Locality for singular stochastic PDEs

I. BAILLEUL[1] & Y. BRUNED

Abstract. We develop in this note the tools of regularity structures to deal with singular stochastic PDEs that involve non-translation invariant differential operators. We describe in particular the renormalised equation for a very large class of spacetime dependent renormalization schemes.

1 – Introduction

In contrast to paracontrolled calculus [18, 1, 3] that was developed in a manifold, hence non-translation invariant, setting, after being set first in the Euclidean torus, the theory of regularity structures [20, 12, 15, 10] has so far been developed in the translation-invariant setting of locally Euclidean spaces and its analytic side was devised for the analysis of equations involving constant coefficients differential operators. The extension of the theory to a manifold setting calls for a development of the theory to deal in a first step with non-translation invariant differential operators in a locally Euclidean setting. It is the purpose of the present note to make that first step, assuming from the reader that she/he is already acquainted with the fundamentals of the theory of regularity structures, such as exposed for instance in Hairer's lecture notes [21, 22], the book [17] by Friz & Hairer, or Chandra & Weber's article [16], where accessible accounts of part of the theory of regularity structures. Bailleul & Hoshino's '*Tourist Guide*' [8] provides a dense self-contained presentation of the analytic and algebraic sides of the theory.

Denote by $(x_0, x_1) \in \mathbb{R} \times \mathbf{T}$ a typical spacetime point over the one dimensional periodic torus $\mathbf{T}$. (We choose for convenience of notations to work on the one dimensional torus rather than on a multidimensional torus.) Let

$$\mathcal{L}_i v := a_i(\cdot)\partial^2_{x_1} v + b_i(\cdot)\partial_{x_1} v, \qquad (1 \leq i \leq k_0),$$

stand for a finite family of second order differential operators with smooth coefficients. We consider systems of parabolic equations of the form

$$\big(\partial_{x_0} - \mathcal{L}_i\big)u_i = f(u)\xi + g(u, \partial_{x_1} u), \qquad (1 \leq i \leq k_0), \tag{1.1}$$

with $u := (u_1, \dots, u_{k_0})$ and each u_i taking values in a finite dimensional space $\mathbb{R}^{d_i}$, and $\xi = (\xi_1, \dots, \xi_{n_0})$ an n_0-dimensional spacetime 'noise'. An initial condition u_0 of positive Hölder regularity is given and fixed throughout that work.

Non-translation invariance refer to the fact that the operators $\mathcal{L}_i$ in the system (1.1) are not supposed to have constant coefficients. Working in a non-translation invariant setting does not influences the algebraic side of the theory of regularity structures encoding the mechanics of the local description of modelled distributions/functions. We can use in the non-translation invariant setting the same algebraic structure, with no extended decorations, as in the translation-invariant setting. The translation-invariant character of the operators involved in the equations studied so far using the theory of regularity structures manifests itself in two different ways. On a technical level, a number of operators have 'radial' kernels $q(y - x)$, rather than just kernels $q(x, y)$, as a consequence of translation-invariance. This simplifies a number of things, to start with the definition of β-regularizing singular kernels – Assumption 5.1 in [20], on which the construction of the regularity structure counterpart of the associated operator rests entirely. We tackle that technical point by adopting the heat kernel setting to the analytic side of the theory of regularity structures adopted in Bailleul & Hoshino's '*Tourist Guide*'. It rests on a family of estimates that play the role the estimates involved in the definition of β-regularizing kernels, and that can be proved to hold in a non-translation invariant setting. This is exposed in Section 2.

On a more fundamental level, translation-invariance manifests itself in the renormalization process in the fact that renormalization *constants* are involved rather than renormalization *functions*,

[1] I.B. acknowledges support from the CNRS and the ANR-16-CE40-0020-01 grant.

as expected in a non-translation-invariant setting, as found for instance in the paracontrolled approach to singular stochastic PDEs developed in [1, 2, 3]. The very formulation of the renormalization scheme of Bruned, Hairer and Zambotti [12] does not make obvious sense anymore in a non-translation-invariant setting. Denote by $\mathsf{M} = (\mathsf{\Pi}, \mathsf{g})$ the canonical smooth admissible model associated with a given (system of) singular stochastic PDE(s) driven by a smooth noise, acting on the regularity structure $\big((\mathcal{T}, \Delta), (\mathcal{T}^+, \Delta^+)\big)$ associated with that system after Bruned, Hairer and Zambotti's work [12]. The renormalized admissible smooth model built in [12] is associated with a character ℓ of an algebra to which one attaches a linear map $M_\ell : \mathcal{T} \to \mathcal{T}$, and a renormalized interpretation operator

$$\mathsf{\Pi}_\ell := \mathsf{\Pi} \circ M_\ell.$$

The map $\mathsf{\Pi}_\ell$ defines a unique admissible model. Such a renormalization scheme cannot work in a non-translation invariant setting, even by replacing ℓ by a character-valued function $\ell(x)$ of the state space variable x, as such a map cannot account for some possible internal renormalization indexed by functions of the internal variables of the multiple integral expression defining the renormalized interpretation map, not by the base/external variable x. So a different approach of renormalization is needed in the non-translation invariant setting. We devised in our previous work [4] an alternative approach to the renormalized equation that by-passes the use of extended decorations. We show in Section 3 that the natural extension of this approach to the renormalized equation works perfectly in the non-translation invariant setting. This alternative involves state space dependent preparation maps and their associated models, both of which are discussed in Section 3.1. The renormalized equation is dealt with in Section 3.2 following the pattern of proof of our previous work [4], as it applies almost verbatim to the non-translation invariant setting. We prove as Theorem 10 of Section 3.2 the following statement, which is the main result of the present work. The notions involved here are explained in the body of the text.

Theorem – *Let $R : \mathbb{R}^2 \times \mathcal{T} \to \mathcal{T}$ be a strong preparation map such that*

$$R\tau = \tau, \qquad \textit{for} \quad \tau \in T_X \oplus \bigoplus_{a \in \mathfrak{T}^+ \times \{0\}} \mathcal{I}_a(T).$$

Let M^R stand for its associated admissible model, with associated reconstruction map $\mathsf{R}^{\mathsf{M}^R}$. Let u stand for the modelled distribution solution of the regularity structure lift of system (1.1) *with initial condition u_0. Then*

$$u := \mathsf{R}^{\mathsf{M}^R} \mathsf{u}$$

is a solution of the renormalized system

$$(\partial_{x_0} - \mathcal{L}_i) u_i = F_i(u, \partial_{x_1} u)\, \xi + \sum_{l=0}^{n_0} F_i\Big(\big(R(\cdot)^* - \mathrm{Id}\big)\zeta_l\Big)(u, \partial_{x_1} u)\, \xi_l, \qquad (1 \le i \le k_0). \tag{1.2}$$

In a setting where the noise ξ is random, one can in particular associate to the character-valued function

$$\ell(x, \tau) := \mathbb{E}\big[(\mathsf{\Pi}\tau)(x)\big],$$

a strong preparation map $R_\ell : \mathbb{R}^2 \times \mathcal{T} \to \mathcal{T}$ in such a way that the associated renormalized system (1.2) takes the same form

$$(\partial_{x_0} - \mathcal{L}_i) u_i = F_i(u, \partial_{x_1} u)\, \xi + \sum_{\tau \in \mathcal{B}^-} \ell(\cdot, \tau)\, \frac{F_i(\tau)\big(u, \partial_{x_1} u\big)}{S(\tau)}, \qquad (1 \le i \le k_0),$$

as it does when the random noise ξ from which $\mathsf{\Pi}$ is built is translation-invariant in law and the above function $\ell(x, \tau)$ is a constant function of x for each τ. We do not touch here on the question of showing that such renormalization schemes provide limit admissible models when used on models built from regularized random noises such as space or spacetime white noise.

