Mean field singular stochastic PDEs

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Abstract. We study some systems of interacting fields whose evolution is given by singular stochastic partial differential equations of mean field type. We provide a robust setting for their study leading to a well-posedness result and a propagation of chaos result. The case of interacting systems with a common noise is also considered.

Contents

1 – Introduction

Let $(\xi^i)_{i\geq 1}$ stand for a sequence of independent, identically distributed, random spacetime distributions on the 2-dimensional torus T^2. We will denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space on which these random variables are defined. We assume that the ξ^i are almost surely continuous functions of time with values in the space of $(\alpha-2)$-Hölder regular distributions over T^2, with $2/3 < \alpha < 1$, with null spatial mean. The archetype of such a noise is given by (the time independent) space white noise. We study a system of interacting fields whose evolution is given by the following system of '*singular*' stochastic partial differential equations (SPDEs)

$$(\partial_t - \Delta)u^i = f(u^i, \mu_t^n)\, \xi_t^i + g(u^i, \mu_t^n), \qquad (1 \leq i \leq n), \tag{1.1}$$

where

$$\mu_t^n := \frac{1}{n}\sum_{i=1}^n \delta_{u_t^i}$$

is the running time empirical measure of the system – a probability measure on a function space. Some (possibly random) initial conditions in that function space are given.

Recall the rule of thumb: *One can make sense of the product of two distributions with given Hölder regularities if and only if the sum of their regularity exponents is positive.* The term '*singular*' in the expression 'singular SPDE' refers to the fact that the regularity of the noise is too low for the regularizing effect of the heat resolvent to give sufficient regularity to the u^i to make sense of the products $f(u^i, \mu_t^n)\, \xi^i$. The diffusivity term $f(u^i, \mu_t^n)$ is expected to have at best parabolic regularity α, while the product $f(u^i, \mu_t^n)\, \xi^i$ is well-defined if and only if $\alpha + (\alpha - 2) > 0$. This condition does not hold in our case where $\alpha < 1$. The settings of regularity structures and paracontrolled calculus have been developed in the last ten years to deal precisely with this kind of problem and one can indeed use either of them to make sense of equation (1.1) as an equation of the form

$$(\partial_t - \Delta)\mathsf{u} = \mathsf{f}(\mathsf{u})\, \xi^{[1,n]} + \mathsf{g}(\mathsf{u}), \tag{1.2}$$

for some n-dimensional unknown u and noise $\xi^{[1,n]}$, and identify conditions on f and g under which (1.2) has a unique solution over a given time interval. This way of proceding does not take profit from the specific structure of the mean field type equation (1.1). It is in particular unclear how to prove a propagation of chaos result for the interacting field system from this point of view. The necessity of a point of view tailor-made to mean field-type dynamics gets even clearer if one looks at what should most naturally be the limit dynamics of a given field of system (1.1) when n tends to ∞, say the field with label $i = 1$. Based on symmetry/exchangeability considerations this field is expected to be a solution of the equation

$$(\partial_t - \Delta)u = f(u, \mathcal{L}(u_t))\, \xi_t + g(u, \mathcal{L}(u_t)), \tag{1.3}$$

where $\mathcal{L}(u_t)$ stands for the law of the random variable u_t and ξ stands for a random distribution with the same law as the ξ^i. Our first aim in this work is to develop a setting within which one can make sense of system (1.1) and equation (1.3) in a unified way, for a large class of spacetime noises ξ.

Denote by z and z' generic spacetime points. The choice of functions f and g in equations of the form (1.1) and (1.3) is guided by the physics of the phenomenon modeled by system (1.1). To make things concrete we consider in this introduction the case where $f(u, \mu)$ and $g(u, \mu)$ depend linearly on their measure argument and are of the form

$$z \mapsto \iint F\big(u(z), v(z')\big) k(z, z') dz' \mu(dv) = \mathbb{E}\Big[\int F\big(u(z), V(z')\big) k(z, z') dz' \Big] \tag{1.4}$$

for u a function on T^2, for a random function V with law μ and a real-valued function F on $\mathbf{R}^2$. Think of the kernel k as a parameter that captures the range of the interaction between the different fields in the system, with extreme cases $k(z, z') = 1$ and $k(z, z') = \delta_z(z')$, and intermediate cases represented by C^2 kernels for instance. The physics behind the two extreme cases is very different and we will technically deal with them in a different way. We will be able to work with functions that depend polynomially on their measure argument. Our main result reads informally as follows. We fix some initial conditions.

1 – Theorem. One can design a setting where equation (1.3) *makes sense.*

- *(a) Under proper regularity and growth assumptions on f and g there exists a positive time T such that system* (1.1) *and equation* (1.3) *have unique solutions on the time interval* $[0, T]$.
- *(b) The law of any fixed tuple of fields in the field system* (1.1) *converges to a tuple of independent, identically distributed, solutions of* (1.3) *as n tends to ∞, on the time interval* $[0, T]$.

So there is propagation of chaos for system (1.1), with mean field dynamics given by the mean field type equation (1.3).

