic Besov-Hölder spaces on space time are denote by* $\mathcal{C}^\gamma((a,b)\times M)$, *the Besov spaces on the manifold* M *are denoted by* $B^a_{p,q}(M)$. *The cotangent space to* M *is denoted by* T^*M *and the conormal to a submanifold* E *of* M *is denoted by* $N^*(E)$.

2 – Long time well-posedness and a priori estimate

We prove the existence of a unique solution to equation (1.3) over any fixed time interval $[0,T]$, for an arbitrary initial condition in $C^{-1/2-\varepsilon}(M)$, for $\varepsilon > 0$ small enough. We adopt here Jagannath & Perkowski's robust approach [57]. They use a clever change of variable to reformulate the equation as a non-singular parabolic partial differential equation (2.4) with random coefficients of regularity no worse than $-1/2-\varepsilon$. This allows to solve the equation locally in time by a fixed point argument set in a classical functional space without resorting to regularity structures or paracontrolled calculus. Section 2.2 is dedicated to proving an L^p estimate on the solution to equation (2.4) that is independent of the initial condition. This plays a crucial role in proving the existence of an invariant probability measure for (1.3) by an argument using compactness.

2.1 Local in time well-posedness

This section is dedicated to proving the local well-posedness of a solution to equation (1.3), uniformly over $r > 0$. We follow for that purpose Jagannath & Perkowski's remarkable work [57]. They noticed therein that a clever reformulation of the equation brings its study back to the study of a nonsingular stochastic PDE for which local in time well-posedness follows from an elementary fixed point argument.

Some distributions in the list (2.1) below involve an operator $\odot$, called *resonant operator*, that we introduce formally in Appendix A; its precise definition here does not matter other than the fact that it is well-defined and continuous from $B^{\alpha_1}_{p_1,q_1}(M)\times B^{\alpha_2}_{p_2,q_2}(M)$ into some Besov space if and only if $\alpha_1+\alpha_2>0$, in which case it takes values in $B^{\alpha_1+\alpha_2}_{p,q}(M)$, for some integrability exponents p,q whose precise value does not matter here. For $\Lambda_1\in B^{\alpha_1}_{p_1,q_1}(M)$ and $\Lambda_2\in B^{\alpha_2}_{p_2,q_2}(M)$, the product $\Lambda_1\Lambda_2$ is well-defined if and only if $\Lambda_1\odot\Lambda_2$ is well-defined.

– *The enhanced noise.* We regularize ξ in space (only) using the heat kernel and set

$$\xi_r := e^{r(\Delta-1)}\xi;$$

this is a Brownian motion in time with values in a space of regular functions. The fact that ξ appears in an additive form in (1.3) does not make it necessary to regularize it in time. Regularizing ξ only in space makes clear the Markovian character of the renormalized equation (1.5). Denote by $\mathcal{L}^{-1}$, respectively $\underline{\mathcal{L}}^{-1}$, the resolvent operator of $(\partial_t-\Delta+1)$ with null initial condition at time 0, respectively at time $-\infty$. Explicitely, $\mathcal{L}^{-1}f(t,.) = \int_0^t e^{(t-s)(\Delta-1)}f(s,.)ds$ and $\underline{\mathcal{L}}^{-1}f(t) = \int_{-\infty}^t e^{(t-s)(\Delta-1)}f(s,.)ds$ for any function f on $\mathbb{R}\times M$. The operator $\underline{\mathcal{L}}^{-1}$ provides stationary solutions. Recall from (1.4) the definitions of the constants a_r and b_r and set

$$\mathring{\mathrm{I}}_r := \underline{\mathcal{L}}^{-1}(\xi_r)$$

and

$$\mathrm{V}_r := (\mathring{\mathrm{I}}_r)^2 - a_r, \qquad \mathrm{Y}_r := \underline{\mathcal{L}}^{-1}(\mathrm{V}_r)$$

$$\<3>_r := (\<1>_r)^3 - 3a_r\<1>_r, \qquad \<30>_r := \underline{\mathcal{L}}^{-1}(\<3>_r)$$

and

$$\widehat{\xi}_r := \Big(\xi_r, \<2>_r,\, \<3>_r,\, \<30>_r \odot \<1>_r,\, \<20>_r \odot \<2>_r - \frac{b_r}{3},\, \big|\nabla \<20>_r\big|^2 - \frac{b_r}{3},\, \<30>_r \odot \<2>_r - b_r \<1>_r\Big). \tag{2.1}$$

One has $\xi_r \in \mathcal{C}^{-5/2-\varepsilon}([0,T]\times M)$ and the restriction to any time interval $[0,T]$ of the other components of $\widehat{\xi}_r$ is seen as an element of the product space

$$C_TC^{-1-2\varepsilon}(M) \times \mathcal{C}^{-3/2-3\varepsilon}([0,T]\times M) \times \mathcal{C}^{-4\varepsilon}([0,T]\times M)^3 \times \mathcal{C}^{-1/2-5\varepsilon}([0,T]\times M). \tag{2.2}$$

Note that only $\<2>_r$ is an element of a space of the form $C_TC^\gamma(M)$, the other terms in $\widehat{\xi}_r$ are elements of a parabolic space of negative regularity, which is less precise than being an element in a space of the form $C_TC^\gamma(M)$ for a negative exponent γ. This is sufficient for our needs. The enhancement $\widehat{\xi}_r$ can be seen as a placeholder for a number of products that are not well-defined in the zero regularization limit. We will see in Section 4 that $\widehat{\xi}_r$ converges in all the $L^p(\Omega)$ spaces, $1 \leq p < \infty$, as $r > 0$ goes to 0, to a limit that does not depend on the mollification used to define ξ_r from ξ. Using the operator $\underline{\mathcal{L}}^{-1}$ rather than the operator $\mathcal{L}^{-1}$ in the definitions of $\<20>_r$ and $\<30>_r$ builds some random distributions that are stationary in time. This property will be useful in Section 5 to get a compactness statement on the family of laws of the solutions to (1.5).

– *Jagannath & Perkowski's formulation of equation* (1.5). Set

$$v_{r,\mathrm{ref}} := 3\,\underline{\mathcal{L}}^{-1}\Big(e^{3\<20>_r}\Big\{\<30>_r\<2>_r - b_r\big(\<1>_r + \<30>_r\big)\Big\}\Big).$$

This is an element of $C_TC^{1-\varepsilon}(M)$. Jagannath & Perkowski' starting point for the analysis of the renormalized form (1.5) of equation (1.3) is that u_r is a solution to (1.5) if and only if

$$v_r := e^{3\<20>_r}\Big(u_r - \<1>_r + \<30>_r\Big) - v_{r,\mathrm{ref}} \tag{2.3}$$

is a solution of a particular equation of the form

$$(\partial_t - \Delta + 1)v_r = -6\nabla\<20>_r \cdot \nabla v_r - e^{-6\<20>_r}v_r^3 + Z_{2,r}v_r^2 + Z_{1,r}v_r + Z_{0,r}, \tag{2.4}$$

where $Z_{2,r}, Z_{1,r}, Z_{0,r}$ are elements of $C_TC^{-1/2-\eta}(M)$, for all $\eta > 0$, that depend continously on $\widehat{\xi}_r$ – see equation (2.4) in [57] (We deduce the regularity properties of the Z_i from the fact that $\mathcal{L}^{-1}$ sends continuously $\mathcal{C}^\gamma([0,T]\times M)$ into $C_TC^{\gamma+2}(M)$ when $-2 < \gamma < 0$.)

We now solve equation (2.4) with an arbitrary initial condition in $C^{-1/2-\varepsilon}(M)$ – [57] only considered the case of an initial condition that differs from $\<1>_r(0)$ by an element of $C^{3/2-\varepsilon}(M)$. For that purpose, and for exponents $\alpha > 0, \beta \in \mathbb{R}$, we introduce the spaces $(\!|\alpha,\beta|\!)$ made up of all functions $v \in C((0,T], C^\beta(M))$ such that

$$t^\alpha \|u(t)\|_{L^\infty} \underset{t\to 0}{\longrightarrow} 0$$

and

$$\|v\|_{(\!|\alpha,\beta|\!)} := \max\left\{\sup_{0<t\leq T} t^\alpha\|v(t)\|_{C^\beta},\ \sup_{0\leq t\neq s\leq T}\frac{\|t^\alpha v(t) - s^\alpha v(s)\|_{L^\infty}}{|s-t|^{\beta/2}}\right\} < \infty.$$

(The use of such weighted spaces is suggested in [57]; we use here the same spaces as in Section 6 of Gubinelli & Perkowski's work [47].) The free propagation map

$$(\mathcal{F}a)(t) := e^{t(\Delta-1)}a$$

sends for instance $C^\beta(M)$ into $(\!|\alpha,\beta+2\alpha|\!)$, for all $\beta \in \{\mathbb{R}\backslash\mathbb{N}\}$ and $\alpha > 0$, and one has for all $0 \leq \delta < \min(\beta, 2\alpha)$

$$\|\mathcal{L}^{-1}(f)\|_{(\!|\alpha-\delta/2,\beta-\delta|\!)} \lesssim \|f\|_{(\!|\alpha,\beta-2|\!)} \tag{2.5}$$

This inequality allows to trade some explosion rate against some regularity. We also have for the same range of exponents and all $f \in (\!|\alpha,\beta-2|\!)$

$$\|f\|_{(\!|\alpha-\delta/2,\beta-\delta|\!)} \lesssim \|f\|_{(\!|\alpha,\beta|\!)}. \tag{2.6}$$

These statements correspond in our setting to Lemma 6.6 and Lemma 6.8 in Gubinelli & Perkowski's work [47] – a proof is given in our companion work [10]. Note that since the different components of the enhanced noise are stationary they do not take value 0 at time 0. The initial condition for v_r is thus different from the initial condition for u_r. We keep the notation ϕ for the initial condition for u_r and write ϕ' for the initial condition for v_r. We will repeatedly use the estimate

$$\|fg\|_{C^{\alpha\wedge\beta}}\leqslant\|f\|_{C^\alpha}\|g\|_{C^\beta}, \tag{2.7}$$

if $\alpha+\beta>0$ which follows immediately from Proposition 24.

Proposition 3 – *Pick $\varepsilon'=4\varepsilon$ and set $\alpha_0:=3/4+(\varepsilon+\varepsilon')/2$. For any $\phi'\in C^{-1/2-\varepsilon}(M)$ there exists a positive time T^* such that for all $0<T<T^*$ equation (2.4) has a unique solution*

$$v_r\in C_TC^{-1/2-\varepsilon}(M)\cap (\!|\alpha_0,1+\varepsilon'|\!)$$

with initial condition ϕ'. This solution depends continuously on $\widehat{\xi}_r$ and $\phi'\in C^{-1/2-\varepsilon}(M)$, and for any small positive λ these exist $T_\lambda\in(\lambda,T^*)$ such that $u\in C\big([\lambda,T_\lambda],C^{3/2-4\varepsilon}(M)\big)$.

Proof – First, remark that $\lim_{t\to0^+}t^{\alpha_0}\mathcal{F}(\phi')=0$, since $\|\mathcal{F}(\phi')\|_{L^\infty}\le t^{-1/4-\varepsilon/2}\|\phi'\|_{-1/2-\varepsilon}$, so $\mathcal{F}(\phi')\in C_TC^{-1/2-\varepsilon}(M)\cap(\!|\alpha_0,1+\varepsilon'|\!)$. We use a standard Picard iteration argument for the map

$$F(v):=\mathcal{F}(\phi')+\mathcal{L}^{-1}\Big(-6\nabla\Upsilon_r\cdot\nabla v-e^{-6\Upsilon_r}v^3+Z_{2,r}v^2+Z_{1,r}v+Z_{0,r}\Big). \tag{2.8}$$

Denote B_R the ball of radius $R=4\|\phi'\|_{\mathcal{C}^{-1/2-\varepsilon}}$ in $C_TC^{-1/2-\varepsilon}(M)\cap(\!|\alpha_0,1+\varepsilon'|\!)$. Let $v_1,v_2\in B_R$. Our first goal is to get a bound of the form

$$\|F(v_2)-F(v_1)\|_{(\!|\alpha_0,1+\varepsilon'|\!)}\lesssim_{\widehat{\xi}_r}T^{\varepsilon/2}(R+R^2)\|v_2-v_1\|_{(\!|\alpha_0,1+\varepsilon'|\!)}.$$

meaning F is a contraction for the $(\!|\alpha_0,1+\varepsilon'|\!)$ norm by choosing T small enough. We have

$$\begin{aligned}\|F(v_2)-F(v_1)\|_{(\!|\alpha_0,1+\varepsilon'|\!)}\le{}&\big\|\mathcal{L}^{-1}\big(6\nabla\Upsilon_r\cdot\nabla(v_2-v_1)\big)\big\|_{(\!|\alpha_0,1+\varepsilon'|\!)}+\big\|\mathcal{L}^{-1}\big(e^{-6\Upsilon_r}(v_2^3-v_1^3)\big)\big\|_{(\!|\alpha_0,1+\varepsilon'|\!)}\\&+\big\|\mathcal{L}^{-1}\big(Z_{2,r}(v_2^2-v_1^2)\big\|_{(\!|\alpha_0,1+\varepsilon'|\!)}+\big\|\mathcal{L}^{-1}\big(Z_{1,r}(v_2-v_1)\big)\big\|_{(\!|\alpha_0,1+\varepsilon'|\!)}.\end{aligned}$$

Since $\nabla\Upsilon_r\in C_TC^{-\eta}(M)$ for all $\eta>0$, we first use the estimate (2.5) with $\delta=0$ and (2.7) to get

$$\begin{aligned}\big\|\mathcal{L}^{-1}\big(\nabla\Upsilon_r\cdot\nabla(v_2-v_1)\big)\big\|_{(\!|\alpha_0,1+\varepsilon'|\!)}&\lesssim\sup_{0<t\le T}t^{\alpha_0}\big\|\big(\nabla\Upsilon_r\cdot\nabla(v_2-v_1)\big)\big\|_{C^{-\eta}}\lesssim_{\widehat{\xi}_r}\sup_{0<t\le T}t^{\alpha_0}\|\nabla(v_2-v_1)\|_{C^{2\eta}}\\&\lesssim T^{\alpha_0-\alpha_1}\|v_2-v_1\|_{(|\alpha_1,1+2\eta|)},\quad(\alpha_1=3/4+(\varepsilon+2\eta)/2)\\&\lesssim T^{\varepsilon'/2-\eta}\|v_2-v_1\|_{(\!|\alpha_1,1+2\eta|\!)},\quad(\eta=\varepsilon'/4)\\&\lesssim T^{\varepsilon'/4}\|v_2-v_1\|_{(\!|\alpha_0,1+\varepsilon'|\!)},\quad(\text{by (2.6) with }\delta=\varepsilon'-2\eta).\end{aligned}$$

Now using again (2.5), (2.6), (2.7) and the fact that $\exp\big(-6\Upsilon_r\big)\in C^{1-\eta}(M)$ for all $\eta>0$, we have, for $\varepsilon'=4\varepsilon$

$$\begin{aligned}&\big\|\mathcal{L}^{-1}\big(\exp(-6\Upsilon_r)(v_1^3-v_2^3)\big)\big\|_{(\!|\alpha_0,1+\varepsilon'|\!)}\lesssim_{\widehat{\xi}_r}\sup_{0<t\le T}t^{\alpha_0}\big\|(v_2-v_1)(v_2^2+v_1^2+v_2v_1)(t)\big\|_{C^\eta}.\\&\lesssim T^{\alpha_0-3\alpha_0'}\|v_2-v_1\|_{(\!|\alpha_0',\eta|\!)}\big(\|v_2\|^2_{(\!|\alpha_0',\eta|\!)}+\|v_1\|^2_{(\!|\alpha_0',\eta|\!)}+\|v_2\|_{(\!|\alpha_0',\eta|\!)}\|v_1\|_{(\!|\alpha_0',\eta|\!)}\big),\quad(\alpha_0'=1/4+(\varepsilon+\eta)/2)\\&\lesssim T^{(\varepsilon'-2\varepsilon-3\eta)/2}\|v_2-v_1\|_{(\!|\alpha_0',\eta|\!)}\big(\|v_2\|^2_{(\!|\alpha_0',\eta|\!)}+\|v_1\|^2_{(\!|\alpha_0',\eta|\!)}+\|v_2\|_{(\!|\alpha_0',\eta|\!)}\|v_1\|_{(\!|\alpha_0',\eta|\!)}\big)\\&\lesssim T^{\varepsilon/2}\|v_2-v_1\|_{(\!|\alpha_0,1+\varepsilon'|\!)}\big(\|v_2\|_{(\!|\alpha_0,1+\varepsilon'|\!)}+\|v_1\|^2_{(\!|\alpha_0,1+\varepsilon'|\!)}+\|v_2\|_{(\!|\alpha_0,1+\varepsilon'|\!)}\|v_1\|_{(\!|\alpha_0,1+\varepsilon'|\!)}\big),\end{aligned}$$

choosing $\eta=\varepsilon/3$ and using (2.6) in the last inequality. Next, with the same argument we have for $\varepsilon'=4\varepsilon$, and setting $\alpha_0'=\frac12+(\varepsilon+\eta)/2,\alpha_0''=\frac14+(\varepsilon+\eta)/2$ in the third inequality

$$\begin{aligned}\big\|\mathcal{L}^{-1}\big(Z_{2,r}(v_2^2-v_1^2)\big)\big\|_{(\!|\alpha_0,1+\varepsilon'|\!)}&\lesssim_{\widehat{\xi}_r}\sup_{0<t\le T}t^{\alpha_0}\big\|(v_2^2-v_1^2)(t)\big\|_{C^{1/2+\eta}}\\&\lesssim\sup_{0<t\le T}t^{\alpha_0}\|v_2(t)-v_1(t)\|_{C^{1/2+\eta}}\big(\|v_2(t)\|_\eta+\|v_1(t)\|_\eta\big)\\&\le T^{\alpha_0-\alpha_0'-\alpha_0''}\|v_2-v_1\|_{(\!|\alpha_0',1/2+\eta|\!)}\big(\|v_2\|_{(\!|\alpha_0'',\eta|\!)}+\|v_1\|_{(\!|\alpha_0'',\eta|\!)}\big)\\&\le T^{(\varepsilon'-2\varepsilon-2\eta)/2}\|v_2-v_1\|_{(\!|\alpha_0,1+\varepsilon'|\!)}\big(\|v_2\|_{(\!|\alpha_0,1+\varepsilon'|\!)}+\|v_1\|_{(\!|\alpha_0,1+\varepsilon'|\!)}\big)\\&\le T^{\varepsilon/2}\|v_2-v_1\|_{(\!|\alpha_0,1+\varepsilon'|\!)}\big(\|v_2\|_{(\!|\alpha_0,1+\varepsilon'|\!)}+\|v_1\|_{(\!|\alpha_0,1+\varepsilon'|\!)}\big),\end{aligned}$$

using (2.6) in the fourth inequality and choosing $\eta = \varepsilon/2$ in the last inequality. Similarly, we get

$$\big\|\mathcal{L}^{-1}\big(Z_{1,r}(v_2 - v_1)\big)\big\|_{(\!|\alpha_0, 1+\varepsilon'|\!)} \lesssim_{\widehat{\xi}_r} T^{\varepsilon/2}\|v_2 - v_1\|_{(\!|\alpha_0,1+\varepsilon'|\!)}.$$

Therefore

$$\|F(v_2) - F(v_1)\|_{(\!|\alpha_0,1+\varepsilon'|\!)} \lesssim_{\widehat{\xi}_r} T^{\varepsilon/2}(R + R^2)\|v_2 - v_1\|_{(\!|\alpha_0,1+\varepsilon'|\!)}.$$

The rest is to estimate $F(v_1) - F(v_2)$ in $C_T C^{-1/2-\varepsilon}$. Since $v_1, v_2 \in B_R$, for any $s \in [0,T]$, $i = 1,2$ and $0 \le \delta < 1 + \varepsilon'$, it follows from (2.6) that $\|v_i\|_{(|\alpha_0-\delta/2,1+\varepsilon'-\delta|)} \le R$, hence we have

$$\|v_i(s)\|_{C^{-1/2-\varepsilon}} \le R \text{ and } \|v_i\|_{C^{1+\varepsilon'-\delta}} \le Rs^{-(\alpha_0-\delta/2)}. \tag{2.9}$$

Recall the following estimate which is used many times below: if $\|u(s)\|_{C^\beta} \lesssim s^{-\gamma}$, for $\gamma < 1$, then

$$\|\mathcal{L}^{-1}u\|_{C^\beta} \lesssim \int_0^T s^{-\gamma} \lesssim T^{1-\gamma}. \tag{2.10}$$

Using (2.9) and the fact that $\nabla\Upsilon_r \in C_T C^{-\varepsilon'/4}(M)$, we have

$$\big\|\nabla\Upsilon_r \cdot \nabla(v_2 - v_1)(s)\big\|_{C^{-1/2-\varepsilon}} \lesssim_{\widehat{\xi}_r} \big\|\nabla(v_2 - v_1)(s)\big\|_{C^{\varepsilon'}} \lesssim \|v_2 - v_1\|_{C^{1+\varepsilon'}} \lesssim s^{-\alpha_0}\|v_2 - v_1\|_{(|\alpha_0,1+\varepsilon'|)},$$

hence by (2.10)

$$\big\|\mathcal{L}^{-1}\big(\nabla\Upsilon_r \cdot \nabla(v_2 - v_1)\big)\big\|_{C^{-1/2-\varepsilon}} \lesssim_{\widehat{\xi}_r} T^{1-\alpha_0}\|v_2 - v_1\|_{(|\alpha_0,1+\varepsilon'|)} = T^{1/4-\varepsilon/2-\varepsilon'/2}\|v_2 - v_1\|_{(|\alpha_0,1+\varepsilon'|)}.$$

Again, by (2.9) and the fact that $\Upsilon_r \in C_T C^{1-\eta}(M)$, and $\|uv\|_{C^\beta} \le \|u\|_{L^\infty}\|v\|_{C^\beta} + \|v\|_{L^\infty}\|u\|_{C^\beta}$, for $\beta \in (0,1)$,

$$\begin{aligned}
\big\|\big(\exp(-6\Upsilon_r)(v_1^3 - v_2^3)\big)(s)\big\|_{-1/2-\varepsilon} &\lesssim_{\widehat{\xi}_r} \big\|(v_2 - v_1)(v_2^2 + v_1^2 + v_2 v_1)(t)\big\|_{C^{-1/2-\varepsilon}} \\
&\lesssim \quad \|v_2 - v_1\|_{C^{-1/2-\varepsilon}}\|v_2^2 + v_1^2 + v_2 v_1\|_{C^{1/2+2\varepsilon}} \\
&\lesssim \quad \|v_2 - v_1\|_{C^{-1/2-\varepsilon}} \sum_{i,j=1}^{2} \|v_i\|_{C^\varepsilon}\|v_j\|_{C^{1/2+2\varepsilon}} \\
&\lesssim \quad \|v_2 - v_1\|_{C^{-1/2-\varepsilon}} s^{-3/4-5\varepsilon/2}R^2
\end{aligned}$$

where we use (2.9) for both $\|v_i\|_{C^\varepsilon}$ and $\|v_j\|_{C^{1/2+2\varepsilon}}$ in the last inequality. Hence by (2.10) we have

$$\big\|\mathcal{L}^{-1}\big(\big(\exp(-6\Upsilon_r)(v_1^3 - v_2^3)\big)\big)\big\|_{-1/2-\varepsilon} \lesssim_{\widehat{\xi}_r} T^{1/4-5\varepsilon/2}R^2\|v_2 - v_1\|_{C^{-1/2-\varepsilon}}.$$

Now, with the same argument and (2.6), for $\eta > \varepsilon$, we have

$$\begin{aligned}
\big\|\big(Z_{2,r}(v_2^2 - v_1^2)\big)(s)\big\|_{C^{-1/2-\varepsilon}} &\lesssim_{\widehat{\xi}_r} \big\|(v_2^2 - v_1^2)(s)\big\|_{C^{1/2+\eta}} \\
&\lesssim (\|v_1\|_{C^\eta} + \|v_2\|_{C^\eta})\|v_1 - v_2\|_{1/2+\eta} \\
&\lesssim s^{-(\alpha_0-(1+\varepsilon'-\eta)/2)}Rs^{-1/2-\varepsilon/2-\eta/2}\|v_1 - v_2\|_{(\!|\frac{1}{2}+\frac{\varepsilon+\eta}{2},\frac{1}{2}+\eta|\!)} \\
&\lesssim s^{-(3/4+\varepsilon+\eta)}R\|v_1 - v_2\|_{(\!|\alpha_0,1+\varepsilon'|\!)},
\end{aligned}$$

hence, choosing $\eta = 2\varepsilon$ yields

$$\big\|\mathcal{L}^{-1}\big(Z_{2,r}(v_2^2 - v_1^2)\big)\big\|_{-1/2-\varepsilon} \lesssim_{\widehat{\xi}_r} T^{1/4-3\varepsilon}R\|v_1 - v_2\|_{(\!|\alpha_0,1+\varepsilon'|\!)}.$$

Similarly, we get

$$\big\|\mathcal{L}^{-1}\big(Z_{1,r}(v_2 - v_1)\big)\big\|_{C^{-1/2-\varepsilon}} \lesssim_{\widehat{\xi}_r} T^{1/2-3\varepsilon/2}\|v_1 - v_2\|_{(\!|\alpha_0,1+\varepsilon'|\!)}.$$

Therefore

$$\begin{aligned}
\|F(v_2) - F(v_1)\|_{C^{-1/2-\varepsilon}} \lesssim_{\widehat{\xi}_r} &\, T^{1/4-5\varepsilon/2}R^2\|v_2 - v_1\|_{C^{-1/2-\varepsilon}} \\
&+ \big(T^{1/4-\varepsilon/2-\varepsilon'/2} + T^{1/4-3\varepsilon}R + T^{1/2-3\varepsilon/2}\big)\|v_1 - v_2\|_{(\!|\alpha_0,1+\varepsilon'|\!)}.
\end{aligned}$$

Combining the estimates above we infer that for $T > 0$ sufficiently small, depending on $\|\phi'\|_{\mathcal{C}^{-1/2-\varepsilon}}$, $\widehat{\xi}_r$, the map F is a contraction on the ball of radius $4\|\phi'\|_{\mathcal{C}^{-1/2-\varepsilon}}$ in $C_T C^{-1/2-\varepsilon}(M) \cap (\!|\alpha_0, 1+\varepsilon'|\!)$. The unique fixed point is our solution on $[0,T]$. Taking the supremum of all such T gives the maximal existence time T^*. Once we know that v takes values in $(\!|\alpha_0, 1+\varepsilon'|\!)$ we can restart the fixed point procedure from a positive time, with an initial condition that is now of Hölder regularity

$(1+\varepsilon')$. It is elementary to adapt the preceding estimates to see that now the solution will take values in $C^{3/2-4\varepsilon}(M)$.

For the continuous dependence on $\widehat{\xi}_r$ and the initial data, we define

$$F(\widehat{\xi}_r,\phi,v) := e^{t(\Delta-1)}(\phi)+\mathcal{L}^{-1}\Big(-6\nabla\Upsilon_r(\xi_r)\cdot\nabla v - e^{-6\Upsilon_r(\xi_r)}v^3+Z_{2,r}(\xi_r)v^2+Z_{1,r}(\xi_r)v+Z_{0,r}(\xi_r)\Big).$$

Let $K>0$ be a uniform constant satisfying

$$\|e^{t(\Delta-1)}\phi\|_{C^{-1/2-\varepsilon}} \leq K\|\phi\|_{C^{-1/2-\varepsilon}} \text{ and } \|e^{t(\Delta-1)}\phi\|_{(\!|\alpha_0,1+\varepsilon'|\!)} \leq K\|\phi\|_{C^{-1/2-\varepsilon}}. \tag{2.11}$$

Take the ball B_R in $C^{-1/2-\varepsilon}(M)$. Since F depends linearly on $\widehat{\xi}_r$ and $\exp(-6\Upsilon_r)$, by the same arguments above, for any $\phi\in B_R$ and we can choose $T=T(R,\widehat{\xi}_r,\widehat{\xi}'_r)$ small enough such that $C(T)<1/2$ and

$$\|F(\widehat{\xi}_r,\phi,v_1)-F(\widehat{\xi}'_r,\phi,v_2)\|_{C_TC^{1/2-\varepsilon}} \leq C(T)\big(\|v_1-v_2\|_{C_TC^{1/2-\varepsilon}}+\|v_1-v_2\|_{(\!|\alpha_0,1+\varepsilon'|\!)}+\|\widehat{\xi}_r-\widehat{\xi}'_r\|\big)$$

and

$$\big\|F(\widehat{\xi}_r,\phi,v_1)-F(\widehat{\xi}'_r,\phi,v_2)\big\|_{(\!|\alpha_0,1+\varepsilon|\!)} \leq C(T)\big(\|v_1-v_2\|_{(\!|\alpha_0,1+\varepsilon'|\!)}+\|\widehat{\xi}_r-\widehat{\xi}'_r\|\big).$$

Now for $\phi_1,\phi_2\in B_R$ we have

$$\begin{aligned}
&\|v_r(\widehat{\xi}_r,\phi_1)-v_r(\widehat{\xi}'_r,\phi_2)\|_{C_TC^{1/2-\varepsilon}} = \big\|F(\widehat{\xi}_r,\phi_1,v_r(\phi_1))-F(\widehat{\xi}'_r,\phi_2,v_r(\phi_2))\big\|_{C_TC^{1/2-\varepsilon}}\\
&\leq \big\|F(\widehat{\xi}_r,\phi_1,v_r(\phi_1))-F(\widehat{\xi}_r,\phi_2,v_r(\phi_1))\big\|_{C_TC^{1/2-\varepsilon}}+\big\|F(\widehat{\xi}_r,\phi_2,v_r(\phi_1))-F(\widehat{\xi}'_r,\phi_2,v_r(\phi_2))\big\|_{C_TC^{1/2-\varepsilon}}\\
&\leq K\|\phi_1-\phi_2\|_{C^{-1/2-\varepsilon}}+C(T)\Big(\|v_r(\phi_1)-v_r(\phi_2)\|_{C_TC^{1/2-\varepsilon}}+\|v_r(\phi_1)-v_r(\phi_2)\|_{(\!|\alpha_0,1+\varepsilon'|\!)}+\|\widehat{\xi}_r-\widehat{\xi}'_r\|\Big).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\|v_r(\phi_1)-v_r(\phi_2)\|_{(\!|\alpha_0,1+\varepsilon'|\!)} &= \big\|F(\widehat{\xi}_r,\phi_1,v_r(\phi_1))-F(\widehat{\xi}'_r,\phi_2,v_r(\phi_2))\big\|_{(\!|\alpha_0,1+\varepsilon'|\!)}\\
&\leq K\|\phi_1-\phi_2\|_{C^{-1/2-\varepsilon}}+C(T)\big(\|v_r(\phi_1)-v_r(\phi_2)\|_{(\!|\alpha_0,1+\varepsilon'|\!)}+\|\widehat{\xi}_r-\widehat{\xi}'_r\|\big),
\end{aligned}$$

so

$$\begin{aligned}
&\|v_r(\phi_1)-v_r(\phi_2)\|_{C_TC^{1/2-\varepsilon}}+\|v_r(\phi_1)-v_r(\phi_2)\|_{(\!|\alpha_0,1+\varepsilon'|\!)}\\
&\qquad\leq 2K\|\phi_1-\phi_2\|_{C_TC^{1/2-\varepsilon}}+C(T)\|v_r(\phi_1)-v_r(\phi_2)\|_{C_TC^{-1/2-\varepsilon}}\\
&\qquad\quad+2C(T)\|v_r(\phi_1)-v_r(\phi_2)\|_{(\!|\alpha_0,1+\varepsilon'|\!)}+2\|\widehat{\xi}_r-\widehat{\xi}'_r\|,
\end{aligned}$$

and we read on the estimate

$$\begin{aligned}
&\|v_r(\phi_1)-v_r(\phi_2)\|_{C_TC^{1/2-\varepsilon}}+\|v_r(\phi_1)-v_r(\phi_2)\|_{(\!|\alpha_0,1+\varepsilon'|\!)}\\
&\qquad\leq \frac{1}{1-2C(T)}\Big(2K\|\phi_1-\phi_2\|_{C^{-1/2-\varepsilon}}+2C(T)\|\widehat{\xi}_r-\widehat{\xi}'_r\|\Big)
\end{aligned}$$

the continuous dependence of the solution on $\widehat{\xi}$ and the initial data. ▷

We see from the proof that T^* depends only on $\widehat{\xi}_r$ and $\phi'\in C^{-1/2-\varepsilon}(M)$. The following additional piece of information will be useful when proving the coming down from infinity property by energy methods in the next section.