Notations – *Define the parabolic distance d on $\mathbb{R} \times \mathbb{R}$ by*

$$d(x, x') := \sqrt{|x_0 - x_0'|} + |x_1 - x_1'|,$$

for arbitrary points $x = (x_0, x_1), x' = (x'_0, x'_1)$ in $\mathbb{R}^2$, and define a scaling operator

$$\mathfrak{s}_\lambda(x) := \big(\lambda^2 x_0, \lambda x_1\big), \tag{1.3}$$

for all $\lambda > 0$ and all $x = (x_0, x_1) \in \mathbb{R}^2$. Set

$$|k|_\mathfrak{s} := 2k_0 + k_1, \qquad (\forall\, k \in \mathbb{N} \times \mathbb{N}),$$

We will write $O_{t,0^+}(t^\infty)$ to denote a t-dependent bounded quantity that is $O(t^n)$ for all $n \geq 0$, as t goes to 0. We will also write $O_{x,\infty}(|x|^{-\infty})$ to denote a x-dependent bounded quantity that is $O\big(|x|^{-n}\big)$ for all $n \geq 0$, as x goes to ∞. Denote by $D_{t,x,x'}$ the differential operator on $\mathbb{R} \times (\mathbb{R}^2)^2$.

We refer to Section 2 of our previous work [4] for a detailed presentation of the regularity structure $\mathscr{T} = \big((\mathcal{T}, \Delta), (\mathcal{T}^+, \Delta^+)\big)$ associated with system (1.1). We only mention here that the (possibly multidimensional) abstract noise symbol is denoted by $\zeta = (\zeta_0, \zeta_1, \dots, \zeta_{n_0})$, with $\zeta_0 = \mathbf{1}$. In accordance, we set $\xi_0 := 1$, the constant function equal to 1. Edges of trees in $\mathcal{T}$ or $\mathcal{T}^+$ are decorated with labels $a \in \mathfrak{T}^+ \times \mathbb{N}^2$, where $\mathfrak{T}^+ = (\mathfrak{t}_1, \dots, \mathfrak{t}_{k_0})$ stands for a finite set with k_0 elements representing which operator $(\partial_{x_0} - \mathcal{L}_i)^{-1}$ is attached to that edge. Recall that $S(\tau)$ stands for the symmetry factor of a decorated tree τ. Assume we are given a family

$$(F^l_k)_{1\leq k\leq k_0, 1\leq l\leq n_0}$$

of functions of a finite number of abstract variables Z_a indexed by $a \in \mathfrak{L}^+ \times \mathbb{N}^2$. We define in the usual way partial derivatives D_a in the variable Z_a, and set for all $k \in \mathbb{N}^2$

$$\partial^k := \sum_a Z_{a+k}\, D_a.$$

We define inductively a family $F = (F_i)_{1\leq i\leq k_0}$ of functions of the variables Z_a, indexed by T, setting for $\tau = X^k \zeta_l \prod_{j=1}^n \mathcal{I}_{a_j}(\tau_j)$, with $a_j = (\mathfrak{t}_{l_j}, k_j)$, for all $1 \leq i \leq k_0$,

$$F_i(\zeta_l) := F^l_i, \qquad F_i(\tau) := \partial^k D_{a_1} ... D_{a_n} F_i(\zeta_l) \prod_{j=1}^n F_{l_j}(\tau_j). \tag{1.4}$$

2 – The analytic setting

Denote by

$$\mathcal{L}_{x_1} v := a(\cdot)\partial^2_{x_1} v + b(\cdot)\partial_{x_1} v,$$

a second order differential operator on $\mathbb{R}$ with smooth coefficients. The notation $\mathcal{L}_{x_1}$ emphasizes the fact that it only acts on the x_1 argument. The analysis of the regularity structure lift of the resolvent operator $(\partial_{x_0} - \mathcal{L}_{x_1})^{-1}$ done by Hairer in Section 5 of [20] rests entirely on a notion of β-regularizing singular kernel whose definition takes profit of translation-invariance. We follow the alternative heat kernel approach of Bailleul & Hoshino's '*Tourist Guide*' [8] to extend the analysis to a non-translation invariant setting.

Define the non-positive symmetric fourth order elliptic differential operator on $\mathbb{R} \times \mathbb{R}$

$$\mathcal{G} := \partial^2_{x_0} - \mathcal{L}^2_{x_1}.$$

Denote by $K(t, x, x')$ the kernel of the heat semigroup $e^{t\mathcal{G}}$ – i.e. the kernel of the fundamental solution of the equation

$$(\partial_t - \mathcal{G}_x)\, K = 0,$$

with boundary condition

$$\lim_{t\to 0^+} K(t, x, x') = \delta_{x'},$$

using the notation $\mathcal{G}_x$ to emphasixe that the operator $\mathcal{G}$ acts on the x-variable of K. Bailleul & Hoshino's '*Tourist Guide*' [8] makes it clear that the analytic side of the machinery of regularity structures rests on the uniform estimate

$$\int_{\mathbb{R}^2} \big|\partial^n_x K(t, x, x')\big|\, d(x, x')^c\, dx \lesssim t^{\frac{c-|n|_\mathfrak{s}}{4}}, \qquad (\forall x' \in \mathbb{R}^2,\, n \in \mathbb{N} \times \mathbb{N},\, c \in \mathbb{R}_+). \tag{2.1}$$

(The notation $p_t(x,x')$ was used in [8] rather than $K(t,x,x')$ – the latter is more convenient in the present work.) This estimate plays the role of the decompositions of the β-regularizing singular kernels used by Hairer in Section 5 of [20].

The present section is dedicated to giving a precise description of the heat kernel $K(t,x,x')$ of $\mathcal{G}$ that implies the estimates (2.1). Our main result is given in Proposition 5 below. Its proof follows the philosophy of Melrose calculus [24] such as popularized by Grieser in his notes [19]. This point of view shares similarities with Hadamard's classical construction of the parametrix for the heat kernel of second order differential operators with regular enough coefficients.

We describe in Section 2.1 a family $(\mathcal{S}_\alpha)_{\alpha\geq 0}$ of classes of t-dependent operators $\mathbb{R}^2$ defined by 'scaling properties' of their kernels. Like in pseudo-differential calculus, one can associate a symbol to such operators, called here 'leading term', and a calculus on symbols that captures the essential action of the composition of such operators. The heat kernel K of $\mathcal{G}$ is constructed in Section 2.2 using a Volterra series representation, taking advantage of the calculus on symbols. The convergence result stated in our main result, Proposition 5, guarantees that K has a 'scaling property' from which the family of estimates (2.1) follows as a direct consequence.

2.1 The heat calculus associated with $\mathcal{G}$

Recall from (1.3) the definition of the parabolic scaling operator $\mathfrak{s}$.