While equation (1.3) and system (1.1) share the common feature of being singular, in the sense that they involve some ill-defined products, the mean field interaction in (1.3) causes a different kind of problem. A close situation was studied by Bailleul, Catellier & Delarue in their analysis of mean field type random rough differential equations [4]. We design in the present work an approach similar to [4] for the study of equation (1.3), using the language of paracontrolled calculus to build our setting. The original form of paracontrolled calculus was introduced by Gubinelli, Imkeller & Perkowski in [11]; one can find a nice short account of the basics of paracontrolled calculus in Gubinelli & Perkowski's lecture notes [12]. Recall that we work with a noise with null spatial mean. Denote by $\omega \in \Omega$ a generic chance element and write $X(\omega)$ for $-(\partial_t - \Delta)^{-1}(\xi(\omega))$, and $\overline{X}$ for an independent copy of the random variable X. As in [4] we use a notion of paracontrolled field that is tailor made to capture not only the paracontrolled structure of u needed to make sense of its product with ξ but also of the structure needed to describe the mean field specific spacetime function

$$(t, x) \mapsto f\big(u_t, \mathcal{L}(u_t)\big)(x).$$

This comes under the form of a definition saying that a random field $u(\omega)$ is ω-paracontrolled by a reference field $X(\omega)$ of parabolic Hölder regularity α if one has almost surely

$$u(\omega) \simeq \mathsf{P}_{(\delta_z u)(\omega)} X(\omega) + \overline{\mathbb{E}}\big[\mathsf{P}_{(\delta_\mu u)(\omega, \cdot)} \overline{X}(\cdot)\big] \tag{1.5}$$

up to a remainder of parabolic regularity 2α, for some random functions $(\delta_z u)(\omega)$ and $(\delta_\mu u)(\omega,\cdot)$ that depend on ω and an additional independent chance element that is averaged out in the $\overline{\mathbb{E}}$ expectation, where $\overline{X}(\cdot) = (\partial_t - \Delta)^{-1}(\overline{\xi}(\cdot))$ and $\overline{\xi}$ has the same law as ξ and is independent of ξ, and $\cdot$ stands for the chance element argument. A precise definition, conveying in particular the meaning of the notations $\delta_z u, \delta_\mu u$, is given in Section 4.2. This definition will play a key role in our construction of a robust setting where to make sense of equation (1.3) and prove a well-posedness result for it.

Setting up a framework for the study of a given singular stochastic PDE driven by a random noise $\xi(\omega)$ usually requires that we enhance the noise with the additional datum of quantities that do not make sense analytically ω-wise. In the archetypal example of the 2-dimensional parabolic Anderson model equation

$$(\partial_t - \Delta)v = v\xi,$$

where ξ is a space white noise that is almost surely of space Hölder regularity $-1-\eta$ for all $\eta > 0$, enhancing the noise consists in building a random variable that plays the role of the ω-wise ill-defined product of $\xi(\omega)$ and $\Delta^{-1}(\xi(\omega))$. This random variable, suggestively denoted by $\big(\xi\Delta^{-1}(\xi)\big)(\omega)$, is given by the $L^2(\Omega,\mathbb{P})$ limit of the renormalized regularized quantity

$$\xi^\varepsilon\Delta^{-1}(\xi^\varepsilon) - C^\varepsilon,$$

where ξ^ε stands for a smooth regularization of ξ that converges to ξ in the space of distributions with Hölder regularity $-1-\eta$, and C^ε is an explicit constant that diverges to $+\infty$ as a multiple of $|\log\varepsilon|$. The fact that the naive approximation $\xi^\varepsilon\Delta^{-1}(\xi^\varepsilon)$ is not converging leads to the interpretation of the solution v to the parabolic Anderson model equation as a limit in probability of solutions v^ε to the renormalized equation

$$(\partial_t - \Delta)v^\varepsilon = v^\varepsilon\xi^\varepsilon - C^\varepsilon v^\varepsilon,$$

rather than as a limit of solutions to the parabolic Anderson model equation driven by the regularized noise ξ^ε. We talk in this setting of the pair of random variables $(\xi, \xi\Delta^{-1}(\xi))$ as an '*enhanced noise*'. A richer enhancement of the noise ξ is needed in the analysis of the mean field equation (1.3). Not only do we need to add the random variable $\big(\xi\Delta^{-1}(\xi)\big)(\omega)$ to our notion of enriched noise, but the description (1.5) of an ω-controlled field should make it plain that we also need to add a doubly random variable that plays the role of the analytically ill-defined product of $\xi(\omega)$ and $(\partial_t - \Delta)^{-1}(\overline{\xi}(\varpi))$, where $(\omega,\varpi) \in \Omega^2$ and we work with the product probability $\mathbb{P}^{\otimes 2}$ on $(\Omega^2, \mathcal{F}^{\otimes 2})$. Luckily, the independence of ξ and $\overline{\xi}$ allows to define a doubly random variable $\big(\xi(\partial_t - \Delta)^{-1}(\overline{\xi})\big)(\omega,\varpi)$ as the $L^2(\Omega^2,\mathbb{P}^{\otimes 2})$ limit of the regularized quantity

$$\xi^\varepsilon(\partial_t - \Delta)^{-1}(\overline{\xi}^\varepsilon)$$

without the need of any *renormalization*. This will lead us to the interpretation of a solution to equation (1.3) as the limit in probability as $\varepsilon > 0$ goes to 0 of the solution u^ε to the renormalized equation

$$(\partial_t - \Delta)u^\varepsilon = f(u^\varepsilon, \mathcal{L}(u^\varepsilon_t))\,\xi^\varepsilon_t - C^\varepsilon(ff')\big(u^\varepsilon, \mathcal{L}(u^\varepsilon_t)\big) + g(u^\varepsilon, \mathcal{L}(u^\varepsilon_t)),$$

where f' stand for the derivative of f with respect to its first argument. Building on the setting that we use to analyse Equation (1.3) we are also able to deal with systems of interacting fields and mean field equations subject to a common unaveraged noise λ

$$(\partial_t - \Delta)u = f_1\big(u, \mathcal{L}(u_t)\big)\xi_t + f_2\big(u, \mathcal{L}(u_t)\big)\lambda_t + g\big(u, \mathcal{L}(u_t)\big).$$

See Theorem 31 for a description of what happens in this case.