Lemma 4 – *Let $0<t_0<t_1$. For $\beta=3/2-\varepsilon$ and any $\kappa\leq\beta/2$, then $t\mapsto v_r(t,x)$ is $\kappa-$ Hölder continuous as a function from $[t_0,t_1]$ to L^∞.*

Proof – By the change of variable $t\mapsto t-t_0$, we can assume $t_0=0, t_1=T>0$ and $v_r\in C_TC^{3/2-\varepsilon}(M)$. We now show the Hölder regularity of v_r at time 0, the adaptation to arbitrary times is straightforward. We have

$$v_r(t,\cdot)=e^{t(\Delta-1)}v_r(0)+\int_0^t e^{(t-s)(\Delta-1)}\Big(-6\nabla\Upsilon_r\cdot\nabla v_r-e^{-6\Upsilon_r}v_r^3+Z_{2,r}v_r^2+Z_{1,r}v_r+Z_{0,r}\Big)(s)ds.$$

We first remark that

$$|v_r(t,\cdot)-v_r(0)|\leq|e^{t(\Delta-1)}v_r(0)-v_r(0)|+\left|\int_0^t e^{(t-s)(\Delta-1)}(f_r(s))ds\right|, \tag{2.12}$$

where $f_r = -6\nabla\Upsilon_r \cdot \nabla v_r - e^{-6\Upsilon_r} v_r^3 + Z_{2,r}v_r^2 + Z_{1,r}v_r + Z_{0,r}$. It follows from the time regularity of the heat flow that $\|(1 - e^{t(\Delta-1)})h\|_{L^\infty} \lesssim t^{\frac{\beta}{2}}\|h\|_{C^\beta}$ for $0 \le \beta \le 2$, hence

$$\|e^{t(\Delta-1)}v_r(0) - v_r(0)\|_{L^\infty} \lesssim t^{\beta/2}\|v_r(0)\|_{C^\beta}. \tag{2.13}$$

Since $v_r \in C_T C^\beta$, with $\beta = 3/2 - \varepsilon$, $\nabla\Upsilon_r \in C_T C^{-\varepsilon}$, $e^{-6\Upsilon_r} \in C_T C^{1-\varepsilon}$, $\mathbf{Z}_r = \{Z_{0,r}, Z_{1,r}, Z_{2,r}\} \subset C_T C^{-\alpha}$ with $\alpha = 1/2 + \varepsilon'$, using $\|gh\|_{C^{\alpha'\wedge\beta'}} \lesssim \|g\|_{C^{\alpha'}}\|h\|_{C^{\beta'}}$ for $\alpha' + \beta' > 0$ we have

$$\|f_r(s,\cdot)\|_{C^{-\alpha}} \lesssim_{\mathbf{Z}_r} \|v_r\|_{C_T C^\beta} \lesssim 1.$$

Then the estimate $\|e^{t(\Delta-1)}h\|_{C^{\eta+\gamma}} \lesssim t^{-\gamma/2}\|h\|_{C^\eta}$, for $\gamma \ge 0$ implies

$$\left\| \int_0^t e^{(t-s)(\Delta-1)}(f_r(s))ds \right\|_{C^{\varepsilon'}} ds \le \int_0^t (t-s)^{-(\alpha-\varepsilon')/2}\|f_r(s)\|_{C^{-\alpha}} ds \lesssim t^{1-\alpha/2-\varepsilon'/2}.$$

By choosing $\varepsilon' \ge \varepsilon/2$ we have $\kappa := 1 - \alpha/2 - \varepsilon'/2 \le \beta/2$, therefore u is κ-Hölder from $[0,T]$ to L^∞. ▷

As in Proposition 6.8 of Mourrat & Weber's work [67] it follows from this property that the function

$$t \in (0,T] \mapsto \|v_r(t)\|_{L^p}^p$$

satisfies the equation

$$\begin{aligned}\frac{1}{p}\big(\|v_r(t)\|_{L^p}^p - \|v_r(s)\|_{L^p}^p\big) &= \int_s^t \big(v_r^{p-1}, \Delta v_r\big) - \int_s^t \Big(\int_M e^{-6\Upsilon_r(s_1)}v_r^{p+2}(s_1)\Big)ds_1 \\ &\quad + \int_s^t \Big(\frac{1}{p}\,(-6\nabla\Upsilon_r, \nabla v_r^p) + (Z_2, v_r^{p+1}) + (Z_1, v_r^p) + (Z_0, v_r^{p-1})\Big).\end{aligned} \tag{2.14}$$

2.2 Long time existence and coming down from infinity

We show in this section that the superlinear attractive drift $-\exp\big(-6\Upsilon_r\big)\,v_r^3$ in equation (2.4) entails an a priori bound on the $L^p(M)$ norm of the solution away from the initial time that is independent of the initial condition. This bound entails the long time existence of the solution v_r to (2.4) and is the key to proving the existence of an invariant probability measure for the dynamics (1.3) via a compactness argument. This point will be developed in Section 5.

We rewrite equation (2.4) in the form

$$\big(\partial_t - (\Delta - 1) + B_r\nabla\big)v_r = -A_r v_r^3 + Z_{2,r}v_r^2 + Z_{1,r}v_r + Z_{0,r}, \tag{2.15}$$

with

$$B_r := 6\nabla\Upsilon_r \in C_T C^{-\eta}(M), \quad A_r := e^{-6\Upsilon_r} \in C_T C^{1-\eta}(M),$$

and $Z_{i,r} \in C_T C^{-1/2-\eta}(M)$, for all $\eta > 0$.

Theorem 5 – *The solution $v_r(t) \in C^{3/2-\varepsilon}(M)$ exists for all times $t > 0$. Pick an even integer $p \ge 8$. There is a random variable $C\big(p, \widehat{\xi}_{r|[0,t]}\big)$ that depends only on the restriction to the interval $[0,t]$ of $\widehat{\xi}_r$ such that one has*

$$\|v_r(t)\|_{L^p(M)} \le C(p, \widehat{\xi}_{r|[0,t]}) \max\left\{\frac{1}{\sqrt{t}}, 1\right\} \tag{2.16}$$

for all $t > 0$, independently of the initial condition $\phi' \in C^{-1/2-\varepsilon}(M)$.

The upper bound in (2.16) is in particular independent of the initial condition in (2.15); this phenomenon is called *coming down from infinity*. We note for later use that keeping track of the implicit constants in the computations below gives an upper in (2.16) takes for $1 \le t \le 2$ the form

$$(1 + \|\widehat{\xi}_r\|)^\gamma \Big(\exp\big(\gamma'\|\Upsilon_r\|_{L^\infty([0,2]\times M)}\big) + 1\Big) \tag{2.17}$$

for some positive constants $\gamma = \gamma(p), \gamma' = \gamma'(p)$, up to a multiplicative constant. We denoted here by $\|\widehat{\xi}_r\|$ the norm of $\widehat{\xi}_r$ seen as an element of the product space where $\widehat{\xi}_r$ takes its values. We use a priori energy estimates to prove Theorem 5, following the strategy initiated by Mourrat & Weber in their proof of a similar result in [68], Theorem 7.1 therein. Gubinelli & Hofmanová also used energy estimates in their work [45] on the Φ_3^4 measure on $\mathbb{R}^3$. See also the proof of Proposition 3.7 in the work [78] of Tsatsoulis & Weber for an implementation of that strategy in the 2-dimensional torus.

We will use in the remainder of this section the shorthand notation

$$B_p^\gamma(M) := B_{p,\infty}^\gamma(M)$$

for any $\gamma \in \mathbb{R}$ and $1 \leq p \leq \infty$. Set

$$F_r(t) := \|v_r(t)\|_{p+2}^p + \|v_r(t)\|_{B^{1+2\varepsilon}_{\frac{p+2}{3}}}^{\frac{p}{3}}.$$

We prove below that one has for all $0 < T_0 \leq s < t \leq T < T^* \wedge 1$

$$\|v_r(t)\|_{L^p}^p + \int_s^t F_r(s_1)^{\frac{p+2}{p}} ds_1 \lesssim_{\widehat{\xi}_r} 1 + F_r(s). \tag{2.18}$$

This inequality shows that, all $0 < T_0 \leq s < t \leq T$

$$\int_s^t F_r(s_1)^{\frac{p+2}{p}} ds_1 \lesssim_{\widehat{\xi}_r} 1 + F_r(s). \tag{2.19}$$

It then follows from a modified version of Mourrat & Weber's comparison test recalled in Proposition 25 of Appendix A that there is an integer $N \geq 1$ and sequence of times $T_0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ such that for all $n \in \{0, \dots, N-1\}$,

$$F_r(t_n) \lesssim_{\widehat{\xi}_r} 1 + {t_{n+1}}^{-\frac{p}{2}},$$

for an implicit constant that does not depend on T_0, T. Pick $t \in [T_0, T]$. There exists $n \in \{0, \cdots, N-1\}$ such that $t \in [t_n, t_{n+1}]$. Moreover, by (2.18) with $s = t_n$, we have

$$\|v_r(t)\|_{L^p}^p \lesssim_{\widehat{\xi}_r} 1 + F_r(t_n) \lesssim_{\widehat{\xi}_r} 1 + {t_{n+1}}^{-\frac{p}{2}} \lesssim_{\widehat{\xi}_r} 1 + t^{-\frac{p}{2}}.$$

This bound holds for T_0 arbitrarily small and $T = 1$, and can be repeated on $[1, 2]$, etc, so that the uniform estimate (2.16) follows. Recall $p > 6$ so the space $L^p(M)$ is continuously embedded into the space $C^{-1/2-\varepsilon}(M)$. Given that T^* depends only on the restriction to $[0, T^*]$ of $\widehat{\xi}_r$ and the initial condition $\phi' \in C^{-1/2-\varepsilon}(M)$ the uniform estimate (2.16) and the continuous injection of $L^p(M)$ into $C^{-1/2-\varepsilon}(M)$ imply we can extend the solution though $T^* \wedge 1$, hence $T^* > 1$. Then Then we can repeat the same argument on the interval $[1, 2]$ and so on to get the long time existence of v_r.

On a technical level, our proof of Theorem 5 will only use the fractional Leibniz rule from Proposition 26 and the elementary interpolation result from Proposition 27, both recalled in Appendix A. Last, recall Young inequality that gives the existence for any positive δ of a constant C_δ such that one has

$$ab \leq \delta a^{p'} + \delta^{-\frac{q'}{p'}} b^{q'},$$

for all positive a, b and exponent $1 < p' < \infty$ with conjugate exponent q'. The proof of (2.18) requires two intermediate results stated as lemmas.

Lemma 6 – *For every* $0 < s < t \leq T$, *we have*

$$\|v_r(t)\|_{L^p}^p + \int_s^t \|v_r(s_1)\|_{L^{p+2}}^{p+2} ds_1 \lesssim_{\widehat{\xi}_r} 1 + \|v_r(s)\|_{L^{p+2}}^p + \int_s^t \|v_r(s_1)\|_{B^{1+\varepsilon}_{\frac{p+2}{3}}}^{\frac{p+2}{3}} ds_1. \tag{2.20}$$

Proof – Pairing the equation with v^{p-1} with respect to the L^2 scalar product yields identity (2.14). As A is positive and bounded below and $(v^{p-1}, -\Delta v)$ is positive, since p is an even integer greater than 4, we obtain

$$\|v_r(t)\|_{L^p}^p + \int_s^t \|v_r\|_{L^{p+2}}^{p+2} \lesssim_{\widehat{\xi}_r} \|v_r(s)\|_{L^p}^p + \int_s^t (B_r, \nabla v_r^p) + (Z_{2,r}, v_r^{p+1}) + (Z_{1,r}, v_r^p) + (Z_{0,r}, v_r^{p-1}). \tag{2.21}$$

where the implicit constant is $p\Big(\exp\big(6\|\Upsilon_r\|_{L^\infty([0,2]\times M)}\big) + 1\Big)$.

We bound the different terms in the right hand side of (2.21). Recall that since B_r is an element of $B^{-\varepsilon'}_{\infty,\infty}(M)$ for all $\varepsilon' > 0$ it is an element of $B^{-\varepsilon}_{1,\infty}(M)$. By the fractional Leibniz rule from

Proposition 26 and Young inequality we have for $\big|(B_r, \nabla v_r^p)\big|$, up to a $\widehat{\xi}_r$-dependent multiplicative constant, the upper bound

$$\|\nabla v_r^p\|_{B_1^\varepsilon} \lesssim \|v_r^p\|_{B_1^{1+\varepsilon}} \lesssim \|v_r^{p-1}\|_{L^{\frac{p+2}{p-1}}} \|v_r\|_{B^{1+\varepsilon}_{\frac{p+2}{3}}} \lesssim \|v_r\|_{L^{p+2}}^{p-1} \|v_r\|_{B^{1+\varepsilon}_{\frac{p+2}{3}}} \lesssim \delta \|v_r\|_{L^{p+2}}^{p+2} + \delta^{-\frac{p-1}{3}} \|v_r\|_{B^{1+\varepsilon}_{\frac{p+2}{3}}}^{\frac{p+2}{3}},$$

where $\delta > 0$ is arbitrarily small. For the other terms, we have first

$$\big|(Z_{2,r}, v_r^{p+1})\big| \lesssim_{\widehat{\xi}_r} \|v_r^{p+1}\|_{B_1^{\frac{1+\varepsilon}{2}}} \lesssim \|v^p\|_{L^{\frac{p+2}{p}}} \|v_r\|_{B^{\frac{1+\varepsilon}{2}}_{\frac{p+2}{2}}} \lesssim \|v_r\|_{L^{p+2}}^p \|v_r\|_{B^{\frac{1+\varepsilon}{2}}_{\frac{p+2}{2}}}.$$

Here we interpolate the last term to obtain

$$\|v_r\|_{B^{\frac{1+\varepsilon}{2}}_{\frac{p+2}{2}}} \lesssim \|v_r\|_{L^{p+2}}^{\frac12} \|v_r\|_{B^{1+\varepsilon}_{\frac{p+2}{3}}}^{\frac12},$$

and we deduce that

$$\big|(Z_{2,r}, v_r^{p+1})\big| \lesssim_{\widehat{\xi}_r} \|v_r\|_{L^{p+2}}^{p+\frac12} \|v_r\|_{B^{1+\varepsilon}_{\frac{p+2}{3}}}^{\frac12} \leqslant \delta \|v_r\|_{L^{p+2}}^{p+2} + C_\delta \|v_r\|_{B^{1+\varepsilon}_{\frac{p+2}{3}}}^{\frac{p+2}{3}},$$

using Young inequality in the second inequality, here $C_\delta = \delta^{-\frac{2p+1}{6}}$.
Similar estimates hold for the $Z_{1,r}$ and $Z_{0,r}$ terms. We have

$$\begin{aligned}
\big|(Z_{1,r}, v_r^p)\big| &\lesssim_{\widehat{\xi}_r} \|v_r^p\|_{B_1^{1+\varepsilon}} \lesssim \|v_r^{p-1}\|_{L^{\frac{p+2}{p-1}}} \|v_r\|_{B^{1+\varepsilon}_{\frac{p+2}{3}}} \lesssim \|v_r\|_{L^{p+2}}^{p-1} \|v_r\|_{B^{1+\varepsilon}_{\frac{p+2}{3}}} \\
&\lesssim \delta \|v_r\|_{L^{p+2}}^{p+2} + \delta^{-\frac{p-1}{3}} \|v_r\|_{B^{1+\varepsilon}_{\frac{p+2}{3}}}^{\frac{p+2}{3}}.
\end{aligned}$$

and

$$\begin{aligned}
\big|(Z_{0,r}, v_r^{p-1})\big| &\lesssim_{\widehat{\xi}_r} \|v_r^{p-1}\|_{B_1^{1+\varepsilon}} \lesssim \|v_r^{p-2}\|_{L^{\frac{p+2}{p-2}}} \|v_r\|_{B^{1+\varepsilon}_{\frac{p+2}{4}}} \lesssim \|v_r\|_{L^{p+2}}^{p-2} \|v_r\|_{B^{1+\varepsilon}_{\frac{p+2}{3}}} \\
&\lesssim \delta \|v_r\|_{L^{p+2}}^{\frac{(p+2)(p-2)}{p-1}} + \delta^{-\frac{p-1}{3}} \|v_r\|_{B^{1+\varepsilon}_{\frac{p+2}{3}}}^{\frac{p+2}{3}} \lesssim 1 + \delta^{\frac{p-1}{p-2}} \|v_r\|_{L^{p+2}}^{p+2} + \delta^{-\frac{p-1}{3}} \|v_r\|_{B^{1+\varepsilon}_{\frac{p+2}{3}}}^{\frac{p+2}{3}}.
\end{aligned}$$

One can then absorb the δ terms of these upper bounds in the corresponding L^{p+2} term in the left hand side of (2.21) to get the result by integrating in time on the interval (s,t). Since we choose $\delta \lesssim (1+\|\widehat{\xi}_r\|)^{-1}$, then $C_\delta, \delta^{-\frac{p-1}{3}} \gtrsim (1+\|\widehat{\xi}_r\|)^\gamma$ for some $\gamma > 0$ depending on p. Combining with the implicit constant in (2.21) we obtain that the implicit constant in (2.20) is of form

$$(1+\|\widehat{\xi}_r\|)^\gamma \Big(\exp\big(\gamma' \|\Upsilon_r\|_{L^\infty([0,2]\times M)}\big) + 1 \Big)$$

for some $\gamma, \gamma' > 0$ depending on p. The implicit constants in the next steps will be obtained in the same way. ▷

Lemma 7 – *For $0 \leq s < t < T < T^* \wedge 1$ we have*

$$\int_s^t \|v_r(s_1)\|_{B^{1+2\varepsilon}_{\frac{p+2}{3}}}^{\frac{p+2}{3}} ds_1 \lesssim_{\widehat{\xi}_r} 1 + F_r(s) + \int_s^t \|v_r(s_1)\|_{L^{p+2}}^{p+2} ds_1. \tag{2.22}$$

Note that we have a $B^{1+\varepsilon}_{\frac{p+2}{3}}$ norm involved in (2.20):

$$\|v_r(t)\|_{L^p}^p + \int_s^t \|v_r(s_1)\|_{L^{p+2}}^{p+2} ds_1 \lesssim \|v_r(s)\|_{L^{p+2}}^p + \int_s^t \|v_r(s_1)\|_{B^{1+\varepsilon}_{\frac{p+2}{3}}}^{\frac{p+2}{3}} ds_1$$

while we estimate in (2.22) a stronger $B^{1+2\varepsilon}_{\frac{p+2}{3}}$ norm. We postpone for a second the proof of Proposition 7 and explain now how we get the estimate (2.18)

$$\|v_r(t)\|_{L^p}^p + \int_s^t F_r(s_1)^{\frac{p+2}{p}} ds_1 \lesssim_{\widehat{\xi}_r} 1 + F_r(s)$$

from (2.20) and (2.22). Since

$$F_r(s_1)^{\frac{p+2}{p}} \lesssim \|v_r(s_1)\|_{L^{p+2}}^{p+2} + \|v_r(s_1)\|_{B^{1+2\varepsilon}_{\frac{p+2}{3}}}^{\frac{p+2}{3}}$$

we see as a consequence of (2.22) that one gets (2.18) if one proves that

$$\|v_r(t)\|_{L^p}^p + \int_s^t \|v_r(s_1)\|_{L^{p+2}}^{p+2}\, ds_1 \lesssim_{\widehat{\xi}_r} 1 + F_r(s). \tag{2.23}$$

We start from the inequality (2.20) and use the interpolation estimate

$$\|v_r\|_{B^{1+\varepsilon}_{\frac{p+2}{3}}} \lesssim \|v_r\|_{L^{\frac{p+2}{3}}}^{1-\theta_\varepsilon} \|v_r\|_{B^{1+2\varepsilon}_{\frac{p+2}{3}}}^{\theta_\varepsilon} \leqslant \delta \|v_r\|^3_{L^{\frac{p+2}{3}}} + C_\delta \|v_r\|^{\sigma_\varepsilon}_{B^{1+2\varepsilon}_{\frac{p+2}{3}}} \leqslant \delta\|v_r\|^3_{L^{p+2}} + C_\delta \|v_r\|^{\sigma_\varepsilon}_{B^{1+2\varepsilon}_{\frac{p+2}{3}}},$$

with

$$\theta_\varepsilon := \frac{1+\varepsilon}{1+2\varepsilon} < 1, \quad \sigma_\varepsilon := \frac{3\theta_\varepsilon}{2+\theta_\varepsilon} < 1.$$

We feed this upper bound inside (2.20); the contribution of the small factor involving the L^{p+2} norm of v_r can be absorbed in the corresponding term in the left hand side of (2.20), so we have

$$\|v_r(t)\|_{L^p}^p + \int_s^t \|v_r(s_1)\|_{L^{p+2}}^{p+2}\, ds_1 \lesssim \|v_r(s)\|_{L^{p+2}}^p + \int_s^t \|v_r(s_1)\|^{\sigma_\varepsilon \frac{p+2}{3}}_{B^{1+2\varepsilon}_{\frac{p+2}{3}}}\, ds_1. \tag{2.24}$$

We use Young inequality once more to bound

$$\|v_r(s_1)\|^{\sigma_\varepsilon \frac{p+2}{3}}_{B^{1+2\varepsilon}_{\frac{p+2}{3}}} \lesssim \delta \|v_r(s_1)\|^{\frac{p+2}{3}}_{B^{1+2\varepsilon}_{\frac{p+2}{3}}} + C_\delta.$$

By choosing δ small enough we can absorb the L^{p+2} term that comes from (2.22) in the left hand side of (2.24) and use that $\|v_r(s)\|^p_{L^{p+2}} \leq F_r(s)$ to get (2.23) from (2.24).

Proof of Lemma 7 – We proceed in two steps.

Step 1. We first prove that one has

$$\|v_r(t)\|_{B^{1+2\varepsilon}_{\frac{p+2}{3}}} \lesssim_{\widehat{\xi}_r} \|e^{(t-s)(\Delta-1)} v_r(s)\|_{B^{1+2\varepsilon}_{\frac{p+2}{3}}} + \left(\int_s^t \|v_r(s_1)\|^{p+2}_{L^{p+2}}\, ds_1\right)^{\frac{3}{p+2}} + \left(\int_s^t \|v_r(s_1)\|^{\frac{p+2}{3}}_{B^{1+\varepsilon}_{\frac{p+2}{3}}}\, ds_1\right)^{\frac{3}{p+2}}. \tag{2.25}$$

We look at each term in the expression for $v_r(t) - e^{(t-s)(\Delta-1)} v_r(s)$

$$\int_s^t e^{(t-s_1)(\Delta-1)} \Big(- A_r(s_1) v_r(s_1)^3 + B_r(s_1)\nabla v_r(s_1) + Z_{2,r}(s_1) v_r(s_1)^2 + Z_{1,r}(s_1) v_r(s_1) + Z_{0,r}(s_1)\Big) ds_1.$$

One has

$$\begin{aligned}
\Big\| \int_s^t e^{(t-s_1)(\Delta-1)} \big(A_r(s_1) v_r(s_1)^3\big)\, ds_1 \Big\|_{B^{1+2\varepsilon}_{\frac{p+2}{3}}} &\lesssim \int_s^t \big\| e^{(t-s_1)(\Delta-1)} \big(A_r(s_1) v_r(s_1)^3\big) \big\|_{B^{1+2\varepsilon}_{\frac{p+2}{3}}} ds_1 \\
&\lesssim \int_s^t (t-s_1)^{-\frac{1+2\varepsilon}{2}} \big\| A_r(s_1) v_r(s_1)^3 \big\|_{L^{\frac{p+2}{3}}}\, ds_1 \\
&\lesssim_{\widehat{\xi}_r} \int_s^t (t-s_1)^{-\frac{1+2\varepsilon}{2}} \|v_r(s_1)\|^3_{L^{p+2}}\, ds_1 \\
&\lesssim \left(\int_s^t \|v_r(s_1)\|^{p+2}_{L^{p+2}} \right)^{\frac{3}{p+2}},
\end{aligned}$$

where we used Hölder inequality, the integrability in time of $(t-s_1)^{-\frac{(1+2\varepsilon)(p+2)}{2(p-1)}}$ and the fact that $s \leq s_1 \leq t < T < 1$. Similarly, we have

$$\begin{aligned}
\Big\| \int_s^t e^{(t-s_1)(\Delta-1)} \big(B_r(s_1) \nabla v_r(s_1)\big) ds_1 \Big\|_{B^{1+2\varepsilon}_{\frac{p+2}{3}}} &\lesssim \int_s^t \Big\| e^{(t-s_1)(\Delta-1)} \big(B_r(s_1)\nabla v_r(s_1)\big) \Big\|_{B^{1+2\varepsilon}_{\frac{p+2}{3}}} ds_1 \\
&\lesssim \int_s^t (t-s_1)^{-\frac{1+3\varepsilon}{2}} \|B(s_1)\nabla v_r(s_1)\|_{B^{-\varepsilon}_{\frac{p+2}{3}}}\, ds_1 \\
&\lesssim_{\widehat{\xi}_r} \int_s^t (t-s_1)^{-\frac{1+3\varepsilon}{2}} \|v_r(s_1)\|_{B^{1+\varepsilon}_{\frac{p+2}{3}}}\, ds_1
\end{aligned}$$

$$\lesssim \left(\int_s^t \|v_r(s_1)\|^{\frac{p+2}{3}}_{B^{1+\varepsilon}_{\frac{p+2}{3}}}\right)^{\frac{3}{p+2}}.$$

Next we have

$$\begin{aligned}\left\|\int_s^t e^{(t-s_1)(\Delta-1)}\big(Z_{2,r}(s_1)\,v_r(s_1)^2\big)ds_1\right\|_{B^{1+2\varepsilon}_{\frac{p+2}{3}}} &\lesssim \int_s^t \left\|e^{(t-s_1)(\Delta-1)}\big(Z_{2,r}(s_1)\,v_r(s_1)^2\big)\right\|_{B^{1+2\varepsilon}_{\frac{p+2}{3}}} ds_1\\ &\lesssim \int_s^t (t-s_1)^{-\frac{1+2\varepsilon+\frac{1+\varepsilon}{2}}{2}} \big\|Z_{2,r}(s_1)\,v_r^2(s_1)\big\|_{B^{-\frac{1+\varepsilon}{2}}_{\frac{p+2}{3}}} ds_1.\end{aligned}$$

Using the interpolation result from Proposition 27 and Young inequality we have

$$\big\|Z_{2,r}v_r^2\big\|_{B^{-\frac{1+\varepsilon}{2}}_{\frac{p+2}{3}}} \lesssim_{\widehat{\xi}_r} \|v_r^2\|_{B^{\frac{1+\varepsilon}{2}}_{\frac{p+2}{3}}} \lesssim \|v_r\|_{L^{p+2}}\|v_r\|_{B^{\frac{1+\varepsilon}{2}}_{\frac{p+2}{2}}} \lesssim \|v_r\|^{\frac{3}{2}}_{L^{p+2}}\|v_r\|^{\frac{1}{2}}_{B^{1+\varepsilon}_{\frac{p+2}{3}}} \lesssim \|v_r\|^{p+2}_{L^{p+2}} + \|v_r\|^{\frac{p+2}{3}}_{B^{1+\varepsilon}_{\frac{p+2}{3}}},$$

and the desired estimate follows as in the previous terms. One proceeds in exactly the same way to prove similar estimates on the $Z_{1,r}$ and $Z_{0,r}$ terms. We leave the details to the interested reader.

Step 2. We first rewrite (2.25) at $t = s_1$ and rise this inequality to the power $\frac{p+2}{3}$. It yields the upper bound

$$\begin{aligned}\|v_r(s_1)\|^{\frac{p+2}{3}}_{B^{1+2\varepsilon}_{\frac{p+2}{3}}} &\lesssim_{\widehat{\xi}_r} \|e^{(s_1-s)(\Delta-1)}v_r(s)\|^{\frac{p+2}{3}}_{B^{1+2\varepsilon}_{\frac{p+2}{3}}} + \int_s^{s_1} \|v_r\|^{p+2}_{L^{p+2}} + \int_s^{s_1} \|v_r\|^{\frac{p+2}{3}}_{B^{1+\varepsilon}_{\frac{p+2}{3}}}\\ &\lesssim_{\widehat{\xi}_r} \|e^{(s_1-s)(\Delta-1)}v_r(s)\|^{\frac{p+2}{3}}_{B^{1+2\varepsilon}_{\frac{p+2}{3}}} + \int_s^{t} \|v_r\|^{p+2}_{L^{p+2}} + \int_s^{t} \|v_r\|^{\frac{p+2}{3}}_{B^{1+\varepsilon}_{\frac{p+2}{3}}},\end{aligned}$$

which holds for $t \geq s_1$, using the fact that $\int_s^{s_1} \|v_r\|^\alpha$ is increasing in s_1 whatever the exponent α the norm on v_r. Integrating on $s_1 \in [s,t]$, we obtain

$$\begin{aligned}\left(\int_s^t \|v_r(s_1)\|^{\frac{p+2}{3}}_{B^{1+2\varepsilon}_{\frac{p+2}{3}}} ds_1\right)^{\frac{3}{p+2}} &\lesssim_{\widehat{\xi}_r} \left(\int_s^t \big\|e^{(s_1-s)(\Delta-1)}v_r(s)\big\|^{\frac{p+2}{3}}_{B^{1+2\varepsilon}_{\frac{p+2}{3}}} ds_1\right)^{\frac{3}{p+2}}\\ &+ (t-s)^{\frac{3}{p+2}}\left[\left(\int_s^t \|v_r(s_1)\|^{p+2}_{L^{p+2}} ds_1\right)^{\frac{3}{p+2}} + \left(\int_s^t \|v_r(s_1)\|^{\frac{p+2}{3}}_{B^{1+\varepsilon}_{\frac{p+2}{3}}} ds_1\right)^{\frac{3}{p+2}}\right],\end{aligned} \tag{2.26}$$

as the function $[\dots]$ after $(t-s)^{\frac{3}{p+2}}$ in the right hand side is increasing. We bound the first term in (2.26) by $\|v_r(s)\|^{\frac{p+2}{3}}_{B^{(1+2\varepsilon)(1-\frac{3}{p+2})}_{\frac{p+2}{3}}}$ using the fact that the linear continuous map

$$e^{(s_1-s)(\Delta-1)} : B^{(1+2\varepsilon)(1-\frac{3}{p+2})}_{\frac{p+2}{3}}(M) \to B^{1+2\varepsilon}_{\frac{p+2}{3}}(M)$$

has a norm bounded above by $(s_1-s)^{-\frac{3(1+2\varepsilon)}{2(p+2)}} \leq (s_1-s)^{-1/2}$, a quantity that is integrable over the interval (s,t). The interpolation estimate

$$\|v_r(s)\|^{\frac{p+2}{3}}_{B^{(1+2\varepsilon)(1-\frac{3}{p+2})}_{\frac{p+2}{3}}} \lesssim \|v_r(s)\|_{L^{\frac{p+2}{3}}}\|v(s)\|^{\frac{p+2}{3}-1}_{B^{1+2\varepsilon}_{\frac{p+2}{3}}} \lesssim \|v_r(s)\|^p_{L^{\frac{p+2}{3}}} + \|v_r(s)\|^{\frac{p}{3}}_{B^{1+2\varepsilon}_{\frac{p+2}{3}}} = F_r(s),$$

gives $F_r(s)$ as a final upper bound for this term. Now, with v_r evaluated at time s_1 and $\theta_\varepsilon = \frac{1+\varepsilon}{1+2\varepsilon}$, we have

$$\|v_r\|_{B^{1+\varepsilon}_{\frac{p+2}{3}}} \lesssim \|v_r\|^{1-\theta_\varepsilon}_{L^{\frac{p+2}{3}}}\|v_r\|^{\theta_\varepsilon}_{B^\gamma_{\frac{p+2}{3}}} \lesssim \|v_r\|^{\frac{3\theta_\varepsilon}{2+\theta_\varepsilon}}_{B^{1+2\varepsilon}_{\frac{p+2}{3}}} + \|v_r\|^3_{L^{p+2}} \leqslant \delta\|v_r\|_{B^{1+2\varepsilon}_{\frac{p+2}{3}}} + C_\delta\big(1+\|v_r\|^3_{L^{p+2}}\big),$$

for some $C_\delta > 0$ and δ small enough so that the term related to $\delta\|v_r\|_{B^{1+2\varepsilon}_{\frac{p+2}{3}}}$ can be absorbed by the left hand side of (2.26). This gives inequality (2.22). ▷

3 – Scaling fields, regularity and microlocal extension

We state in this section an extension result, Theorem 11, that provides conditions under which a distribution on a manifold defined outside a submanifold can be extended to the whole manifold. The quantification of this extension result involves the notion of scaling field that is introduced in Section 3.1. Such vector fields are also known as Euler vector fields. Some function spaces associated with a given scaling field are introduced in Section 3.2, they generalize the weakly homogeneous distributions introduced by Meyer in [65] and Theorem 11 is proved in Section 3.3. This statement is put to work in the particular setting of a configuration space in Section 3.4 to give a useful extension result for a class of Feynman graphs.

We work in Section 3.1 to 3.3 in the setting of a smooth manifold $\mathcal{X}$ where a smooth submanifold $\mathcal{Y} \subset \mathcal{X}$ is given.

3.1 Scaling fields

Definition – *Let $\mathfrak{I}_{\mathcal{Y}}$ be the ideal of smooth real valued functions on $\mathcal{X}$ that vanish on $\mathcal{Y}$. Set for $k \geqslant 1$:*

$$\mathfrak{I}^k_{\mathcal{Y}} := \{f_1 \dots f_k \,;\, (f_1, \dots, f_k) \in \mathfrak{I}_{\mathcal{Y}} \times \cdots \times \mathfrak{I}_{\mathcal{Y}}\}.$$

A vector field ρ defined on a neighbourhood of $\mathcal{Y}$ is called a **scaling field for** $\mathcal{Y} \subset \mathcal{X}$ *if for all $f \in \mathfrak{I}_{\mathcal{Y}}$*

$$f \in \rho f + \mathfrak{I}^2_{\mathcal{Y}}.$$

This type of vector field is also called an *Euler vector field* in the litterature. Denote by n the dimension of $\mathcal{X}$ and by d the dimension of $\mathcal{Y}$. If ρ is a scaling field for $\mathcal{Y} \subset \mathcal{X}$ there exists a neighbourhoof of $\mathcal{Y}$ that is stable by the backward semiflow $(e^{-s\rho})_{s\geq 0}$ of ρ and every point $y \in \mathcal{Y}$ has a neighbourhood U_y in $\mathcal{X}$ on which coordinates

$$h = (h_1, \dots, h_n) : U \to \mathbb{R}^n$$

are defined and such that

$$U_y \cap \mathcal{Y} = h^{-1}(\mathbb{R}^d)$$

with $\mathbb{R}^d \subset \mathbb{R}^n$, and

$$\rho = \sum_{i=1}^{n-d} h^i \partial_{h^i}.$$

(A proof of existence of a stable neighbourhood can be found in Lemma 2.4 of [28] and the normal form theorem can be found in Proposition 2.5 of [28] – see also Lemma 2.1 in [63].) The example of the configuration space of ℓ points in $\mathbb{R}^k$ will be particularly relevant for us. The scaling field ρ whose flow reads

$$e^{-t\rho}(x_1, \dots, x_\ell) = \big(x_1, e^{-t}(x_2 - x_1) + x_1, \dots, e^{-t}(x_\ell - x_1) + x_1\big)$$

will move all points towards the deepest diagonal and its dynamics is tangent to all the larger diagonals. We will mainly work in the sequel with product (sub)manifolds

$$\mathcal{X} = \mathbb{R}^p \times X,$$
$$\mathcal{Y} = (\{0\} \times \mathbb{R}^q) \times Y$$

with $(\{0\} \times \mathbb{R}^q) \subset \mathbb{R}^p$ and $Y \subset X$, and scaling fields of the form

$$\rho = \sum_{j=1}^{q} 2t_j \partial_{t_j} + \rho_Y \tag{3.1}$$

for the canonical coordinates $(t_j)_{1\leq j\leq q}$ on $\mathbb{R}^q$ and a scaling field ρ_Y for $Y \subset X$. This is an example of a weighted vector field on a weighted manifold. We give a formal definition in the case where our submanifold $\mathcal{Y}$ is the transverse intersection $\mathcal{Y}_1 \cap \mathcal{Y}_2$ where $\mathcal{Y}_1 = (\{0\} \times \mathbb{R}^q) \times X$ and $\mathcal{Y}_2 = (\mathbb{R}^p \times Y)$. Then one has a description of the ideal $\mathfrak{I}_{\mathcal{Y}}$ as the product $\mathfrak{I}_{\mathcal{Y}_1}\mathfrak{I}_{\mathcal{Y}_2}$ so the ideal $\mathfrak{I}_{\mathcal{Y}}$ has a bifiltration

$$\mathfrak{I}^k_{\mathcal{Y}_1}\mathfrak{I}^l_{\mathcal{Y}_2} \subset \cdots \subset \mathfrak{I}_{\mathcal{Y}_1}\mathfrak{I}_{\mathcal{Y}_2} = \mathfrak{I}_{\mathcal{Y}}.$$

Now assume we want to put a weight 2 to powers of $\mathfrak{I}_{\mathcal{Y}_1}$ and weight 1 to powers of $\mathfrak{I}_{\mathcal{Y}_2}$. We want to give an intrinsic characterization for vector fields of the form given by equation (3.1).