Definition 1 – *Given $\alpha \geq 0$, define $\mathcal{S}_\alpha$ as the family of smooth functions K on $(0,\infty)\times(\mathbb{R}^2)^2$ such that*

- *one has the off diagonal decay*
$$D^m_{t,x,x'}K\big(t,x,x'\big) = O_{t,0^+}(t^\infty), \tag{2.2}$$
for all $x\neq x'$ and all $m\in\mathbb{N}\times(\mathbb{N}^2)^2$,
- *one has*
$$K\big(t,x,x'\big) = t^{-\frac{7}{4}+\alpha}\widetilde{K}\Big(t,\mathfrak{s}_{t^{-1/4}}(x-x'),x'\Big),$$
for a smooth function $\widetilde{K}:(0,\infty)\times(\mathbb{R}^2)^2\to\mathbb{R}$ such that the function
$$(s,z,x')\mapsto \widetilde{K}(s^4,z,x')$$
is smooth on $[0,\infty)\times(\mathbb{R}^2)^2$, with
$$D^m_{s,z,x'}\widetilde{K}\big(s^4,z,x'\big) = O_{z,\infty}\big(|z|^{-\infty}\big), \tag{2.3}$$
for each $m\in\mathbb{N}\times(\mathbb{N}^2)^2$, uniformly in $s\in[-T,T]$ and $x'\in\mathbb{R}^2$, for any fixed $T>0$.

We choose the letter '$\mathcal{S}$' for this class of kernels to emphasize the scaling property encoded in its definition. One has
$$\widetilde{K}(t,z,x') = t^{\frac{7}{4}+\alpha}\,K\Big(t,x'+\mathfrak{s}_{t^{1/4}}(z),x'\Big).$$
The reason for the choice of exponent 7/4 will appear clearly in the proof of Proposition 3 – this is the only choice that gives spacetime convolution the property stated therein. Note the inclusion $\mathcal{S}_\beta\subset\mathcal{S}_\alpha$, for $0\leq\alpha\leq\beta$. Note also that $\widetilde{K}(s^4,z,x')$ is smooth up to $\{s=0\}$ included.

Definition – *The* **leading term** K *of K is defined for $z\in\mathbb{R}^2$ and $x'\in\mathbb{R}^2$ as*
$$\mathsf{K}(z,x') := \widetilde{K}(0,z,x').$$

The next elementary statement will be crucial in the construction and the description of the heat kernel of $\mathcal{G}$. The gain of exponent involved in this statement is the source of the convergence of the series representation (2.10) of the heat kernel of $\mathcal{G}$ given in Proposition 5.

Lemma 2 – *Pick $\alpha\geq 0$. Given $K\in\mathcal{S}_\alpha$, its leading term satisfies $\mathsf{K}=0$ iff $K\in\mathcal{S}_{\alpha+1/4}$.*

Proof – One has $\mathsf{K}=0$ if $K\in\mathcal{S}^{\alpha+1/4}$, by definition of the class $\mathcal{S}_\beta$ for every $\beta>\alpha$. If now $\mathsf{K}=0$, one can use the fact that $\widetilde{K}(s^4,z,x')$ is a smooth function of s up to $\{s=0\}$ to write the Taylor formula

$$K\big(s^4, z, x'\big) = (s^4)^{-7/4+\alpha} s \int_0^1 \widetilde{K}\big(as, z, x'\big)\, da.$$

One reads on this identity the fact that $K \in \mathcal{S}_{\alpha+1/4}$. ▷

It will be useful below to notice here that for $K \in \mathcal{S}_1$ and f smooth the function Kf defined by the formula

$$(Kf)(t, x) := \int K(t, x, x') f(x')\, dx'$$

is a smooth function of $(t^{1/4}, x)$, for $t \in [0, \infty)$ and $x \in \mathbb{R}^2$, and that we have

$$(Kf)(0, x) = f(x) \int \mathsf{K}(z, x)\, dz. \tag{2.4}$$

It follows that if $K \in \mathcal{S}_\alpha$ with $\alpha > 1$, one has

$$(Kf)(0, x) = 0; \tag{2.5}$$

write $K = t^{\alpha-1} K'$, with $K' \in \mathcal{S}_1$.

We define as usual the spacetime convolution of two smooth functions A, B defined on $(0, \infty) \times (\mathbb{R}^2)^2$ setting

$$(A * B)(t, x, x') := \int_0^t \int_{\mathbb{R}^2} A(t-s, x, w) B(s, w, x')\, dwds$$

whenever the integral is absolutely convergent. We will write

$$A^{*n} := A * \cdots * A$$

for the n-fold convolution of A with itself.

Proposition 3 – *Pick $\alpha, \beta > 0$ and $A \in \mathcal{S}_\alpha$ and $B \in \mathcal{S}_\beta$, with leading terms A and B respectively. Then $A * B$ is an element of $\mathcal{S}_{\alpha+\beta}$ with leading term denoted by $\mathsf{A} \star \mathsf{B}$, equal to*

$$\mathsf{A} \star \mathsf{B}(z, x') = \int_0^1 \int_{\mathbb{R}^2} (1-a)^{-7/4-\alpha} a^{-7/4-\beta}\, \mathsf{A}\Big(\mathfrak{s}_{(1-a)^{-1/4}}(z - v), x'\Big)\, \mathsf{B}\Big(\mathfrak{s}_{a^{-1/4}}(v), x'\Big)\, dvda.$$

Proof – We split the integral giving $A * B$ into a part 'near' x' and a part 'near' x. This is done by inserting a smooth non-negative function χ in the integral, identically equal to 1 near x'. A change of variables $a := s/t$ and $v := \mathfrak{s}_{t^{-1/4}}(w - x')$ and $z := \mathfrak{s}_{t^{-1/4}}(x - x')$ gives for

$$\int_0^t \int_{\mathbb{R}^2} A(t-s, x, w) B(s, w, x') \chi(w)\, dwds$$

the formula

$$\begin{aligned} & t^{-7/4-\alpha-\beta} \int (1-a)^{-7/4-\alpha} a^{-7/4-\beta} \times \\ & \widetilde{A}\Big(t(1-a), \mathfrak{s}_{(1-a)^{-1/4}}(z-v), x' + \mathfrak{s}_{t^{1/4}}(v)\Big) \widetilde{B}\Big(ta, \mathfrak{s}_{a^{-1/4}}(v), x'\Big) \chi\big(x' + \mathfrak{s}_{t^{1/4}}(v)\big)\, dvda, \end{aligned}$$

with an integral over $[0, 1] \times \mathbb{R}^2$. We see on this formula that the exponents add up, which justifies our choice of exponent $7/4$ in defining the classes $\mathcal{S}$. We split the integral on the two sub-intervals $[0, 1/2]$ and $[1/2, 1]$ and treat the interval $[0, 1/2]$ first. Changing variable $v' := \mathfrak{s}_{a^{-1/4}}(v)$ provides an $a^{3/4}$ from the Jacobian and shows that the exponent of a in the integral is strictly bigger than 1 as $\beta > 0$, so the integral is indeed converging. Rapid decay of the function of z defined by this half integral comes from the rapid decay of $\widetilde{A}$ and $\widetilde{B}$ together with the inequality

$$\big|\mathfrak{s}_{(1-a)^{-1/4}}\big(z - \mathfrak{s}_{a^{1/4}}(v')\big)\big| \gtrsim \big||z| - |v'|\big|.$$