Organization of this work. We treat the elementary case of systems (1.1) and equation (1.3) with additive noise ($f = 1$) in Section 2. Very robust results can be obtained in this simple setting, leading in particular to a simple proof of propagation of chaos for the corresponding system of interacting fields for an essentially arbitrary random noise with values in $C_TC^{\alpha-2}$. No tools from paracontrolled calculus are needed to deal with this case. We use the language of paracontrolled calculus to study more general equations or systems. We recall what we need from this domain in Section 3.1 and study equation (1.3) in the simple setting of a diffusivity with form (1.4) and C^2 kernel k in Section 3.3. The notion of mean field enhancement of the noise is introduced in Section 4.1, with an associated notion of paracontrolled structure described in Section 4.2. The well-posed character of equation (1.3) is the object of Section 4.3. The quantitative regularity result that we obtain for the solution u of equation (1.3) as a function fo the enhanced noise entails in Section 5 a propagation of chaos result for system (1.1). Section 6 is dedicated to the study of mean field equations/systems with a common unaveraged noise.

Notations. We gather here a number of notations that we will use frequently.

– *We fix throughout this work some regularity exponents*

$$\frac{2}{3} < \beta < \alpha < 1.$$

– *For $\gamma \in \mathbf{R}$, we denote by $C^\gamma = C^\gamma(\mathsf{T}^2)$ the Besov space $B^\gamma_{\infty\infty}(\mathsf{T}^2)$, with norm $\|\cdot\|_\gamma$. For any Banach space E and $\gamma \geq 0$ we set*

$$C^\gamma_T E := C^\gamma([0,T], E)$$

and write $L^\infty_T E$ for $L^\infty([0,T];E)$. We will also need the parabolic Hölder space $\mathscr{C}^\alpha_T$ on $[0,T]\times\mathsf{T}^2$, which is isometric to $C^{\alpha/2}_T L^\infty(\mathsf{T}^2)\cap C_T C^\alpha(\mathsf{T}^2)$ equipped with its natural norm. We will denote $(P_t)_{t\geq 0}$ the semigroup generated by the Laplace-Beltrami operator Δ on an ad hoc function space. Recall the elementary estimate

$$\|P_t u\|_{C^{\gamma+\delta}} \lesssim_T t^{-\delta/2}\|u\|_{C^\gamma},$$

for $\delta > 0$ and $0 < t \leq T$.

– *We denote by $L^p(\Omega, E)$ the space of E-valued random variables in $L^p(\Omega, \mathcal{F}, \mathbb{P})$.*

– *For an integrability exponent $1 \leq p < \infty$ we denote by $\mathcal{P}_p(E)$ the set of probability measures on E that has a moment of order p and by $\mathcal{W}_{p,E}$ the $p-$Wasserstein metric on $\mathcal{P}_p(E)$. We define a distance on $L^\infty_T\mathcal{P}_p(C^\alpha)$ setting*

$$d_{L^\infty_T\mathcal{W}_{p,C^\alpha}}(\mu,\mu') := \sup_{t\in[0,T]} \mathcal{W}_{p,C^\alpha}(\mu_t,\mu'_t).$$

– *We denote by $\mathcal{L}(Z)$ the law of a random variable Z.*

– *For a measure μ on a metric space E and $\phi \in C_b(E)$ write $\mu(\phi)$ for $\int \phi\, d\mu$.*

2 – Additive noise

Fix $0 < T_0 < \infty$ and $1 \leq p < \infty$. Let $\zeta \in C_{T_0}C^{\alpha-2}$ be an *arbitrary random element.* Following Coghi, Deuschel, Friz & Maurelli [9] we begin our work by studying the case of a mean field type equation with additive noise

$$(\partial_t - \Delta)u = \zeta + g(u, \mathcal{L}(u_t)) \tag{2.1}$$

and random initial condition u_0, assuming that the random variable (ζ, u_0) is an element of $L^p\big(\Omega, C_{T_0}C^{\alpha-2}\times C^\alpha\big)$. No singular product is involved in the study of this equation and we will be able to solve it with classical tools. We prove in Section 2.1 that equation (2.1) is well-posed under proper Lipschitz assumptions on g and that the law of its solution is a Lipschitz continuous function of the law of (ζ, u_0) in the Wasserstein p-space. This strong result leads in Section 2.2 to a propagation of chaos result for an associated field system.

2.1 – Additive mean field equation. For $\mu \in \mathcal{P}_p(C_{T_0}C^\alpha)$ and $t \in [0, T_0]$, we write μ_t for the image measure of μ in C^α by the t-time coordinate map $u \in C_{T_0}C^\alpha \mapsto u_t \in C^\alpha$.

Assumption (H_g) – *There exists a constant L such that for every $v_1, v_2 \in C^\alpha$ and $\nu_1, \nu_2 \in \mathcal{P}_p(C^\alpha)$ we have*

$$\big\|g(v_1,\nu_1) - g(v_2,\nu_2)\big\|_{C^{\alpha-2}}^p \le L^p\big(\|v_1 - v_2\|_{C^\alpha}^p + \mathcal{W}_{p,C^\alpha}(\nu_1,\nu_2)^p\big).$$

*2 – Proposition. Suppose Assumption **(H_g)** holds. For any $\mu \in \mathcal{P}_p(C_{T_0}C^\alpha), u_0 \in C^\alpha$ and $\zeta \in C_{T_0}C^{\alpha-2}$ the equation*

$$(\partial_t - \Delta)u = \zeta + g(u,\mu) \tag{2.2}$$

with initial condition u_0 has a unique solution $u \in C_{T_0}C^\alpha$.