Definition – *For all* (k,l) *and for all* $f \in \mathfrak{I}^k_{\mathcal{Y}_1}\mathfrak{I}^l_{\mathcal{Y}_2}$, *a* **parabolic scaling field** ρ *satisfies*

$$\rho f - (2k+l)f \in \left(\mathfrak{I}^{k+1}_{Y_1}\mathfrak{I}^l_{Y_2} + \mathfrak{I}^k_{Y_1}\mathfrak{I}^{l+1}_{Y_2}\right).$$

In the sequel, we will simply call 'scaling fields' some parabolic weighted scaling fields as we will only work with such fields. The **weighted co-dimension** of $\mathcal{Y}$ is here defined as

$$\mathrm{codim}_w(\mathcal{Y}) := 2(p-q) + \dim(X) - \dim(Y).$$

3.2 Function spaces associated with scaling fields

We assume from now on that $\mathcal{X}$ has a Riemannian structure and denote by $K_t(x,y)$ its heat kernel. Let now $U \subset \mathcal{X}$ be an open set and Γ be a closed conic set in $T^*U\backslash\{0\}$. We denote by $\mathcal{D}'_\Gamma(U)$ the space of distributions on U whose wave front set is contained in Γ. This is a locally convex topological vector space endowed with a natural *normal* topology invented by Y. Dabrowski, see [25, p. 823] and [18] for results about why this topology is well–behaved wrt natural operations on distributions. For the convenience of the reader, we recall the seminorms defining the normal topology:

$$\|\Lambda\|_{N,V,\chi,\kappa} = \sup_{\xi\in V} |(1+|\xi|)^N \widehat{(\kappa_*\Lambda)\chi}(\xi)|$$

for all chart $\kappa : \Omega \subset U \mapsto \mathbb{R}^{\dim(\mathcal{X})}$, integer N, $\chi \in C^\infty_c(\kappa(\Omega))$, cone $V \subset \mathbb{R}^{n*}$ such that

$$\mathrm{supp}(\chi) \times V \cap \kappa^{-1*}\Gamma = \emptyset, \text{ where } \kappa^{-1*}\Gamma = \{(\kappa(x); ({}^t d\kappa)^{-1}(\xi)); (x;\xi) \in \Gamma\}.$$

And we also need the seminorms of the strong topology of distributions:

$$\sup_{\chi\in B} |\langle \Lambda, \chi\rangle|$$

where B is a bounded set of $C^\infty_c(\mathcal{X})$ which means that there is some compact K such that $\mathrm{supp}(B) \subset K$ and for any differential operator P, $\sup_{\chi\in B} \|P\chi\|_{L^\infty(K)} < +\infty$. To be bounded in $\mathcal{D}'_\Gamma(U)$ will always mean that all the above seminorms are bounded.

The following elementary example will play an important role in the sequel. Assume $k \in \mathbb{N}$ is of the form $d_1 + d_2 + d_3$ with $d_i \in \mathbb{N}\backslash\{0\}$ and $\mathbb{R}^k = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \mathbb{R}^{d_3}$, and denote here by ρ the linear vector field on $\mathbb{R}^k$ whose restriction to $\mathbb{R}^{d_1}$ is null, whose restriction to $\mathbb{R}^{d_2}$ is the identity and whose restriction to $\mathbb{R}^{d_3}$ is twice the identity. So for $z = (x,y,t) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \mathbb{R}^{d_3}$ one has

$$\rho = y\partial_y + 2t\partial_t.$$

This vector field over $\mathbb{R}^k$ will be our model scaling field in a parabolic setting.

Lemma 8 – *The family of distribution*

$$\delta(z' - e^{-s\rho}z), \qquad (1 \le s \le +\infty)$$

on $\mathbb{R}^k \times \mathbb{R}^k$ *is bounded in* $\mathcal{D}'_{\Gamma_\rho}(\mathbb{R}^k \times \mathbb{R}^k)$, *where*

$$\Gamma_\rho = \bigcup_{1\le s\le+\infty} \left\{\big((z, e^{-s\rho}z), (\lambda, e^{s\rho}\lambda)\big); (z,\lambda) \in T^*\mathbb{R}^k\right\} \subset T^*(\mathbb{R}^k \times \mathbb{R}^k).$$

This estimate can also be used to give an upper bound on the wave front set of the resolvent $(\rho+z)^{-1}$ which implies the radial type estimates for ρ. This is very similar in spirit to the radial estimates from some works of Melrose [64], Vasy [79] or Dyatlov & Zworski [34].

Proof – Note that the distributions $\delta(z' - e^{-s\rho}z) \in \mathcal{D}'(\mathbb{R}^k \times \mathbb{R}^k)$ are nothing but the Schwartz kernels of the transfer operator $\varphi \in C^\infty(\mathbb{R}^k) \mapsto e^{-s\rho*}\varphi \in C^\infty(\mathbb{R}^k)$, so we will use the identification $[e^{-s\rho*}] = \delta(z' - e^{-s\rho}z)$. The fact that this family of distributions is bounded (weak boundedness implies strong boundedness by uniform boundedness) automatically follows from the continuity of the pull-back of a distribution by a smooth family of diffeomorphisms and the strong convergence of $\delta(z' - e^{-s\rho}z)$ to $\delta(z' - (x,0,0))$ when s goes to infinity, for $z = (x,y,\mathtt{t})$. (We use a slightly different calligraphic symbol $\mathtt{t}$ to keep the letter t for the time parameter in our equations.)
Fix an arbitrary compact subset $K \subset \mathbb{R}^{d_1+d_2+d_3}$ that is stable by the scaling maps

$$(x,y,\mathtt{t}) \mapsto \big(x, e^{-s}y, e^{-2s}\mathtt{t}\big), \quad (s \geqslant 0).$$

Then we shall restrict the Schwartz kernel $[e^{-s\rho}]$ to $K\times K$. It means we estimate this wave front set near the diagonal but for arbitrary large times s. Choose some test functions χ_1,χ_2 in $C^\infty_K(\mathbb{R}^{d_1+d_2+d_3})$, supported in K. In local coordinates we have

$$\begin{aligned}
&\int_{\mathbb{R}^{d_1+d_2+d_3}} e^{i\xi_2\cdot x+i\eta_2\cdot y+i\tau_2\cdot \mathsf{t}}\chi_2(x,y,\mathsf{t})e^{-s\rho*}(\chi_1 e^{i\xi_1\cdot x+i\eta_1\cdot y+i\tau_1\cdot\mathsf{t}})\,dxdyd\mathsf{t}\\
=\ & e^{-s(d_2+2)}\int_{\mathbb{R}^{d_1+d_2+d_3}} e^{i\xi_2\cdot x+i\eta_2\cdot y+i\tau_2\mathsf{t}}\chi_2(x,y,\mathsf{t})\big(\chi_1(x,e^{-s}y,e^{-2s}\mathsf{t})e^{i\xi_1\cdot x+ie^{-s}\eta_1\cdot y+e^{-2s}\tau_1\cdot\mathsf{t}}\big)\,dxdyd\mathsf{t}\\
=\ & \widehat{\chi_s}\left(\xi_1+\xi_2,\eta_2+e^{-s}\eta_1,\tau_2+e^{-2s}\tau_1\right)
\end{aligned}$$

where

$$\chi_s(x,y,\mathsf{t})=\chi_2(x,y,\mathsf{t})\chi_1\big(x,e^{-s}y,e^{-2s}\mathsf{t}\big)$$

is a bounded family of smooth compactly supported functions (this is crucial) when $s\in[0,+\infty)$. We then have for any $N\geq 1$ the upper bound

$$\big|\widehat{\chi_s}(\xi,\eta,\tau)\big|\leqslant C_N\big(1+|\xi|+|\eta|+|\tau|\big)^{-N} \tag{3.2}$$

where the constant C_N does not depend on $s\in[0,+\infty)$. Hence in any closed conic set V which does not meet the subset

$$\Lambda=\Big\{\big(\xi,-\xi,\eta,-e^{-s}\eta,\tau,-e^{-2s}\tau\big)\in(\mathbb{R}^k\times\mathbb{R}^k)^*, s\geqslant 1\Big\},$$

there exists some $\varepsilon>0$ such that for all $\big(\xi_1,\xi_2,\eta_1,\eta_2,\tau_1,\tau_2\big)\in V\subset(\mathbb{R}^k\times\mathbb{R}^k)^*$, we have for all $s\geqslant 1$ the inequality

$$\big|(\xi_1+\xi_2,e^{-s}\eta_1+\eta_2,e^{-2s}\tau_1+\tau_2)\big|\geqslant\varepsilon\big(|\xi_1|+|\xi_2|+|\eta_1|+|\eta_2|+|\tau_1|+|\tau_2|\big).$$

This implies the following Fourier bound

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^k} e^{i\xi_2\cdot x+i\eta_2\cdot y+i\tau_2\cdot\mathsf{t}}\chi_2(x,y,\mathsf{t})\ e^{-s\rho*}(\chi_1 e^{i\xi_1\cdot x+i\eta_1\cdot y+i\tau_1\cdot\mathsf{t}})\,dxdyd\mathsf{t}\Big|\\
&\qquad\leqslant C_N\left(1+|\xi_1+\xi_2|+|\eta_2+e^{-s}\eta_1|+|\tau_2+e^{-2s}\tau_1|\right)^{-N}\\
&\qquad\leqslant C_N\varepsilon^{-N}\big(1+|\xi_1|+|\xi_2|+|\eta_1|+|\eta_2|+|\tau_1|+|\tau_2|\big)^{-N}
\end{aligned}$$

for all $s\geqslant 1$ and $(\xi_1,\xi_2,\eta_1,\eta_2,\tau_1,\tau_2)\in V\subset(\mathbb{R}^k\times\mathbb{R}^k)^*$. The previous bound analyzes the wave front set of the family $\delta(z'-e^{-s\rho}(z))$ near $T^*(K\times K)\subset T^*(\mathbb{R}^k\times\mathbb{R}^k)$. Since K is arbitrary the family of distributions $\delta(z'-e^{-s\rho}(z))$ is bounded in $\mathcal{D}'_{\Gamma_\rho}(\mathbb{R}^k\times\mathbb{R}^k)$, with $\Gamma_\rho\subset T^*(\mathbb{R}^k\times\mathbb{R}^k)$ given by

$$\begin{aligned}
\Gamma_\rho=&\underbrace{\Big\{\big(x,x,0,0,0,0;\xi,-\xi,0,\eta_2,0,\tau_2\big);(\xi,\eta_2,\tau_2)\neq(0,0,0)\Big\}}_{\text{the radial set which is the conormal of the singular set of }\rho}\\
&\cup\Big\{\big(x,y,\mathsf{t},x,e^{-s}y,e^{-2s}\mathsf{t};\xi,\eta,\tau,-\xi,-e^{s}\eta,-e^{2s}\tau\big);s\geqslant 1,(\xi,\eta,\tau)\neq(0,0,0)\Big\}.
\end{aligned}$$

▷

Set

$$\pi(z):=(x,0,0). \tag{3.3}$$

Lemma 8 is useful to give a description of the Taylor subtraction operation, the Taylor subtractors of order 0 and 1 read

$$R_0:\varphi\mapsto\varphi-\varphi\circ\pi,\ \ R_1:\varphi\mapsto\varphi-\varphi(x,0,0)-y\cdot\partial_y\varphi(x,0,0)-t\partial_t\varphi(x,0,0),$$

We call these operators R_0 and R_1, with the letter R chosen for '*remainder*'. Denote generically by $[\Lambda]$ the Schwartz kernel of an operator Λ.

Proposition 9 – *The operators R_0,R_1 have Schwartz kernel*

$$[R_0]=\int_0^\infty[\rho(e^{-s'\rho})^*]ds',\quad [R_1]=\int_0^\infty\left[(1-e^{s'}+e^{s'}\rho)\rho e^{-s'\rho*}\right]ds'-[\rho]$$

and

$$[(e^{-s\rho})^*R_0]=\int_0^\infty\left[\rho(e^{-(s+s')\rho})^*\right]ds',\quad [e^{-s\rho*}R_1]=\int_0^\infty\left[(1-e^{s'}+e^{s'}\rho)\rho(e^{-(s'+s)\rho})^*\right]ds'-([e^{-s\rho})^*\rho]$$

and the families of distributions $\big([(e^{-s\rho})^*R_0]\big)_{0\leq s\leq +\infty}$ *and* $\big([(e^{-s\rho})^*R_1]\big)_{0\leq s\leq +\infty}$ *are bounded in* $\mathcal{D}'_{\Gamma_\rho}(\mathbb{R}^k\times\mathbb{R}^k)$.

Proof – We write a detailed proof for R_0; the proof for R_1 is very similar and left to the reader. For a test function χ with compact support on $\mathbb{R}^k\times\mathbb{R}^k$ write $\rho_1\chi$ for the action of the vector field on the first component of χ. We have

$$\big\langle[\rho(e^{-s\rho})^*],\chi\big\rangle = -\int_{\mathbb{R}^k}(\rho_1\chi)(z,e^{-s\rho}z)\,dz$$

and since $(\rho_1\chi)(z,e^{-s\rho}z)$ vanishes along the singular set $\{y=0,t=0\}$ the integrand is of order e^{-s}, so the integral is converging. The wavefront bound follows from the wave front bound on the propagator $[(e^{-s\rho})^*]$ and the fact that the wave front of a distribution is stable under the action the vector field ρ. ▷

We come back to the general setting of an open subset $U\subset\mathcal{X}$ and assume we are given a closed conic set Γ in $T^*U\backslash\{0\}$. It is a classical fact that for $\alpha<0$ the Besov space $C^\alpha(\mathcal{X})=B^\alpha_{\infty,\infty}(\mathcal{X})$ can be characterized as the set of distributions Λ on $\mathcal{X}$ such that

$$\sup_{x\in\mathcal{X}}\sup_{0<t\leq 1} t^{-\alpha/2}\big|\langle\Lambda,K_t(x,\cdot)\rangle\big|<\infty.$$

A distribution $\Lambda\in\mathcal{D}'(U)$ is an element of $\mathcal{D}'_\Gamma(U)$ iff for all pseudodifferential operators Q with Schwarz kernel compactly supported in $U\times U$ and whose symbol vanishes on Γ one has for every compact subset C of $\mathcal{X}$ a finite positive constant $m_{C,Q}$ such that

$$\sup_{x\in C}\sup_{0<t\leq 1}\big|\langle\Lambda,QK_t(x,\cdot)\rangle\big|\leq m_{C,Q}<\infty. \tag{3.4}$$

One can describe an element of $C^\alpha_{\mathrm{loc}}(U)$, with $\alpha<0$, with wave front set in Γ in terms similar to (3.4) as the set of distributions $\Lambda\in C^\alpha_{\mathrm{loc}}(U)$ iff for all pseudodifferential operators Q with Schwarz kernel compactly supported in $U\times U$ and whose symbol vanishes on Γ one has for every compact subset C of $\mathcal{X}$ a finite constant $m_{C,Q}$ such that

$$\sup_{x\in C}\sup_{0<t\leq 1} t^{-\alpha/2}\big|\langle\Lambda,(I+Q)K_t(x,\cdot)\rangle\big|\leq m_{C,Q}<\infty.$$

The I element ensures the element Λ is Hölder whereas the operator Q is here to test the smoothness of Λ outside of the wave front set. We now come back to the setting $\mathcal{Y}\subset\mathcal{X}$ of Section 3.1 and denote by ρ a scaling field for this embedding. Let U stand for an open set of $\mathcal{X}$ which is stable by the semiflow of ρ: $e^{-s\rho}(U)\subset U$, $\forall s\geqslant 0$. Let ρ stand for a scaling field whose backward semiflow leaves Γ fixed

$$(e^{-s\rho})^*\Gamma\subset\Gamma$$

for all $s\geq 0$.

Definition – *For* $\alpha<0$ *and* $a\in\mathbb{R}$ *we define the space* $\mathcal{S}^{\alpha,(a,\rho)}_\Gamma(U)$ *of distributions* $\Lambda\in\mathcal{D}'(U)$ *with the following property. For all pseudodifferential operators* Q *with Schwarz kernel compactly supported in* $U\times U$ *and whose symbol vanishes on* Γ*, for each compact set* $C\subset U$*, there is a finite positive constant* $m_{C,Q}$ *such that*

$$\sup_{s\geq 1}\sup_{x\in C}\sup_{0<t\leq 1} e^{as}t^{-\alpha/2}\big|\big\langle(e^{-s\rho})^*\Lambda,(I+Q)K_t(x,\cdot)\big\rangle\big|\leq m_{C,Q}<\infty.$$

We define $\mathcal{S}^a_\Gamma(U)$ as the union over α of all the spaces $\mathcal{S}^{\alpha,(a,\rho)}_\Gamma(U)$, for $a\in\mathbb{R}$ fixed and ρ a scaling field for the inclusion $\mathcal{Y}\subset\mathcal{X}$ whose backward semiflows leave Γ fixed. The letter '$\mathcal{S}$' is chosen for *scaling*. The exponent a retains the scaling property and Γ information on the wavefront set. Note that the space $\mathcal{S}^a_\Gamma(U)$ is a priori larger than conormal distributions with wavefront set in $N^*(\mathcal{Y}\subset U)$ since elements in $\mathcal{S}^a_\Gamma(U)$ might have some wavefront set contained in the cone Γ which is not necessarily included in $N^*(\mathcal{Y}\subset U)$. The notation does not emphasize the dependence of this space on the inclusion $\mathcal{Y}\subset\mathcal{X}$. This will always be clear for us from the context. An elementary example is given by the principal value of $1/|x|$ in $\mathbb{R}$, where $\mathcal{Y}=\{0\}\subset\mathbb{R}$ and it has scaling exponent $a=-1$ and wavefront set $\Gamma=T^*\mathbb{R}$. Note the fact that for all element $\Lambda\in\mathcal{S}^a_\Gamma(U)$ the family of scaled distributions $(e^{at}e^{-t\rho*}\Lambda)_{t\geqslant 0}$ is bounded in $\mathcal{D}'_\Gamma$. The next propositions give two examples of elements of some space $\mathcal{S}^{0,(a,\rho)}(U)$ for some scaling exponent a and some scaling field ρ. Denote by $\mathbf{d}_2$ the diagonal of M^2 for any set M.

Lemma 10 – *Let M be a closed manifold and $K'_t(x,y)$ be a smooth kernel on $M^2\backslash\mathbf{d}_2$ such that one can associate to any small enough open set U a coordinate system in which one has for all multiindices α,β*

$$\left|\partial^\alpha_{s,t}\partial^\beta_{x,y}K'_{|t-s|}(x,y)\right| \lesssim \left(\sqrt{t-s}+|y-x|\right)^{-a-2|\alpha|-|\beta|}. \tag{3.5}$$

Denote by ρ_2 a scaling field on M^2 for the inclusion $\mathbf{d}_2\subset M^2$ and set

$$\rho = 2(t-s)\partial_s+\rho_2,$$

and for $n\geqslant 2$, we denote by $\pi:(x_1,\dots,x_n)\in M^n\to(x_1,x_2)\in M^2$ the canonical projection on the first two components. Then the family

$$\left(e^{\ell a}(e^{-\ell\rho})^*\pi^*K_{|t-\cdot|}(\cdot,\cdot)\right)_{\ell\geq 0}$$

is bounded in $\mathcal{D}'_{N^*(\{s=t\})}\big((M^n\times\mathbb{R})\backslash(\pi^*\mathbf{d_2}\cap\{s=t\})\big)$, that is

$$\pi^*K_{|t-\cdot|}(\cdot,\cdot)\in\mathcal{S}^{0,(a,\rho)}_{N^*(\{s=t\})}\big((M^n\times\mathbb{R})\backslash(\pi^*\mathbf{d_2}\cap\{s=t\})\big).$$

In the sequel, we denote by $\mathcal{K}^a$ the C^∞–module of kernels $K_t(x,y)$ as above depending on two variables endowed with the weakest topology containing the $C^\infty\left([0,+\infty)\times M^2\backslash\mathbf{d}_2\right)$ topology and which makes all seminorms defined by the estimates (3.5) continuous.

Proof – We first localize in a neighbourhood $U\times U$ of the diagonal since K is smooth off–diagonal. It is enough to prove the claim for $K(x,y)\chi_1(y)\chi_2(x)$ where $\chi_i\in C^\infty_c(U)$ and the uase partition of unity go get the global result. In $U\times U$ we pull-back everything to the configuration space, which we write with a slight abuse of notations

$$\pi^*(K\chi_1\chi_2)(t,s,x_1,\dots,x_n)=K(t,s,x_1,x_2)\chi_1(x_1)\chi_2(x_2).$$

We already know that this kernel satisfies some bound of the form

$$|K(t,s,x_1,x_2)\chi_1(x_1)\chi_2(x_2)|\lesssim\left(\sqrt{|t-s|}+|x_1-x_2|\right)^{-a}$$

Somehow we would like to flow both sides of the inequality by the parabolic dynamics $(e^{-t\rho})^*$ and bound the term $e^{-t\rho*}\left(\sqrt{|t-s|}+|x_1-x_2|\right)^{-a}$ asymptotically when t goes to $+\infty$. We use for that purpose the normal form Theorem for the space part of the Euler vector fields

$$\rho_{[n]}=\sum_{k=2}^n h_k\cdot\partial_{h_k},$$

for some new coordinates $(h_k)_{k=2}^n$ that vanish at order 1 along the deep space diagonal $\mathbf{d}_n$. The fact that x_1-x_2 vanishes at first order along $\mathbf{d}_n$ implies by Taylor expansion at first order that

$$x_1-x_2=A(h)+\mathcal{O}(|h|^2) \tag{3.6}$$

where $A(h)$ is a linear function of $(h_k)_{k=2}^n$. One then has

$$\left(e^{-t\rho_{[n]}}\right)^*(x_1-x_2)=(e^{-t\rho_{[n]}})^*A(h)+\mathcal{O}(e^{-2t}|h|^2)=A(e^{-t}h)+\mathcal{O}(e^{-2t}|h|^2),$$

and an exponential lower bound of the form

$$e^{-t}|x_1-x_2|\lesssim\left|(e^{-t\rho_{[n]}})^*(x_1-x_2)\right|$$

which yields the desired bound

$$\left|e^{-u\rho*}D^\alpha_tD^\beta_x\pi^*(K\chi_1\chi_2)(t,s,x_1,\dots,x_n)\right|\lesssim e^{u(a+2|\alpha|+|\beta|)}\left(\sqrt{|t-s|}+|x_1-x_2|\right)^{-a-2|\alpha|-|\beta|}$$

and proves the claim. The above bound allows for instance to justify that the singularities when $x_1\neq x_2$ are conormal along the equal time region $t=s$ since we are smooth on each half region $t\geqslant s$ and $s>t$. ▷

In the sequel we endow the vector space of kernels $K_t(x,y)$ as above, depending on two variables, of the weakest topology containing the $C^\infty\left([0,+\infty)\times M^2\backslash\mathbf{d}_2\right)$ topology that makes all seminorms defined by the estimates (3.5) continuous.

3.3 The canonical extension

Let us use a unique notation $\mathbf{0}$ for the zero section of any vector bundle.

Definition – *Let $\mathcal{X}$ be a smooth manifold and $\mathcal{Y}\subset\mathcal{X}$. A closed conic set $\Gamma\subset T^*(\mathcal{X}\backslash\mathcal{Y})\backslash\mathbf{0}$ is said to satisfy the* **conormal landing condition** *if its closure $\widetilde{\Gamma}$ in $T^*(\mathcal{X})\backslash\mathbf{0}$ satisfies $\widetilde{\Gamma}\subset\big(\Gamma\cup N^*(\mathcal{Y})\big)$.*

Theorem 11 – *Let $\mathcal{X}$ be a smooth weighted manifold and $\mathcal{Y}\subset\mathcal{X}$ and $\Gamma\subset T^*(\mathcal{X}\backslash\mathcal{Y})\backslash\mathbf{0}$ be a closed conic set that satisfy the conormal landing condition. Assume we are given a family $(\Lambda_\varepsilon)_{0<\varepsilon\leq 1}$ of distributions on $\mathcal{X}$ that converge as ε goes to 0 to an element $\Lambda\in\mathcal{S}^a_\Gamma(\mathcal{X}\backslash\mathcal{Y})$.*

(a) *If*

$$a>-\mathrm{codim}_w(\mathcal{Y}\subset\mathcal{X})$$

then Λ has a unique extension into a distribution over $\mathcal{X}$ such that the convergence of Λ_ε to Λ occurs in $\mathcal{S}^a_{\Gamma\cup N^*(\mathcal{Y})}(\mathcal{X})$.

(b) *If*

$$a>-\mathrm{codim}_w(\mathcal{Y}\subset\mathcal{X})-1$$

there exists a family $\Lambda_{\mathcal{Y},\varepsilon}$ of distributions supported on $\mathcal{Y}$, with wavefront set in $N^*(\mathcal{Y})$ such that $\Lambda_\varepsilon-\Lambda_{\mathcal{Y},\varepsilon}$ has a limit in $\mathcal{D}'(\mathcal{X})$ and the convergence occurs in $\mathcal{S}^{a'}_{\Gamma\cup N^*(\mathcal{Y})}(\mathcal{X})$ for all

$$a'<a.$$

Proof – We follow the proof of similar results proved in an elliptic setting in [25] – see Theorem 1.10, Theorem 4.4 and Section 6 therein. We give here the main arguments to emphasize the differences with [25] that come from our parabolic setting.

Let ρ be a scaling field for the inclusion $\mathcal{Y}\subset\mathcal{X}$ such that $\Lambda\in\mathcal{S}^{\alpha,(a,\rho)}_{\Gamma\cup N^*(\mathcal{Y})}(\mathcal{X})$ for some $\alpha\in\mathbb{R}$, and let χ be a smooth function equal to 1 in a neighborhood of $\mathcal{Y}$ stable by the backward semiflow of ρ and such that χ vanishes outside some larger neighborhood.

1. We first use the normal form theorem to reduce our problem to the model case of a distribution on $\mathbb{R}^k$, with $k=d_1+d_2+d_3$ with coordinates (x,y,t), the scaling field $\rho=y\partial_y+2t\partial_t$ is linear and globally defined, and the extension is done with respect to the linear subspace $\mathbb{R}^{d_1}\subset\mathbb{R}^k$. We work in that setting in the remainder of the proof. Let then $(\Theta_\varepsilon)_{0<\varepsilon\leq 1}$ be a family of distributions on $\mathbb{R}^k$. We assume that the Θ_ε converge as ε goes to 0 to an element $\Theta\in\mathcal{S}^a_\Gamma(\mathbb{R}^k\backslash\mathbb{R}^{d_1})$, where $a>-\mathrm{codim}_w(\mathcal{Y}\subset\mathcal{X})$.

2. Pick $0<s_0$ and think of it as being large. We use the identity

$$\mathrm{Id}-(e^{s_0\rho})^*\chi=\mathrm{Id}-\chi+\int_0^{s_0}(e^{s\rho})^*\,(-\rho\chi)\,ds$$

to define an extension of our distribution Θ. Set for convenience $\overline{\chi}:=-\rho\chi$ whose support does not meet $\mathcal{Y}$. We have for any test function $f\in\mathcal{D}(\mathbb{R}^k)$

$$\big\langle\Theta(1-(e^{s_0\rho})^*\chi),f\big\rangle=\big\langle\Theta(1-\chi),f\big\rangle+\int_0^{s_0}e^{-s(d_2+2d_3)}\,\big\langle\overline{\chi}(e^{-s\rho})^*\Theta,(e^{-s\rho})^*f\big\rangle\,ds.$$

The exponential factor comes from the Jacobian of the flow of $e^{-s\rho}$. Note that $d_2+2d_3=\mathrm{codim}_w(\mathcal{Y}\subset\mathcal{X})$. If $\Gamma\subset T^*\mathbb{R}^k\backslash\mathbf{0}$ stands for a closed conic set invariant by the lifted dynamics of $(e^{-s\rho})^*$ such that $\Gamma\cap T^*\mathbb{R}^{d_1}\subset N^*(\mathbb{R}^{d_1})$, our choice of scaling exponent a ensures that the family

$$(\Theta^{(s)})_{s\geq 0}:=(e^{as}(e^{-s\rho})^*\Theta)_{s\geq 0}$$

is bounded in $\mathcal{D}'_\Gamma(\mathbb{R}^k)$. One then has for the Schwartz kernels

$$\big[\Theta\,(1-(e^{s_0\rho})^*\chi)\,\big](z,z')=\big[\Theta(1-\chi)\big](z,z')+\int_0^{s_0}e^{-s(a+d_2+2d_3)}\big(\overline{\chi}\,\Theta^{(s)}\big)(z)\delta\big(z'-e^{-u\rho}(z)\big)\,ds$$

We know from the hypocontinuity theorem on the Hörmander product of distributions [18, Thm 6.1 p. 219] that the family

$$(\overline{\chi}\Theta^{(s)})(z)\delta(z'-e^{-u\rho}(z))$$

with $s \geq s_1$ large enough, is bounded in $\mathcal{D}'_{\overline{\Gamma}}(\mathbb{R}^k)$, where

$$\overline{\Gamma} := (\Gamma \times \mathbf{0}) \cup \Gamma_\rho \cup \big((\Gamma \times \mathbf{0}) + \Gamma_\rho\big).$$

For the moment this means that the s_0-dependent family of distributions associated with the kernels

$$(\star) := \int_0^{s_0} e^{-s(a+d_2+2d_3)} \big(\overline{\chi}\,\Theta^{(s)}\big)(z)\delta(z' - e^{-s\rho}(z))\,ds$$

is bounded in $\mathcal{D}'_{\overline{\Gamma}}(\mathbb{R}^k \times \mathbb{R}^k)$ uniformly in $s_0 \geq s_1$; in particular the integral converges in $\mathcal{D}'_{\overline{\Gamma}}(\mathbb{R}^k \times \mathbb{R}^k)$ when s_0 goes to $+\infty$. Now we interpret the integral over the variable z as a pushforward along the fibers of the linear projection

$$\pi : (z, z') \mapsto z'.$$

The pushforward Theorem yields that $\pi_*(\star)$ is bounded in $\mathcal{D}'_{\pi_*\overline{\Gamma}}(\mathbb{R}^k)$ where

$$\pi_*\overline{\Gamma} = (\pi_*\Gamma_\rho) \cup \pi_*\left(\Gamma \times \mathbf{0} + \Gamma_\rho\right)$$

and

$$\pi_*\Gamma_\rho = \big\{\big((x,0,0),(0,\eta,\tau)\big)\big\}$$
$$\pi_*\left(\Gamma \times \mathbf{0} + \Gamma_\rho\right) = \Big\{\big(e^{-s\rho}(z), (e^{-s\rho})^*(\lambda)\big); (z,\lambda) \in \Gamma, 0 \leq s \leq +\infty\Big\} \subset \Gamma \cup N^*(\mathcal{Y})$$

since the cone Γ is invariant by the lifted flow of $e^{-s\rho}$ provided $s < +\infty$ and the limit points of the form $\lim_{s\to+\infty}\big(e^{-s\rho}(z), (e^{-s\rho})^*(\lambda)\big)$ for $(z,\lambda) \in \Gamma$ must belong to the conormal $N^*(\mathcal{Y})$ by the conormal landing condition on Γ. It is at this precise place we are using the conormal landing condition assumption on Γ. The distributions $\Theta(1 - (e^{s\rho})^*\chi)$ are thus converging in $\mathcal{D}'_{\Gamma\cup N^*\mathbb{R}^{d_1}}(\mathbb{R}^k)$ to

$$\langle \Theta^+, f\rangle = \langle \Theta(1-\chi), f\rangle + \int_0^\infty e^{-s(d_2+2d_3)} \langle \overline{\chi}(e^{-s\rho})^*\Theta, f \circ e^{-s\rho}\rangle\,ds$$

The uniqueness of the extension Θ^+ follows from the continuity of all the operations involved in above. To see the scaling property of the extension we note that the family

$$\Big(\Theta^{\ell'} := e^{a\ell}(e^{-\ell\rho})^*\Theta\Big)_{\ell\geq 0}$$

is bounded in $\mathcal{D}_\Gamma(\mathbb{R}^k)$, since $\Theta \in \mathcal{S}^a_\Gamma(\mathbb{R}^k)$ means that the family $\big(e^{a\ell}(e^{-\ell\rho})^*\Theta\big)_{\ell\geq 0}$ is bounded in $\mathcal{D}'_\Gamma(\mathbb{R}^k)$ by definition of $\mathcal{D}'_\Gamma(\mathbb{R}^k)$ and $\mathcal{S}'_\Gamma(\mathbb{R}^k)$, and observe that

$$e^{sa}(e^{-s\rho})^*\Theta^+ = \pi_*\left(\int_0^\infty e^{-s(a+d_2+2d_3)}\left(\Theta^{s+a'}\overline{\chi}\otimes 1\right)[e^{-s\rho}]\,ds\right).$$

3. In the borderline case our proof follows closely [25, Prop 4.9 p. 841] except we work in a parabolic setting. We proceed as above with $(e^{-s\rho})^*$ replaced by $(e^{-s\rho})^*R_0$ if $-\mathrm{codim}_w(\mathcal{Y} \subset \mathcal{X}) - 1 < a \leqslant -\mathrm{codim}_w(\mathcal{Y} \subset \mathcal{X})$ and $(e^{-s\rho})^*R_1$ if $a = -\mathrm{codim}_w(\mathcal{Y} \subset \mathcal{X}) - 1$. So our extension reads

$$\langle \Theta^+, f\rangle = \langle \Theta(1-\chi), f\rangle + \int_0^\infty e^{-s(d_2+2d_3)}\big\langle \overline{\chi}(e^{-s\rho})^*\Theta, \big(R_i f\big)\circ e^{-s\rho}\big\rangle\,ds,$$

where $R_i f, i = 0,1$ is obtained from f by Taylor subtraction. The integral converges absolutely since $[e^{-s\rho}R_i] = \mathcal{O}_{\mathcal{D}'_\Gamma}(e^{-s(1+i)})$ and the map

$$\Theta \mapsto \Theta^+$$

is continuous from $\mathcal{S}^a_\Gamma(\mathcal{X}\setminus\mathcal{Y})$ to $\mathcal{S}^{a'}_{\Gamma\cup N^*(\mathcal{Y})}(\mathcal{X})$ for all $a' < a$ as we will see below when we check the weak homogeneity of the extension Θ^+. This shows that when $-\mathrm{codim}_w(\mathcal{Y} \subset \mathcal{X}) - 1 < a \leqslant -\mathrm{codim}_w(\mathcal{Y} \subset \mathcal{X})$, one can take

$$\Lambda_{\mathbb{R}^{d_1},\varepsilon}(f) = \int_0^\infty e^{-s(d_2+2d_3)}\big\langle \Theta_\varepsilon, \overline{\chi}\circ e^{-s\rho}\,\Pi(f)\big\rangle\,ds$$

and when $a = -\mathrm{codim}_w(\mathcal{Y} \subset \mathcal{X}) - 1$, we choose

$$\Lambda_{\mathbb{R}^{d_1},\varepsilon}(f) = \int_0^\infty e^{-s(d_2+2d_3)}\big\langle \Theta_\varepsilon, \overline{\chi}\circ e^{-s\rho}\,\Pi(f) + t.(\partial_t f)(.,0,0) + y.(\partial_y f)(.,0,0)\big\rangle\,ds.$$

For simplicity, in the remainder of the proof we shall specialize to the case $-\mathrm{codim}_w(\mathcal{Y}\subset\mathcal{X})-1 < a \leqslant -\mathrm{codim}_w(\mathcal{Y}\subset\mathcal{X})$. By the wave front set condition on Θ_ε, one can always decompose $\Lambda_{\mathbb{R}^{d_1},\varepsilon}$ under the product form

$$\Lambda_{\mathbb{R}^{d_1},\varepsilon} = c_\varepsilon \Lambda_{\mathbb{R}^{d_1}}$$

where

$$\Lambda_{\mathbb{R}^{d_1}} = \Pi$$

is a distribution independent of ε, supported on $\mathbb{R}^{d_1}$, with wavefront set contained in $N^*(\mathbb{R}^{d_1})$, and the function c_ε is given by

$$c_\varepsilon(x) = \int_0^\infty e^{-s(d_2+2d_3)}\Theta_\varepsilon(x,y,t)\overline{\chi}\big(e^{-s\rho}(x,y,t)\big)\, dsdydt.$$

To check the weak homogeneity write

$$e^{s'a}\Big\langle e^{-s'\rho}\Theta^+, f\Big\rangle = \int_0^\infty \Big\langle \Theta^{(s)}\overline{\chi}, e^{-(s-s')\rho}\varphi - \Pi(\varphi)\Big\rangle\, ds$$

and observe that the support of $e^{-(s-s')\rho}\varphi$ meets the support of $\overline{\chi}$ only if $s \geqslant C+s'$ for a constant C that depends only on the support of $\overline{\chi}$. So the integral can be split in

$$-\int_0^{C+s'} \Big\langle \Theta^{(s)}\overline{\chi}, \Pi(\varphi)\Big\rangle\, ds + \int_{C+s'}^\infty \Big\langle \Theta^{(s)}\overline{\chi}, e^{-(s-s')\rho}\varphi - \Pi(\varphi)\Big\rangle\, ds$$

A change of variable shows that the second term is uniformly bounded in s' whereas the first term is bounded above by $(C+s')\|\varphi\|_{C^0}$. This concludes the proof that $e^{ts}e^{-t\rho}\overline{U} = \mathcal{O}_{\mathcal{D}'}(t)$. ▷

In the case (b), note that the extension is no longer unique. Any two extensions differ by some conormal distribution supported on $\mathcal{Y}$ whose order is 0 if $-\mathrm{codim}_w(\mathcal{Y}\subset\mathcal{X})-1 < a \leqslant -\mathrm{codim}_w(\mathcal{Y}\subset\mathcal{X})$ and of order 1 if $a = -\mathrm{codim}_w(\mathcal{Y}\subset\mathcal{X})-1$.