The part of the integral corresponding to the interval $[1/2, 1]$ is treated similarly. Use another change of variable to deal with

$$\int_0^t \int_{\mathbb{R}^2} A(t-s, x, w) B(s, w, x')\, (1-\chi)(w)\, dwds.$$

Dominated convergence gives the formula for $\mathsf{A} \star \mathsf{B}$. ▷

Given $x' \in \mathbb{R}^2$, define the *leading part of $\mathcal{G}$ with coefficients frozen at x'* by

$$\mathcal{G}_z^{x'} := \partial_{z_0}^2 - a(x_1')^2 \partial_{z_1}^4.$$

Note that this operator is *not* equal to the operator obtained from $\mathcal{G}$ by freezing its coefficients at x'; we have only kept from the latter the terms of higher derivatives in z_0 and z_1. This operator is in particular independent of the drift b in the definition of $\mathcal{L}_{x_1}$. Set also for $z = (z_0, z_1) \in \mathbb{R}^2$

$$z \cdot \partial_z := 2z_0 \partial_{z_0} + z_1 \partial_{z_1}.$$

Proposition 4 – *Pick $\alpha \geq 1$ and $K \in \mathcal{S}_\alpha$. Then*

$$E := (\partial_t - \mathcal{G})K \in \mathcal{S}_{\alpha-1}$$

and the leading term E of E satisfies the equation

$$\mathsf{E}(z, x') = \Big(-\frac{7}{4} + \alpha - \frac{1}{4}\, z \cdot \partial_z - \mathcal{G}_z^{x'} \Big) \mathsf{K}(z, x'). \tag{2.6}$$

The letter E is chosen for ‘error’ as it quantifies how K fails to satisfy the equality $(\partial_t - \mathcal{G})K = 0$. Since $\mathsf{K}(\cdot, x')$ only captures the main, zero-th order, behaviour of the kernel K near x' in terms terms of the scaling parameter ‘t’, it is not surprising that only the leading part $\mathcal{G}^{x'}$ of $\mathcal{G}$ appears in relation (2.6). The drift b in $\mathcal{L}_{x_1}$ plays in particular no role in this relation while it does in the definition of $\mathcal{G}$ and E.

Proof – Set $\ell := \frac{7}{4} - \alpha$ and write $K(t, x, x') = t^{-\ell} \widetilde{K}\big(t, \mathfrak{s}_{t^{-1/4}}(x - x'), x'\big)$, with $\widetilde{K}$ a function of (t, z, x') as in Definition 1. We have

$$\begin{aligned} \partial_t K &= -\ell t^{-\ell-1} \widetilde{K} + t^{-\ell}\, \partial_t \widetilde{K} - \frac{t^{-\ell-1}}{4}\, \mathfrak{s}_{t^{-1/4}}(x - x') \cdot \partial_z \widetilde{K} \\ &=: t^{-\ell-1} \Big(-\ell - \frac{1}{4}\, z \cdot \partial_z \Big) \widetilde{K} + E_1, \end{aligned} \tag{2.7}$$

with

$$E_1 = t^{-\ell} \partial_t \widetilde{K} \in \mathcal{S}_{\alpha - 3/4}$$

since

$$\partial_t = 1/(4t^{3/4})\, \partial_{t^{1/4}},$$

and $\widetilde{K}$ is a smooth function of $t^{1/4}$ from (2.3). Since $\partial_{x_0} = t^{-1/2} \partial_{z_0}$ and $\partial_{x_1} = t^{-1/4} \partial_{z_1}$, one has

$$\mathcal{G}K =: t^{-\ell-1} \mathcal{G}_z^{x'} \widetilde{K} + E_2, \tag{2.8}$$

with

$$\begin{aligned} E_2 &= t^{-\ell-1} \Big(\partial_{z_0}^2 - a(x_1)^2 \partial_{z_1}^4 - \mathcal{G}_z^{x'} \Big) \widetilde{K} \\ &\quad + \Big(t^{-\ell-3/4} \big(2ab + 2a\partial_{x_1} a \big)(x_1)\, \partial_{z_1}^3 \widetilde{K} + t^{-\ell-1/2} \big(2a\partial_{x_1} b + b\partial_{x_1} a + a\partial_{x_1}^2 a \big)(x_1)\, \partial_{z_1}^2 \widetilde{K} \\ &\qquad + t^{-\ell-1/4} \big(a\partial_{x_1} b + b\partial_{x_1} b \big)(x_1)\, \partial_{z_1} \widetilde{K} \Big) \\ &=: t^{-\ell-1} \Big(\partial_{z_0}^2 - a(x_1)^2 \partial_{z_1}^4 - \mathcal{G}_z^{x'} \Big) \widetilde{K} + (\cdots)(x_1) \\ &= t^{-\ell-1} \Big(a(x_1')^2 - a(x_1)^2 \Big) \partial_{z_1}^4 \widetilde{K} + (\cdots)\big(x_1' + t^{1/4} z_1\big) \\ &= t^{-\ell-3/4} \underline{a}(x_1, x_1')\big(z_1' - z_1\big) \partial_{z_1}^4 \widetilde{K} + (\cdots)\big(x_1' + t^{1/4} z_1\big)\, \partial_{z_1}^2 \widetilde{K}, \end{aligned}$$

using the division property

$$a(x_1')^2 - a(x_1)^2 =: \underline{a}(x_1, x_1')\big(x_1' - x_1\big) = t^{1/4} b(x_1, x_1')\big(z_1' - z_1\big),$$

for a smooth function $\underline{a}$. We see on the above identity for E_2 that $E_2 \in \mathcal{S}_{\alpha-3/4}$, so this term makes no contribution to the leading term of R, and one gets from (2.7) and (2.8) the equation giving E in terms of K. ▷

2.2 Heat kernel estimates

Fix $x'\in\mathbb{R}^2$ and let $K_1(t,\cdot,x')$ stand for the kernel of the function $(e^{t\mathcal{G}^{x'}})^*\delta_{x'}$ – the density at 'time' t of the $\mathcal{G}^{x'}$-semigroup started from x'. Given that the operator $\mathcal{G}^{x'}$ is translation invariant, $K_1(t,\cdot,x')$ is obtained from the inverse Fourier transform of the Schwartz function

$$(\lambda_0,\lambda_1)\mapsto \exp\Big(-t\big(\lambda_0^2+a(x_1')^2\lambda_1^4\big)\Big),$$

and one has the scaling identity

$$K_1(t,x,x')=t^{-3/4}K_1\big(1,\mathfrak{s}_{t^{-1/4}}(x-x'),x'\big).$$

So $K_1\in\mathcal{S}_1$ and it has leading term

$$\mathsf{K}_1(z,x')=K_1(1,z,x')$$

that satisfies the identity

$$\Big(-\frac{1}{4}-\frac{1}{4}\,z\cdot\partial_z-\mathcal{G}_z^{x'}\Big)\mathsf{K}_1=0. \tag{2.9}$$

The kernel K_1 satisfies from identity (2.4) the convergence

$$\lim_{t\to0^+}K_1(t,x,x')=\delta_{x'}(x).$$

One has

$$E_1:=(\partial_t-\mathcal{G})K_1\in\mathcal{S}_0$$

from Proposition 4, but since it has null leading term by (2.9), Lemma 2 tells us that E_1 is actually an element of $\mathcal{S}_{1/4}$. The convolution $K_1*E_1^{*n}$ defines then an element of $\mathcal{S}_{1+n/4}$ for any $n\geq1$, from Proposition 3.