Proof – Set

$$Z_t := \int_0^t P_{t-s}(\zeta_s)\, ds$$

and recall the well-known Schauder type bound

$$\|Z\|_{C_{T_0}C^\alpha} \lesssim_{T_0} \|\zeta\|_{C_{T_0}C^{\alpha-2}}. \tag{2.3}$$

One can rewrite equation (2.2) in integral form

$$u_t = P_t(u_0) + Z_t + \int_0^t P_{t-s}g(u_s,\mu_s)ds. \tag{2.4}$$

The estimate (2.3) ensures that the map

$$\Phi : u \in C_{T_0}C^\alpha \mapsto P_t(u_0) + Z_t + \int_0^t P_{t-s}g(u_s,\mu_s)ds \in C_{T_0}C^\alpha$$

is well-defined. For $u, u' \in C_{T_0}C^\alpha$, using Assumption *($H_g$)* and (2.3), we have

$$\|\Phi(u)_t - \Phi(u')_t\|_{C^\alpha} \le \int_0^t \big\|P_{t-s}g(u_s,\mu_s) - P_{t-s}g(u'_s,\mu_s)\big\|_{C^\alpha} ds \le \int_0^t L\|u_s - u'_s\|_{C^\alpha} ds.$$

Denote by $\Delta_k(0,t)$ the simplex $\{0 \le s_1 \le \cdots \le s_k \le t\}$ and write ds for $ds_1 \dots ds_k$. An iteration of the previous bound gives

$$\|\Phi^{\circ k}(u)_t - \Phi^{\circ k}(u')_t\|_{C^\alpha} \le L^k \int_{\Delta_k(0,t)} \|u_{s_k} - u'_{s_k}\|_{C^\alpha} ds \le \frac{(LT)^k}{k!}\|u - u'\|_{C_TC^\alpha}.$$

The map $\Phi^{\circ k}$ is thus contracting for k large enough, so it has a unique fixed point. ▷

We denote by $u^\mu(\zeta, u_0)$ the solution to equation (2.2). We now work with (ζ, u_0) random, an element of $L^p\big(\Omega, C_{T_0}C^{\alpha-2} \times C^\alpha\big)$.

3 – Proposition. For every $\mu \in \mathcal{P}_p(C_{T_0}C^\alpha)$ the law of $u^\mu(\zeta, u_0)$ belongs to $\mathcal{P}_p(C_{T_0}C^\alpha)$.

Proof – Write $\delta_{\mathbf{0}}$ for Dirac distribution on the null function $\mathbf{0}$. We have from the integral formulation (2.4) the estimate

$$\begin{aligned}
\|u_t^\mu\|_{C^\alpha} &\le C\Big(\|u_0\|_{C^\alpha} + \|Z_t\|_{C^\alpha} + \int_0^t \|g(u_s^\mu,\mu_s)\|_{C^\alpha} ds\Big) \\
&\le C\left(\|u_0\|_{C^\alpha} + \|Z_t\|_{C^\alpha} + \int_0^t \|g(0,\delta_{\mathbf{0}})\| + L\Big(\|u_s\|_{C^\alpha} + \mathcal{W}_{p,C^\alpha}(\mu_s,\delta_{\mathbf{0}})\Big)\mathrm{d}s\right) \\
&\le C\Big(\|u_0\|_{C^\alpha} + \|Z_t\|_{C^\alpha} + T_0\|g(0,\delta_{\mathbf{0}})\|_{C^\alpha} + T_0\mathcal{W}_{p,C_{T_0}C^\alpha}(\mu,\delta_{\mathbf{0}})\Big) + CL\int_0^t \|u_s\|_{C^\alpha}\mathrm{d}s,
\end{aligned}$$

for some positive constant C. We get the inequality

$$\|u_t\|_{C^\alpha} \le C\Big(\|u_0\|_{C^\alpha} + \|Z_t\|_{C^\alpha} + T_0\|g(0,\delta_{\mathbf{0}})\|_{C^\alpha} + T_0\mathcal{W}_{p,C_TC^\alpha}(\mu,\delta_{\mathbf{0}})\Big)e^{CLt}$$

from Gronwall lemma, from which the conclusion follows. ▷

Set

$$\Psi : \left\{ \begin{array}{ccc} \mathcal{P}_p(C_{T_0}C^\alpha) \times L^p(\Omega, C_{T_0}C^{\alpha-2} \times C^\alpha) & \to & \mathcal{P}_p(C_{T_0}C^\alpha) \\ \big(\mu, (\zeta, u_0)\big) & \mapsto & \mathcal{L}\big(u^\mu(\zeta,u_0)\big) \end{array} \right.$$

We define a ***solution to equation*** (2.1) ***with initial condition*** u_0 as a fixed point of the map

$$\Psi\big(\cdot,(\zeta,u_0)\big) : \mathcal{P}_p(C_{T_0}C^\alpha) \to \mathcal{P}_p(C_{T_0}C^\alpha).$$

4 – Theorem. Suppose ***Assumption (H$_g$)*** *holds. Then equation* (2.1) *has a unique solution denoted by* $u(\zeta,u_0)$. *We have the Lipschitz estimate*

$$\mathcal{W}_{p,C_{T_0}C^\alpha}\big(\mathcal{L}(u(\zeta,u_0)),\mathcal{L}(u(\zeta',u_0'))\big) \lesssim_{g,p,T_0} \mathcal{W}_{p,C_{T_0}C^{\alpha-2}\times C^\alpha}\big(\mathcal{L}(\zeta,u_0),\mathcal{L}(\zeta',u_0')\big). \tag{2.5}$$