3.4 Configuration space

In the sequel and unless otherwise specified, we will denote by $\mathcal{M}$ the space–time $\mathbb{R}\times M$ and space–time points $(t,x)\in\mathbb{R}\times M$ are shortly written as $m\in\mathcal{M}$. The configuration space $\mathcal{M}^p$ of p points in $\mathcal{M}$ will play a particular role in the sequel. For $I\subset\{1,\dots,p\}$ we denote by

$$\begin{aligned}\mathbf{d}_I &:= \big\{m=(m_1,\dots,m_p)\,;\, m_i=m_j, \text{ for } i\neq j \text{ iff } (i,j)\in I^2\big\}\\ \mathcal{T}_I &:= \big\{m=(m_1,\dots,m_p)\,;\, t_i=t_j, \text{ for } i\neq j \text{ iff } (i,j)\in I^2\big\}\end{aligned}$$

the corresponding diagonal in the product space. One has $\mathbf{d}_I\subset\mathcal{T}_I$ and $\mathbf{d}_I\cap\mathbf{d}_J=\emptyset$ (resp $\mathcal{T}_I\cap\mathcal{T}_J=\emptyset$) if $I\neq J$. We denote by

$$\mathbf{d} = \big\{(m_1,\dots,m_1)\in\mathcal{M}^p\,;\, m_1\in\mathcal{M}\big\}$$

the deepest diagonal in $\mathcal{M}^p$. In the sequel, we will work on some submanifold $\mathcal{T}_J\subset\mathcal{M}^p$ for some fixed $J\subset\{1,\dots,p\}$ where all time variables indexed by J coincide and we will use a particular class of scaling fields that will leave all $\mathcal{T}_J$ and $\mathbf{d}_I$ stable.

Definition 12 – *Pick some open chart $U\subset M, \kappa : U\mapsto\mathbb{R}^d$ such that $\kappa(U)\subset\mathbb{R}^d$ is an open convex ball – this is always possible up to making things smaller. We define a scaling field $\rho_{[p]}$ in U from its flow given by for any $(x_1,\dots,x_p)\in\kappa(U)^p$ by*

$$(x_1,\dots,x_p)\in U^p\mapsto\big(x_1, e^{-t}(x_2-x_1)+x_1,\dots,e^{-t}(x_p-x_1)+x_1\big)\in\kappa(U)^p.$$

We define the local scaling field on $(\mathbb{R}\times U)^p$, with local coordinates $(s_i,x_i)_{1\leq i\leq p}$, setting

$$\rho = 2\Big(\sum_{j=2}^p (s_j-s_1)\partial_{s_j}\Big) + \rho_{[p]} =: 2\rho_{times} + \rho_{[p]}.$$

We obtain global scaling fields by gluing together the above local objects. Consider a cover $\cup_i U_i^p$ of some neighborhood of the space diagonal and choose $\chi\in C_c^\infty(\cup_i U_i^p)$ such that $\chi=1$ near the space diagonal. For a subordinated partition of unity $\sum\chi_i=1$ of support of χ, we set $\rho = 2\rho_{times}+\chi\sum_i\chi_i\rho_i$ where each local scaling field $\rho_i\in C^\infty(T(U_i^p))$ is constructed in charts as above.

We will typically be given a family $(\Lambda_\varepsilon)_{0<\varepsilon\leq 1}$ of distributions on $\mathcal{T}_J \setminus \big(\cup_{I\subset\{\{1,\dots,p\}\}}\mathbf{d}_I\big)$ that converge to a limit as a distribution outside all the diagonals of $\mathcal{T}_J$. We will use Theorem 11 to extend it to the whole of $\mathcal{T}_J$ by an inductive procedure under some scaling-type assumptions. The inductive structure of the extension procedure will come from the geometric form of *Popineau & Stora's lemma*, which we recall here. We associate to $I \subset \{1,\dots,p\}$ the open set

$$\mathcal{O}_I := \Big\{ m = (m_1,\dots,m_p) \in \mathcal{M}^p\,;\, m_i \neq m_j \ \forall\,(i,j) \in I \times I^c \Big\}.$$

Lemma 13 – *One has*

$$\mathcal{M}^p \backslash \mathbf{d} = \bigcup_{I\subset\{1,\dots,p\}} \mathcal{O}_I$$

and there is an associated smooth partition of the unity, $1 = \sum_{I\subset\{1,\dots,p\}} \eta_I \in C^\infty(\mathcal{M}^p \setminus \mathbf{d})$*, with the family* $(\eta_I \circ e^{-s\rho})_{s\geq 0}$ *bounded in* $C^\infty(\mathcal{M}^p\backslash\mathbf{d})$ *for some scaling field* ρ *for the inclusion* $\mathbf{d} \subset \mathcal{M}^p$*.*

The proof is simple and can be found in [26, Lemma 6.3]. The proof of the claim on the family $(\eta_I \circ e^{-s\rho})_{s\geq 0}$ can be found in [27, Lemma 6.3.1 p. 131]. The above partition of unity induces naturally a partition of unity on $\mathcal{T}_J \setminus \mathbf{d}$ with the same properties.

In the simplest cases the distributions Λ_ε will be given as products of distributions, with each factor depending possibly only on a subset of the variables $\mathcal{M}^p$. The easiest case in which to make sense of such products relies on Hörmander's product theorem [18, Thm 6.1 p. 219] and gives the following statement.

Lemma 14 – *If* $\Lambda_1 \in \mathcal{D}'(\mathcal{M}^p)$ *depends only on the first* $1 \leq k < p$ *components of* $\mathcal{M}^p$ *and* $\Lambda_2 \in \mathcal{D}'(\mathcal{M}^p)$ *depends only on the last* $p-(k-1)$ *components, so they have only one component in common, and*

$$WF(\Lambda_1) \subset \bigcup_{I\subset\{1,\dots,k\}} N^*(\mathbf{d}_I) \cup N^*(\mathcal{T}_I)\,,$$
$$WF(\Lambda_2) \subset \bigcup_{J\subset\{k,\dots,p\}} N^*(\mathbf{d}_J) \cup N^*(\mathcal{T}_J)\,,$$

then the product $\Lambda_1\Lambda_2$ *is well-defined in* $\mathcal{D}'(\mathcal{M}^p)$ *and*

$$WF(\Lambda_1\Lambda_2) \subset \big(WF(\Lambda_1) + WF(\Lambda_2)\big) \cup WF(\Lambda_1) \cup WF(\Lambda_2).$$

Proof – Denote by λ a generic element of $T^*\mathcal{M}$. If $(\lambda_1,\dots,\lambda_k,0,\dots,0)$ and $(0,\dots,,0,\mu_k,\mu_{k+1},\dots,\mu_p)$ stand for non-null elements of $T^*(\mathcal{M}^p)$ such that

$$\sum \lambda_i = 0, \quad \sum \mu_j = 0,$$

then the convex sum

$$(\lambda_1,\dots,\lambda_k,0,\dots,0) + (0,\dots,,0,\mu_k,\mu_{k+1},\dots,\mu_p) = (\lambda_1,\dots,\lambda_k+\mu_k,\mu_{k+1},\dots,\mu_p)$$

cannot vanish. This implies that $WF(\Lambda_1) \cup (-WF(\Lambda_2)) = \{0\}$. ▷

We give another important consequence of Theorem 11 before talking about Feynman amplitudes. We are particularly interested in the case where $\mathcal{M} = \mathbb{R} \times M$. Denote by $\mathbf{d}_{M^p\subset\mathcal{M}^p}$ the canonical injection of the deepest diagonal of M^p into $\mathcal{M}^p$.

Proposition 15 – *Let* Γ_1, Γ_2 *be closed conic sets in* $T^*(\mathcal{M}^p)$ *satisfying the following* **condition** *with respect to* $\mathbf{d}_{M^p\subset\mathcal{M}^p}$

$$\mathbf{0} \notin (\Gamma_1 + \Gamma_2) \cap \big(T^*(\mathcal{M}^p)\backslash\mathbf{d}_{M^p\subset\mathcal{M}^p}\big).$$

Let ρ *stand for a parabolic scaling field for the inclusion* $\mathbf{d}_{M^p\subset\mathcal{M}^p} \subset \mathcal{M}^p$ *whose backward semi-flow leave both* Γ_1 *and* Γ_2 *fixed. Assume we are given two distributions* $\Lambda_1 \in \mathcal{S}^{(s_1,\rho)}_{\Gamma_1}(\mathcal{M}^p), \Lambda_2 \in \mathcal{S}^{(s_2,\rho)}_{\Gamma_2}(\mathcal{M}^p)$*, so the product* $\Lambda_1\Lambda_2$ *is well-defined on* $\mathcal{M}^p\backslash\mathbf{d}_{M^p\subset\mathcal{M}^p}$*. This product has a unique extension as an element of* $\mathcal{S}^{(s_1+s_2,\rho)}_{\Gamma}(\mathcal{M}^p)$ *with*

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup (\Gamma_1+\Gamma_2) \cup N^*(M^p \subset \mathcal{M}^p).$$

We invite the reader to check that the condition $\mathbf{0} \notin (\Gamma_1+\Gamma_2)\cap\big(T^*(\mathcal{M}^p)\backslash\mathbf{d}_{M^p\subset\mathcal{M}^p}\big)$ ensures that $\Gamma_1+\Gamma_2$ satisfies the conormal landing condition for the inclusion $\mathbf{d}_{M^p\subset\mathcal{M}^p} \subset \mathcal{M}^p$. The statement means that for any mollification $\Lambda_1^\varepsilon, \Lambda_2^\varepsilon$ of these distributions the product $\Lambda_1^\varepsilon \Lambda_2^\varepsilon$ is converging in $\mathcal{S}_\Gamma^{(s_1+s_2,\rho)}(\mathcal{M}^p)$ to a limit independent of the mollification.

Let $\mathcal{G} = (V, E)$ be an oriented finite graph with p vertices (V) and edge set E with no two edges with the same vertices. We are also given $J \subset V$ which describes coinciding times. We associate to each vertex $v \in V$ a variable $z_v = (t_v, x_v) \in (\mathbb{R} \times M)$ and to each edge $e \in E$ its two vertices $v(e)_-, v(e)_+$ according to its orientation. We assume

$$V = V' \sqcup V_A$$

where V_A is a disjoint union of $n_\mathcal{G}$ triples of vertices, with each triple made up of an unordered pair of vertices and another vertex. We write $((v_1^j, v_2^j), v_*^j)$ such triples and denote by $n_\mathcal{G}$ the number of such triples. The set V' is a disjoint union of singletons. We assume there is no edge in the graph relating two points of an unordered pair or one of these points to the single vertex of the associated triple. We assume we are given for each edge $e \in E$ a kernel $K_e \in \mathcal{K}^{a_e}$ for some scaling exponent $a_e \in \mathbb{R}$ where the space $\mathcal{K}^{a_e}$ is defined in Lemma 10. We also assume we are given a distribution

$$[\odot](x, y, z) \in \mathcal{S}^{-6}_{N^*(\{x=y\}) \cup N^*(\{z=y\}) \cup N^*(\{x=y=z\})}(M^3)$$

where the scaling is with respect to $\mathbf{d} \subset M^3$. We view $[\odot]$ as a distribution on $\mathcal{M}^3$ still denoted by $[\odot]$ – just pull–back by the canonical projection from $\mathcal{M}^3$ to M^3. Denote by $\mathbf{d}_{V'}$ the diagonals of $(\mathbb{R} \times M)^p$, for $V' \subset V$. The amplitude $\mathcal{A}^\mathcal{G}$ associated with the graph $\mathcal{G}$ is the distribution on $\mathcal{T}_J \backslash \bigcup_{V' \subset V} \mathbf{d}_{V'}$ defined by the product

$$\mathcal{A}_\mathcal{G}(z_1, \dots, z_p) := \prod_{e \in E} K_e\big(z_{v(e)_-}, z_{v(e)_+}\big) \prod_{1 \le j \le n_\mathcal{G}} [\odot]\big((x_{v_1^j}, x_{v_2^j}), x_{v_*^j}\big)$$

with the second product corresponding to all of V_A, see figure 1 for a fully detailed example. We talk of $\mathcal{A}_\mathcal{G}$ as the *Feynman amplitude* associated with $\mathcal{G}$. The following fact is a direct consequence of Lemma 14.

Lemma 16 – *If $\mathcal{G} = (V, E)$ is a tree, for every $e \in E$, each two point kernel K_e belongs to the module $\mathcal{K}^{a_e}, a_e \in \mathbb{R}$ endowed with the topology of Lemma 10 and each three point kernel*

$$[\odot](x, y, z) \in \mathcal{S}^{-6}_{N^*(\{x=y=z\})}(M^3)$$

where the scaling is with respect to $\mathbf{d} \subset M^3$. Then the multilinear map that sends $\big([\odot], (K_e)_{e \in E}\big) \in \mathcal{S}^{-6}_{N^*(\{x=y=z\})}(M^3) \times \prod_{e \in E} \mathcal{K}^{a_e} \longmapsto \mathcal{A}_\mathcal{G} \in \mathcal{D}'_\Gamma\big(\mathcal{T}_J\big)$ to the Feynman amplitude $\mathcal{A}_\mathcal{G} \in \mathcal{D}'_\Gamma\big(\mathcal{T}_J\big)$ where $\Gamma = \cup_{V' \subset V} N^*(\mathbf{d}_{V'}) \cup N^*\left(\mathcal{T}_{V'}\right)$, is continuous.

The weak homogeneity exponent -6 for $[\odot]$ comes from the fact that $[\odot] \in \mathcal{D}'(M^3)$ is the Schwartz kernel of the resonant product and that our manifold M has dimension 3. Lemma 14 also allows to restrict the analysis of Feynman amplitudes to connected irreducible subgraphs since the amplitude of a reducible graph reads

$$\mathcal{A}_{\mathcal{G}_1}(\mathbf{z}_1, z_i) K(z_i, z_j) \mathcal{A}_{\mathcal{G}_2}(z_j, \mathbf{z}_2)$$

for collective variables $(\mathbf{z}_1, z_i), (z_j, \mathbf{z}_2)$ partitioning $\{z_v\}_{v \in V}$, corresponding to a partition $V = V_1 \cup V_2$ of V, where $K(z_i, z_j)$ has wave front contained in $N^*\big(\mathbf{d}_2((\mathbb{R} \times M)^2)\big) \cup N^*\big(\{t_i = t_j\}\big)$ and

$$WF(\mathcal{A}_{\mathcal{G}_1}) \subset \bigcup_{V_1' \subset V_1} N^*(\mathbf{d}_{V_1'}) \cup N^*\left(\mathcal{T}_{V_1'}\right),\ WF(\mathcal{A}_{\mathcal{G}_2}) \subset \bigcup_{V_2' \subset V_2} N^*(\mathbf{d}_{V_2'}) \cup N^*\left(\mathcal{T}_{V_2'}\right).$$

Applying Lemma 14 twice allows to show that the product is well-defined so the only difficulty is to treat the amplitudes $\mathcal{A}_{\mathcal{G}_1}$ and $\mathcal{A}_{\mathcal{G}_2}$.

Our next goal is to describe a recursive algorithm that controls the convergence as distribution over the whole space $\mathcal{T}_J$. We prove in the next statement that the multilinear Feynman map

$$\big([\odot], (K_e)_{e \in E}\big) \in \mathcal{S}^{-6}_{N^*(\{x=y=z\})}(M^3) \times \prod_{e \in E} \mathcal{K}^{a_e} \longmapsto \mathcal{A}_\mathcal{G} \in \mathcal{S}^a_\Gamma\big(\mathcal{T}_J\big) \tag{3.7}$$

where

$$\Gamma = \bigcup_{V' \subset V} \big(N^*(\mathbf{d}_{V'}) \cup N^*\left(\mathcal{T}_{V'}\right)\big), \quad a = -6 n_\mathcal{G} - \sum_{e \in E} a_e,$$

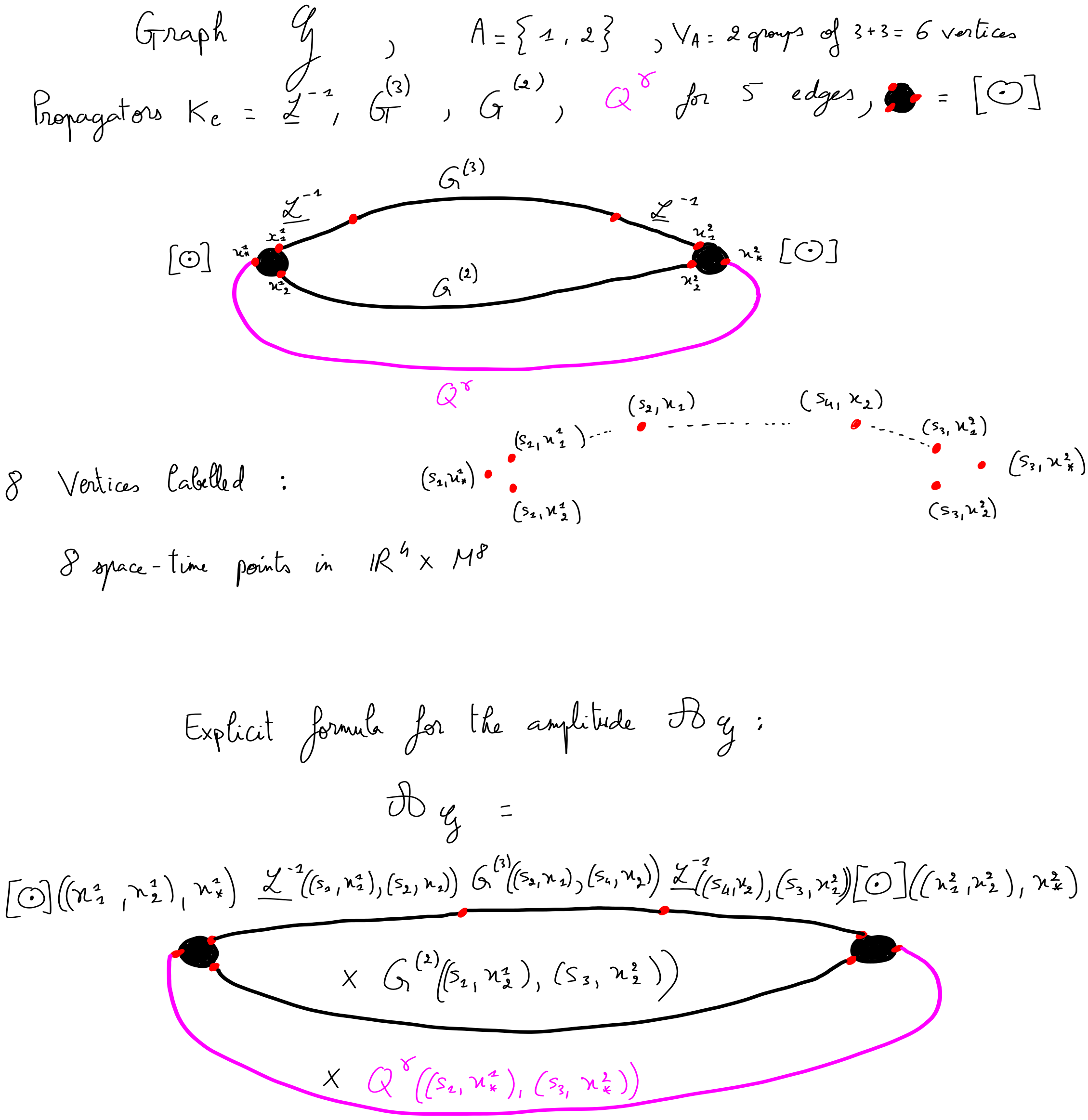


FIGURE 1. Feynman rules for some quintic graph.

is continuous under suitable conditions on the weak homogeneity a_e of the two point kernels $(K_e)_{e\in E}$. We only consider below some subgraphs $\mathcal{G}'=(V',E')$ of $\mathcal{G}$ which contain all the points of a given triple if ever they contain one of them. Recall all our analysis takes place in the submanifold $\mathcal{T}_J$ of $\mathcal{M}^p$. We denote by $\mathbf{d}_{V'}$ the deepest diagonal of $(\mathbb{R}\times M)^{|V'|}$, and with a slight abuse of notation also denote by $\mathbf{d}_{V'}$ the diagonal $\mathbf{d}_{V'}\cap\mathcal{T}_J$. Set

$$a_{\mathcal{G}} := \sum_{e\in E} a_e$$

and

$$\Gamma_{\mathcal{G}} := \bigcup_{V' \subset V} \left(N^*(\mathbf{d}_{V'} \subset \mathcal{T}_J) \cup N^*\left(\mathcal{T}_{V'} \subset \mathcal{T}_J\right)\right).$$

Theorem 17 – *The following holds.*

(a) *If every connected irreducible subgraph $\mathcal{G}' = (V', E')$ of $\mathcal{G}$ with non-trivial fundamental group satisfies*

$$\sum_{e \in E'} a_e + \operatorname{codim}_w(\mathbf{d}_{V'}) > 0. \tag{3.8}$$

Then the Feynman map defined by equation 3.7 is continuous.

(b) *If every connected irreducible strict subgraph of $\mathcal{G}$ with non-trivial fundamental group satisfies condition* (3.8) *and $\mathcal{G}$ satisfies*

$$\sum_{e \in E} a_e + \operatorname{codim}_w(\mathbf{d}_V) > -1$$

then there exists a family $\Lambda_{\mathcal{Y},\varepsilon}$ of distributions supported on $\mathbf{d}_V$, with wavefront set in $N^*(\mathbf{d}_V)$ such that $\Lambda_\varepsilon - \Lambda_{\mathcal{Y},\varepsilon}$ has a limit in $\mathcal{D}'\big((\mathbb{R} \times M)^p\big)$ and the convergence occurs in $\mathcal{S}^{a'}_{\Gamma_{\mathcal{G}} \cup N^*(\mathbf{d}_V) \cup N^*(\mathcal{T}_V)}\big((\mathbb{R} \times M)^p\big)$ for all

$$a' < -\operatorname{codim}_w\big(\mathbf{d}_V \subset \big((\mathbb{R} \times M)^p\big)\big).$$

Proof – For a subset I of $\{1, \dots, p\}$ we agree to denote by $\mathcal{G}_I$ the subgraph of $\mathcal{G}$ with vertex set I and edge set the edges $e \in E$ with vertices $v(e)_\pm$ both in I.

(a) We use repeatedly item (a) of Theorem 11 in an induction procedure to extend $\mathcal{A}_{\mathcal{G}}$ from the open set $\mathcal{T}_J \backslash \bigcup_{I \subset \{1,\dots,p\}} \mathbf{d}_I$ to $\mathcal{T}_J$, by extending it first to $\mathcal{T}_J \backslash \bigcup_{|I| \geq 3} \mathbf{d}_I$ then $\mathcal{T}_J \backslash \bigcup_{|I| \geq 4} \mathbf{d}_I$ and so on. Here, we make an essential use of the fact that the flow by scaling fields ρ from definition 12 preserves both time and space–time diagonals $\mathcal{T}_J, \mathbf{d}_I$. Moreover, the symplectic lifts of these scaling fields preserve the conormals of all time and space–time diagonals $N^*\left(\mathcal{T}_J\right), N^*\left(\mathbf{d}_I\right)$. This is crucial to satisfy the stability assumptions of the cones by the scaling fields in Proposition 15. At step ℓ one extends the distribution by adding the disjoint diagonals $\mathbf{d}_I$ with $|I| = \ell + 1$ in any order $\mathbf{d}_{I_1}, \mathbf{d}_{I_2}, \dots$ by viewing $\mathbf{d}_{I_k}$ as a submanifold of

$$\mathcal{E}^k_\ell := \left(\mathcal{T}_J \backslash \bigcup_{|I| \geq \ell} \mathbf{d}_I\right) \cup \bigcup_{j=1}^{k} \mathbf{d}_{I_j}.$$

We associate to every $I' \subset I_k$ the open set

$$\mathcal{U}_{I'} := \Big\{m = (m_1, \dots, m_p) \in \mathcal{E}^k_\ell \,;\, m_i \neq m_j \ \forall (i,j) \in I' \times (I_k \backslash I')\Big\}.$$

As in the geometric Popineau & Stora lemma one has

$$\mathcal{E}^{k-1}_\ell = \bigcup_{I' \subset I_k} \mathcal{U}_{I'}$$

and there is an associated smooth partition of the unity, $1 = \sum_{I' \subset I_k} \eta_{I'}$, with the family $(\eta_{I'} \circ e^{-s\rho})_{s \geq 0}$ bounded in $C^\infty\left(\mathcal{T}_J \backslash \bigcup_{|I| \geq \ell} \mathbf{d}_I\right)$ for some scaling field ρ for the inclusion $\mathbf{d}_{I_k} \subset \mathcal{E}^k_\ell$. We have on each open set $\mathcal{U}_{I'}$ the identity

$$\mathcal{A}_{\mathcal{G}_{|\mathcal{U}_{I'}}} = \mathcal{A}_{\mathcal{G}_{I'}}(z_j)_{j \in I'}\, \mathcal{A}_{\mathcal{G}_{I_k \backslash I'}}(z_j)_{j \in I_k \backslash I'} \prod_{(i,j) \in I' \times (I_k \backslash I')} K_{e_{ij}}(z_i, z_j),$$

where $K_{e_{ij}}$ is the propagator connecting i and j. We know by induction that the two amplitudes $\mathcal{A}_{\mathcal{G}_J}(z_j)_{j \in I'}$ and $\mathcal{A}_{\mathcal{G}_J}(z_j)_{j \in I_k \backslash I'}$ are well-defined distributions with wave front sets

$$WF(\mathcal{A}_{\mathcal{G}_{I'}}) \subset \bigcup_{J' \subset I'} N^*(\mathbf{d}_{J'}) \cup N^*\left(\mathcal{T}_{J'}\right)$$

and

$$WF(\mathcal{A}_{\mathcal{G}_{I_k \backslash I'}}) \subset \bigcup_{J'' \subset I_k \backslash I'} N^*(\mathbf{d}_{J''}).$$

One can then use Proposition 15 for the finite sum

$$\sum_{I'\subset I_k} \eta_{I'} \mathcal{A}_{\mathcal{G}_{I'}}(z_j)_{j\in I'}\, \mathcal{A}_{\mathcal{G}_{I_k\setminus I'}}(z_j)_{j\in I_k\setminus I'} \prod_{(i,j)\in I'\times(I_k\setminus I')} K_{e_{ij}}(z_i,z_j)$$

to see that it has a unique extension as an element of $\mathcal{S}^{-a}_{\Gamma_{\mathcal{G}}}(\mathcal{T}_J)$, with $a = 6n_{\mathcal{G}} + \sum_{e\in E} a_e$ and

$$\Gamma_{\mathcal{G}} = \bigcup_{V'\subset V} \big(N^*\big(\mathbf{d}_{V'}\subset \mathcal{T}_J\big) \cup N^*\left(\mathcal{T}_{V'}\subset \mathcal{T}_J\right)\big)$$

that depends continuously on its restriction to $\mathcal{E}^{k-1}_\ell$. (Each element in the sum is indeed weakly homogeneous of degree strictly greater than $-\mathrm{codim}_w(\mathbf{d}_{I_k})$.)

(b) We proceed as in **(a)** all the way down to the final step where we extend the distribution to the deepest diagonal using item (b) of Theorem 11. ▷

4 – Random fields from renormalization

Recall the definition (2.1) of the enhancement $\widehat{\xi_r}$ of the regularized noise ξ_r

$$\widehat{\xi_r} := \Big(\xi_r, \RSV_r,\ \RSW_r,\ \RSVthree_r \odot \RSI_r,\ \RSY_r \odot \RSV_r - \frac{b_r}{3},\ \big|\nabla \RSY_r\big|^2 - \frac{b_r}{3},\ \RSVthree_r \odot \RSV_r - b_r \RSI_r\Big).$$

We use the index r to emphasize the regularization of the noise. We show in this section how to use Theorem 17 to prove the convergence of $\widehat{\xi_r}$ in its natural space (2.2).

– *The benefits of hypercontractivity.* We will freely use the fact that each element of the enhancement, or polynomial quantities built from them, are polynomials of the noise, so hypercontractivity entails that for any smoothing operator A

$$\mathbb{E}\big[\|Af\|^{2p}_{L^{2p}}\big] \lesssim \mathbb{E}\big[\|Af\|^2_{L^2}\big]^p$$

where the norms refer to space or spacetime Lebesgue spaces depending on the setting. This directly gives for instance that

$$\mathbb{E}\big[\|f\|^{2p}_{B^s_{2p,2p}(M)}\big] \leq \mathbb{E}\big[\|f\|^2_{H^s(M)}\big]^p \tag{4.1}$$

for a random space distribution f on M. It follows then from Besov embedding that f is almost surely in the Hölder space $C^{s-d/(2p)}(M)$ if the upper bound in (4.1) is finite. Variations on this fact explain why we only estimate in the sequel some Sobolev norms. Consider as an example the quantity $\underline{\mathcal{L}}^{-1}(\xi)$ denoted here $\RSI$, *without regularization*. One has for all $t>0$

$$\mathbb{E}\big[\|\RSI(t)\|^2_{H^s(M)}\big] = \frac{1}{2}\,\mathrm{Tr}_{L^2(M)}\big((1-\Delta)^{s-1}\big).$$

Since $(1-\Delta)^{s-1}$ is a pseudodifferential operator of order $2(s-1)$ and we are in space dimension 3 it is trace class if $2(s-1) < -3$, that is if $s < -1/2$. From what we said above, we recover in this way the fact that $\RSI(t)$ takes almost surely its values in the Hölder space $C^{-1/2-\varepsilon}(M)$ for every $\varepsilon > 0$. One has

$$\mathbb{E}\big[\|\RSI(t_1) - \RSI(t_2)\|^2_{H^s(M)}\big] = \frac{1}{2}\,\mathrm{Tr}_{L^2(M)}\big((1-\Delta)^{s-1}\big) - \frac{1}{2}\,\mathrm{Tr}_{L^2(M)}\big((1-\Delta)^{s-1} e^{|t_1-t_2|(\Delta-1)}\big).$$

Since

$$\mathrm{Id} - e^{|t_1-t_2|(\Delta-1)} = \mathcal{O}\big(|t_1-t_2|^{\varepsilon/2}\big) \in \Psi^{\varepsilon}(M)$$

we have from the composition theorem for pseudo-differential operators that

$$(1-\Delta)^{s-1}\big(\mathrm{Id} - e^{|t_1-t_2|(\Delta-1)}\big) = \mathcal{O}\big(|t_1-t_2|^{\varepsilon/2}\big) \in \Psi^{2(s-1)+\varepsilon}(M)$$

so if $2(s-1)+\varepsilon < -3$ we have trace class operators and the continuity of the trace shows that

$$\mathbb{E}\big[\|\RSI(t_1) - \RSI(t_2)\|^2_{H^s(M)}\big] \lesssim |t_1-t_2|^{\varepsilon/2}.$$

Together with Kolmogorov's classical regularity theorem, what we said above justifies that $\RSI$ takes almost surely its values in $C^\eta_T C^{-1/2-\varepsilon}(M)$, for all small enough $\eta>0$ and all $\varepsilon>0$, and its norm in the corresponding space has moments of any finite order.