Proposition 5 – *The series*

$$K:=K_1+\sum_{n\geq1}(-1)^nK_1*E_1^{*n} \tag{2.10}$$

converges in $C^\infty\big((0,\infty)\times(\mathbb{R}^2)^2\big)$ and defines an element of $\mathcal{S}_1$. This is the heat kernel of the operator $\mathcal{G}$.

Proof – It is elementary to check that

$$\big(\partial_t-\mathcal{G}\big)(K_1*E_1)=E_1+E_1*E_1,$$

and that K_1*E_1 tends to 0 as to t decreases to 0^+ as a consequence of identity (2.5). One thus has a telescopic sum when applying $(\partial_t-\mathcal{G})$ to the partial sums of K, so the proof of Proposition 5 boils down to proving the convergence in $C^\infty\big((0,\infty)\times(\mathbb{R}^2)^2\big)$ of the sum defining K.

• Given that $E_1\in\mathcal{S}_{1/4}$, one defines an element E of $\mathcal{S}_1$ setting

$$E:=E_1^{*4},$$

so writing

$$E(t,x,x')=t^{-3/4}\widetilde{E}\big(t,\mathfrak{s}_{t^{-1/4}}(x-x'),x'\big)$$

and changing coordinates in the iterated integral that defines E^{*n}, one has for $E^{*n}(t,x,x')$ the expression

$$\int_{0\leq t_{n-1}\leq\cdots\leq t_1\leq t}\int_{(\mathbb{R}^2)^{n-1}}\prod_{i=0}^{n-2}\widetilde{E}\Big(t_i-t_{i+1},z_i,x-\sum_{j=0}^{i-1}\mathfrak{s}_{(t_i-t_{i+1})^{-1/4}}(z_{i+1})\Big)\widetilde{E}\Big(t_{n-1},z_{n-1},x'\Big)\,dzdt,$$

with $t_0=t$ and $dz=dz_1\dots dz_{n-1}$ and $dt=dt_1\dots dt_{n-1}$. The super-polynomial bound on $\widetilde{E}$ that one gets from the version of (2.3) with no derivatives ensures the convergence of the integral on $(u_1,\dots,u_{n-1})$ and provides as a pointwise upper bound for $E^{*n}(t,x,x')$ a constant multiple of $(Ct)^{n-1}/(n-1)!$, for a positive constant C independent of x and x'. It follows as an elementary consequence that the $K_1*E_1^{*i}*E^n$, for $1\leq i\leq3$, also satisfy the same bound, which gives the uniform convergence of the sum defining K.

• Work with $E_1^{*4+\ell}$ instead of E_1^{*4} to prove convergence on compact time intervals of $(0,\infty)$ of the $\ell^{\rm th}$ derivatives of K, inductively on ℓ, using the full power of the uniform estimate (2.3). We leave the details to the reader.

• One sees on the defining formula (2.10) for K that one can decompose it as

$$K = K_1 + \sum_{n=1}^{k} (-1)^n K_1 * E_1^{*n} + O(t^k)$$

in C^ℓ, for all k and ℓ, so K satisfies indeed the off-diagonal decay estimate (2.2) and its associated function $\widetilde{K}$ satisfies the rapid decay estimate (2.3).

• Uniqueness of the heat kernel of $\mathcal{G}$ is automatic from the symmetry of the operator $\mathcal{G}$ and the preceding existence result, by an elementary duality argument since the operator $\mathcal{G}$ is symmetric. ▷

It follows directly from the scaling properties involved in the definition of the class $\mathcal{S}_1$ that the kernel K satisfies the scaling bounds (2.1). We record that fact in a statement.

Corollary 6 – *The heat kernel of the operator $\mathcal{G}$ satisfies the scaling estimates*

$$\int_{\mathbb{R}^2} \big|\partial_x^n K(t,x,x')\big|\, d(x,x')^c\, dx \lesssim t^{\frac{c-|n|_{\mathfrak{s}}}{4}}, \qquad (\forall x'\in\mathbb{R}^2,\, n\in\mathbb{N}\times\mathbb{N},\, c\in\mathbb{R}_+).$$

It is straightforward to adapt all the proofs of the results of this section to the setting of $\mathbb{R}\times\mathbb{R}^d$, for any space dimension d. The exponent $7/4$ in the definition of the class $\mathcal{S}$ is changed into $1+\frac{2+d}{4}$.

3 – Renormalization schemes and renormalised equation

The results of Section 2 allow to put the analysis of the system (1.1) in the heat kernel setting from [8] and develop the analytic side of the theory of regularity structures as described therein, using as an algebraic background the same setting as for the analysis of a corresponding system of translation-invariant equations. We take care in this section of the renormalization problem.

We use in this section the notations on regularity structures set in the 'Notations' paragraph at the end of the introduction, Section 1. Let then $\mathscr{T}$ be the regularity structure associated with the system (1.1) of singular stochastic PDEs. Recall from the introduction that there is no hope to build a renormalization admissible model from another admissible model $\sf\Pi$ using formulas of the form ${\sf\Pi}\circ M$, for a state space dependent linear map $M:\mathbb{R}^2\times\mathcal{T}\to\mathcal{T}$. Our key tool for the analysis of renormalization of system (1.1) is the following obvious extension of the notion of preparation map introduced by Bruned in [9] and used crucially by the authors in [4].

Definition – *A* **preparation map** *is a map*

$$R:\mathbb{R}^2\times\mathcal{T}\to\mathcal{T}$$

such that all the maps $R(x,\cdot)$ fix polynomials and such that

- *for each $\tau\in\mathcal{T}$ there exist finitely many $\tau_i\in\mathcal{T}$ and smooth real-valued bounded functions λ_i on $\mathbb{R}^2$ such that for every $x\in\mathbb{R}^2$*

$$R(x,\tau) = \tau + \sum_i \lambda_i(x)\tau_i, \quad \textit{with}\quad \mathsf{deg}(\tau_i)\geq\mathsf{deg}(\tau) \quad \textit{and} \quad |\tau_i|_\Xi < |\tau|_\Xi, \tag{3.1}$$

- *one has for all $x\in\mathbb{R}^2$*

$$\big(R(x,\cdot)\otimes\mathrm{Id}\big)\Delta = \Delta R(x,\cdot). \tag{3.2}$$

We will see in Section 3.1 that one can associate to any (state space dependent) preparation map a smooth admissible model on $\mathscr{T}$. Its use in the analysis of the associated renormalized equation is analyzed in Section 3.2, where our main statement is proved in Theorem 10. The result is specialized to BHZ-type renormalization schemes in Section 3.3.

Example 7 – *The archetype of state space dependent preparation map is defined from a map δ_r, with the index 'r' for 'root', defined similarly as the splitting map δ, but extracting from any $\tau \in \mathcal{T}$ only one diverging subtree of τ with the same root as τ at a time, and summing over all possible such subtrees – see Definition 4.2 in* [9]*. Given a map $\ell : \mathbb{R}^2 \times \mathcal{T}^- \to \mathbb{R}$ such that $\ell(x,\cdot)$ is a character of the algebra $\mathcal{T}^-$, for every $x \in \mathbb{R}^2$, the map*

$$R_\ell(x) := \big(\ell(x,\cdot) \otimes \mathrm{Id}\big)\delta_r \tag{3.3}$$

is a preparation map.