Proof – Fix (ζ,u_0) and use the shorthand notation $\Psi_{\zeta,u_0}(\cdot)$ for $\Psi\big(\cdot,(\zeta,u_0)\big)$. For $\mu,\mu'\in\mathcal{P}_p(C_TC^\alpha)$ write u^μ and $u^{\mu'}$ for $u^\mu(\zeta,u_0)$ and $u^{\mu'}(\zeta,u_0)$, respectively. One has

$$u_t^\mu - u_t^{\mu'} = \int_0^t \Big(P_{t-s}g(u_s^\mu,\mu_s) - P_{t-s}g(u_s^{\mu'},\mu_s')\Big)ds,$$

and

$$\big\|u_t^\mu - u_t^{\mu'}\big\|_{C^\alpha}^p \le C\int_0^t \Big(\big\|u_s^\mu - u_s^{\mu'}\big\|_{C^\alpha}^p + \mathcal{W}_p\big(\mu_{[0,s]},\mu'_{[0,s]}\big)^p\Big)ds,$$

for some constant C, so we get from Gronwall lemma the estimate

$$\mathcal{W}_{p,C_tC^\alpha}\Big(\mathcal{L}(u^\mu_{[0,t]}),\mathcal{L}(u^{\mu'}_{[0,t]})\Big)^p \le Ce^{CT_0}\int_0^t \mathcal{W}_{p,C_sC^\alpha}\big(\mu_{[0,s]},\mu'_{[0,s]}\big)^p ds.$$

A direct iteration gives

$$\begin{aligned}\mathcal{W}_{p,C_{T_0}C^\alpha}\big(\Psi_{\zeta,u_0}^{\circ k}(\mu^1),\Psi_{\zeta,u_0}^{\circ k}(\mu^2)\big)^p &\le (Ce^{CT_0})^k\int_{\Delta_t^k}\mathcal{W}_{p,C_{s_k}C^\alpha}\big(\mu_{[0,s_k]},\mu'_{[0,s_k]}\big)^p ds\\ &\le (Ce^{CT_0})^k\,\frac{1}{k!}\,\mathcal{W}_{p,C_{T_0}C^\alpha}\big(\mu,\mu'\big)^p,\end{aligned}$$

so the map $\Psi_{\zeta,u_0}^{\circ k}$ is contracting for k sufficiently large and equation (2.1) has a unique solution.

Let now $\zeta,\zeta'\in C_{T_0}C^{\alpha-2}$ be two noises and $u_0,u_0'\in C^\alpha$ be two initial conditions. Pick $\mu\in\mathcal{P}_p(C_{T_0}C^\alpha)$ and write u and u' for $u(\zeta,u_0)$ and $u'(\zeta,u_0)$, respectively. We can assume without loss of generality that ζ,ζ',u_0,u_0' are such that the p-th moment of $\|u-u'\|_{C_{T_0}C^\alpha}$ is equal to the p-Wasserstein distance between $\mathcal{L}(u(\zeta,u_0))$ and $\mathcal{L}(u(\zeta',u_0'))$. Since

$$u_s - u_s' = P_s(u_0-u_0') + Z_s - Z_s' + \int_0^s \Big(P_{s-r}(g(u_r,\mu_r)) - P_{s-r}(g(u_r',\mu_r))\Big)dr,$$

we have

$$\begin{aligned}\sup_{s\in[0,t]}\|u_s-u_s'\|_{C^\alpha} &\le \|u_0-u_0'\|_{C^\alpha} + \|Z-Z'\|_{C_TC^\alpha} + C\int_0^t \|u_s-u_s'\|_{C^\alpha}ds\\ &\lesssim \|u_0-u_0'\|_{C^\alpha} + \|\zeta-\zeta'\|_{C_TC^{\alpha-2}} + C\int_0^t \|u_s-u_s'\|_{C^\alpha}ds\end{aligned}$$

and

$$\mathbb{E}\Big[\sup_{s\in[0,t]}\|u_s-u_s'\|_{C^\alpha}^p\Big] \lesssim_p \|u_0-u_0'\|_{C^\alpha}^p + \mathbb{E}\big[\|\zeta-\zeta'\|_{C_TC^{\alpha-2}}^p\big] + \int_0^t \mathbb{E}\Big[\sup_{r\in[0,s]}\|u_r-u_r'\|_{C^\alpha}^p\Big]ds.$$

We get the Lipschitz estimate (2.5) from Gronwall lemma. ▷

Note that we do not assume that the noise ζ and the initial condition u_0 are independent.

2.2 – Propagation of chaos. Let now $(\zeta^i,u_0^i)_{i\ge1}$ be a sequence of independent, identically distributed, random variables with common distribution the law of (ζ,u_0). Denote by $(\Omega,\mathcal{F},\mathbb{P})$ the probability space on which this sequence of random variables is defined, with $\omega\in\Omega$ a generic element of Ω. Fix $\omega\in\Omega$. For an integer $n\ge1$ consider the interacting system of fields

$\big(u^{1,n}(\omega),\dots,u^{n,n}(\omega)\big)$ with initial conditions $\big(u_0^1(\omega),\dots,u_0^n(\omega)\big)$ and dynamics

$$\begin{aligned}(\partial_t-\Delta)u^{i,n}(\omega) &= \zeta^i(\omega)+g\big(u^{i,n}(\omega),\mu_t^n(\omega)\big),\\ \mu_t^n(\omega) &:= \frac{1}{n}\sum_{k=1}^n \delta_{u_t^{k,n}(\omega)},\end{aligned}\tag{2.6}$$

for $1\leq i\leq n$. H. Tanaka [18] was the first to notice that system (2.6) is actually, *for each* $\omega\in\Omega$, an equation of the form (2.1) set on the finite probability space $\{1,\dots,n\}$ equipped with the uniform probability measure λ_n. Following [5], we call this observation '*Tanaka's trick*'. Random variables on the space $\{1,\dots,n\}$ are n-tuples indexed by $1\leq i\leq n$. Denote by $\mathcal{L}_{\lambda_n}(X)$ the law under λ_n of an arbitrary random variable X defined on $\{1,\dots,n\}$. Denote also by

$$U_n : j\mapsto j$$

the canonical random variable on $\{1,\dots,n\}$. Tanaka's trick says that a solution to the system

$$(\partial_t-\Delta)u^i(\omega)=\zeta^i(\omega)+g\big(u^i(\omega),\mathcal{L}_{\lambda_n}(u^{U_n(\cdot)}(\omega))\big),\qquad(1\leq i\leq n)$$

with parameter ω and chance element $i\in\{1,\dots,n\}$, is precisely given by the n-tuple

$$\big(u^{1,n}(\omega),\dots,u^{n,n}(\omega)\big)$$

of solutions to the field system (2.6).