– *Organisation of this section.* We deal with the convergence of the Wick monomials $\curlyvee_r$ and Ψ_r in Section 4.1. The convergence of the quartic terms $\Upsilon^{\Psi}_r \odot \mathfrak{l}_r$, $\Upsilon_r \odot \curlyvee_r - \frac{b_r}{3}$, $\left|\nabla \Upsilon_r\right|^2 - \frac{b_r}{3}$ are the subject of Section 4.2. Section 4.3 is dedicated to the convergence of the quintic term $\Upsilon^{\Psi}_r \odot \curlyvee_r - b_r \mathfrak{l}_r$. The reader will find in Mourrat, Weber & Xu's work [69] a detailed partial construction of these objects in the model setting of the flat 3-dimensional torus. There analysis relies crucially on Fourier representations of these quantities which has no direct counterpart in a manifold setting. One then needs a different kind of analysis in such a non-homogeneous setting. It is the aim of our companion work [10] to provide a self-contained description of the tools that we use for that purpose. In the sequel, we will systematically first estimate regularities using Wick renormalization. Then we shall compare the usual Wick renormalization which is not *locally covariant* with a *locally covariant* renormalization in which we only subtract *universal quantities* at the cost of producing objects which do not belong to homogeneous Wiener chaoses. Let us discuss the notion of *local covariance* with some simple example which also explains why the usual Wick renormalization fails to be locally covariant. A *function valued invariant of Riemannian structure* is a function c which assigns to each manifold M and Riemannian structure g on M some function $c[g]$ on M which is *locally covariant* in the following sense: given $f : M' \hookrightarrow M$ a diffeomorphism onto an open submanifold of M, and a Riemannian structure g on M, then c must satisfy the equation $f^*c[g] = c[f^*g]$. Moreover, we require that c depends smoothly on the Riemannian metric g. In fact, one can prove such locally covariant function c should depend at every point on finite jets of the metric and c has an invariance property under changes of coordinates [35, section 8 p. 76]. This definition is partly motivated by [37, 2.4 p. 160] and [5, p. 282] on local index theory and also by the notion of local covariance arising in algebraic Quantum Field Theory [76, 60, 21]. In the usual Wick renormalization for some massive GFF ϕ on some Riemannian 3-manifold (M, g) with covariance $(-\Delta + 1)^{-1}$, one first mollifies ϕ via heat regularization. This yields a random smooth function $\phi_r := e^{r(\Delta-1)}\phi$. To renormalize the square ϕ_r^2, one subtracts to the square $(\phi_r)^2(x)$ of the mollified field at x, the counterterm $a_r(x) = e^{2r(\Delta-1)}(-\Delta+1)^{-1}(x,x)$ and it is well–known that the difference $(\phi_r)^2(.) - a_r(.)$ will converge as random distribution when $r \to 0^+$. However, the counterterm $a_r(x) = e^{2r(\Delta-1)}(-\Delta+1)^{-1}(x,x)$ that we subtracted is **nonlocal** in the metric g at x, it depends on the global Riemannian geometry of (M, g) and not on finite jets of the metric g at x. Hence such a_r is not *locally covariant* in the above sense. Now we observe that the diagonal value $e^{2r(\Delta-1)}(-\Delta+1)^{-1}(x,x)$ has an asymptotic expansion of the form:

$$e^{2r(\Delta-1)}(-\Delta+1)^{-1}(x,x) \sim \frac{1}{4\pi^{\frac{3}{2}} r^{\frac{1}{2}}} + \mathcal{O}(1).$$

If instead of subtracting the diagonal value of $e^{2r(\Delta-1)}(-\Delta+1)^{-1}$, one subtracted its **singular part**: $(\phi_r)^2(.) - \frac{1}{4\pi^{\frac{3}{2}} r^{\frac{1}{2}}}$ then one would still get a random distribution at the limit when $r \to 0^+$ but the covariantly renormalized Wick square $:\phi^2:$ would no longer have zero expectation. So one may wonder why is it so important to subtract only locally covariant quantities ? The answer lies in the deep notion of locality in quantum field theory. It is a folklore result in quantum fields on curved backgrounds that subtracting non locally covariant counterterms is incompatible with locality in the sense of Atiyah-Segal. Let us quote the beautiful discussion on the regularization of tadpoles and the relation with Atiyah-Segal gluing in [58, 1.2 p. 1852]: "*In various treatments of scalar theory, tadpole diagrams were set to zero (this corresponds to a particular renormalization scheme – in flat space, this is tantamount to normal ordering, ...). However, in our framework this prescription contradicts locality in Atiyah-Segal sense,... One good solution is to prescribe to the tadpole diagrams the zeta-regularized diagonal value of the Green's function. We prove that assigning to a surface its zeta-regularized tadpole is compatible with locality,... However there are other consistent prescriptions (for instance, the tadpole regularized via point-splitting and subtracting the singular term, ...). This turns out to be related to Wilson's idea of RG flow in the space of interaction potentials,...*" We refer the interested reader to [58, Section 5 p. 1885] for further details on this central topic of quantum fields on curved backgrounds. So in the present paper, we follow a similar strategy as in the previous example and try to subtract only locally covariant quantities, in fact we shall see that we subtract universal quantities that do not even depend on the metric g.

4.1 The Wick monomials

– The argument used for the study of $\mathring{\imath}$, without regularization, works for the study of $\curlyvee_r$, with regularization. Since ξ_r is regularized in space $\mathring{\imath}_r$ is here a function on $[0,T]\times M$. Set

$$a_r(z) := \mathbb{E}\big[(\mathring{\imath}_r)^2(z)\big], \qquad \overline{\curlyvee}_r(z) := (\mathring{\imath}_r)^2(z) - a_r(z).$$

Abuse notations and denote by the same letter the operator $(1-\Delta)^s$ and its kernel. Denote also by $G_r^{(2)}(t_1,t_2)$ the operator with kernel

$$\frac{1}{4}\Big(\frac{e^{(2r+|t_2-t_1|)(\Delta-1)}}{1-\Delta}(x,y)\Big)^2.$$

The operators $G_r^{(2)}(t,t)$ take values in $\Psi^{-1}(M)$ and $G_r^{(2)}(t_1,t_2)$ is of order $|t_2-t_1|^{\gamma/2}$ in $\Psi^{-1+\gamma}(M)$ since $e^{|t_2-t_1|(\Delta-1)}(1-\Delta)^{-1}$ is of order $|t_2-t_1|^{\gamma}$ in $\Psi^{-2+2\gamma}(M)$. This holds uniformly in $r\in[0,1]$. As one has from Wick formula

$$\begin{aligned}\mathbb{E}\big[\|\overline{\curlyvee}_r(t)\|^2_{H^\gamma(M)}\big] &= \frac{1}{2}\int_{M^2} e^{4r(\Delta-1)}(1-\Delta)^{\gamma-2}(x,y)\,dxdy\\ &= \frac{1}{2}\,\mathrm{Tr}_{L^2(M)}\big((1-\Delta)^\gamma G_r^{(2)}(t,t)\big)\end{aligned}$$

for each t, and the composition theorem for pseudodifferential operators tells us that

$$(1-\Delta)^\gamma G_r^{(2)}(t,t)\in\Psi^{2\gamma-1}(M)$$

uniformly in t, it has finite L^2 trace as soon as $\gamma<-1$ in our 3-dimensional space setting. Similarly we have

$$\begin{aligned}&\mathbb{E}\big[\|\overline{\curlyvee}_r(t_2)-\overline{\curlyvee}_r(t_1)\|^2_{H^\gamma(M)}\big]\\ &\quad=\frac{1}{2}\,\mathrm{Tr}_{L^2(M)}\big((1-\Delta)^\gamma\big\{G_r^{(2)}(t_1,t_1)+G_r^{(2)}(t_2,t_2)\big\}\big)-\mathrm{Tr}_{L^2(M)}\big((1-\Delta)^\gamma(G_r^{(2)}(t_1,t_2)\big)\end{aligned}$$

and we see that it is of order $|t_2-t_1|^{\eta/2}$ for all $\gamma<-1-\eta$. We conclude as above from Kolmogorov's regularity theorem that $\overline{\curlyvee}_r\in C_T^\eta C^{-1-\varepsilon}(M)$ for all $\eta>0,\varepsilon>0$, uniformly in $r\in(0,1]$, and we further get from the r-uniform and continuity of the different quantities as functions of r the convergence of $\overline{\curlyvee}_r$ in $C_T^\eta C^{-1-\varepsilon}(M)$ to a limit which we denote by $\overline{\curlyvee}$.

In the present paper, we choose to define the **renormalization in a locally covariant way** wrt the Riemannian metric g. Therefore, we shall subtract from $\curlyvee_r$ only the **singular part** of the *function* $a_r(z)=\mathbb{E}\left(\curlyvee_r(z)\right)$. We actually prove that this singular part a_r is actually some universal constant. The *function* $a_r(z)$ differs from the *constant* a_r, but as we have for $z=(t,x)$

$$a_r(z)=\frac{e^{2r(\Delta-1)}}{1-\Delta}(x,x)=\int_{2r}^1 e^{a(\Delta-1)}(x,x)\,da+b(x)$$

for a smooth function b, the small time asymptotics of the heat kernel tells us that $e^{a(\Delta-1)}(x,x)=\frac{1}{(4\pi a)^{\frac{3}{2}}}+\mathcal{O}(a^{-\frac{1}{2}})$ hence the function $a_r(\cdot)-a_r$ is bounded, uniformly in $r\in(0,1]$. To prove that the difference $a_r(\cdot)-a_r$ is smooth, we rely on the description of the heat kernel of [41, def 2.1 p. 6]. In local coordinates in some open subset U, the heat kernel can be represented as $a^{-\frac{3}{2}}A(a,\frac{x-y}{\sqrt{a}},y)$ where $A\in C^\infty([0,+\infty)_{\frac{1}{2}}\times\mathbb{R}^3\times U)$. Hence $a_r(x)-a_r=\int_{2r}^1 a^{-\frac{3}{2}}\left(A(\sqrt{a},0,x)-A(0,0,x)\right)da$ converges together with all its derivatives in x since $\partial_x^\beta a^{-\frac{3}{2}}\left(A(\sqrt{a},0,x)-A(0,0,x)\right)=\mathcal{O}(a^{-\frac{1}{2}})$ for all multi-indices β. Hence the convergent integral defining $a_r(x)-a_r$ depends smoothly on $x\in U$.

The convergence of $\curlyvee_r$ in $C_T^\eta C^{-1-\varepsilon}(M)$ to a limit which we denote by $\curlyvee$ follows as a consequence. We note that as a random variable, $\curlyvee$ is not a homogeneous element in the chaos of order 2 of the Gaussian noise since we did not subtract the full expectation. However it differs from a homogeneous element by a deterministic smooth function hence $\curlyvee$ has moments of any order $1\le r<\infty$ that are equivalent to its second moment. In the sequel, we will always prove stochastic estimates for homogeneous elements in Wiener chaoses and then justify why the locally covariant renormalization, subtracting only universal local quantities, still yields a stochastic object with the same analytic properties.

– We proceed similarly to study the convergence of

$$\Psi_r = (\mathsf{I}_r)^3 - 3a_r \mathsf{I}_r$$

in the parabolic space $\mathcal{C}^{-3/2-3\varepsilon}([0,T]\times M)$ to a limit Ψ, for all $\varepsilon>0$. From what we just saw it suffices to prove the convergence of $(\mathsf{I}_r)^3 - 3a_r(\cdot)\mathsf{I}_r$ in that space.

In order to prove the parabolic regularity of some random fields we introduce some local Sobolev seminorms defined from some cut-off function and using some Laplace type operators which are not necessarily the massive Laplacian $1-\Delta$. We introduce some *probe operator* $\mathcal{Q}^\gamma$ whose kernel

$$[\mathcal{Q}^\gamma]((t,x),(s,y)) = \Big(\eta\kappa^*\left(-\partial_t^2+P^2\right)^{\frac{\gamma}{2}}\kappa_*\eta\Big)(t-s,x,y)$$

depends on space-time variables, where P is minus the flat Laplace operator $-\sum_{i=1}^d \partial_{x^i}^2$ and $\kappa : U\subset M\mapsto \mathbb{R}^d$ is some open chart and $\eta\in C_c^\infty(U)$ and $\eta\geqslant 0$. The operator $\mathcal{Q}^\gamma$ belongs to $\mathcal{S}_\Gamma^{-5-2\gamma}(\mathbb{R}^2\times M^2)$ with

$$\Gamma = N^*\big(\{t=s\}\times \mathbf{d}_2\subset \mathbb{R}^2\times M^2\big).$$

The parabolic Sobolev seminorm is defined as

$$\|F\|^2_{\frac{\gamma}{2},\gamma,\kappa_i,\eta_i} := \big\|\left(-\partial_t^2+P^2\right)^{\frac{\gamma}{4}}\kappa_{i*}\left(\chi_i F\right)\big\|^2_{L^2(\mathbb{R}\times\mathbb{R}^d)} \tag{4.2}$$

where $\cup_{i\in I}U_i$ is an open cover of M by charts $\kappa_i : U_i\subset M\mapsto\mathbb{R}^d$ and $\sum_{i\in I}\eta_i = 1$ a subordinated partition of unity, then the formula

$$\|F\|^2_{\mathcal{H}^\gamma} := \sum_{i\in I}\|F\|^2_{\frac{\gamma}{2},\gamma,\kappa_i,\eta_i}$$

defines a Sobolev norm in the parabolic scaling. Any other choice of cover and Laplace type operators leads to an equivalent norm.

Denote by $G_r^{(3)}$ the operator with kernel

$$\frac{1}{8}\Big(\frac{e^{(2r+|t-s|)(\Delta-1)}}{1-\Delta}(x,y)\Big)^3.$$

This operator belongs to $\mathcal{S}_\Gamma^{-3}(\mathbb{R}^2\times M^2)$ with

$$\Gamma = N^*\big(\{t=s\}\times\mathbf{d}_2\subset\mathbb{R}^2\times M^2\big)\cup N^*\big(\{t=s\}\subset\mathbb{R}^2\times M^2\big)$$

We have

$$\|\mathcal{Q}^{\gamma/2}\Psi_r\|^2_{\mathcal{H}^\gamma}\leq \mathrm{tr}_{L^2(\mathbb{R}\times M)}(G_r^{(3)}\mathcal{Q}^\gamma)$$

with a finite trace, uniformly in $0\leq r\leq 1$, if and only if $-8-2\gamma>-5$, that is iff $\gamma<-\frac{3}{2}$. Here again, we would like to add that with our locally covariant renormalization procedure we are not doing true Wick ordering, hence the element $\Psi_r = (\mathsf{I}_r)^3 - 3a_r\mathsf{I}_r$ does not belong to some homogeneous chaos of order 3. However, Ψ_r differs from the Wick ordered element by a deterministic smooth function times I_r which is almost surely in $\mathcal{C}^{-1/2-\varepsilon}([0,T]\times M)$ hence a fortiori in $\mathcal{C}^{-3/2-3\varepsilon}([0,T]\times M)$. Furthermore the random variable $\Psi\in\mathcal{C}^{-3/2-3\varepsilon}([0,T]\times M)$ has moments of any fixed order $1\leq r<\infty$ equivalent to its second moment.

A reader with a probabilistic background may pause for a second and think of the fact that the chaos decomposition of a monomial expression of the noise of degree ℓ given by an integral is equivalent to the decomposition of the domain of integration $\mathcal{M}^p$, with $p\geq\ell$, as

$$\mathcal{M}^p = \Big(\mathcal{M}^p\backslash\bigsqcup_{|J|=2}^{\ell}\mathbf{d}_J\Big)\sqcup\bigsqcup_{j=2}^{\ell}\big(\bigsqcup_{|J|=j}\mathbf{d}_J\big),$$

for the diagonals $\mathbf{d}_J$ involving the integration variables corresponding to where the noise sits in the integral. The first term corresponds to the term in the chaos of order ℓ, the integral over $\bigsqcup_{|J|=j}\mathbf{d}_J$ corresponds to the term in the chaos of order $\ell-j$.

4.2 The quartic terms

Let us start by introducing some notation. From Appendix A, we define a family $(P_k^i,\tilde{P}_k^i)_{k\in\mathbb{N},i\in I}$ of generalized Littlewood-Paley-Stein projectors indexed by the frequency 2^k and the chart index

$i \in I$. The localized resonant product $\odot_i$, where i is the chart index, is defined in the appendix as $u \odot_i v = \sum_{|k-\ell|\leqslant 1} P_k^i(u)\tilde{P}_\ell^i(v)$. The goal of the present subsection is to deal with the terms

$$\tau_1 = \Psi_r \odot_i \Upsilon_r, \quad \overline{\tau}_2 = \Upsilon_r \odot_i \Upsilon_r - \chi_i \frac{b_r}{3}, \quad \overline{\tau}_3 = \chi_i \big|\nabla \Upsilon_r\big|^2 - \chi_i \frac{b_r}{3}$$

where $\chi_i \in C_c^\infty(U_i)$ is the cut–off function used to define $\odot_i$, and find for each of them the range of exponents γ such that their expected γ-Sobolev parabolic squared seminorm is finite. For that purpose, we shall use the local Sobolev seminorms $\|.\|_{\frac{\gamma}{2},\gamma,\kappa_i,\eta_i}$ defined in equation 4.2.

Note that τ_1 and $\overline{\tau}_2$ are defined with the localized resonant term $\odot_i$ made with these projectors and recall from Appendix A that these resonant terms are not commutative, in the sense that $A \odot_i B \neq B \odot_i A$. However, we do not have to worry about the definitions of τ_1 and $\overline{\tau}_2$, since the analytic properties of P_k^i and $\tilde{P}_k^i$ are similar, which entails that for every A, B, the construction of the renormalized part of $B \odot_i A$ is totally equivalent to that of $A \odot_i B$ which we provide here.

Denote by $[\odot_i](x, y, z)$ the kernel of the localized resonant operator $\odot_i$ on the chart $U_i \subset M$. Write

$$\begin{aligned}
\underline{\mathcal{L}}^{-1}\big((t,x),(s,y)\big) &= \mathbf{1}_{(-\infty,t]}(s)\, e^{(t-s)(\Delta-1)}(x,y)\\
G_r^{(p)}\big((t,x),(s,y)\big) &= 2^{-p}\left(\left\{e^{|t-s|(\Delta-1)}(1-\Delta)^{-1}\right\}(x,y)\right)^p, \qquad (1 \le p \le 3)\\
[\odot_i](x,y,z) &= \sum_{|k-\ell|\leqslant 1} P_k^i(x,y)\,\tilde{P}_\ell^i(x,z)\\
\mathcal{Q}^\gamma\big((t,x),(s,y)\big) &= \left(\eta_i\kappa^*\left(-\partial_t^2 + P^2\right)^{\frac{\gamma}{2}}\kappa_*\eta_i\right)(t-s,x,y)
\end{aligned}$$

These kernels are in some spaces of the form $\mathcal{S}_\Gamma^a$ for different ambiant spaces, scaling exponents a and wavefront sets as follows.

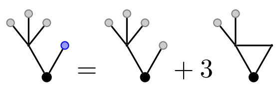
- The kernel $\underline{\mathcal{L}}^{-1}$ has scaling exponent -3 and wavefront set
$$N^*\left(\{t=s\} \times \mathbf{d}_2 \subset \mathbb{R}^2 \times M^2\right).$$
- The kernel $G_r^{(p)}$ have scaling exponent $-p$ and wavefront set
$$N^*\left(\{t=s\} \subset \mathbb{R}^2 \times M^2\right) \cup N^*\left(\{t=s\} \times \mathbf{d}_2 \subset \mathbb{R}^2 \times M^2\right).$$
- The kernel $[\odot_i]$ has scaling exponent -6 and wavefront set
$$N^*\left(\{x=y=z\} \subset M^3\right)$$
- The kernel $\mathcal{Q}^\gamma$ has scaling exponent $-5-2\gamma$ and wavefront set
$$N^*\big(\{t=s\} \times \mathbf{d}_2 \subset \mathbb{R}^2 \times M^2\big)$$

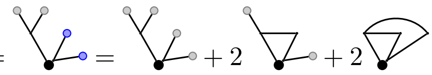
In this list the kernels on $(\mathbb{R} \times M)^2$ satisfy a local diagonal bound of the form

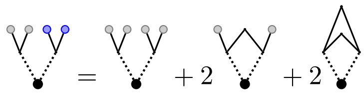
$$|\partial^\alpha_{\sqrt{t},\sqrt{s},x,y}K| \lesssim \left(\sqrt{|t-s|} + |x-y|\right)^{-a-|\alpha|}$$

for the corresponding scaling exponent a. We also often see the kernels $\underline{\mathcal{L}}^{-1}$, $\mathcal{Q}^\gamma$ and $G_r^{(1)}$ as some time dependent space operators $X(t_1 - t_2)$ whose kernels are then given by $X\big((t_1,x),(t_2,y)\big)$. The proof of the above microlocal bounds is done in detail in our companion work [10].

We use a pictorial representation of the τ_i in which the black dot $\bullet$ represents a resonant operator and the noises are coloured circles. In a given graph noises of the same color are integrated outside all diagonals of the corresponding set of variables. In the present subsection, it will be convenient to first discuss stochastic estimates for Wick ordered elements which live in homogeneous Wiener chaoses, then explain why our locally covariant renormalization yields stochastic elements that differ from the Wick renormalized ones only up to higher regularity elements. The Wiener chaos decomposition of the τ_i is

$\tau_1 :=$ 

$\tau_2 :=$ 

$\tau_3 :=$ 

We write $\tau_k = \tau_{k4} + \tau_{k2} + \tau_{k0}$ with τ_{kj} in the homogeneous chaos of degree $j \in \{0,2,4\}$. By orthogonality

$$\mathbb{E}\big[\|\tau_k\|^2_{\frac{\gamma}{2},\gamma,\kappa_i,\eta_i}\big] = \mathbb{E}\big[\|\tau_{k4}\|^2_{\frac{\gamma}{2},\gamma,\kappa_i,\eta_i}\big] + \mathbb{E}\big[\|\tau_{k2}\|^2_{\frac{\gamma}{2},\gamma,\kappa_i,\eta_i}\big] + \|\tau_{k0}\|^2_{\frac{\gamma}{2},\gamma,\kappa_i,\eta_i}.$$

Note that we are not interested in the τ_k themselves but in their renormalized versions

$$\overline{\tau}_1 = \tau_1, \quad \overline{\tau}_2 = \tau_2 - \chi_i\frac{b_r}{3}, \quad \overline{\tau}_3 = \tau_3 - \chi_i\frac{b_r}{3},$$

generically written $\overline{\tau}_k = \tau_k - c_{k,r}$, with $c_{k,r}$ for 'counterterm', for which we still have the orthogonality relation

$$\mathbb{E}\big[\|\tau_k\|^2_{\frac{\gamma}{2},\gamma,\kappa_i,\eta_i}\big] = \mathbb{E}\big[\|\tau_{k4}\|^2_{\frac{\gamma}{2},\gamma,\kappa_i,\eta_i}\big] + \mathbb{E}\big[\|\tau_{k2}\|^2_{\frac{\gamma}{2},\gamma,\kappa_i,\eta_i}\big] + \|\tau_{k0} - c_{k,r}\|^2_{\frac{\gamma}{2},\gamma,\kappa_i,\eta_i} \tag{4.3}$$

Note that each element $\tau_{kj}, j = 0,2,4$, in the Wiener chaos can be written as $F_j(:\xi^{\otimes j}:)$ where each $F_j : C^\infty(\mathbb{R}\times M)^{\otimes j} \mapsto \mathcal{D}'(\mathbb{R}\times M)$ is a multilinear functional valued in distributions involving iterated integrals of the heat operator, products represented by the trees. The following elementary well-known result tells us that we only need to bound very symmetric Feynman diagrams to bound the preceding expectations.

Lemma 18 – *Let $F \in L^2(M^n)$ be a function of n variables on some compact Riemannian manifold (M,g), ξ the white noise on (M,g). Then:*

$$\mathbb{E}\Big[\big\langle F, :\xi^{\otimes n}:\big\rangle^2\Big] \leqslant \|F\|^2_{L^2(M^n)}.$$

Proof – We define the symmetrization operator S_n as $S_n(\varphi)(x_1,\dots,x_n) = \frac{1}{n!}\sum_{\sigma\in S_n}\varphi(x_{\sigma(1)},\dots,x_{\sigma(n)})$. Note the important fact that S_n is self–adjoint on $L^2(M^n)$ hence S_n is the orthogonal projector on symmetric L^2 functions. By the Itô isometry property,

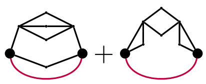
$$\mathbb{E}\left(\langle F, :\xi^{\otimes n}:\rangle^2\right) = \mathbb{E}\left(\langle S_nF, :\xi^{\otimes n}:\rangle^2\right) = \|S_nF\|^2_{L^2(M^n)}$$

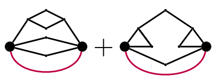
hence to prove the Lemma it suffices to note that $\|S_nF\|_{L^2(M^n)} \leqslant \|F\|_{L^2(M^n)}$ which is obvious since S_n is an orthogonal projector. ▷

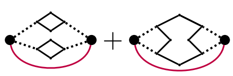
So each expectation in (4.3) can be bounded by a quantity of the form

$$\int_{(\mathbb{R}\times M)^{j+2}} F_j(x_1;y_1,\dots,y_j)F_j(x_2;y_1,\dots,y_j)\mathcal{Q}^\gamma(x_1,x_2)\,dydx$$

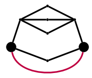
which can be represented by some mirror symmetric Feynman diagram. Using a purple edge for the kernel of the operator $\mathcal{Q}^\gamma$ we have

$\mathbb{E}\big[\|\tau_1\|^2_{\frac{\gamma}{2},\gamma,\kappa_i,\eta_i}\big] \lesssim$ 

$\mathbb{E}\big[\|\overline{\tau}_2\|^2_{\frac{\gamma}{2},\gamma,\kappa_i,\eta_i}\big] \lesssim$  $+ \big\|\tau_{20} - \chi_i\frac{b_r}{3}\big\|^2_{\frac{\gamma}{2},\gamma,\kappa_i,\eta_i}$

$\mathbb{E}\big[\|\overline{\tau}_3\|^2_{\frac{\gamma}{2},\gamma,\kappa_i,\eta_i}\big] \lesssim$  $+ \big\|\tau_{30} - \chi_i\frac{b_r}{3}\big\|^2_{\frac{\gamma}{2},\gamma,\kappa_i,\eta_i}$

We will use the notation $\mathcal{G}_{kj}$ for positive quantity represented by the mirror graph associated with τ_{kj} and the notation $\mathcal{A}_{kj}$ for the associated distribution on the corresponding configuration space ; we use Theorem 17 to prove their convergence. We use the word '*amplitude*' to talk about any of the $\mathcal{A}_{kj}$. The terms involving τ_{20} and τ_{30} are treated differently from the mirror graphs. We are now going to state all the amplitudes $\mathcal{A}_{kj}$ and to bound them. Recall that the vertex set is partitioned as $V = V' \sqcup V_A$. We always denote by x_a, x_b, etc. the space points associated with the vertices in V', while in our setting V_A will always contain two triplets $((v_1^j, v_2^j), v_*^j)$ whose associated space points are denoted $((x_1^j, x_2^j), x_*^j)$.

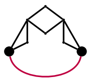
4.2.1 Bounds for τ_1. The graph $\mathcal{G}_{14} =$  has four closed, irreducible, connected subgraphs, for each of which we need to compute its weak homogeneity. We do the computation for $\mathcal{G}_{14}$ itself and let the reader deal with its three subgraphs. The amplitude of $\mathcal{G}_{14}$ is as follows:

$$\mathcal{A}_{14} = \prod_{j=1}^{2} [\odot_i](x_*^j, x_1^j, x_2^j)\mathcal{Q}^\gamma\big((s_1, x_*^1), (s_2, x_*^2)\big)G_r^{(1)}\big((s_1, x_1^1), (s_2, x_1^2)\big)$$
$$\underline{\mathcal{L}}^{-1}\big((s_1, x_2^1), (s_3, x_a)\big)G_r^{(3)}\big((s_3, x_a), (s_4, x_b)\big)\underline{\mathcal{L}}^{-1}\big((s_2, x_2^2), (s_4, x_b)\big).$$

By summing the weak homogeneity of all analytical objects appearing in the amplitude $\mathcal{A}_{14}$, the kernels $[\odot_i], G_r^{(1)}, \underline{\mathcal{L}}^{-1}, G_r^{(3)}, \mathcal{Q}^\gamma$, we get for any $\gamma < 0$

$$\sum \text{weak homogeneities} = 2(-6) + (-1) + 2(-3) - 3 - 5 - 2\gamma = -27 - 2\gamma$$
$$> -\text{codim}_w\big(\{s_1 = s_2 = s_3 = s_4, x_*^1 = \cdots = x_b\}\big) = -6 - 21 = -27,$$

which is sharp. One get the same condition on γ when checking the condition on the subgraphs of $\mathcal{G}_{14}$ so Theorem 17 entails that $\mathcal{G}_{14} < +\infty$ for all $\gamma < 0$.

The verification for the graph $\mathcal{G}_{12} =$  is similar. We have

$$\mathcal{A}_{14} = \prod_{j=1}^{2} [\odot_i](x_*^j, x_1^j, x_2^j)\mathcal{Q}^\gamma\big((s_1, x_*^1), (s_2, x_*^2)\big)G_r^{(1)}\big((s_1, x_1^1), (s_3, x_a)\big)\underline{\mathcal{L}}^{-1}\big((s_1, x_2^1), (s_3, x_a)\big)$$
$$G_r^{(2)}\big((s_3, x_a), (s_4, x_b)\big)G_r^{(1)}\big((s_2, x_1^2), (s_4, x_b)\big)\underline{\mathcal{L}}^{-1}\big((s_2, x_2^2), (s_4, x_b)\big).$$

For instance the subamplitude $[\odot_i](x_*^1, x_1^1, x_2^1)G_r^{(1)}\big((s_1, x_1^1), (s_3, x_a)\big)\underline{\mathcal{L}}^{-1}\big((s_1, x_2^1), (s_3, x_a)\big)$ associated with a triangle attached to a $\bullet$ are weakly homogeneous of degree

$$-6 - 3 - 1 = -10 > -\text{codim}_w\big(\{s_1 = s_3, x_*^1 = x_1^1 = x_2^1 = x_a\}\big) = -2 + 3(-3) = -11.$$

It yields the same range of Sobolev regularity $\gamma < 0$. We invite the reader to check all the subgraphs by themselves. In the locally covariant renormalization picture, the random distribution τ_1 will differ from the Wick renormalized one by a quantity of the form

$$f\left(\big(\underline{\mathcal{L}}^{-1}\Upsilon\big) \odot_i \Upsilon\right) + f\left(\int [\odot_i](x_*, x_1, x_2)\underline{\mathcal{L}}^{-1}\big((t, x_2), (s, x_a)\big)G_r^{(1)}\big((t, x_1), (s, x_a)\big)dsdx_1dx_2dx_a\right)$$

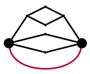
where f is some deterministic smooth function. The resonant product between $\big(\underline{\mathcal{L}}^{-1}\Upsilon\big) \in \mathcal{C}^{\frac{3}{2}-\varepsilon}, \forall\varepsilon > 0$ and $\Upsilon \in \mathcal{C}^{-\frac{1}{2}-\varepsilon}$, $\forall\varepsilon > 0$ is well–defined in $\mathcal{C}^{1-\varepsilon}$, $\forall\varepsilon > 0$. The second term, which rewrites as $f\left(\sum_{|k-\ell|\leqslant 1}\int_{-\infty}^t \left(P_k^i \circ e^{(t-s)(\Delta-1)} \circ G^{(1)}(t-s) \circ \tilde{P}_\ell^i\right)(x_*, x_*)\, ds\right)$ is a well–defined smooth function that we can control by the following argument. The operator $\underline{\mathcal{L}}^{-1} \in \Psi_H^{-1}$ and $G^{(1)} \in \Psi_P^{-2}$ where the heat calculus Ψ_H and parabolic calculus Ψ_P are defined in our companion paper [10]. Then the composition $\left(\int_{-\infty}^t \underline{\mathcal{L}}^{-1}(t-s) \circ G^{(1)}(t-s)ds\right)$ belongs to Ψ_P^{-3}, hence it belongs to $\Psi^{-4}(M)$ uniformly in t and is trace class on M. The series of pseudodifferential operators

$$\sum_{|k-\ell|\leqslant 1}\int_{-\infty}^t \left(P_k^i \circ e^{(t-s)(\Delta-1)} \circ G^{(1)}(t-s) \circ \tilde{P}_\ell^i\right)(x, x)\, ds$$
$$= \sum_{|k-\ell|\leqslant 1}\int_{-\infty}^t \Big\{\left(P_k^i \circ \tilde{P}_\ell^i \circ e^{(t-s)(\Delta-1)} \circ G^{(1)}(t-s)\right)(x, x)$$
$$+ \left(P_k^i \circ [e^{(t-s)(\Delta-1)} \circ G^{(1)}(t-s), \tilde{P}_\ell^i]\right)(x, x)\Big\}\, ds$$

will also converge in $\Psi^{-4}(M)$ since the series $\sum_{|k-\ell|\leqslant 1} P_k^i \circ \tilde{P}_\ell^i$ converges in $\Psi^0_{0,1}(M)$ and the commutator $[e^{(t-s)(\Delta-1)} \circ G^{(1)}(t-s), \tilde{P}_\ell^i]$ is bounded in $\Psi^{-5}(M)$ uniformly in (ℓ,i) since the sequence $(\tilde{P}_\ell^i)_{\ell,i}$ is bounded in $\Psi^0(M)$. Therefore the second series involving the commutator term converges in $\Psi^{-5}(M)$. We use for that purpose a commutator identity that says that for every pseudodifferential operator $A \in \Psi^m(M)$, the series

$$\sum_{|k-\ell|\leqslant 1} \left(P_k^i A \tilde{P}_\ell^i - A P_k^i \tilde{P}_\ell^i \right)$$

converges as a pseudodifferential operator of order $m-1$ that we prove in our companion work. Finally, this implies that the term $\int [\odot_i]\big(x_*, x_1, x_2\big) \underline{\mathcal{L}}^{-1}\big((t,x_2),(s,x_a)\big) G_r^{(1)}\big((t,x_1),(s,x_a)\big) ds dx_1 dx_2 dx_a$ is a well–defined smooth function.