3.1 Renormalized canonical models associated with preparation maps

Let R be a preparation map and $\sf\Pi$ stand for the canonical model on $\mathscr{T}$ associated with a smooth n_0-dimensional noise ξ. Set

$$\mathsf{\Pi}^R\zeta = \widehat{\mathsf{\Pi}}^R\zeta = \xi,$$

and define inductively the maps $\mathsf{\Pi}^R$ and $\widehat{\mathsf{\Pi}}^R$ by the relations

$$\mathsf{\Pi}^R = \widehat{\mathsf{\Pi}}^R R, \quad \widehat{\mathsf{\Pi}}^R(\tau\bar\tau) = (\widehat{\mathsf{\Pi}}^R\tau)(\widehat{\mathsf{\Pi}}^R\bar\tau), \quad \widehat{\mathsf{\Pi}}^R(\mathcal{I}_a\tau) = D^aK * (\mathsf{\Pi}^R\tau). \tag{3.4}$$

It follows from this definition and the fact that R lives fixed of elements of the for $\mathcal{I}_a\tau$ that the map $\mathsf{\Pi}^R$ satisfies the admissibility condition

$$\mathsf{\Pi}^R(\mathcal{I}_a\tau) = D^aK * (\mathsf{\Pi}^R\tau).$$

Recall that the co-action Δ satisfies the induction relation

$$\begin{aligned} \Delta(\bullet) &:= \bullet \otimes \mathbf{1}, \qquad \text{for } \bullet \in \big\{\mathbf{1}, X_i, \zeta\big\}, \\ \Delta(\mathcal{I}_a\tau) &:= (\mathcal{I}_a \otimes \mathrm{Id})\Delta + \sum_{|\ell+m|<\deg(\mathcal{I}_a\tau)} \frac{X^\ell}{\ell!} \otimes \frac{X^m}{m!}\mathcal{I}^+_{a+\ell+m}(\tau). \end{aligned} \tag{3.5}$$

and set

$$\mathsf{\Pi}^R_x\tau = \big(\mathsf{\Pi}^R \otimes (\mathsf{g}^R_x)^{-1}\big)\,\Delta$$

with

$$(\mathsf{g}^R_x)^{-1}\big(\mathcal{I}^+_a\tau\big) = -\big(D^aK * \mathsf{\Pi}^R_x\tau\big)(x), \tag{3.6}$$

and

$$\widehat{\mathsf{g}^R_{yx}} = \big(\mathrm{Id} \otimes \mathsf{g}^R_{yx}\big)\,\Delta, \tag{3.7}$$

with

$$\mathsf{g}^R_{yx} = \Big((\mathsf{g}^R_y)^{-1} \circ S^+ \otimes (\mathsf{g}^R_x)^{-1}\Big)\Delta^+.$$

An induction on the degree of τ shows that $\mathsf{\Pi}^R_x$ is the solution of the induction relation

$$\begin{aligned} (\mathsf{\Pi}^R_x\tau)(y) &= \Big(\widehat{\mathsf{\Pi}}^R_x\big(R(y)\tau\big)\Big)(y), \\ \widehat{\mathsf{\Pi}}^R_x(\tau\bar\tau) &= (\widehat{\mathsf{\Pi}}^R_x\tau)\,(\widehat{\mathsf{\Pi}}^R_x\bar\tau), \\ \big(\widehat{\mathsf{\Pi}}^R_x(\mathcal{I}_a\tau)\big)(y) &= \big(D^aK * \mathsf{\Pi}^R_x\tau\big)(y) - \sum_{|k|\le\deg(\mathcal{I}_a\tau)} \frac{(y-x)^k}{k!}\big(D^{a+k}K * \mathsf{\Pi}^R_x\tau\big)(x). \end{aligned}$$

This was done in Section ?? of [9] in the case where the preparation map R does not depend on its $\mathbb{R}^2$ argument. The proof carries over verbatim to the present setting. For showing the analytical bounds for $\widehat{\mathsf{g}^R_{yx}}$, one needs the following recursive identity, whose proof follows the same lines as the proof of Lemma 11 in [5] noting that

$$\widehat{g^R_{yx}}(\tau) = \sum_i c_i(x)\tau_i$$

for coefficients $c_i(x)$ depending on x, as a consequence of the definition of the character $(g^R_x)^{-1}$.

Lemma 8 – *One has the identity*

$$\widehat{\mathsf{g}^R_{yx}}\,(\mathcal{I}_a\tau) = \mathcal{I}_a\Big(\widehat{\mathsf{g}^R_{yx}}\tau\Big) - \sum_{|\ell|<\deg(\mathcal{I}_a\tau)} \frac{(X+x-y)^\ell}{\ell!}\,\mathsf{\Pi}^R_x\Big(\mathcal{I}_{a+\ell}\big(\widehat{\mathsf{g}^R_{yx}}\tau\big)\Big)(y). \tag{3.8}$$

Equipped with this statement, it is straightforward to follow carefully the proof of Proposition 10 in [5] and prove the following result.

Proposition 9 – *The pair* $\mathsf{M}^R = \big(\mathsf{g}^R, \Pi^R\big)$ *defines a smooth admissible model on* $\mathscr{T}$.

> **Proof –** The only point that needs to be checked is the analytical bounds for Π^R. It is performed by induction on the size of the trees. Given a tree τ, one has by definition
> $$\big(\Pi_x^R \tau\big)(y) = \Big(\widehat{\Pi}_x^R\big(R(y)\tau\big)\Big)(y) = \sum_i \lambda_i(y)\big(\widehat{\Pi}_x^R \tau_i\big)(y),$$
> where
> $$R(y)\tau = \sum_i \lambda_i(y)\tau.$$
> We apply the induction hypothesis on $\big(\widehat{\Pi}_x^R \tau_i\big)(y)$ to get the correct bound
> $$\big|\big(\widehat{\Pi}_x^R \tau_i\big)(y)\big| \lesssim |y-x|^{\mathsf{deg}(\tau_i)}, \quad \mathsf{deg}(\tau_i) \geq \mathsf{deg}(\tau).$$
> Then working with a smooth model makes the terms $\lambda_i(y)$ bounded. This allows us to conclude that
> $$\big|\big(\widehat{\Pi}_x^R \tau_i\big)(y)\big| \lesssim \sum_i \lambda_i(y)|y-x|^{\mathsf{deg}(\tau_i)} \lesssim |y-x|^{\mathsf{deg}(\tau)}.$$
> The bound for $(\widehat{\Pi}_x^R \mathcal{I}_a(\tau))(y)$ follows from the one of $(\Pi_x^R \tau)(y)$ and the fact that we subtract in the definition of $(\widehat{\Pi}_x^R \mathcal{I}_a(\tau))(y)$ the correct Taylor expansion. ▷

The weaker conclusion in Proposition 10 of [2] that M^R defines an admissible model on a modified version $\mathscr{T}^{(\varepsilon)}$ of $\mathscr{T}$ comes from the fact that we work therein with non smooth models for which we use a density argument that requires the introduction of $\mathscr{T}^{(\varepsilon)}$. We only work in the present work with the smooth canonical model Π so there is no need to introduce $\mathscr{T}^{(\varepsilon)}$.