Recall that a *sequence* $(\mu^n)_{n\geq1}$ *of probability measures on* E^n, invariant by the action on E^n of the permutation group of n elements, is said to be μ*-chaotic* if for every $1\leq k\leq n$ and $\phi_1,\dots\phi_k\in C_b(E)$, we have

$$\mu^n\big(\phi_1\otimes\cdots\otimes\phi_k\otimes\mathbf{1}^{\otimes(n-k)}\big)\underset{n\to\infty}{\longrightarrow}\prod_{i=1}^k\mu(\phi_i).$$

A well-known criterion of μ-chaoticity is given by the convergence in law of the empirical mean of an iid n-sample of μ^n to the measure μ itself – see for instance Proposition 2.2 in Sznitman's lecture notes [17]. Now the law of large numbers tells us that the empirical mean

$$\frac{1}{n}\sum_{i=1}^n\delta_{(\zeta^i,u_0^i)(\omega)}$$

converges $\mathbb{P}$-almost surely in $\mathcal{W}_{p,C_{T_0}C^{\alpha-2}\times C^\alpha}$ to $\mathcal{L}(\zeta,u_0)$. The following fact is thus a consequence of the Lipschitz estimate (2.5) and Sznitman's criterion. In the next statement we write $u\in L^p(\Omega,C_{T_0}C^\alpha)$ for the solution to equation (2.1).

5 – *Corollary. For any integer $k\geq1$, the law of the k-tuple $(u^{1,n},\dots,u^{k,n})$ converges weakly to $\mathcal{L}(u)^{\otimes k}$ when n tends to ∞.*

3 – Basics on paracontrolled calculus and long range mean field equations

The study of equation (1.3) with a non-constant diffusivity $f(\cdot)$ requires that we use one of the languages that have been developed in the last ten years for the study of a large class of singular stochastic PDEs. The problem involved in this class of equations is best illustrated on the toy example of the parabolic Anderson model equation

$$(\partial_t-\Delta)u=u\xi$$

set on T^2, with ξ a space white noise. Recall ξ has almost surely Hölder space regularity $-1-\varepsilon$ for all $\varepsilon>0$. One expects from the Schauder estimates satisfied by the resolvent of the heat operator that u has parabolic regularity $(\alpha-2)+2=\alpha$. This regularity is not sufficient for making sense of the product $u\xi$ since $\alpha+(\alpha-2)<0$. There are at least two languages one can use to circumvent this problem and set a robust solution theory for this equation and a whole class of equations involving the same pathology. We choose to work here with the language of paracontrolled calculus first introduced by Gubinelli, Imkeller & Perkowski in [11]. We recall in Section 3.1 the notions and results from paracontrolled calculus that we will use; we refer the reader to [12, 10, 14] for accounts of the basics on the subject. These results are sufficient to deal with the soft case of a mean field equation (1.3) with diffusivity given by the model function (1.4) with a C^2 kernel k. We deal with that case in Section 3.3 as a warm-up for Section 4.

3.1 – Basics on paracontrolled calculus. We will use the notations $h_1 < h_2$ and $h_1 \odot h_2$ for the paraproduct and the resonant operators on *space distributions* h_1, h_2, defined from the Littlewood-Paley projectors. From its definition $h_1 < h_2$ is well-defined for all distributions h_1, h_2 on T^2 and has high Fourier modes that are modulations of the high Fourier modes of h_2 by low Fourier modes of h_1. On that ground, it makes sense to think of $h_1 < h_2$ as a distribution that 'looks like' h_2. Recall from Lemma 2.4 of [11] that the corrector

$$\mathsf{C}(a,b,c) := (a < b) \odot c - a\,(b \odot c)$$

has a continuous extension from $C^2 \times C^2 \times C^2$ to $C^{\alpha_1} \times C^{\alpha_2} \times C^{\alpha_3}$ with values in $C^{\alpha_1+\alpha_2+\alpha_3}$ if $\alpha_2 + \alpha_3 < 0$ and $0 < \alpha_1 + \alpha_2 + \alpha_3 < 1$. The following continuity estimate from [2], Proposition 14 therein, will also be useful. One has

$$\big\| a < (b < c) - (ab) < c \big\|_{C^{\alpha_2+\alpha_3}} \lesssim \|a\|_{L^\infty} \|b\|_{C^{\alpha_2}} \|c\|_{C^{\alpha_3}}, \tag{3.1}$$

for all $a \in L^\infty, b \in C^{\alpha_2}$ with α_2 in $(0,1)$ and $c \in C^{\alpha_3}$ with $-3 < \alpha_3 < 3$. (The regularity exponent 3 has no particular meaning; it is purely technical.)