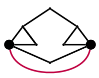
4.2.2 Elementary bounds for τ_2. Notice that $\mathcal{G}_{24} =$  has a series-parallel structure but not $\mathcal{G}_{22}$. We check for the reader's convenience what happens for $\mathcal{G}_{24}$. We have

$$\begin{aligned}\mathcal{A}_{24} = & \prod_{j=1}^{2} [\odot_i](x_*^j, x_1^j, x_2^j) \mathcal{Q}^\gamma\big((s_1, x_*^1), (s_2, x_*^2)\big) G_r^{(2)}\big((s_1, x_1^1), (s_2, x_1^2)\big) \\ & \underline{\mathcal{L}}^{-1}\big((s_1, x_2^1), (s_3, x_a)\big) G_r^{(2)}\big((s_3, x_a), (s_4, x_b)\big) \underline{\mathcal{L}}^{-1}\big((s_2, x_2^2), (s_4, x_b)\big) .\end{aligned}$$

Here again this graph has four closed, irreducible, connected subgraphs and we calculate the weak homogeneity of $\mathcal{A}_{24}$ itself by summing the weak homogeneity of all analytical objects, the kernels $G_r^{(2)}, [\odot_i], \underline{\mathcal{L}}^{-1}, \mathcal{Q}^\gamma$, appearing in the amplitude

$$\begin{aligned}\sum \text{weak homogeneities} &= 2(-2) + 2(-6) + 2(-3) - 5 - 2\gamma = -27 - 2\gamma \\ &> -\text{codim}_w\big(\{s_1 = s_2 = s_3 = s_4, x_*^1 = \cdots = x_b\}\big) = -6 - 21 = -27\end{aligned}$$

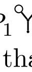
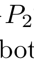
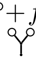
hence $\gamma < 0$. Repeating this verification for all subgraphs yields the result that $\mathcal{G}_{24} < \infty$ for all $\gamma < 0$.

The verification for the graph $\mathcal{G}_{22} =$  of amplitude

$$\begin{aligned}\mathcal{A}_{22} = & \prod_{j=1}^{2} [\odot_i](x_*^j, x_1^j, x_2^j) \mathcal{Q}^\gamma\big((s_1, x_*^1), (s_2, x_*^2)\big) G_r^{(1)}\big((s_1, x_1^1), (s_2, x_1^2)\big) \\ & \underline{\mathcal{L}}^{-1}\big((s_1, x_2^1), (s_3, x_a)\big) G_r^{(1)}\big((s_1, x_1^1), (s_3, x_a)\big) G_r^{(1)}\big((s_3, x_a), (s_4, x_b)\big) \\ & \underline{\mathcal{L}}^{-1}\big((s_2, x_2^2), (s_4, x_b)\big) G_r^{(1)}\big((s_2, x_1^2), (s_4, x_b)\big) .\end{aligned}$$

is similar, for instance the subamplitude

$$[\odot_i](x_*^1, x_1^1, x_2^1)\, G_r^{(1)}\big((s_1, x_1^1), (s_2, x_1^2)\big)$$

is weakly homogeneous of degree $-6-1 = -7 > \text{codim}_w\big(\{s_1 = s_2, x_*^1 = x_1^1 = x_2^1 = x_1^2\}\big) = -2-9 = -11$ and yields the same range of regularity exponent. In a similar way as what we did for τ_1, the locally covariant renormalized τ_2 will differ from the Wick renormalized τ_2 by homogeneous terms in Wiener chaoses of order 2 and 0 which have the form P_1$+P_2$$+f_3$ where P_1, P_2 are smoothing operators and f_3 is a smooth function. We used the fact that both  and  differ from their Wick renormalized version by a smooth function and also that for any smooth function $f \in C^\infty(M)$, the multiplication operator by the localized resonant product $u \in \mathcal{D}'(M) \mapsto f \odot_i u \in C^\infty(M)$ is smoothing.

4.2.3 The divergent part of τ_{20}. Recall that we denote by $[\odot_i]$ the kernel of $\odot_i$. An immediate calculation yields for $z=(t,x)$

$$\begin{aligned}\tau_{20}(z)=\mathbb{E}[\tau_2(z)] &= 2\int_{-\infty}^{t}\int_{M^3}[\odot_i](x,x_1,x_3)\,\mathcal{L}^{-1}(t-s,x_1,x_2)\,G_r^{(2)}(s-t,x_2,x_3)d_gx_{123}ds \\ &= 2\sum_{|k-\ell|\leqslant 1}\int_{-\infty}^{t}P_k^i(x,.)\circ G_r^{(2)}(s-t)\circ e^{(t-s)(\Delta-1)}\circ\tilde{P}_\ell^i(.,x)\,ds,\end{aligned} \tag{4.4}$$

with the shorthand notation $d_gx_{123}=d_gx_1d_gx_2d_gx_3$ for the Riemannian volume form on M^3, with d_gx_i the corresponding volume form on M.

§1. Preliminary analysis. We reformulate the integral

$$\int_{-\infty}^{t}G_r^{(2)}(s,t)\circ e^{(s-t)(\Delta-1)}\,ds \tag{4.5}$$

as the composition of two operators in the parabolic calculus $(\Psi_P^\alpha)_{\alpha\in\mathbb{R}}$ defined in our companion work [10]. This calculus extends the heat calculus $(\Psi_H^\alpha)_{\alpha\in\mathbb{R}}$ as defined in Grieser's note [41] in order to include the kernels $G_r^{(p)}, p\in\{1,2,3\}$ needed to define φ_3^4 amplitudes. Since $G_r^{(2)}\in\Psi_P^{-\frac{3}{2}}$ uniformly in $r>0$ and $e^{t(\Delta-1)}\in\Psi_H^{-1}$, we prove in [10] that the composition $\int_{-\infty}^{t}G_r^{(2)}(s,t)\circ e^{(s-t)(\Delta-1)}\,ds$ defines an element in $\Psi_P^{-\frac{5}{2}+\gamma}$ for any $\gamma>0$, uniformly in $r>0$. Hence it is a pseudodifferential operator depending continuously on t of order $-3+2\gamma$, by the comparison Theorem viewing parabolic operators as parameter dependent pseudodifferential operators in [10]. It follows that the t-indexed family of operators (4.5) is bounded in $\Psi^{-3+2\gamma}(M)$ and fails to be trace class when the regularization parameter r tends to 0. Our goal in the sequel of this section is to extract the singular part of this operator.

First we need to disentangle this operator in (4.4) from the Littlewood-Paley-Stein projectors $P_k^i,\tilde{P}_\ell^i$. We use for that purpose a commutator identity that says that for every pseudodifferential operator $A\in\Psi^m(M)$, the series

$$\sum_{|k-\ell|\leqslant 1}\Big(P_k^iA\tilde{P}_\ell^i-AP_k^i\tilde{P}_\ell^i\Big)$$

converges as a pseudodifferential operator of order $m-1$. The operator

$$\sum_{|k-\ell|\leqslant 1}\Big(P_k^iG_r^{(2)}(s,t)e^{(s-t)(\Delta-1)}\tilde{P}_\ell^i-G_r^{(2)}(s,t)e^{(s-t)(\Delta-1)}P_k^i\tilde{P}_\ell^i\Big) \tag{4.6}$$

is in particular in $\Psi^{-4+2\gamma}(M)$ uniformly in $r\in[0,1]$, for any $0<\gamma<1/2$, so it is trace class.

Denote by δ_x is the unique distribution depending on the Riemannian volume form d_gy such that

$$\int_M\delta_x(y)f(y)\,d_gy=f(x),$$

for all bounded measurable functions f. With this notation one has

$$(4.4)=2\int_{-\infty}^{t}\sum_{|k-\ell|\leqslant 1}\Big\langle P_k^i(\delta_x),G_r^{(2)}(s,t)e^{(s-t)(\Delta-1)}\tilde{P}_\ell^i(\delta_x)\Big\rangle\,ds$$

and we see from the preceding regularity result for the commutator (4.6) that the singular part of (4.4) coincides with the singular part of

$$\bigg\langle\bigg(\Big\{\int_{-\infty}^{t}G_r^{(2)}(s,t)e^{(s-t)(\Delta-1)}ds\Big\}\delta_x\bigg)\odot_i\delta_x,1\bigg\rangle$$

Given that the localized paraproduct of any two distributions is always well-defined, and that (cf Appendix A)

$$\chi_i(uv)=u\odot_i v+u\prec_i v+u\succ_i v,$$

this localized singular part coincides with the singular part of

$$\chi_i(x)\Big\{\int_{-\infty}^{t}G_r^{(2)}(s,t)e^{(s-t)(\Delta-1)}ds\Big\}(x,x)$$

Recall that

$$G_r^{(2)}(s,t) = \left(\left\{\frac{1}{2}e^{(|t-s|+2r)(\Delta-1)}(1-\Delta)^{-1}\right\}(x,y)\right)^2.$$

§2. Explicit computation of the divergent part. We proceed in two steps by localizing first in space and time and then by using the heat kernel asymptotics.

§2.1. Space and time localization. Fix $z=(t,x)$. First we localize in space near both x and in the integral near t. Let χ stand for a smooth indicator function of a neighbourhood of x in M – without loss of generality the domain of a chart. Since the quantity

$$\int_{-\infty}^{t}\int G_r^{(2)}(s,t)(x,y)(1-\chi(y))e^{(s-t)(\Delta-1)}(y,x)\,dyds$$

has a well-defined limit when $r>0$ goes to 0 we concentrate on

$$\begin{aligned}(\star) &:= \int_{-\infty}^{t}\int G_r^{(2)}(s,t)(x,y)\chi(y)e^{(s-t)(\Delta-1)}(y,x)\,dyds \\ &= \frac{1}{4}\int_0^\infty\left(\int_{[a+2r,+\infty)^2}\int e^{s_1(\Delta-1)}(x,y)e^{s_2(\Delta-1)}(x,y)\chi(y)e^{a(\Delta-1)}(x,y)\,dyds_1ds_2\right)da \simeq \int_0^1(\cdot)\end{aligned}$$

up to some smoothing operator that has a well-defined limit as $r>0$ goes to 0.

§2.2. Use the heat kernel asymptotics. We have near x

$$e^{s(\Delta-1)}(x,y) = \frac{1}{(4\pi s)^{\frac{3}{2}}}e^{-\frac{|x-y|^2_{g(x)}}{4s}}e^{-s} + R\big(s,x,\frac{x-y}{\sqrt{s}}\big) =: K^0(s,x,y) + R\Big(s,x,\frac{x-y}{\sqrt{s}}\Big)$$

for a remainder term $R\in\Psi_H^{-2}(M)$ in the heat calculus as defined in [41], that is $R\big(s,x,\frac{x-y}{\sqrt{s}}\big)$ has the same estimates as s times the heat kernel itself. It follows from that fact that replacing any of the heat kernels $e^{s_1(\Delta-1)}, e^{s_2(\Delta-1)}, e^{a(\Delta-1)}$ by a remainder term R gives a contribution to $(\star)$ that remains uniformly bounded for $r\in[0,1]$. We can therefore keep in our computations only the leading term of the heat expansion. Integrating first with respect to y the stationary phase in a chart on $\mathrm{supp}(\chi)$ gives the asymptotics

$$\begin{aligned}&\int K^0(s_1,x,y)K^0(s_2,x,y)\chi(y)K^0(a,x,y)\,dy \\ &= \frac{e^{-(s_1+s_2+a)}(4\pi)^{-\frac{9}{2}}}{(s_1s_2u)^{\frac{3}{2}}}\int e^{-\frac{|x-y|^2_{g(x)}}{4}(s_1^{-1}+s_2^{-1}+u^{-1})}\chi(y)\det(g)^{\frac{1}{2}}(y)\,dy \\ &= \frac{e^{-(s_1+s_2+a)}(4\pi)^{-3}}{(s_1^{-1}+s_2^{-1}+a^{-1})^{\frac{3}{2}}(s_1s_2a)^{\frac{3}{2}}} + \mathcal{O}\Big((s_1s_2a)^{-\frac{3}{2}}(s_1^{-1}+s_2^{-1}+a^{-1})^{-\frac{5}{2}}\Big)\end{aligned}$$

with an error term $\mathcal{O}\Big((s_1s_2u)^{-\frac{3}{2}}(s_1^{-1}+s_2^{-1}+u^{-1})^{-\frac{5}{2}}\Big)$ bounded uniformly in the x variable. The change of variable $s_1=s_1'(a+2r), s_2=s_2'(a+2r)$, with $s_1',s_2'\geq 1$ in the integral

$$\begin{aligned}&\int_0^1\int_{[a+2r,1]^2}(s_1s_2a)^{-\frac{3}{2}}\left(s_1^{-1}+s_2^{-1}+a^{-1}\right)^{-\frac{5}{2}}ds_1ds_2da \\ &\qquad\leq \int_0^1\int_{[a+2r,1]^2}(s_1s_2a)^{-\frac{3}{2}}(s_1^{-1}+s_2^{-1}+a^{-1})^{-\frac{5}{2}}ds_1ds_2da\end{aligned}$$

shows that this integral is finite.

In the end we are left with computing the singular part of

$$\begin{aligned}&\int_0^1\int_{[a+2r,+\infty]^2}\frac{e^{-(s_1+s_2+a)}}{(s_2a+s_1a+s_1s_2)^{\frac{3}{2}}}ds_1ds_2da \\ &= \int_0^1(a+2r)^{-1}\int_{[1,+\infty]^2}\frac{1}{\big((\alpha+\beta)+\alpha\beta\big)^{\frac{3}{2}}}d\alpha d\beta da + \mathcal{O}(1) \\ &= \frac{2\pi}{3}\int_0^1(a+2r)^{-1}\,da + \mathcal{O}(1) = -\frac{2\pi}{3}\log r + \mathcal{O}(1).\end{aligned}$$

which finally tells us that the diverging part of τ_{20} is exactly given by $\chi_i \frac{b_r}{3}$, $\forall i \in I$. What is left of τ_{20} after the subtraction of the divergent part defines a smooth function which is an element of $\mathcal{H}^{<0}$.

4.2.4 Bounds on $\overline{\tau}_3$. We first briefly discuss the counterterm for $\overline{\tau}_3$ and we will do the stochastic etimates in a second step. Note that for any test function φ we have from an integration by parts

$$\int_M \chi_i |\nabla \Upsilon_r|^2 \varphi = -\int_M \left(\Delta \Upsilon_r\right) \Upsilon_r \chi_i \varphi - \int_M \langle \nabla \Upsilon_r, \nabla(\chi_i\varphi)\rangle \Upsilon_r.$$

The singular part of $|\nabla \Upsilon_r|^2$ as a random distribution is thus the same as the singular part of $\left(\Delta \Upsilon_r\right)\Upsilon_r$. The latter is equal to

$$\begin{aligned}
\mathbb{E}\left[\left((1-\Delta)\Upsilon_r\right)\Upsilon_r\right] &= \int_{(-\infty,t]^2} \mathrm{tr}_{L^2}\left((1-\Delta)e^{(t-s_1)(\Delta-1)} \circ G_r^{(2)}(s_1-s_2) \circ e^{(t-s_2)(\Delta-1)}\right) ds_1 ds_2 \\
&= \int_{(-\infty,t]^2} \mathrm{tr}_{L^2}\left((1-\Delta)e^{(t-s_1)(\Delta-1)} \circ G_r^{(2)}(s_1-s_2) \circ e^{(t-s_2)(\Delta-1)}\right) ds_1 ds_2 \\
&= \int_{(-\infty,t]^2} \mathrm{tr}_{L^2}\left(G_r^{(2)}(s_1-s_2) \circ (1-\Delta)e^{(2t-s_1-s_2)(\Delta-1)}\right) ds_1 ds_2
\end{aligned}$$

so changing variables for $s_1' = t - s_1$ and $s_2' = s_1 - s_2$ gives

$$\begin{aligned}
&= \int_{(-\infty,t]^2} \mathrm{tr}_{L^2}\left(G_r^{(2)}(s_2') \circ (1-\Delta)e^{(2s_1'+s_2')(\Delta-1)}\right) ds_1' ds_2' \\
&= \int_{(-\infty,t]^2} \mathrm{tr}_{L^2}\left(G_r^{(2)}(s_2') \circ \frac{d}{s_1'} e^{(2s_1'+s_2')(\Delta-1)}\right) ds_1' ds_2' \\
&= \int_{-\infty}^{t} \mathrm{tr}_{L^2}\left(e^{s_2'(\Delta-1)} \circ G_r^{(2)}(s_2')\right) ds_2'
\end{aligned}$$

where we use time permutation symmetry of $G_r^{(2)}$ and cyclicity of the L^2-trace. This quantity is equal to the divergent part of τ_{20}. We next discuss the regularity of $\overline{\tau}_3$. First, we need to isolate the resonant part in the scalar product. Since we work in the manifold setting, note that we need to define carefully the resonant scalar product of two vector fields in $C^\infty(TM)$. For $s_1, s_2 \in C^\infty(TM)^2$, using the notations and conventions of the Appendix A, we define $\langle s_1 \odot_i s_2\rangle_{TM}$ as:

$$\langle s_1 \odot_i s_2\rangle_{TM} := \kappa_i^* \left((\kappa_{i*}g)^{\mu\nu} \kappa_{i*}\psi_i \left(\kappa_{i*}(\chi_i s_1)_\mu \odot \kappa_{i*}(\tilde{\chi}_i s_2)_\nu\right)\right)$$

where $\kappa_i, \psi_i, \chi_i, \tilde{\chi}_i$ come from our definition of resonant product, $(\kappa_{i*}g)$ is the metric g induced by the charts $\kappa_i : U_i \mapsto \kappa_i(U_i) \subset \mathbb{R}^d$. Similarly we have

$$\langle s_1 \prec_i s_2\rangle_{TM} := \kappa_i^* \left((\kappa_{i*}g)^{\mu\nu} \kappa_{i*}\psi_i \left(\kappa_{i*}(\chi_i s_1)_\mu \prec \kappa_{i*}(\tilde{\chi}_i s_2)_\nu\right)\right)$$

and we recover the usual decomposition: $\langle s_1, s_2\rangle_{TM} = \sum_i \langle s_1 \odot_i s_2\rangle_{TM} + \langle s_1 \prec_i s_2\rangle_{TM} + \langle s_1 \succ_i s_2\rangle_{TM}$ for the scalar product on sections of TM. We need to prove that such resonant scalar product satisfies some approximate integration by parts identity and we are careful since the Laplacian is no longer translation invariant. For every $Y \in C^\infty(M)$ a calculation relying on the definitions of both resonant product and resonant scalar product:

$$\langle \nabla Y \odot_i \nabla Y\rangle_{TM} = \chi_1(Y \odot_1 Y) + (PY) \odot_2 Y + Y \odot_i \Delta Y$$

where χ_1 is a smooth function, P is a differential operator of order 1 on M with smooth coefficients, $\odot_1, \odot_2$ are localized resonant type products which might differ from the original $\odot_i$ in the choice of smooth cut–off functions involved in the definition but have the exact same analytical properties from Proposition 24 and the last term involves the localized resonant product of Y with ΔY. From the point of view of regularities, for $Y = \Upsilon_r \in \mathcal{C}^{1-2\varepsilon}([0,T] \times M)$, $\chi_1(\Upsilon_r \odot_1 \Upsilon_r) \in C_T\mathcal{C}^{2-4\varepsilon}(M)$ and $(P\Upsilon_r) \odot_2 \Upsilon_r \in C_T\mathcal{C}^{1-4\varepsilon}(M)$ and finally we are reduced to the study of $\Upsilon_r \odot_i \Delta\Upsilon_r$. This is now really very similar to what we did for the graph τ_2 except there is an extra propagator $\Delta\underline{\mathcal{L}}^{-1}$ in all the Feynman amplitudes. However microlocal estimates from our companion work [10] actually show that $\Delta\underline{\mathcal{L}}^{-1} \in \mathcal{S}_\Gamma^{-1}\left(\mathbb{R}^2 \times M^2\right)$ is weakly homogeneous of degree -5 with wave front set in

$\Gamma = N^*\big(\{t=s\}\times \mathbf{d}_2 \subset \mathbb{R}^2\times M^2\big)$. Now we repeat the stochastic estimates on $\mathbb{R}^6\times M^{10}$ taking this new kernel and its weak homogeneity into account. For instance, the amplitude controlling the homogeneous chaos of order 4 in the chaos decomposition of $\Upsilon_r \odot_i \Delta\Upsilon_r$ now reads:

$$\begin{aligned}\mathcal{A}_{34} = \prod_{j=1}^{2}&[\odot_i](x_*^j,x_1^j,x_2^j)\mathcal{Q}^\gamma\big((s_1,x_*^1),(s_2,x_*^2)\big)\\ &\Delta\underline{\mathcal{L}}^{-1}\big((s_1,x_1^1),(s_5,x_c)\big)G_r^{(2)}\big((s_5,x_c),(s_6,x_d)\big)\Delta\underline{\mathcal{L}}^{-1}\big((s_2,x_1^2),(s_6,x_d)\big)\\ &\underline{\mathcal{L}}^{-1}\big((s_1,x_2^1),(s_3,x_a)\big)G_r^{(2)}\big((s_3,x_a),(s_4,x_b)\big)\underline{\mathcal{L}}^{-1}\big((s_2,x_2^2),(s_4,x_b)\big)\,.\end{aligned}$$

Here again this graph has four closed, irreducible, connected subgraphs and we calculate the weak homogeneity of $\mathcal{A}_{34}$ itself by summing the weak homogeneity of all analytical objects, the kernels $G_r^{(2)},[\odot_i],\underline{\mathcal{L}}^{-1},\Delta\underline{\mathcal{L}}^{-1},\mathcal{Q}^\gamma$, appearing in the amplitude

$$\begin{aligned}&\sum \text{weak homogeneities} = 2(-2)+2(-6)+2(-3)-5-2\gamma+2(-5) = -37-2\gamma\\ &\quad > -\mathrm{codim}_w\big(\{s_1=s_2=s_3=s_4=s_5=s_6,x_*^1=\cdots=x_d\}\big) = -10-27=-37\end{aligned}$$

hence $\gamma<0$. Repeating this verification for all subgraphs and for all amplitudes controlling the term homogeneous of order 2 in the chaos decomposition yields the result that $\Upsilon_r \odot_i \Delta\Upsilon_r \in \mathcal{C}^{-4\varepsilon}([0,T]\times M)$ almost surely.

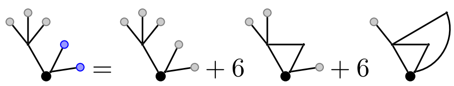
The locally covariant renormalization for $\overline{\tau_3}$ differs from the Wick one by $f_1.\nabla\Upsilon + f_2$ where f_1 is a smooth vector and f_2 a smooth function which has higher regularity than $|\nabla\Upsilon|^2$.

4.3 The quintic term

We deal in this section with the term

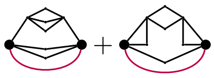
$$\overline{\tau}_4 = \Psi_r \odot_i \Upsilon_r - \chi_i b_r \mathrm{I}_r,$$

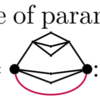
where $\odot_i$ is the localized resonant product and $\chi_i\in C_c^\infty(U_i)$ as dicussed in Appendix A. We will be relatively brief since most of the arguments and machinery were already introduced before. The chaos decomposition of τ_4 reads

$\tau_4 =$ 

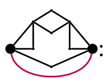
$=: \tau_{45}+\tau_{43}+\tau_{41}.$

and

$\mathbb{E}\big[\|\overline{\tau}_4\|^2_{\frac{\gamma}{2},\gamma,\kappa_i,\eta_i}\big] \lesssim$  $+\ \mathbb{E}\big[\|\tau_{41}-b_r\mathrm{I}\|^2_{\frac{\gamma}{2},\gamma,\kappa_i,\eta_i}\big]$

The first two graphs are easily treated by the same techniques as for the quartic graph: One extracts closed, connected irreducible graphs and finds the range of parameter γ so that they satisfy the criterion of Theorem 17. More precisely, we have for $\mathcal{G}_{45} =$ :

$$\begin{aligned}\mathcal{A}_{45} = \prod_{j=1}^{2}&[\odot_i](x_*^j,x_1^j,x_2^j)\mathcal{Q}^\gamma\big((s_1,x_*^1),(s_2,x_*^2)\big)G_r^{(2)}\big((s_1,x_1^1),(s_2,x_1^2)\big)\\ &\underline{\mathcal{L}}^{-1}\big((s_1,x_2^1),(s_3,x_a)\big)G_r^{(3)}\big((s_3,x_a),(s_4,x_b)\big)\underline{\mathcal{L}}^{-1}\big((s_2,x_2^2),(s_4,x_b)\big)\end{aligned}$$

and for $\mathcal{G}_{43} =$ :

$$\begin{aligned}\mathcal{A}_{43} = \prod_{j=1}^{2}&[\odot_i](x_*^j,x_1^j,x_2^j)\mathcal{Q}^\gamma\big((s_1,x_*^1),(s_2,x_*^2)\big)G_r^{(1)}\big((s_1,x_1^1),(s_2,x_1^2)\big)\\ &\underline{\mathcal{L}}^{-1}\big((s_1,x_2^1),(s_3,x_a)\big)G_r^{(1)}\big((s_1,x_1^1),(s_3,x_a)\big)G_r^{(2)}\big((s_3,x_a),(s_4,x_b)\big)\\ &\underline{\mathcal{L}}^{-1}\big((s_2,x_2^2),(s_4,x_b)\big)G_r^{(1)}\big((s_2,x_1^2),(s_4,x_b)\big)\end{aligned}$$

The reader can check that the scaling degrees of the subgraphs yield the range $\gamma < -\frac{1}{2}$.

Lemma 19 – *For any $\gamma < -\frac{1}{2}$, there is a finite constant C such that one has*

$$\mathbb{E}\left[\|\tau_{41} - \chi_i b_r \text{?}\|^2_{\frac{\gamma}{2},\gamma,\kappa_i,\eta_i}\right] \leq C$$

uniformly in $0 < r \leq 1$.

Proof – This is a consequence of item **(b)** in Theorem 17 and what we did in Section 4.2.3. For $\mathcal{A}_{\mathcal{G}_{41}}$, we first isolate the divergent subamplitude, it reads:

$$\mathcal{A}_r\big(t, s, x_*, x_1, x_2, x_a\big) := \sum_i [\odot_i](x_*, x_1, x_2)\underline{\mathcal{L}}^{-1}\big((t, x_1), (s, x_a)\big)G_r^{(2)}\big((t, x_2), (s, x_a)\big)$$

We need to introduce some distribution on $M^4 \times \mathbb{R}^2$ denoted by $\delta_{\mathbf{d}}$ which is supported by $\{t = s, x_* = x_1 = x_2 = x_a\}$ and such that for all $\varphi \in C_c^\infty\left(M^4 \times \mathbb{R}^2\right)$

$$\langle \delta_{\mathbf{d}}, \varphi\rangle = \int_{M\times\mathbb{R}} \varphi(x, x, x, x, t, t)\, dxdt$$

where we recall that dx_1 denotes abusively the Riemannian volume hence $\delta_{\mathbf{d}}$ depends only on the Riemannian volume form on M. We already know that $\mathcal{A}_r$ belongs to $\mathcal{S}^a_\Gamma(M^4 \times \mathbb{R}^2 \setminus \mathbf{d})$ where $\mathbf{d} = \{t = s, x_* = x_1 = x_2 = x_a\}$ is the spacetime diagonal,

$$\begin{aligned}\Gamma = N^*\{t = s\} \cup N^*\{t = s, x_1 = x_a\} \cup N^*\{t = s, x_2 = x_a\} \cup N^*\{x_* = x_1 = x_2\}\\ \cup N^*\{t = s, x_1 = x_2 = x_a\} \cup N^*\{t = s, x^* = x_1 = x_2\} \cup N^*\{t = s, x^* = x_1 = x_2 = x_a\}\end{aligned}$$

and $a = -2 - 6 - 3 = -11$ uniformly in $r \in [0, 1]$ by our bounds on the kernels $G_r^{(2)}, \mathcal{L}^{-1}, [\odot_i]$. But the weighted codimension of $\mathbf{d}$ equals $9 + 2 = 11$, so we are in case **(b)** of Theorem 11 and our extension requires a renormalization as in Hadamard's finite parts. We need to see that

(a) *one can renormalize $\mathcal{A}_r$ by subtracting some explicit distributional counterterm $\chi_i c_r$ proportional to $\delta_{\mathbf{d}}$.*

(b) The global renormalized amplitude of $\mathcal{A}_{\mathcal{G}_{41}}$:

$$(\mathcal{A}_r - c_r\chi_i\delta_{\mathbf{d}})(z_1, z_2, z_3) \otimes G_r^{(1)}(z_1, z_4) \otimes (\mathcal{A}_r - c_r\chi_i\delta_{\mathbf{d}})(z_4, z_5, z_6)\mathcal{Q}^\gamma(z_3, z_6)$$

will then be well-defined at the limit when $r > 0$ goes to 0 for all values of the parameter γ, by Lemma 14. Note that $c_r = c_r(\cdot)$ is a priori a function on $M \times \mathbb{R}$.

(c) Moreover, one can check the identity

$$\begin{aligned}&\mathbb{E}\left[\|\tau_{41} - \chi_i b_r \text{?}\|^2_{\frac{\gamma}{2},\gamma,\kappa_i,\eta_i}\right]\\ =\ &\Big\langle(\mathcal{A}_r - c_r\chi_i\delta_{\mathbf{d}})(z_1, z_2, z_3) \otimes G_r^{(1)}(z_1, z_4) \otimes (\mathcal{A}_r - c_r\chi_i\delta_{\mathbf{d}})(z_4, z_5, z_6)\mathcal{Q}^\gamma(z_3, z_6), 1\Big\rangle\end{aligned}$$

and that the right hand side converges when $\gamma < -\frac{1}{2}$ using the criterion of Theorem 17.

(d) *One can actually take $c_r(\cdot)$ constant equal to b_r.*

The first step in our proof is to implement in real conditions the abstract extension Theorem 11 with counterterms and we also need to control the wave front of the extension. The second step is to explicitely compute the abstract counterterm c_r whose existence is given by Theorem 11 in terms of trace densities of some operators, this computation is similar as the one we did for the quartic term.

Let us study the renormalization problem locally in $U^4 \times \{t = s\}$, since the diagonal $\mathbf{d}$ can be covered by such sets, we can recover the global extension just from working with the local extensions. Let $U \subset M$ be a chart domain and $x_* : U \mapsto x_*(U) \subset \mathbb{R}^3$ some coordinate functions on U so that $x_*(U)$ is some convex ball of $\mathbb{R}^3$. Let $\chi : M^4 \to \mathbb{R}$ with support in U^4, identically equal to 1 in a neighbourhood of $\{(x_*, x_*, x_*, x_*) \in U^4, x_* \in U\}$. (Here we make an abuse of notation denoting by x_* both elements of U and their coordinates.) As in the extension Theorem 11, we use the parabolic scaling defined by the scaling field $\rho = 2(t_2 - t_1)\partial_{t_2} + (x_1 - x_*).\partial_{x_1} + (x_2 - x_*).\partial_{x_2} + (x_a - x_*).\partial_{x_a}$ whose semiflow $e^{-t\rho}, t \geqslant 0$ leaves U^4 stable. Then we have the continuous partition of unity formula $\chi = \int_0^\infty e^{t\rho*}\psi dt$ for some function $\psi \in C^\infty(U^4)$ vanishing near $\{(x_*, x_*, x_*, x_*) \in U^4, x_* \in U\}$.