3.2 Renormalised equation

In the section, we derive the renormalized equation associated to the models described in the previous section; it involves the functions $F_i(\tau)$ defined in (1.4). We recall here from Section 4.2 of Hairer' seminal work [20] the definition of the lift F_i of a smooth function F_i. One has for any
$$\mathsf{a} =: a_{\mathbf{1}}\mathbf{1} + \mathsf{a}' \in T,$$
with $\langle \mathsf{a}', \mathbf{1}\rangle = 0$,
$$\mathsf{F}_i(\mathsf{a}) = \sum_k \frac{D^k F(a_{\mathbf{1}})}{k!}\,(\mathsf{a}')^k.$$
We need to add an extra assumption to our preparation map R introduced in our previous work [4]. We first recall the definition of the product $\star : \mathcal{T}\times\mathcal{T} \to \mathcal{T}$. It is defined for all $\sigma = X^k \prod_i \mathcal{I}_{a_i}(\sigma_i) \in T$ and $\tau \in \mathcal{T}$ by the formula
$$\sigma \star \tau := \uparrow_{N_\tau}^k \left(\prod_i \mathcal{I}_{a_i}(\sigma_i) \curvearrowright \tau\right)$$
where
$$\uparrow_{N_\tau}^k \tau := \sum_{\sum_{v\in B} k_v = k} \; \prod_{v\in N_T} \uparrow_v^{k_v} \tau.$$
and $\curvearrowright$ corresponds to the simultaneous grafting of the τ_i via edges decorated by the labels a_i.

Definition – *A* **strong preparation map** *is a preparation map satisfying*
$$R^*\left(\sigma \star \tau\right) = \sigma \star \left(R^*\tau\right) \tag{3.9}$$
for all $\sigma \in \mathcal{T}$ *and* $\tau \in \mathcal{T}$.

This identity implicitly holds pointwise on all of $\mathbb{R}^2$. Models of the form (3.3) provide examples of strong preparation maps. If we consider $\sigma \in \mathcal{T}^+$ and $\tau \in \mathcal{T}$ then (3.9) is the dual version of the 'commutation' relation (3.2) between R and Δ. The definition of a strong preparation map

requires that (3.9) holds *for all* $\sigma \in \mathcal{T}$. This property of a preparation map ensures the crucial relation
$$\mathsf{F}_i\big(R(x,\cdot)^*\tau\big) = \partial^k D_{a_1}\cdots D_{a_n}\mathsf{F}_i\big(R(x,\cdot)^*\zeta_l\big)\prod_{j=1}^n \mathsf{F}_{l_j}(\tau_j), \tag{3.10}$$
when $\tau = X^k\zeta_l\prod_{j=1}^n \mathcal{I}_{a_j}(\tau_j)$ and $a_j = (\mathfrak{t}_{n_j}, k_j)$, for all $x\in\mathbb{R}^2$. Since the model M^R takes values in the space of continuous functions, the reconstruction operator $\mathsf{R}^{\mathsf{M}^R}$ associated with it is given by the explicit formula
$$\big(\mathsf{R}^{\mathsf{M}^R}\mathsf{v}\big)(x) = \big(\Pi_x^R\mathsf{v}(x)\big)(x)$$
for any modelled distribution v with positive regularity, so
$$\big(\mathsf{R}^{\mathsf{M}^R}\mathsf{v}\big)(x) = \Big(\widehat{\Pi}_x^R\big(R(x)\mathsf{v}(x)\big)\Big)(x). \tag{3.11}$$
The multiplicative character of the map $\widehat{\Pi}_x^R(\cdot)(x)$ is the crucial feature of this factorization of $\mathsf{R}^{\mathsf{M}^R}$. For preparation maps R for which
$$R(\mathcal{I}_a\tau) = \mathcal{I}_a\tau \tag{3.12}$$
for all $\tau\in\mathcal{T}$ and $a\in\mathfrak{T}^+\times\mathbb{N}^2$, one has
$$\Pi_x^R(\mathcal{I}_a\tau) = \widehat{\Pi}_x^R(\mathcal{I}_a\tau).$$
Denote by $T_X\subset T$ the linear space spanned by the polynomials in T. Under assumption (3.12), modelled distributions v with values in the subspace $T_X\oplus\bigoplus_{a\in\mathfrak{T}^+\times\{0\}}\mathcal{I}_a(T)$ of T satisfy in that case the identity
$$\big(\mathsf{R}^{\mathsf{M}}\mathsf{v}\big)(x) = \Big(\widehat{\Pi}_x^R\mathsf{v}(x)\Big)(x).$$
This is the case of the modelled distribution solution of the regularity structure lift of the system (1.1).

Theorem 10 – *Let $R:\mathbb{R}^2\times\mathcal{T}\to\mathcal{T}$ be a strong preparation map such that*
$$R\tau = \tau, \qquad \textit{for}\quad \tau\in T_X\oplus\bigoplus_{a\in\mathfrak{T}^+\times\{0\}}\mathcal{I}_a(T).$$
Let M^R stand for its associated admissible model. Let u stand for the modelled distribution solution of the regularity structure lift of system (1.1) *with initial condition u_0. Then*
$$u := \mathsf{R}^{\mathsf{M}^R}\mathsf{u}$$
is a solution of the renormalized system
$$(\partial_{x_0}-\mathcal{L}_i)u_i = F_i(u,\partial_{x_1}u)\,\xi + \sum_{l=0}^{n_0} F_i\Big(\big(R(\cdot)^*-\mathrm{Id}\big)\zeta_l\Big)(u,\partial_{x_1}u)\,\xi_l, \qquad (1\le i\le k_0). \tag{3.13}$$

We use the notation $R(\cdot)$ to emphasize the dependence of R on its $\mathbb{R}^2$ argument. The most general renormalized equations involves the noise ξ in their counter-terms – recall $\xi_0 = 1$. It is only for strong preparation maps R such that $R(x)^*\zeta_l = \zeta_l$ for all $l\neq 0$, for all x, that the counter-terms do not involve the noise.

Proof – The proof follows verbatim the proof of Theorem 9 in [4]. We check that the fact that R depends on $x\in\mathbb{R}^2$ does not change its mechanics. As we are working with an admissible model we have
$$(\partial_{x_0}-\mathcal{L}_i)u_i = \mathsf{R}^{\mathsf{M}^R}(\mathsf{v}_i), \quad \mathsf{v}_i = \sum_{\deg(\tau)<\gamma-2}\frac{\mathsf{F}_i(\tau)(\mathsf{u},D\mathsf{u})}{S(\tau)}\,\tau,$$
for a sum over the canonical basis of $\mathcal{T}$. The function v_i is a modelled distribution of regularity γ. One has from (3.11), by construction,
$$\big(\mathsf{R}^{\mathsf{M}^R}\mathsf{v}_i\big)(x) = \Big(\widehat{\Pi}_x^R\big(R(x)\mathsf{v}_i(x)\big)\Big)(x),$$
and
$$\big\langle R(x)\mathsf{v}_i(x),\tau\big\rangle = \big\langle \mathsf{v}_i(x)\,,\,R(x)^*\tau\big\rangle = \mathsf{F}_i\big(R(x)^*\tau\big)\big(\mathsf{u}(x),D\mathsf{u}(x)\big),$$

with $\mathsf{F}_i\big(R(x)^*\tau\big)$ given by (3.10). Emphasize the crucial fact that R and v_i are evaluated at the same point x. One can then write

$$R(x)\mathsf{v}_i(x) = \sum \frac{\mathsf{F}_i\big(R(x)^*\tau\big)\big(\mathsf{u}(x), D\mathsf{u}(x)\big)}{S(\tau)}\,\tau,$$

and using the Faà di Bruno formula as in the proof of Theorem 9 in [4], one gets

$$R(x)\mathsf{v}_i(x) = \sum_{l=1}^{n_0} \mathsf{F}_i\big(R(x)^*\zeta_l\big)\big(\mathsf{u}(x), D\mathsf{u}(x)\big)\zeta_l.$$