*Definition – Pick a reference distribution $\Lambda \in C^\rho$, with $\rho \in \mathbf{R}$. A **distribution** v **on** T^2 is said to be **paracontrolled by** Λ if there exists a positive regularity exponent γ and functions $v' \in C^\gamma$ and $v^\# \in C^{\gamma+\rho}$ such that*

$$v = (v' < \Lambda) + v^\#.$$

We denote by $\mathcal{D}^\gamma(\Lambda)$ the space of all such couples $(v', v^\#)$; it is equipped with the norm

$$\|(v', v^\#)\|_{\mathcal{D}^\gamma} := \big\|v'\big\|_{C^\gamma} + \|v^\#\|_{C^{\gamma+\rho}}. \tag{3.2}$$

For reference distributions $\Lambda_1, \Lambda_2 \in C^\rho$ and $\mathbf{v}_1 = (v_1', v_1^\#) \in \mathcal{D}^\gamma(\Lambda_1)$ and $\mathbf{v}_2 = (v_2', v_2^\#) \in \mathcal{D}^\gamma(\Lambda_2)$ we set

$$d_{\mathcal{D}^\gamma}(\mathbf{v}_1, \mathbf{v}_2) := \big\|v_1' - v_2'\big\|_{C^\gamma} + \big\|v_1^\# - v_2^\#\big\|_{C^{\gamma+\rho}}.$$

The expression '*Gubinelli derivative of* v' is sometimes used to talk about v'. Note that the exponent γ in $\mathcal{D}^\gamma(\Lambda)$ does *not* refer to the regularity of v but rather to the regularity exponents of v' and $v^\sharp$. Indeed the distribution v is C^ρ. Let a and b be two functions on T^2 with $a \in \mathcal{D}^\beta(b)$ for $\beta > 0$, with Gubinelli derivative a'. Bony's paralinearization result implies that if h stands for a C_b^3 function from $\mathbf{R}$ into itself then $h(a) \in \mathcal{D}^\beta(b)$; we denote by $h(a)' = h'(a)a'$ its Gubinelli derivative and by $h(a)^\sharp$ its remainder term. (See e.g. Section 2.3 of [11].)

We will denote by $k_1 \prec k_2$ the modified paraproduct on *spacetime distributions* introduced in Section 5 of [11]. It is a parabolic version of the paraproduct operator $<$ that has the same analytic properties in the scale of Besov parabolic function spaces as the operator $<$ in the scale of spatial Besov function spaces. When applied to parabolic distributions $k_1 \in C_T^{\alpha/2} L^\infty, k_2 \in C_T C^\beta$ the two paraproducts are related by the continuity relation

$$\big\|k_1 < k_2 - k_1 \prec k_2\big\|_{C_T C^{\alpha+\beta}} \lesssim \|k_1\|_{C_T^{\alpha/2} L^\infty} \|k_2\|_{C_T C^\beta}. \tag{3.3}$$

We further note the useful estimate

$$\big\|(\partial_t - \Delta)(k_1 \prec k_2) - k_1 \prec \big((\partial_t - \Delta)k_2\big)\big\|_{C_T C^{\alpha+\beta-2}} \lesssim \|k_1\|_{\mathscr{C}_T^\alpha} \|k_2\|_{C_T C^\beta}.$$

(These two results are the content of Lemma 5.1 of [11].) We use the $\prec$ paraproduct and a slightly different notion of size to deal with parabolic functions paracontrolled by a reference parabolic function Ξ.

Definition – Pick a reference function $\Xi \in \mathscr{C}_T^\rho$, with $\rho > 0$. A ***parabolic function*** *u* ***on*** *$[0,T] \times \mathsf{T}^2$ is said to be* ***paracontrolled by*** *Ξ if there exists a function $u' \in \mathscr{C}_T^\beta$, with $\beta > 0$, such that*

$$u^{\#} := u - u' \prec \Xi \in \mathscr{C}_T^\rho$$

and

$$\sup_{t\in(0,T]} t^{\beta/2} \|u_t^{\#}\|_{C^{\beta+\rho}} < +\infty.$$

We denote by $\mathcal{D}_T^{\rho,\beta}(\Xi)$ the space of all such couples $(u', u^\sharp)$; it is equipped with the norm

$$\big\|(u', u^\sharp)\big\|_{\mathcal{D}_T^{\rho,\beta}} := \|u'\|_{\mathscr{C}_T^\beta} + \|u^{\#}\|_{\mathscr{C}_T^\rho} + \sup_{t\in(0,T]} t^{\beta/2}\|u_t^{\#}\|_{C^{\beta+\rho}}.$$

For two reference functions $\Xi_1, \Xi_2 \in \mathscr{C}_T^\rho$ and $\mathbf{u}_1 = (u_1', u_1^{\#}) \in \mathcal{D}_T^{\rho,\beta}(\Xi_1)$ and $\mathbf{u}_2 = (u_2', u_2^{\#}) \in \mathcal{D}_T^{\rho,\beta}(\Xi_2)$ we set

$$d_{\mathcal{D}_T^{\rho,\beta}}(\mathbf{u}_1, \mathbf{u}_2) := \|u_1' - u_2'\|_{\mathscr{C}_T^\beta} + \|u_1^{\#} - u_2^{\#}\|_{\mathscr{C}_T^\rho} + \sup_{t\in(0,T]} t^{\beta/2}\|u_1^{\#} - u_2^{\#}\|_{\beta+\rho}.$$

3.2 – Noise enhancement and product definition. Fix a positive time horizon T_0, set

$$\mathscr{L} := (\partial_t - \Delta)$$

and write $\mathscr{L}^{-1}$ for the resolvent operator with null initial condition at time 0. Define

$$\begin{aligned}\mathsf{L} : C_{T_0}C^\infty \times C([0,T_0], \mathbf{R}) &\longrightarrow C_{T_0}C^\infty \times C_{T_0}C^\infty \\ (\ell, c) &\longrightarrow \big(\ell, \mathscr{L}^{-1}(\ell) \odot \ell - c\big).\end{aligned}$$