Let $\varphi \in C^\infty_c(U^4)$ be a test function, we denote by $\prod_{i=*,1,2,a}\det(g)^{\frac{1}{2}}_{x_i}$ the density of the Riemannian volume on U^4 endowed with the product metric wrt the measure $d^3x_*d^3x_1d^3x_2d^3x_a$. Then we decompose the pairing as

$$\begin{aligned}&\left\langle \mathcal{A}_r,\varphi\right\rangle = \underbrace{\left\langle \mathcal{A}_r(1-\chi),\varphi\right\rangle}\\ &\underbrace{+\int_0^\infty\left(\int_{U^4\times\mathbb{R}^2} e^{-8u}\left(e^{-u\rho*}\mathcal{A}_r\right)\psi e^{-u\rho*}\left(\varphi\prod_{i=*,1,2,a}\det(g)^{\frac{1}{2}}_{x_i}-\left(\varphi\prod_{i=*,1,2,a}\det(g)^{\frac{1}{2}}_{x_i}\right)(x_*,x_*,x_*,x_*,t,t)\right)dx_{*12a}dtds\right)du}\\ &+\int_0^\infty\left(\int_{U\times\mathbb{R}}\mathcal{A}_r\left(e^{u\rho*}\psi\right)\varphi(x_*,x_*,x_*,x_*,t,t)\det(g)^2_{x_*}dx_*dt\right)du.\end{aligned}$$

Observe that $\left(e^{-u\rho*}\mathcal{A}_r\right)=\mathcal{O}_{\mathcal{D}'}(e^{8u})$ and also $\left(\varphi\prod_{i=*,1,2,a}\det(g)^{\frac{1}{2}}_{x_i}-\varphi(x_*,x_*,x_*,x_*,t,t)\det(g)^2_{x_*}\right)$ vanishes on $\{(x_*,x_*,x_*,x_*,t,t);(t,x_*)\in\mathbb{R}\times U\}$ hence

$$e^{-u\rho*}\left(\varphi\prod_{i=*,1,2,a}\det(g)^{\frac{1}{2}}_{x_i}-\varphi(x_*,x_*,x_*,x_*,t,t)\det(g)^2_{x_*}\right)=\mathcal{O}(e^{-u})$$

and the integral over u converges absolutely (There is a subtlety related to compactness but it is easy for the reader to check that the product $\psi e^{-u\rho*}\left(\varphi\prod_{i=*,1,2,a}\det(g)^{\frac{1}{2}}_{x_i}-\varphi(x_*,x_*,x_*,x_*,t,t)\det(g)^2_{x_*}\right)$ forms a bounded family of test functions). Therefore both terms underbraced have well-defined limits when $r\to 0^+$. We need to identify the counterterm in the singular part $\mathcal{S}(\cdot)$ of the non-underbraced term

$$\begin{aligned}&\mathcal{S}\int_0^\infty\left(\int_{U^4\times\mathbb{R}^2}\mathcal{A}_r\left(e^{u\rho*}\psi\right)\varphi(x_*,x_*,x_*,x_*,t,t)\det(g)^2_{x_*}\prod_{i=*,1,2,a}dx_idtds\right)du\\ &=\mathcal{S}\left(\int_{U\times\mathbb{R}}\left(\int_{U^3\times\mathbb{R}}\mathcal{A}_r\chi(x_*,x_1,x_2,x_a)d^3x_1d^3x_2d^3x_ads\right)\det(g)_{x_*}\varphi(x_*,x_*,x_*,x_*,t,t)\det(g)^2_{x_*}dtdx_*\right)\\ &=\int_{U\times\mathbb{R}}\mathcal{S}\left(\int_{U^3\times\mathbb{R}}\mathcal{A}_r\chi(x_*,x_1,x_2,x_a)\det(g)_{x_*}d^3x_1d^3x_2d^3x_ads\right)\varphi(x_*,x_*,x_*,x_*,t,t)\det(g)^2_{x_*}dtdx_*\\ &=\left\langle c_r\delta_{\mathbf{d}},\varphi\right\rangle.\end{aligned}$$

Beware that $\det(g)_{x_*}$ is the square of the volume density at x_* and depends only on x_*, where

$$\mathcal{S}\left(\int_{U^3\times\mathbb{R}}\mathcal{A}_r\chi(x_*,x_1,x_2,x_a)\det(g)^{\frac{3}{2}}_{x_*}dx_1dx_2dx_ads\right)=c_r(t,x_*).$$

There is a subtlety about why we could interchange the extraction of singular parts with the integration, this is due to the fact that the integrand admits an asymptotic expansion. It remains to prove why the renormalized amplitude

$$\overline{\mathcal{A}}_r=\lim_{r\to 0^+}\left(\mathcal{A}_r-c_r(t,x_*)\delta_{\mathbf{d}}\right)$$

has the correct wave front set in Γ. It suffices to prove the property for $\mathcal{A}_r\chi R$ where R is the Taylor subtraction operator from Proposition 9. This follows from the proof of Theorem 11 where one replaces the wave front bound on $[e^{-u\rho}]$ by the identical wave front bound on $[e^{-u\rho}R]$ which coincide by Proposition 9. We then focus on the explicit evaluation of the counterterm and we will establish that it does not depend on x_* so factors out of the pairing. Now the counterterm c_r appears when we consider the divergent part of the term

$$\begin{aligned}c_r(t,x_*) &= \mathcal{S}\int_{-\infty}^t\int_{U^3}\mathcal{A}_r(t,s,x_*,x_1,x_2,x_a)\det(g)^{\frac{3}{2}}_{x_*}\chi(x_*,x_1,x_2,x_a)\,dx_1dx_2dx_ads\\ &= \mathcal{S}\int_{-\infty}^t\int_{U^3}\mathcal{A}_r(t,s,x_*,x_1,x_2,x_a)\det(g)^{\frac{1}{2}}_{x_1}\det(g)^{\frac{1}{2}}_{x_2}\det(g)^{\frac{1}{2}}_{x_a}\chi(x_*,x_1,x_2,x_a)\,dx_1dx_2dx_ads\\ &= \mathcal{S}\int_{-\infty}^t\sum_{0\leqslant|k-\ell|\leqslant 1}P^i_k(x_*,.)\circ\tilde{\chi}\circ\left(G^{(2)}_r(t-s)\circ e^{(t-s)(\Delta-1)}\right)\circ\tilde{\chi}\circ\tilde{P}^i_\ell(.,x_*)\,ds\end{aligned}$$

where the $P_k^i, \tilde{P}_\ell^i$ are the Littlewood-Paley-Stein projectors, the functions $\tilde{\chi} \in C_c^\infty(U)$ are arbitrary test functions s.t. $\tilde{\chi} = 1$ near x_* and where we used the explicit definition of $[\odot_i]$ in terms of the Littlewood-Paley-Stein projectors. Again we used the fact that $\det(g)_{x_*}^{\frac{3}{2}} - \det(g)_{x_1}^{\frac{1}{2}} \det(g)_{x_2}^{\frac{1}{2}} \det(g)_{x_a}^{\frac{1}{2}}$ vanishes when $x_* = x_1 = x_2 = x_a$ hence it does not contribute to the singular part of the above integral and we have the freedom to change the base point of the volume elements. The last term is precisely a trace density whose singular part matches the divergent part of τ_{20} up to a combinatorial factor. $\rhd$

We conclude our discussion by pointing out the difference between the locally covariant and Wick renormalizations. The locally covariant renormalization of $\Psi \odot_i \mathsf{V}$ differs from the Wick renormalization by $\big(\underline{\mathcal{L}}^{-1}\,(f\mathsf{I})\big) \odot_i \mathsf{V} + P(\Psi)$ where f is a smooth function and P a smoothing operator and therefore $\big(\underline{\mathcal{L}}^{-1}\,(f\mathsf{I})\big) \odot_i \mathsf{V} \in \mathcal{C}^{\frac{1}{2}-\varepsilon}$, $\forall \varepsilon > 0$ which is absorbed in the $\mathcal{C}^{-\frac{1}{2}-0}$ regularity of $\Psi \odot_i \mathsf{V}$.

The results of this section justify the convergence in $L^2(\Omega)$, hence in any $L^p(\Omega)$ with $p < \infty$, of $\widehat{\xi}_r$ to a limit random variable $\widehat{\xi}$ in its natural space. The convergence *in probability* of $v_r \in (\!|\alpha_0, 1+\varepsilon'|\!)$ to a limit v in that space follows as a consequence of the pathwise continuity of v_r with respect to $\widehat{\xi}_r$ obtained from the fixed point construction of v_r. Formula (2.3) relating v_r to u_r shows the convergence *in probability* of $u_r \in C_T C^{-1/2-\varepsilon}(M)$ to a limit u in that space.

5 – Invariant measure

We prove in Section 5.1 that the dynamics generated by equation (1.3) is Markovian and that its semigroup has the Feller property. The existence of an invariant measure is obtained from a compactness argument building on the L^p coming down from infinity property of Theorem 5. We prove in Section 5.2 that the invariant probability measure is non-Gaussian.

5.1 A Markovian dynamics

Denote by $\mathcal{F}_t$ the usual augmentation of the σ-algebra generated by the random variables $\xi(f)$, for functions $f \in L^2(\mathbb{R} \times M)$ that are null on $[t, \infty) \times M$. Since ξ_r is white in time the dynamics

$$\partial_t u_r = (\Delta - 1)u_r - u_r^3 + 3(a_r - b_r)u_r + \xi_r$$

generates an $(\mathcal{F}_t)_{t \geq 0}$-Markov process. For looking at the restriction to a finite time interval $[0, T]$ of this process it is convenient to extend functions on $[0, T]$ into functions on $[0, +\infty)$ that are constant on $[T, +\infty)$. For $t \in \mathbb{R}$ and any $(s, x) \in \mathbb{R} \times M$ set

$$\tau_t(s, x) := (s - t, x).$$

Denote by $\theta_s : \Omega \to \Omega$, $s \geq 0$ a family of measurable shifts on $(\Omega, \mathcal{F})$ such that one has

$$(\xi \circ \theta_s, f) = (\xi, f \circ \tau_s)$$

for all s and all L^2 test functions f. The Markov property for

$$u_r : \Omega \times [0, T] \times C^{-1/2-\varepsilon}(M) \to C^{-1/2-\varepsilon}(M)$$

writes

$$\mathbb{E}\big[F\big(u_r(s + \cdot, \phi)\big)\mathbf{1}_E\big] = \mathbb{E}\big[F\big(u_r \circ \theta_s(\cdot, u_r(s, \phi))\big)\mathbf{1}_E\big], \tag{5.1}$$

for any bounded measurable cylindrical functional F on $C\big((0, T], C^{-1/2-\varepsilon}(M)\big)$ and all events $E \in \mathcal{F}_s$, with $0 \leq s \leq T$ arbitrary. We need the following quantitative stability result to pass to the zero r limit in (5.1).

Lemma 20 – *Fix some positive times $t_1 < \cdots < t_k$. There exists two positive constants γ, γ' such that the restriction of the functions*

$$\phi \in C^{-1/2-\varepsilon}(M) \mapsto u_r(t_i, \phi) \in C^{-1/2-\varepsilon}(M), \quad (1 \leq i \leq k)$$

to any centered ball of $C^{-1/2-\varepsilon}(M)$ with radius $R > 0$ is Lipschitz continous, with Lipschitz constant bounded above by an explicit function of R and $\widehat{\xi}_r$.

Proof – This result is obtained from the exact same statement for the functions $v_r(t_i,\cdot)$. The relation

$$u_r(t,\phi_1) - u_r(t,\phi_2) = e^{-3\Upsilon_r(t)}\big(v_r(t,\phi_1') - v_r(t,\phi_2')\big),$$

with

$$\phi_i = \mathfrak{l}_r(0) - \Psi_r(0) + e^{-3\Upsilon_r(0)}\big(\phi_i' + v_{\mathrm{ref}}(0)\big)$$

allows to transport the locally Lipschitz character of v_r to u_r. It suffices to prove the statement with $k=1$ and $t_1 = 1$; we prove in that case that $\phi \in C^{-1/2-\varepsilon}(M) \mapsto v_r(1,\phi) \in C^{-1/2-\varepsilon}(M)$ is locally Lipschitz. Define

$$F(\phi,v) := e^{t(\Delta-1)}(\phi) + \mathcal{L}^{-1}\Big(- 6\nabla\Upsilon_r\cdot\nabla v - e^{-6\Upsilon_r}v^3 + Z_{2,r}v^2 + Z_{1,r}v + Z_{0,r}\Big). \tag{5.2}$$

Let $K>0$ be a uniform constant satisfying

$$\|e^{t(\Delta-1)}\phi\|_{C^{-1/2-\varepsilon}} \leq K\|\phi\|_{C^{-1/2-\varepsilon}} \text{ and } \|e^{t(\Delta-1)}\phi\|_{(|\alpha_0,1+\varepsilon'|)} \leq K\|\phi\|_{C^{-1/2-\varepsilon}}. \tag{5.3}$$

Take the ball B_R in $C^{-1/2-\varepsilon}(M)$. It follows from the proof of Proposition 3 that for any $\phi\in B_R$, there exists $T = T(\hat{\xi}_{r|[0,2]}, R)$ and a constant $C(T) < 1/2$ only depending on T such that

$$\|F(\phi,v_1) - F(\phi,v_2)\|_{C_TC^{-1/2-\varepsilon}} \leq C(T)\big(\|v_1-v_2\|_{C_TC^{-1/2-\varepsilon}} + \|v_1-v_2\|_{(\!|\alpha_0,1+\varepsilon'|\!)}\big)$$

and

$$\|F(\phi,v_1) - F(\phi,v_2)\|_{(\!|\alpha_0,1+\varepsilon|\!)} \leq C(T)\|v_1-v_2\|_{(\!|\alpha_0,1+\varepsilon'|\!)}).$$

Now by the same argument in the proof of Proposition 3, we infer that for $\phi_1,\phi_2\in B_R$,

$$\begin{aligned}&\|v_r(\cdot,\phi_1) - v_r(\cdot,\phi_2)\|_{C_TC^{-1/2-\varepsilon}} \\ &\leq K\|\phi_1-\phi_2\|_{C^{-1/2-\varepsilon}} + C(T)\big(\|v_r(\cdot,\phi_1) - v_r(\cdot,\phi_2)\|_{C_TC^{-1/2-\varepsilon}} + \|v_r(\cdot,\phi_1) - v_r(\cdot,\phi_2)\|_{(\!|\alpha_0,1+\varepsilon'|\!)}\big)\end{aligned}$$

and

$$\|v_r(\cdot,\phi_1) - v_r(\cdot,\phi_2)\|_{(\!|\alpha_0,1+\varepsilon'|\!)} \leq K\|\phi_1-\phi_2\|_{C^{-1/2-\varepsilon}} + C(T)\|v_r(\cdot,\phi_1) - v_r(\cdot,\phi_2)\|_{(\!|\alpha_0,1+\varepsilon'|\!)}.$$

This implies that

$$\begin{aligned}\|v_r(\cdot,\phi_1) - v_r(\cdot,\phi_2)\|_{C_TC^{-1/2-\varepsilon}} &+ \|v_r(\cdot,\phi_1) - v_r(\cdot,\phi_2)\|_{(\!|\alpha_0,1+\varepsilon'|\!)} \\ &\leq 2K\|\phi_1-\phi_2\|_{C_TC^{-1/2-\varepsilon}} + C(T)\|v_r(\cdot,\phi_1) - v_r(\cdot,\phi_2)\|_{C_TC^{-1/2-\varepsilon}} \\ &\quad + 2C(T)\|v_r(\cdot,\phi_1) - v_r(\cdot,\phi_2)\|_{(\!|\alpha_0,1+\varepsilon'|\!)},\end{aligned}$$

hence

$$\|v_r(\cdot,\phi_1) - v_r(\cdot,\phi_2)\|_{C_TC^{-1/2-\varepsilon}} \leq \frac{2K}{1-C}\|\phi_1-\phi_2\|_{C^{-1/2-\varepsilon}}.$$

By the L^p a priori estimate of Theorem 5, if $T<1$ then

$$\|v_r(t,\phi)\|_{C^{-1/2-\varepsilon}} \leq R' := \frac{C(\widehat{\xi}_{r|[0,2]})}{T^{1/2}}, \quad (T/2\leq t\leq 1).$$

Then as above we can get a short time $T' = T'(\widehat{\xi}_{r|[0,2]}, R')$ such that the map $F(\cdot,\cdot)$ is contracting with a constant $C(T') < 1/2$ for any initial condition in $B_{R'}\subset C^{-1/2-\varepsilon}(M)$. Since $v_r(t)\in B_{R'}$ for all $t\in[T/2,1]$ we can divide the interval $[T/2,1]$ into subintervals $[t_j, t_j+T']$ and repeat our process above to get

$$\big\|v_r(t,\phi_1) - v_r(t,\phi_2)\big\|_{C^{-1/2-\varepsilon}} \leq \frac{2K}{1-C(T')}\big\|v_r(t_j,\phi_1) - v_r(t_j,\phi_2)\big\|_{C^{-1/2-\varepsilon}},$$

on each small interval $[t_j, t_j+T']$. Combining all yields that $\phi\mapsto v_r(1,\phi)$ is locally Lipschitz. $\triangleright$

Proposition 21 – *The dynamics of u is Markovian and its associated semigroup $(\mathcal{P}_t)_{t\geq 0}$ on $C^{-1/2-\varepsilon}(M)$ has the Feller property.*

Proof – Given any $\eta>0$ it follows from Lemma 20 there is an $R(\eta)>0$ such that outside an event of probability η the random variables $(u_r\circ\theta_s(\cdot,\cdot))_{0<r\leq 1}$ are (r-uniformly) uniformly continuous in their second argument on the centered ball of $C^{-1/2-\varepsilon}(M)$ of radius $R(\eta)$ and $u_r(s,\phi)$ is converging to $u(s,\phi)$ with $|u(s,\phi)|_{C^{-1/2-\varepsilon}}\leq R(\eta)$. The process $u_r\circ\theta_s(\cdot,u_r(s,\phi))$ is thus converging in probability to $u\circ\theta_s(\cdot,u(s,\phi))$, so one can get the Markov property of the limit

process u by passing to the zero r limit in (5.1) along a subsequence r_k where the convergence of u_{r_k} is almost sure, using dominated convergence.

The Feller property of the semigroup $(\mathcal{P}_t)_{t\geq 0}$, that is the fact that it sends the space of continuous functions on $C^{-1/2-\varepsilon}(M)$ into itself, is a direct consequence of the pathwise continuous dependence of the solution u to (1.3) with respect to the initial condition ϕ and dominated convergence in the expression $(\mathcal{P}_t f)(\phi) = \mathbb{E}\big[f(u(t,\phi))\big]$. ▷

Proposition 22 – *The semigroup $(\mathcal{P}_t)_{t\geq 0}$ has an invariant probability measure.*

Proof – Recall we turned equation (1.5) on u_r into equation (2.4) on v_r, with abstract form (2.15). Coming back to

$$u = \mathrm{I} - \Psi + e^{-3\mathrm{Y}}(v + v_{\mathrm{ref}}), \tag{5.4}$$

seen as an element of $C_T C^{-1/2-\varepsilon}(M)$, one can write for any fixed time

$$\begin{aligned}&\big\{\|u(t)\|_{C^{-1/2-\varepsilon}} > 3m\big\}\\ &\qquad\subset \Big\{\|\mathrm{I}(t)\|_{C^{-1/2-\varepsilon}} > m\Big\} \cup \Big\{\|\Psi(t)\|_{C^{-1/2-\varepsilon}} > m\Big\} \cup \Big\{\big\|e^{-3\mathrm{Y}}(v+v_{\mathrm{ref}})\big\|_{C^{-1/2-\varepsilon}} > m\Big\},\end{aligned}$$

with

$$\mathbb{P}\big(\|\mathrm{I}(t)\|_{C^{-1/2-\varepsilon}} > m\big) + \mathbb{P}\big(\|\Psi(t)\|_{C^{-1/2-\varepsilon}} > m\big) = o_m(1)$$

uniformly in $t \geq 0$ by stationarity. We also have

$$\begin{aligned}\mathbb{P}\Big(\big\|e^{-3\mathrm{Y}(t)}&(v+v_{\mathrm{ref}})(t)\big\|_{C^{-1/2-\varepsilon}} > m\Big)\\ &\leq \mathbb{P}\Big(\big\|e^{-3\mathrm{Y}(t)}\big\|_{C^{-1/2-\varepsilon}} \geq c\Big) + \mathbb{P}\Big(\|(v+v_{\mathrm{ref}})(t)\|_{C^{-1/2-\varepsilon}} > \frac{m}{c}\Big)\\ &\leq o_c(1) + \mathbb{P}\Big(\|v(t)\|_{C^{-1/2-\varepsilon}} > \frac{m}{2c}\Big) + \mathbb{P}\Big(\|v_{\mathrm{ref}}(t)\|_{C^{-1/2-\varepsilon}} > \frac{m}{2c}\Big)\\ &\leq o_c(1) + o_{m/c}(1).\end{aligned}$$

The $o_m(1)$ function does not depend on t by stationarity. In the last step we used the ϕ-independent the estimate (2.17) quantifying the upper bound (2.16) in the coming down from infinity property together with the stationarity of v_{ref}. This gives the t-uniform and ϕ-independent estimate

$$\mathbb{P}\big(\|u(t)\|_{C^{-1/2-\varepsilon}} > 3m\big) = o_m(1). \tag{5.5}$$

We have been cautious to construct an enhanced noise whose law is stationary in time. This property together with the independence of the estimate (5.5) with respect to the initial condition allows then to propagate (5.5) uniformly in time by restarting fictively the dynamics every integer time while keeping an upper bound $o_m(1)$ that does not depend on the interval considered. The family of laws $\mathscr{L}(u_r(t,\phi))$ of $u_r(t,\phi)$ is thus tight in $C^{-1/2-2\varepsilon}(M)$, independently of the regularization parameter $r \in [0,1]$ and the initial condition $\phi \in C^{-1/2-\varepsilon}(M)$, uniformly in $t \geq 1$. It follows that for any $\phi \in C^{-1/2-2\varepsilon}(M)$ the probability measures on $C^{-1/2-\varepsilon}(M)$

$$\frac{1}{T-1}\int_1^T \delta_{\mathscr{L}(u(t,\phi))}\,dt, \qquad (T \geq 2)$$

have a weak limit along a subsequence of times T tending to infinity. The Feller property of the semigroup generated by (1.3) ensures that this weak limit is an invariant probability measure of the dynamics. ▷

5.2 Non-triviality of the ϕ_3^4 measure

Some care is needed when working with Jagannath & Perkowski's representation

$$u_r = \mathrm{I}_r - \Psi_r + e^{-3\mathrm{Y}_r}(v_r + v_{r,\mathrm{ref}})$$

of u_r when it comes to taking the expection of some quantities. This is related to the fact that the random variable Y being a quadratic polynomial of a Gaussian noise the random variable $\exp(-3\mathrm{Y})$ may not be integrable. This a priori makes tricky to say anything about the integrability of $u_r(t)$ from its description in terms of $v_r(t)$. To circumvent this problem we follow Jagannath &

Perkowski' suggestion to trade $\exp(-\Upsilon)$ for $\exp(-P_{\geq n}\Upsilon)$ in their change of unknown (2.3). The operator

$$P_{\geq n} = \sum_{i \in I} \sum_{|k| \geq n} P_k^i$$

removes a number of initial terms of a Littlewood-Paley expansion. One can thus *choose n random* so that

$$\|P_{\geq n}\Upsilon\|_{C_T C^{1-\eta}} \leq 1.$$

Set

$$f_n := \sum_{i \in I} \sum_{|k| \leq n-1} P_k^i(\Upsilon) \in C_T C^\infty(M).$$

This change of unknown adds a term $f_n v$ into the equation for v, which only changes Z_1 for a new Z_1 that is still an element of $C_T C^{-1/2-\varepsilon}(M)$ and is a polynomial of the noise. As $P_k^i P_\ell^j = 0$ for $|k - \ell|$ greater than a fixed constant we have

$$\|f_n\|_{C_T C^{1-2\varepsilon}} \lesssim \|\Upsilon\|_{C_T C^{1-2\varepsilon}}$$

independently of our definition of the random integer n. So the new Z_1 has finite moments of any order. We get from the estimates (2.16) and (2.17) quantifying of the coming down, with $\exp(-\Upsilon)$ now replaced by $\exp(-P_{\geq n}\Upsilon)$, and the formula (2.3) relating u and v the fact that

$$u(1) = \mathring{\mathrm{I}}(1) + \Psi(1) + (\star) \tag{5.6}$$

for an element $(\star) \in C^{1-\eta}(M)$ whose norm belongs to all the $L^q(\Omega)$ spaces, $1 \leq q < \infty$, uniformly with respect to the initial condition ϕ of u. We now see clearly that $u(1) \in C^{-1/2-\varepsilon}(M)$ belongs to all the $L^q(\Omega)$ spaces, $1 \leq q < \infty$, *uniformly with respect to the initial condition ϕ*. We assume in the remainder of this section that we work with this version of Jagannath & Perkowski's equation.

The mechanics of the proof that the Φ_3^4 measure is non-Gaussian is well-known. We write it here for completeness and follow for that purpose the lecture notes [44] of Gubinelli – Section 6.4 therein, after Gubinelli & Hofmanová's work [46]. Assume ϕ is random, with law the invariant measure of the dynamics, so $u(1)$ itself has the same law. Consider the heat regularization $e^{r\Delta}u(1)$ of our solution u at time 1. In this subsection, for simplicity, we shall assume that we used true Wick ordering for the renormalization which simplifies the discussion and allows to use true orthogonality properties of the Wiener chaos decomposition. Our argument is of semiclassical nature, we will use the small r asymptotic behaviour of heat kernels to justify nontriviality – so r somehow plays the role of a semiclassical parameter. If the Φ_3^4 measure were Gaussian the random variable $e^{r(\Delta-1)}(u(1))$ would also be Gaussian uniformly when $r > 0$ goes to 0. So its truncated four point function

$$\begin{aligned} C_4^r = C_4\Big(e^{r(\Delta-1)}(u(1)), e^{r(\Delta-1)}(u(1)), e^{r(\Delta-1)}(u(1)), e^{r(\Delta-1)}(u(1))\Big) \\ := \mathbb{E}\big[e^{r(\Delta-1)}(u(1))^4\big] - 3\,\mathbb{E}\big[e^{r(\Delta-1)}(u(1))^2\big]^2 \end{aligned}$$

would be identically null uniformly in $r \in (0, 1]$. Recall from Section 4.2 the notation $G_r^{(p)}$. The sufficient integrability of the different elements of the decomposition

$$u(1) = \mathring{\mathrm{I}}(1) - \Psi(1) + e^{-3P_{\geq n}\Upsilon(1)}\big(v(1) + v_{\mathrm{ref}}(1)\big),$$

allows to plug it inside the formula for the fourth order cumulant and use Wick's Theorem to get

$$\begin{aligned} &C_4\Big(e^{r(\Delta-1)}(u(1)), e^{r(\Delta-1)}(u(1)), e^{r(\Delta-1)}(u(1)), e^{r(\Delta-1)}(u(1))\Big) \\ &= 24\int_{-\infty}^{t} G_r^{(3)}(t-s) \circ e^{(t-s+r)(1-\Delta)}(x,x)ds + 216\int_{(-\infty,t]^2}\int_{y_1,y_2\in U^2} G_0^{(2)}(s_1-s_2,y_1,y_2) \\ &\times e^{(r+t-s_1)(\Delta-1)}(y_1,x)e^{(r+t-s_2)(\Delta-1)}(y_2,x)G_r^{(1)}(t-s_1,y_1,x)G_r^{(1)}(t-s_2,y_2,x)\,dy_{12}ds_{12} \\ &+ \mathbb{E}\left[\Big(e^{r(\Delta-1)}\mathring{\mathrm{I}}\Big)\,P\left(e^{r(\Delta-1)}\Psi(1), e^{r(\Delta-1)}e^{-3P_{\geq n}\Upsilon(1)}\big(v(1)+v_{\mathrm{ref}}(1)\big)\right)\right] \end{aligned}$$

where P is some polynomial functional in its stochastic arguments. We have many cancellations in the above expression since Gaussian cumulants only retain connected Feynman graphs and we also use orthogonality of homogeneous Wiener chaoses of different degrees. The remainder has the corresponding decay

$$\mathbb{E}\left[\mathsf{I}(1)\,P\left(e^{r(\Delta-1)}\Psi(1), e^{r(\Delta-1)}e^{-3P_{\geq n}\Upsilon(1)}\big(v(1)+v_{\mathrm{ref}}(1)\big)\right)\right]=\mathcal{O}(r^{-\frac{1}{4}})$$

since we just need to recall that the remainder only involves the terms,

$$e^{r(\Delta-1)}\mathsf{I}(1)=\mathcal{O}(r^{-\frac{1}{4}}),\quad e^{r(\Delta-1)}\Psi(1)=\mathcal{O}(1),\quad e^{r(\Delta-1)}e^{-3P_{\geq n}\Upsilon(1)}\big(v(1)+v_{\mathrm{ref}}(1)\big)=\mathcal{O}(1)$$

since they are Hölder regular in $\mathcal{C}^{\frac{1}{2}-0}$ and $\mathcal{C}^{1-0}$ respectively. (We used the fact that we can probe the space Hölder regularity by testing against heat kernels: $\sup_{\varepsilon\in(0,1]}\varepsilon^{-\frac{s}{2}}\|e^{\varepsilon(\Delta-1)}u\|_{L^\infty(M)}\lesssim\|u\|_{C^s(M)}$ and also we made an implicit use of Besov embeddings, $\forall\delta>0$, $\|.\|_{C^{s-\frac{d}{p}-\delta}}\lesssim\|.\|_{B^s_{p,p}}$ together with hypercontractive estimates which allows us to consider expectations of Hölder norms.)

Let us study in detail the asymptotics of the first term on the right hand side of the equation for C_4^r which has a Feynman integral interpretation. For every $x\in U$, choose some cut-off function $\chi\in C_c^\infty(U)$ which equals 1 near x that we use to localize the asymptotics as in the calculation of counterterms, then we can extract the small r leading asymptotics as

$$\int_{-\infty}^{t}G_r^{(3)}(t-s)\circ e^{(t-s+r)(1-\Delta)}(x,x)ds$$
$$\simeq\int_0^\infty\int_{y\in U}\chi(y)\left(\int_{[a+r,+\infty)^3}\prod_{i=1}^{3}e^{s_i(\Delta-1)}(x,y)ds_i\right)e^{(a+r)(\Delta-1)}(y,x)da.$$

We compute the integral with respect to y first; this reads

$$\int\left(\prod_{i=1}^{3}K^0(s_i,x,y)\right)K^0(a+r,x,y)\chi(y)dy$$
$$\simeq\frac{e^{-(s_1+s_2+s_3+a+r)}(4\pi)^{-6}}{(s_1s_2s_3(a+r))^{\frac{3}{2}}}\int_U e^{-\frac{|x-y|_{g(x)}}{4}(s_1^{-1}+s_2^{-1}+s_3^{-1}+(a+r)^{-1})}\chi(y)\det(g)_y^{\frac{1}{2}}dy$$
$$\simeq\frac{(4\pi)^{-\frac{9}{2}}e^{-(s_1+s_2+s_3+a+r)}}{(s_1s_2s_3(a+r))^{\frac{3}{2}}(s_1^{-1}+s_2^{-1}+s_3^{-1}+(a+r)^{-1})^{\frac{3}{2}}}$$
$$+\mathcal{O}\left((s_1s_2s_3(a+r))^{-\frac{3}{2}}(s_1^{-1}+s_2^{-1}+s_3^{-1}+(a+r)^{-1})^{-\frac{5}{2}}\right)$$

where we only keep the leading terms in the heat asymptotic expansion and use a stationary phase estimate. It is possible, as we did for the counterterms, to show that the integral with respect to a,s_1,s_2,s_3 of the $\mathcal{O}(\cdots)$ term gives subleading asymptotics compared to the leading term. We are reduced after a change of variables to the asymptotics of the following integral

$$\int_0^1(a+r)^{-\frac{3}{2}}\left(\int_{[1,+\infty)^3}\big(a_2a_3+a_1a_3+a_1a_2+a_1a_2a_3\big)^{-\frac{3}{2}}da_{123}\right)da\simeq cr^{-\frac{1}{2}}$$

for some non-null constant c. The next term in the formula for C_4^r is

$$\int_{(-\infty,t]^2}\int_{y_1,y_2\in U^2}G_0^{(2)}(s_1-s_2,y_1,y_2)\,e^{(r+t-s_1)(\Delta-1)}(y_1,x)\,e^{(r+t-s_2)(\Delta-1)}(y_2,x)$$
$$\times G_r^{(1)}(t-s_1,y_1,x)\,G_r^{(1)}(t-s_2,y_2,x)\,dy_1dy_2ds_{12}$$

which is bounded by a constant multiple of the integral over $(-\infty,t]^2\times U^2$ of

$$\left(\sqrt{|s_1-s_2|}+|y_1-y_2|\right)^{-2}\left(\sqrt{|r+t-s_1|}+|y_1-x|\right)^{-4}\left(\sqrt{|r+t-s_2|}+|y_2-x|\right)^{-4}.$$

Making first the change of variables $s_i\mapsto r^2(s_i-t)+s_i$, $y_i\mapsto r(y_i-x)+x$ and then using polar coordinates gives $\mathcal{O}(|\log r|)$ as an upper bound for htat integral, therefore the cumulant C_4^r

blows-up like $24cr^{-\frac{1}{2}}$ when $r>0$ goes to 0. This shows that C_4^r does not vanish asymptotically and that the Φ_3^4 measure is non-Gaussian.

For a Φ_3^4 measure obtained as above as weak limit of Birkhoff averages, the covariance property under Riemannian isometries is clear from its construction and the fact that the renormalisation constants a_r, b_r do not depend on which Riemannian metric is used: given a field ϕ on (M,g) whose law is a Φ_3^4 measure, let $f: M' \mapsto M$ be a smooth diffeomorphism, then the pulled–back field $f^*\phi$ on (M', f^*g) will have the law of a Φ_3^4 measure of the SPDE (1.3) for the metric f^*g. Such measure gives for the first time a non-perturbative, non-topological interacting quantum field theory on 3-dimensional curved Riemannian spaces. We prove in [6] that the semigroup on $C^{-1/2-\varepsilon}(M)$ generated by the dynamics (1.3) has a unique invariant probability measure. This uniqueness result yields a stronger notion of covariance.

6 – Non-constant coupling function and vector bundle cases

6.1 Non-constant coupling function

Denote by $\mu_{\mathrm{GFF}}(d\phi)$ the Gaussian free field measure on M. We construct in this section a Φ_3^4 measure on M with formal description

$$e^{-\int_M \lambda(x)\phi^4(x)dx}\mu_{\mathrm{GFF}}(d\phi) \tag{6.1}$$

corresponding to a space dependent coupling constant $\lambda(x)$, say a C^∞ function of x s.t. $\lambda>0$ on M. We considered so far the case $\lambda=1$. We show that the results proved to analyse this special case allow to deal with the general case. We construct the measure (6.1) as an invariant measure of a Markovian evolution in $C^{-1/2-\varepsilon}(M)$. The goal of this section is to prove that the **counterterms are local functionals in** the coupling function $\lambda\in C^\infty(M,\mathbb{R}_{>0})$ which is a deep feature of renormalization: *The divergent counterterms you need to subtract at some point x depends only on finite jets of the Lagrangian functional density at the same point x.* We refer to the work [1] of Abdesselam which also deals with space dependent couplings for some hierarchical model in 3d.

Let us make a detailed calculation to determine the fine structure of the divergences that arise in the equation as well as to establish the locality of the counterterms. We use the same regularization ξ_r as above. Start from $\mathcal{L}u=\xi_r-\lambda u^3$, set a first decomposition $u=\<1>_r+Z$, hence $\mathcal{L}(\<1>_r+Z)=\xi_r-\left(\<1>_r+Z\right)^3$ therefore

$$\begin{aligned}\mathcal{L}Z=-\lambda\left(\<1>_r^3+3Z\<1>_r^2+3Z^2\<1>_r+Z^3\right)&=-\lambda\left(\<3>_r+3a_r\<1>_r+3Z\<2>_r+3Za_r+3Z^2\<1>_r+Z^3\right)\\&=-\lambda\left(\<3>_r+3Z\<2>_r+3Z^2\<1>_r+Z^3\right)-3\lambda a_r u\end{aligned}$$

where we compare the covariant Wick renormalized powers $\<2>_r$ with the nonrenormalized powers and the counterterms are shown in red to clearly see the difference.

This motivates to define a new regularized equation for u_r as

$$\mathcal{L}u=\xi_r-\lambda u^3+3\lambda a_r u$$

where adding the counterterm in red has the effect of Wick renormalizing the trees appearing on the r.h.s of the equation for Z. Therefore, in what follows, all trees that appear are covariantly Wick renormalized. We also define the following new stochastic trees decorated by the subscript $\cdot_\lambda$:

$$\<30>_{r,\lambda}:=\underline{\mathcal{L}}^{-1}(\lambda\<3>_r)$$

and

$$\<20>_{r,\lambda}:=\underline{\mathcal{L}}^{-1}(\lambda\<2>_r),$$

this means they have an extra coupling function λ inserted at the vertex to avoid confusion with all previous trees introduced earlier where the coupling function was set to $\lambda=1$.

Now we introduce a further decomposition writing u as $u = \<1>_r \underbrace{- \<30>_{r,\lambda} + R}_{=Z}$ where we used a subscript $._\lambda$ to denote the fact that the nonlinear term u^3 has become λu^3 which affects the trees in the equation.

Then the next remainder R satisfies the new equation:

$$\mathcal{L}R = -3\lambda \<2>_r \left(R - \<30>_{r,\lambda}\right) - 3\lambda \<1>_r \left(R - \<30>_{r,\lambda}\right)^2 - \lambda \left(R - \<30>_{r,\lambda}\right)^3$$

with Wick renormalized r.h.s. Let us anticipate a bit on what follows and try to guess what problematic terms we will encounter next. We spot two problematic terms in the equation for R, first we expect that R has regularity $1-$ and $\<2>_r$ has regularity $-1-$ hence the product $R\<2>_r$ is ill–defined with the borderline regularity. Furthermore the product $3\lambda \<2>_r \<30>_{r,\lambda}$ is clearly ill–defined and requires a renormalization. We will discuss later how to deal with this term.