Using the (crucial) multiplicativity property of $\widehat{\mathsf{\Pi}}_x^R$ we see that

$$\begin{aligned}
&\big((\partial_{x_0} - \mathcal{L}_i)u_i\big)(x) = \big(\mathsf{R}^{\mathsf{M}^R}\mathsf{v}_i\big)(x) = \widehat{\mathsf{\Pi}}_x^R\big(R(x)\mathsf{v}_i(x)\big)(x) \\
&= \sum_{l=0}^{n_0} \widehat{\mathsf{\Pi}}_x^R\Big(\mathsf{F}_i\big(R^*(x)\zeta_l\big)(\mathsf{u}(x), D\mathsf{u}(x))\zeta_l\Big)(x) \\
&= \sum_{l=0}^{n_0} F_i\big(R^*(x)\zeta_l\big)\Big(\big(\widehat{\mathsf{\Pi}}_x^R\mathsf{u}(x)\big)(x), \partial_{x_1}\big(\widehat{\mathsf{\Pi}}_x^R\mathsf{u}(x)\big)(x)\Big)\,\widehat{\mathsf{\Pi}}_x^R\zeta_l \\
&= \sum_{l=0}^{n_0} F_i(R^*(x)\zeta_l)\big(u(x), \partial_{x_1}u(x)\big)\xi_l(x).
\end{aligned}$$

▷

3.3 BHZ renormalization

In the translation-invariant setting, one can recast the renormalization scheme introduced by Bruned, Hairer and Zambotti in [12] in the setting of preparation maps associating to any element ℓ of the renormalization group a strong preparation map R_ℓ whose dual R_ℓ^* is given by

$$R_\ell^*(\tau) = \sum_{\sigma\in\mathcal{B}^-} \ell(\sigma)\,(\tau\star\sigma),$$

where $\mathcal{B}^-$ stands for the canonical basis of $\mathcal{T}^-$ and $\star$ is a product first introduced in Bruned and Manchon's recent work [13] – see also Section 2 of [4]. (The map R_ℓ is of the form (3.3).) If the random noise ξ in (1.1) takes values in the space of smooth functions and has a translation-invariant distribution the BHZ renormalization corresponds to taking

$$\ell(\tau) = \mathbb{E}\big[(\mathsf{\Pi}\tau)(0)\big],$$

with $\mathsf{\Pi}$ the canonical smooth model associated with ξ – see identity (6.25) in [12]. The renormalized system (3.13) then takes the form

$$(\partial_{x_0} - \mathcal{L}_i)u_i = F_i(u, \partial_{x_1}u)\,\xi + \sum_{\tau\in\mathcal{B}^-} \ell(\tau)\,\frac{F_i(\tau)\big(u, \partial_{x_1}u\big)}{S(\tau)}, \qquad (1 \le i \le k_0).$$

The same conclusion holds in the general non-translation invariant setting, where $R(x)^*\zeta_l = \zeta_l$ for all $l \neq 0$ and all $x \in \mathbb{R}^2$, and where the renormalized system takes now the form

$$(\partial_{x_0} - \mathcal{L}_i)u_i = F_i(u, \partial_{x_1}u)\,\xi + \sum_{\tau\in\mathcal{B}^-} \ell(\cdot,\tau)\,\frac{F_i(\tau)\big(u, \partial_{x_1}u\big)}{S(\tau)}, \qquad (1 \le i \le k_0).$$

We do not touch here on the question of showing that such renormalization schemes provide limit admissible models when used on models built from regularized random noises such as space or spacetime white noise. We expect the proof of the convergence result proved by Chandra and Hairer in [15] in a translation invariant setting to have a direct counterpart in the non-translation invariant setting.

Remarks – 1. *How far are we from having a framework for dealing with systems of real-valued functions on a 2 or 3-dimensional closed manifold satisfying singular stochastic PDEs of the form*

(1.1) *with the differential of* u *in the role of* $\partial_{x_1} u$*? The pattern of proof of the estimate* (2.1) *is robust enough to be adapted to a closed manifold setting. The algebraic results of Section 3 are insensitive to the fact that we could be on a manifold. The only point where analysis enters the scene is when using the reconstruction theorem. The proof of the latter given in the 'Tourist guide'* [8] *works verbatim on a manifold setting once one has the estimates* (2.1)*. Alternatively, one can use the manifold version of Caravenna & Zambotti's approach to the reconstruction theorem* [14]*, given by Rinaldi and Sclavi in* [25]*. However, the present setting is not sufficient to deal with more general systems of singular stochastic PDEs, like the scalar* Φ^4_3 *equation on a closed manifold, for which more geometrical background needs to be introduced (jets in particular) to deal with local expansions of order higher than* 1*. See the forthcoming work* [23] *of Hairer & Singh on the subject.*

2. *The group* G^- *of characters of the algebra* $\mathcal{T}^-$ *comes equipped with a convolution product derived from the extraction/contraction coproduct* $\Delta^- : \mathcal{T}^- \to \mathcal{T}^- \otimes \mathcal{T}^-$*:*

$$\ell \circ \bar{\ell} := \big(\ell \otimes \bar{\ell}\big)\, \Delta^-, \quad \ell^{-1}(\cdot) = \ell(S^- \cdot),$$

where S^- *is the antipode associated to* Δ^-*. In the translation-invariant setting of all previous works, this group has an explicit representation in* $L(\mathcal{T})$ *given by*

$$M_\ell := (\ell \otimes \mathrm{Id})\, \delta,$$

for which $\delta : \mathcal{T} \to \mathcal{T}^- \otimes \mathcal{T}$ *is a co-action and*

$$M_\ell M_{\bar{\ell}} = M_{\ell \circ \bar{\ell}}.$$

Although we cannot build such a representation of the group G^- *in a non-translation invariant setting, we can associate*

$$R_\ell(x) = \big(\ell(x, \cdot) \otimes \mathrm{Id}\big)\delta_r$$

to any character and any state space point x*, and set*

$$\ell \circ \bar{\ell} := \big(\ell \otimes \bar{\ell}\big)\, \delta_r,$$

pointwise in x*, with* δ_r *is the map from Example 7. One then has*

$$R_\ell R_{\bar{\ell}} = R_{\ell \circ \bar{\ell}}.$$

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• I. Bailleul – Univ. Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France.
E-mail: ismael.bailleul@univ-rennes1.fr

• Y. Bruned – School of Mathematics, University of Edinburgh, EH9 3FD, Scotland.
E-mail: Yvain.Bruned@ed.ac.uk