As in the work of Jagannath–Perkowski, we introduce a Cole–Hopf transform $v_r := e^{3\<20>_{r,\lambda}} R$ to kill the borderline ill–defined product $R\<2>_r$. A long and tedious calculation yields the following equation for v_r:

$$\begin{aligned}\mathcal{L}v_r &= 9|\nabla \<20>_{r,\lambda}|^2 v_r - 3v_r \<20>_{r,\lambda} - 6\nabla(\<20>_{r,\lambda})\nabla v_r + 3\lambda e^{3\<20>_{r,\lambda}} \<2>_r \<30>_{r,\lambda} \\ &-3\lambda e^{3\<20>_{r,\lambda}} \left(e^{-3\<20>_{r,\lambda}} v_r - \<30>_{r,\lambda}\right)^2 \<1>_r - \lambda e^{3\<20>_{r,\lambda}} \left(e^{-3\<20>_{r,\lambda}} v_r - \<30>_{r,\lambda}\right)^3 .\end{aligned}$$

Let us again try to identify the origin of new divergences in this new equation. It may arise from the product $e^{3\<20>_{r,\lambda}} \left(\lambda \<2>_r \<30>_{r,\lambda}\right)$ where we need to renormalize the quintic term in parenthesis. Once it is renormalized it will be of regularity $-1-0$ and we still need to discuss how we can make sense of the product with $e^{3\<20>_{r,\lambda}}$ which is of regularity $1-$. As usual, by the paraproduct decomposition, it is the resonant term $\lambda \<2>_r \odot \<30>_{r,\lambda}$ which requires a renormalization which is very similar to what we did for the quintic term τ_4 except for the insertion of the coupling function λ at the vertices. Let us indicate what changes must be made on the treatment of τ_{41} in order to renormalize this term with the coupling function λ. The divergent subamplitude $\mathcal{A}_r$ now reads

$$\mathcal{A}_r(t, s, x_*, x_1, x_2, x_a) := [\odot](x_*, x_1, x_2)\underline{\mathcal{L}}^{-1}\big((t, x_1), (s, x_a)\big) G_r^{(2)}\big((t, x_2), (s, x_a)\big) \lambda(x_2)\lambda(x_a)$$

where the function λ appearing in the subamplitude $\mathcal{A}_r$ is taken at two different points x_2 and x_a. Now if one repeats the proof of Lemma 19 with the function λ inserted at the right places, one ends up with a counterterm of the form

$$c_r(t_1, x_*) \quad = \quad \mathcal{S}\int_{-\infty}^{t_1} \sum_{0\leqslant|k-\ell|\leqslant 1, i} P_k^i(x_*, .) \circ \tilde{\chi}\lambda \circ \left(G_r^{(2)}(t-s) \circ e^{(t-s)(\Delta-1)}\right) \circ \lambda\tilde{\chi} \circ \tilde{P}_\ell^i(., x_*) ds$$

where the $P_k^i, \tilde{P}_\ell^i$ are the Littlewood-Paley-Stein projectors, the functions $\tilde{\chi} \in C_c^\infty(U)$ are arbitrary test functions s.t. $\tilde{\chi} = 1$ near x_* and where we used the explicit definition of $[\odot]$ in terms of the Littlewood-Paley-Stein projectors. Observe that changing the position of λ at two places in the above composition of operators inserts two commutator terms and since the commutators lower the pseudodifferential order $[\Psi^{m_1}, \Psi^{m_2}] \in \Psi^{m_1+m_2-1}$, we get

$$\begin{aligned}c_r(t, x_*) &= \mathcal{S}\int_{-\infty}^{t} \sum_{0\leqslant|k-\ell|\leqslant 1, i} \lambda^2(x_*) P_k^i(x_*, .) \circ \tilde{\chi} \circ \left(G_r^{(2)}(t-s) \circ e^{(t-s)(\Delta-1)}\right) \circ \tilde{\chi} \circ \tilde{P}_\ell^i(., x_*) ds \\ &+ \text{ Trace density of some element in } \Psi^{-4}(M) \\ &= \mathcal{S}\int_{-\infty}^{t} \sum_{0\leqslant|k-\ell|\leqslant 1, i} \lambda^2(x_*) P_k^i(x_*, .) \circ \tilde{\chi} \circ \left(G_r^{(2)}(t-s) \circ e^{(t-s)(\Delta-1)}\right) \circ \tilde{\chi} \circ \tilde{P}_\ell^i(., x_*) ds\end{aligned}$$

since elements in $\Psi^{-4}(M)$ are trace class. Therefore as we saw in Section 4.3, we find that the term

$$\lambda \<2>_r \odot \<30>_{r,\lambda} - \lambda^2 b_r \<1>_r$$

converges as $r \to 0^+$ in $\mathcal{C}^{-1/2-5\varepsilon}([0,T]\times M)$ in $L^2(\Omega)$. There is no problem for multiplying it with

$$\exp\left(3\Upsilon_{r,\lambda}\right) \in \mathcal{C}^{1-2\varepsilon}([0,T]\times M)$$

and send r to 0. For this reason, in the ill–defined product $\underbrace{e^{3\Upsilon_{r,\lambda}}}\underbrace{\left(\lambda \vee_r \Psi_{r,\lambda} - \lambda^2 b_r \mathfrak{l}_r\right)}$ where the term in parenthesis is well–defined at the limit $r \to 0^+$ but the product of the two terms underbraced is ill–defined at the limit $r \to 0^+$.

In our companion paper [10], we show that using two renormalizations yields the existence of

$$e^{3\Upsilon_{r,\lambda}}\left(\lambda \vee_r \Psi_{r,\lambda} - \lambda^2 b_r \mathfrak{l}_r - \lambda^2 b_r \Psi_{r,\lambda}\right)$$

as a random variable valued in $\mathcal{C}^{-4\varepsilon}([0,T]\times M) + C_T\mathcal{C}^{\frac{1}{2}-7\varepsilon} + C_T\mathcal{C}^{-1-2\varepsilon}(M)$ when $r \to 0^+$. Now we define a new element ϕ_r by the equation:

$$v_r = Y + \phi_r = \underbrace{\mathcal{L}^{-1}\left(3e^{3\Upsilon_{r,\lambda}}\left(\lambda \vee_r \Psi_{r,\lambda} - \lambda^2 b_r\left(\mathfrak{l}_r + \Psi_{r,\lambda}\right)\right)\right)}_{=Y} + \phi_r$$

where $Y \in \mathcal{C}^{1-\varepsilon}([0,T]\times M)$, $\forall \varepsilon > 0$ as $r \to 0^+$.

We rewrite the equation for v_r in terms of both (Y, ϕ_r), the goal of introducing Y is that it is well–defined thanks to the double renormalizations we just performed and this equation makes appear new divergent terms:

$$\begin{aligned}\mathcal{L}\phi_r &= 9|\nabla \Upsilon_{r,\lambda}|^2(Y+\phi_r) - 3(Y+\phi_r)\Upsilon_{r,\lambda} - 6\nabla(\Upsilon_{r,\lambda})\nabla(Y+\phi_r) + 3\lambda^2 b_r e^{3\Upsilon_{r,\lambda}}\left(\mathfrak{l}_r + \Psi_{r,\lambda}\right)\\ &\quad -3\lambda e^{3\Upsilon_{r,\lambda}}\left(e^{-3\Upsilon_{r,\lambda}}(Y+\phi_r) - \Psi_{r,\lambda}\right)^2 \mathfrak{l}_r - \lambda e^{3\Upsilon_{r,\lambda}}\left(e^{-3\Upsilon_{r,\lambda}}(Y+\phi_r) - \Psi_{r,\lambda}\right)^3.\end{aligned}$$

The first divergent term in purple on the r.h.s comes from the counterterm in the definition of Y. The next terms which are ill–posed read $9|\nabla\Upsilon_{r,\lambda}|^2$ and $\nabla(\Upsilon_{r,\lambda}).\nabla Y$ since Y is a priori in $C_T\mathcal{C}^{1-2\varepsilon}(M)$ and $\nabla(\Upsilon_{r,\lambda}) \in C_T\mathcal{C}^{-2\varepsilon}(M)$ hence the scalar product will be ill–defined. In the companion work [10], we also show how to extract the singular term from this scalar product. More precisely :

$$\nabla(\Upsilon_{r,\lambda}).\nabla Y = |\nabla(\Upsilon_{r,\lambda})|^2 3e^{3\Upsilon_{r,\lambda}}\Psi_{r,\lambda} + \text{well–defined}.$$

We isolated the singular term at the limit $r \to 0^+$ as $|\nabla(\Upsilon_{r,\lambda})|^2$. It remains to explain how to extract the counterterm of $|\nabla(\Upsilon_{r,\lambda})|^2$ while taking into account the presence of the coupling function λ. This was done at the end of subsection 4.2. Similarly, the singular part of $|\nabla(\Upsilon_{r,\lambda})|^2$ is the same as the singular part of $(\Delta\Upsilon_{r,\lambda})\Upsilon_{r,\lambda}$ whose divergent part now reads

$$\mathcal{S}\left(\int_{-\infty}^{t} tr_{L^2}\left(\lambda \circ e^{s_2'(\Delta-1)} \circ \lambda \circ G_r^{(2)}(s_2')ds_2'\right)\right)$$

where the λ in the above expression is viewed as multiplication operator. Note that the difference $\int_{-\infty}^{t} tr_{L^2}\left(\left(\lambda \circ e^{s_2'(\Delta-1)} \circ \lambda - \lambda^2 \circ e^{s_2'(\Delta-1)}\right) \circ G_r^{(2)}(s_2')ds_2'\right)$ is regular when $r \to 0^+$ simply by the fact that $\lambda(x)\lambda(y) - \lambda^2(x)$ vanishes near the diagonal $x = y$ and by simple power counting argument as we did when we studied amplitudes. Finally, the singular part

$$\mathcal{S}\left(\int_{-\infty}^{t} tr_{L^2}\left(\lambda^2 \circ e^{s_2'(\Delta-1)} \circ G_r^{(2)}(s_2')ds_2'\right)\right) = \lambda^2(x)\frac{b_r}{3}$$

as we did already calculate for quartic graphs. This means that

$$\nabla(\Upsilon_{r,\lambda}).\nabla Y - \lambda^2(x) b_r e^{3\Upsilon_{r,\lambda}}\Psi_{r,\lambda}$$

has a well–defined limit when $r \to 0^+$.

In the same way the limit $9|\nabla\Upsilon_{r,\lambda}|^2 - 9\lambda^2\frac{b_r}{3}$ has a well–defined limit. Let us add and subtract the divergent terms in the equation for ϕ_r to single out the divergences which are again represented in *red*, this now reads

$$\begin{aligned}\mathcal{L}\phi_r &= \left(9|\nabla \Upsilon_{r,\lambda}|^2 - 3\lambda^2 b_r\right)(Y+\phi_r) - 3(Y+\phi_r)\Upsilon_{r,\lambda} - 6\nabla(\Upsilon_{r,\lambda})\nabla(Y+\phi_r)\\ &+6\lambda^2(x) b_r e^{3\Upsilon_{r,\lambda}}\Psi_{r,\lambda} + 3\lambda^2 b_r(Y+\phi_r) - 6\lambda^2 b_r e^{3\Upsilon_{r,\lambda}}\Psi_{r,\lambda} + 3\lambda^2 b_r e^{3\Upsilon_{r,\lambda}}\left(\mathsf{I}_r + \Psi_{r,\lambda}\right)\\ &-3\lambda e^{3\Upsilon_{r,\lambda}}\left(e^{-3\Upsilon_{r,\lambda}}(Y+\phi_r) - \Psi_{r,\lambda}\right)^2 \mathsf{I}_r - \lambda e^{3\Upsilon_{r,\lambda}}\left(e^{-3\Upsilon_{r,\lambda}}(Y+\phi_r) - \Psi_{r,\lambda}\right)^3 .\end{aligned}$$

Now note that the sum of all the terms in red equals $3\lambda^2 b_r e^{3\Upsilon_{r,\lambda}} u$, so introduce the function

$$v_{r,\mathrm{ref},\lambda} := 3\underline{\mathcal{L}}^{-1}\left(\lambda e^{3\Upsilon_{r,\lambda}}\left\{\Psi_{r,\lambda}\vee_r - b_r\left(\mathsf{I}_r + \lambda\Psi_r\right)\right\}\right). \tag{6.2}$$

This is an element of $C_T C^{1-\varepsilon}(M)$ that converges in any of these spaces as r goes to 0 in $L^2(\Omega)$. Setting

$$v_r := e^{3\Upsilon_r}\left(u_r - \mathsf{I}_r + \lambda\Psi_r\right) - v_{r,\mathrm{ref},\lambda}$$

we see that u_r is a solution to the renormalized equation

$$\mathcal{L}u = \xi_r - \lambda u^3 + \left(3\lambda a_r - 3\lambda^2 b_r\right)u \tag{6.3}$$

has a well–defined solution in the limit $r \to 0^+$, the solution enjoys all the properties we proved in case λ is constant. if and only if v_r is a solution to an equation of the form

$$\mathcal{L}v_r = b_{r,\lambda}\nabla v_r - a_{r,\lambda}v_r^3 + Z_{2,r,\lambda}v_r^2 + Z_{1,r,\lambda}v_r + Z_{0,r,\lambda}$$

where $b_{r,\lambda} \in C_T C^{-\varepsilon}(M), a_{r,\lambda} \in C_T C^{1-2\varepsilon}(M)$ and $Z_{i,r,\lambda} \in C_T C^{-1/2-\varepsilon}(M)$ all converge in their spaces as $r > 0$ goes to 0 in $L^2(\Omega)$, for all $\varepsilon > 0$. This analysis of equation (6.3) puts us in a position to go in the present setting over all the different steps that we have done above to construct the Φ^4_3 measure when $\lambda = 1$ and provides a construction of the Φ^4_3 measure in a setting where the coupling constant is space dependent.

6.2 The Φ^4_3 vectorial model in the bundle case.

We conclude this section by showing that our result also holds when ϕ is taken to be the section of a vector bundle. This corresponds to the construction of the vectorial ϕ^4_3 measure (which is sometimes called the $O(N)$-model in the physics literature). We summarize here what changes need to be done in the bundle case, while more technical details are postponed in our companion paper. First, we consider a Hermitian vector bundle $E \mapsto M$, smooth (resp. $\mathcal{C}^\alpha$, distributional) sections of E are denoted by $\Gamma^\infty(M,E) = C^\infty(E)$ (resp $\Gamma^\alpha(M,E) = \mathcal{C}^\alpha(E), \Gamma^{-\infty}(M,E) = \mathcal{D}'(E)$) and we are given some generalized Laplacian Δ_g which means Δ_g is a symmetric differential operator acting on $C^\infty(E)$ s.t. its principal symbol is positive definite, symmetric, diagonal, it reads $g_{\mu\nu}(x)\xi^\mu\xi^\nu \otimes Id_{End(E_x)}$ as a function on $C^\infty(T^*M, End(E))$ where $g_{\mu\nu}$ is the induced Riemannian cometric on T^*M. We furthermore assume that $-\langle\varphi, \Delta_g\varphi\rangle_{L^2(E)} \geqslant 0$ for all $\varphi \in C^\infty(E)$. The above in particular implies that $-\Delta_g$ is a nonnegative, elliptic second order operator. The corresponding heat operator now reads $\mathcal{L} = \partial_t + 1 - \Delta_g$. It is well–known from elliptic theory that $P := 1 - \Delta_g$ has self–adjoint extension as $P : H^2(E) \mapsto L^2(E)$, P has compact self–adjoint resolvent with discrete real spectrum in $\mathbb{R}_{\geqslant 0}$ and the eigenfunctions of P form an L^2–basis of the space $L^2(E)$ of L^2 sections of E. In this case, we can define some E-valued white noise as $\xi = \sum_{\lambda\in\sigma(P)} c_\lambda e_\lambda$ where the sum runs over the eigenvalues of P, e_λ are the eigensections of P and $c_\lambda \sim \mathcal{N}(0,1)$ are i.i.d gaussian random variables. The E-valued GFF reads $P^{-\frac{1}{2}}\xi$. The goal is to make sense of the Gibbs measure

$$\frac{e^{-\int_M \lambda\left(\langle\varphi,\varphi\rangle_E\right)^2} d\mu_{GFF}}{\mathbb{E}_{GFF}\left(e^{-\int_M \lambda\left(\langle\varphi,\varphi\rangle_E\right)^2}\right)}$$

where $\langle .,.\rangle_E$ denotes the Hermitian scalar product of E, the interaction term now reads $\left(\langle\varphi,\varphi\rangle_E\right)^2$ in the vectorial case and $\lambda \in C^\infty(M, \mathbb{R}_{>0})$ is the coupling function.

The corresponding vectorial ϕ^4_3 SPDE reads:

$$\mathcal{L}u = -\lambda u\langle u,u\rangle + (\mathrm{rk}(E)+2)(a_r - b_r)u + \sqrt{2}\xi_r \tag{6.4}$$

where u is an E-valued random distribution over space time $\mathbb{R}\times M$, $\xi_r = e^{-rP}\xi$ and a_r, b_r are the *exact same constants than in the scalar case.* All E-valued Besov (resp Hölder, Sobolev) distributions are defined almost exactly like in the scalar case using local charts on M and local trivializations of $E\mapsto M$, we denote them by $\mathcal{B}^s_{p,q}(E)$ (resp $\mathcal{C}^s(E), H^s(E)$). Because the analytical properties of the heat kernel $(e^{-tP})_{t\geqslant 0}$ acting on sections of E are exactly the same as in the scalar case, both inverses $\mathcal{L}^{-1}$ and $\underline{\mathcal{L}}^{-1}$ are well–defined with the exact same definitions and they have the exact same analytical properties as in the scalar case. The symbol $\overset{\circ}{\mid}_r$ still denotes $\underline{\mathcal{L}}^{-1}\xi_r$ Because of the classical results on the asymptotic expansion of the heat kernel in the bundle case [17], the key idea is that the singularities are valued in diagonal elements in $C^\infty(End(E))$, we immediately find that the covariant Wick renormalization for the cubic power reads $\overset{\circ\circ\circ}{\Psi}_r := \left\langle \overset{\circ}{\mid}_r, \overset{\circ}{\mid}_r\right\rangle \overset{\circ}{\mid}_r - (\mathrm{rk}(E)+2)a_r\overset{\circ}{\mid}_r$ for the *same universal constant* a_r as in the scalar case and $\mathrm{rk}(E)$ is the rank of the vector bundle E. Beware that the cubic vertex has a new meaning, it is a Hermitian scalar product in the fibers of E times an element of a fiber of E since u^3 has become $\langle u,u\rangle_E\, u$. The new stochastic tree now reads

$$\overset{\Psi}{\mid}_{r,\lambda} := \underline{\mathcal{L}}^{-1}(\lambda\overset{\circ\circ\circ}{\Psi}_r) := \underline{\mathcal{L}}^{-1}\left(\lambda\left(\left\langle \overset{\circ}{\mid}_r, \overset{\circ}{\mid}_r\right\rangle \overset{\circ}{\mid}_r - (\mathrm{rk}(E)+2)a_r\overset{\circ}{\mid}_r\right)\right).$$

Then we introduce a similar decomposition as: $u = \overset{\circ}{\mid}_r - \overset{\Psi}{\mid}_{r,\lambda} + R$. Writing the equation satisfied by R, we see that the new term in the bundle case we need to eliminate is the borderline ill–defined product $-\lambda\left\langle \overset{\circ}{\mid}_r, \overset{\circ}{\mid}_r\right\rangle R - 2\lambda\left\langle R, \overset{\circ}{\mid}_r\right\rangle \overset{\circ}{\mid}_r$. We define some random endomorphism $\overset{\circ\circ}{\vee}_r$ acting on smooth sections $C^\infty(E)$ as

$$\overset{\circ\circ}{\vee}_r : T\in C^\infty(E)\mapsto \left\langle \overset{\circ}{\mid}_r, \overset{\circ}{\mid}_r\right\rangle T + \left\langle T, \overset{\circ}{\mid}_r\right\rangle \overset{\circ}{\mid}_r - \big(\mathrm{rk}(E)+2\big)a_r T\in\mathcal{D}'(E).$$

Observe that with this definition, it holds $3\overset{\circ\circ\circ}{\Psi}_r = \overset{\circ\circ}{\vee}_r(\overset{\circ}{\mid}_r) - 2(\mathrm{rk}(E)+2)\overset{\circ}{\mid}_r$, which is consistent with the fact that $\overset{\circ\circ}{\vee}_r$ is the renormalized version of $3\overset{\circ}{\mid}{}^2_r$. The bundle morphism $\overset{\circ\circ}{\vee}_r$ is *local* since it is $C^\infty(M)$-linear hence it can be identified canonically with some random element in $\mathcal{D}'(M, End(E))$. Therefore, we define a new *vectorial Cole–Hopf transform* needed in the bundle case in terms of the above random endomorphism $\overset{\circ\circ}{\vee}_r$ as follows :

Definition 23 – *In the above notation, we define the random bundle map:*

$$R := e^{-\underline{\mathcal{L}}^{-1}\lambda\overset{\circ\circ}{\vee}_r}(v_r) \tag{6.5}$$

where similar stochastic estimates in the bundle case allow to prove that $\underline{\mathcal{L}}^{-1}\lambda\overset{\circ\circ}{\vee}_r$ *is a.s. in* $\mathcal{C}^{1-\varepsilon}\left(M, End(E)\right)$ *for all* $\varepsilon>0$.

Accordingly, one also defines the stochastic object

$$v_{r,\mathrm{ref}} := \underline{\mathcal{L}}^{-1}\Big(e^{\underline{\mathcal{L}}^{-1}\lambda\overset{\circ\circ}{\vee}_r}\Big\{\overset{\circ\circ}{\vee}_r(\overset{\Psi}{\mid}_{r,\lambda}) - (\mathrm{rk}(E)+2)b_r\big(\overset{\circ}{\mid}_r + \overset{\Psi}{\mid}_{r,\lambda}\big)\Big\}\Big),$$

that obeys the same estimate in the bundle case and is a.s. in $C_T\mathcal{C}^{1-\varepsilon}$ for all $\varepsilon>0$. Similarly, define

$$\begin{aligned}
\tau_1 : T\in C^\infty(E)&\mapsto \langle \overset{\circ}{\mid}_r\odot\overset{\Psi}{\mid}_{r,\lambda}\rangle T + \langle T, (\overset{\circ}{\mid}_r\rangle\odot\overset{\Psi}{\mid}_{r,\lambda}) + \langle T, (\overset{\Psi}{\mid}_{r,\lambda}\rangle\odot\overset{\circ}{\mid}_r),\\
\tau_2 : T\in C^\infty(E)&\mapsto \overset{\circ\circ}{\vee}_r\odot\big(\underline{\mathcal{L}}^{-1}(\lambda\overset{\circ\circ}{\vee}_r(T))\big) - (\mathrm{rk}(E)+2)b_r T,\\
\tau_3 : T\in C^\infty(E)&\mapsto \nabla\underline{\mathcal{L}}^{-1}(\lambda\overset{\circ\circ}{\vee}_r)\odot\big(\nabla\underline{\mathcal{L}}^{-1}(\lambda\overset{\circ\circ}{\vee}_r(T))\big) - (\mathrm{rk}(E)+2)b_r T,\\
\tau_4 &:= \overset{\circ\circ}{\vee}_r\odot(\overset{\Psi}{\mid}_{r,\lambda}) - (\mathrm{rk}(E)+2)b_r\overset{\circ}{\mid}_r.
\end{aligned}$$

τ_1 is local and belongs to $C_T\mathcal{C}^{0-}(M, End(E))$, while τ_2 and τ_3 are not local, and only belong to $C_T\mathcal{L}(C^\infty(E), \mathcal{C}^{0-}(E))$. Finally, it holds $\tau_4\in C_T\mathcal{C}^{-1/2-}(E)$. Contrary to the second Wick power, we do not need τ_2 and τ_3 to be local, since we do not aim to rise them to some exponent or take their exponential, and always evaluate them at some $T\in\mathcal{C}^\alpha(E)$. The proofs that these objects are correctly renormalized, and that $v_{r,\mathrm{ref}}$ obeys the same estimates can be found in our companion paper. Regarding the global existence estimates, the L^p estimates can be obtained exactly in the same way in the vector bundle case, taking care of pairing the equation with $\langle v,v\rangle^{(p-2)/2}v$ instead of v^{p-1}, and to perform this pairing both in E and in $L^2(M)$.

A – Littlewood-Paley-Stein projectors

There are several ways of defining some Littlewood-Paley type projectors on function spaces over a manifold – see [59, 7, 8, 50, 70] for a sample. We choose here an intermediate road and use the classical Littlewood-Paley projectors over $\mathbb{R}^d$ to define a number of operators on functions spaces over M using local charts. This allows to import at low cost some known regularity properties of the corresponding objects from the flat to the curved setting. We denote as usual by $B^\gamma_{p,q}(M)$ the Besov spaces over M and by C^γ(M) the Besov-Hölder space $B^\gamma_{\infty,\infty}(M)$, with associated norm denoted by $\|\cdot\|_{C^\gamma}$.

Let then denote by

$$a' \prec b' := \sum_{-1\leq j<k-1} (\Delta_j a')(\Delta_k b')$$

the paraproduct of some distributions a' and b' on $\mathbb{R}^d$, and write

$$a' \odot b' := \sum_{|j-k|\leq 1} (\Delta_j a')(\Delta_k b')$$

for the resonance of a' and b' whenever the latter is defined. Let $(U_i,\kappa_i)_i$ denote a finite open cover of M by some charts, with κ_i a smooth diffeomorphism between $U_i\subset M$ and $\kappa_i(U_i)\subset\mathbb{R}^d$. Let $(\chi_i)_i$ be a partition of unity subordinated to $(U_i)_i$, so $\sum_i \chi_i = 1$, with $\chi_i\in C^\infty_c(U_i)$. Choose also for every index i a function $\tilde\chi_i\in C^\infty_c(U_i)$ such that $\tilde\chi_i$ equals 1 on the support of χ_i and some function $\psi_i\in C^\infty_c(\kappa_i(U_i))$ which equals 1 on the support of $\kappa_{i*}(\tilde\chi_i)$. Given some smooth functions a,b on M we have the decomposition

$$\begin{aligned}
ab &= \sum_{i\in I}(a\chi_i)(b\tilde\chi_i) = \sum_{i\in I}\kappa_i^*\big[\kappa_{i*}(a\chi_i)\big]\kappa_i^*\big[\kappa_{i*}(b\tilde\chi_i)\big]\\
&= \sum_{i\in I}\kappa_i^*\big[(\kappa_{i*}(a\chi_i)(\kappa_{i*}(b\tilde\chi_i)\big] = \sum_{i\in I}\kappa_i^*\big[\psi_i\kappa_{i*}(a\chi_i)\kappa_{i*}(b\tilde\chi_i)\big]\\
&= \sum_{i\in I}\kappa_i^*\big[\psi_i\big(\kappa_{i*}(a\chi_i)\prec\kappa_{i*}(b\tilde\chi_i)\big)\big] + \sum_{i\in I}\kappa_i^*\big[\psi_i\big(\kappa_{i*}(a\chi_i)\odot\kappa_{i*}(b\tilde\chi_i)\big)\big]\\
&\quad + \sum_{i\in I}\kappa_i^*\big[\psi_i\big(\kappa_{i*}(a\chi_i)\succ\kappa_{i*}(b\tilde\chi_i)\big)\big]
\end{aligned}$$

Actually, for arbitrary $\chi_i,\tilde\chi_i\in C^\infty_c(U_i)^2$ such that $\tilde\chi_i=1$ on the support of χ_i, we set the generalized Littlewood-Paley-Stein projectors

$$\begin{aligned}
P^i_k(a) &:= \kappa_i^*\big[\psi_i\Delta_k\big(\kappa_{i*}(\chi_i a)\big)\big],\\
\tilde P^i_k(a) &:= \kappa_i^*\big[\tilde\psi_i\Delta_k\big(\kappa_{i*}(\chi_i a)\big)\big],
\end{aligned}$$

where $\tilde\psi_i\in C^\infty_c(\kappa_i(U_i))$ equals 1 on the support of ψ_i. We do not necessarily require that $\sum_{i\in I}\chi_i = 1$.

On the manifold M, recall $i\in I$ denotes a chart index, we define generalized chart localized operations as :

$$a\prec_i b := \sum_{-1\leq j<k-1}\left(P^i_j a\right)\left(\tilde P^i_k b\right),$$

and

$$a\succ_i b := \sum_{i\in I}\sum_{-1\leq j<k-1}\left(P^i_k a\right)\left(\tilde P^i_j b\right),$$

and

$$a\odot_i b := \sum_{i\in I}\sum_{|j-k|\leqslant 1}\left(P^i_j a\right)\left(\tilde P^i_k b\right).$$

In particular when $\sum_{i\in I}\chi_i = 1$, the above operations decompose the product ab on M as:

$$ab = \sum_{i\in I}\left(a\prec_i b + a\odot_i b + a\succ_i b\right).$$

Note the important fact that the definition of the resonant product and paraproducts are asymmetrical, therefore they are noncommutative meaning that $a\prec b\neq b\succ a$, however all the regularity properties are similar as in the flat case. We collect in the next two statements some regularity properties of these operators and refer the reader to our companion work [10] for their proofs.

Proposition 24 – *One has the following continuity estimates. For every chart index i,*

– *For p, p_1, p_2, q, q_1, q_2 in $[1, +\infty]$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$*

(a) *For $\gamma_2 \in \mathbb{R}$*

$$\|a \prec_i b\|_{B^{\gamma_2}_{p,q_2}} \lesssim \|a\|_{L^{p_1}} \|b\|_{B^{\gamma_2}_{p_2,q_2}},$$

(b) *For $\gamma_1 < 0$ and $\gamma_2 \in \mathbb{R}$*

$$\|a \prec_i b\|_{B^{\gamma_1+\gamma_2}_{p,q}} \lesssim \|a\|_{B^{\gamma_1}_{p_1,q_1}} \|b\|_{B^{\gamma_2}_{p_2,q_2}}$$

(c) *For any $\gamma_1, \gamma_2 \in \mathbb{R}$ with $\gamma_1 + \gamma_2 > 0$ one has*

$$\|a \odot_i b\|_{B^{\gamma_1+\gamma_2}_{p,q}} \lesssim \|a\|_{B^{\gamma_1}_{p_1,q_1}} \|b\|_{B^{\gamma_2}_{p_2,q_2}}.$$

We recall from Lemma 7.2 of Mourrat & Weber's work [68] the following comparison test that we used in our proof of Theorem 5.

Proposition 25 – *Let a continuous function $F : [0, T] \to [0, +\infty)$ that satisfies the inequality*

$$\int_s^t F(s_1)^\lambda \, ds_1 \leq c\,(F(s) + 1) \tag{A.1}$$

for all $0 \leq s \leq t \leq T$, for some exponent $\lambda > 1$ and some positive constant c. Then there is a sequence of times $t_0 = 0 < t_1 < \cdots < t_N = T$ such that one has

$$F(t_n) \leq 1 + 2^{\frac{\lambda}{\lambda-1}} \Big(\frac{c}{1 - 2^{-(\lambda-1)}}\Big)^{\frac{1}{\lambda-1}} t_{n+1}^{-\frac{1}{\lambda-1}},$$

for all $0 \leq n \leq N - 1$.

Proof – We include a proof following closely Mourrat-Weber's comparison test in a slightly different setting compared to them since we have $F(s) + 1$ rather than $F(s)$ on the right hand side of the inequality (A.1). We first define $t_0 = 0$, then given some time t_n, consider $t^*_{n+1} = t_n + c2^\lambda(1 + F(t_n))^{1-\lambda}$, if $t^*_{n+1} \geqslant T$ we stop the algorithm and set $N = n + 1$, $t_{n+1} = T$ and verify that the conclusion of the statement holds. Otherwise, choosing t_{n+1} such that $F(t_{n+1}) = \inf_{t_n < s < t^*_{n+1}} F(s)$ yields a bound of the form $F(t_{n+1}) \leqslant \frac{1+F(t_n)}{2}$. By iteration, this yields a bound of the form $F(t_{n+1}) \leqslant \frac{F(t_0)-1}{2^{n+1}} + 1$. Note that for n large enough, since $\lambda > 1$

$$t^*_{n+1} - t_n = c2^\lambda(1 + F(t_n))^{1-\lambda} \geqslant c2^\lambda\big(2 + \frac{F(t_0) - 1}{2^n}\big)^{1-\lambda} \geqslant c2^\lambda(2 + 1/3)^{1-\lambda} \geqslant 2c,$$

hence the algorithm must terminate for n large enough after finite number of iterations. Now we need to check the conclusion $t_n = \sum_{k=0}^{n-1}(t_{k+1} - t_k) \leqslant c2^\lambda \sum_i (1 + F(t_i))^{1-\lambda}$. Note that since $F(t_i) \geqslant (F(t_n) - 1)2^{n-i} + 1$ then $(1 + F(t_i))^{1-\lambda} \leqslant ((F(t_n) - 1)2^{n-i} + 2)^{1-\lambda}$, so we have

$$\begin{aligned} t_{n+1} \leqslant c2^\lambda \sum_{i=0}^n \big((F(t_n) - 1)2^{n-i} + 2\big)^{1-\lambda} &\leqslant c2^\lambda(F(t_n) - 1)^{1-\lambda} \sum_{i=0}^n 2^{(n-i)(1-\lambda)} \\ &\leqslant c2^\lambda(F(t_n) - 1)^{1-\lambda} \frac{1}{1 - 2^{1-\lambda}}, \end{aligned}$$

which yields the estimate from the statement. ▷

Last we recall the fractional Leibniz rule and an elementary interpolation result used in the proof of the coming down property in Section 2.2, we prove these results in [10].

Proposition 26 – *Let $\alpha > 0, r \in \mathbb{N}$ and $p, p_1, p_2, q \in [1, \infty]$ such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

Then

$$\|u^{r+1}\|_{B^\alpha_{p,q}} \lesssim \|u^r\|_{L^{p_1}} \|u\|_{B^\alpha_{p_2,q}}.$$

Proposition 27 *– Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in [1, \infty]$ and $\theta \in [0, 1]$. Define $\alpha = \theta\alpha_1 + (1-\theta)\alpha_2$, and $p, q \in [1, \infty]$ by*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Then

$$\|u\|_{B^{\alpha}_{p,q}} \lesssim \|u\|^{\theta}_{B^{\alpha_1}_{p_1,q_1}} \|u\|^{1-\theta}_{B^{\alpha_2}_{p_2,q_2}}.$$

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