The letter L is chosen for '*lift*'. The ***space*** $\mathfrak{N}$ ***of enhanced noises*** is the closure in $C_{T_0}C^{\alpha-2} \times C_{T_0}C^{2\alpha-2}$ of the range of L. As a shorthand notation, for $c \in C([0,T_0], \mathbf{R})$, we set

$$\mathsf{L}_c(\cdot) := \mathsf{L}(\cdot, c). \tag{3.4}$$

We denote by

$$\widehat{\zeta} = (\zeta, \zeta^{(2)})$$

a generic element of $\mathfrak{N}$ and set here

$$Z := \mathscr{L}^{-1}(\zeta) \in \mathscr{C}_{T_0}^\alpha.$$

The natural norm of $\widehat{\zeta}$ as an element of the product space is denoted by $\|\widehat{\zeta}\|$. The following statement provides a large class of random noises with a natural enhancement as random element of $\mathfrak{N}$. It is proved in Appendix A. We write P_t for $e^{t\Delta}$.

6 – *Theorem. Let $(\xi_t)_{0\le t\le T_0}$ stand for a time-dependent Gaussian random distribution on T^2 with covariance of the form*

$$\mathbb{E}\big[(\xi_t, \phi)(\xi_s, \psi)\big] = c(t,s)\langle \psi \star C, \phi\rangle_{L^2}$$

for some distribution C on T^2. We assume that the Fourier transform of C satisfies for some $\eta < 1 - \alpha$ the condition

$$|\widehat{C}(k)| \lesssim |k|^\eta,$$

and that the function c satisfies the inequality

$$0 \le c(t,t) + c(s,s) - 2c(s,t) \le |t-s|^\delta$$

for some positive exponent δ. Then one defines a random variable $X \odot \xi \in L^1(\Omega, C_TC^{2\alpha-2})$ setting

$$\big(X \odot \xi\big)(t) := \int_0^t \Big(P_{t-s}(\xi_s) \odot \xi_t - \mathbb{E}[P_{t-s}(\xi_s) \odot \xi_t]\Big) ds \tag{3.5}$$

One further has $X \odot \xi \in L^p(\Omega, C_TC^{2\alpha-2})$ for all $1 \le p < \infty$ and if ξ^ε stands for a space regularization of ξ then

$$\mathsf{L}\big(X^\varepsilon \odot \xi^\varepsilon, \mathbb{E}[X^\varepsilon \odot \xi^\varepsilon]\big)$$

converges in $L^p(\Omega, C_T C^{2\alpha-2})$ *to* $X \odot \xi$ *as* $\varepsilon > 0$ *goes to* 0.

The end of this section deals with deterministic enhanced noises. The datum of an element of $\mathfrak{N}$ allows to give a definition of some a priori ill-defined product.

7 – *Definition. Pick* $\widehat{\zeta} \in \mathfrak{N}$ *and* $\beta > 2 - 2\alpha$ *and* $0 < t \leq T_0$. *Let* $u \in C([0,T] \times \mathsf{T}^2)$ *be such that for each* $t \in [0,T]$ *one has* $\mathbf{u}_t \in \mathcal{D}^\beta(Z_t)$. *We define the product* $\mathbf{u}_t\lambda_t$ *as the element of* $\mathcal{D}^\beta(\lambda_t)$ *specified by the decomposition*

$$\mathbf{u}_t\lambda_t := u_t < \lambda_t + (\mathbf{u}_t\lambda_t)^\#,$$

where

$$(\mathbf{u}_t\lambda_t)^\# := \lambda_t < u_t + u_t^\# \odot \lambda_t + \mathsf{C}\big(u_t', Z_t, \lambda_t\big) + u_t'\zeta_t^{(2)}$$

and

$$\|(\mathbf{u}_t\lambda_t)^\#\|_{C^{\alpha+\beta-2}} \lesssim \|\mathbf{u}\|_{\mathcal{D}^\beta(Z_t)}\Big(\|\lambda_t\|_{C^{\alpha-2}} + \|Z_t\|_{C^\alpha}\|\lambda_t\|_{C^{\alpha-2}} + \big\|\zeta_t^{(2)}\big\|_{C^{2\alpha-2}}\Big). \tag{3.6}$$

For $\widehat{\zeta^i} = \big(\zeta^i, \zeta^{i(2)}\big) \in \mathfrak{N}$, $Z^i = \mathscr{L}^{-1}(\zeta^i)$ and $\mathbf{u}_t^i \in \mathcal{D}^\beta(Z_t^i)$, with $i \in \{1,2\}$, set

$$m := \max_{i\in\{2,3\}}\Big\{\big\|\zeta^i\big\|_{C^{\alpha-2}}, \big\|\zeta^{i(2)}\big\|_{C^{2\alpha-2}}, \|\mathbf{u}_t^i\|_{\mathcal{D}^\beta(Z_t^i)}\Big\}.$$

The proof of the following proposition can be found in [11], Theorem 3.7 therein.

8 – *Proposition. We have the local Lipschitz estimate*

$$\Big\|(\mathbf{u}_t^1\zeta_t^1)^\# - \big(\mathbf{u}_t^2\zeta_t^2\big)^\#\Big\|_{C^{\alpha+\beta-2}} \lesssim_m d_{\mathcal{D}^\beta}\big(\mathbf{u}_t^1, \mathbf{u}_t^2\big) + \big\|\widehat{\zeta}^1 - \widehat{\zeta}^2\big\|_{\mathfrak{N}},$$

and the function $t \mapsto \mathbf{u}_t\lambda_t$ *is in* $C_T C^{\alpha-2}$ *for* $\mathbf{u} \in \mathcal{D}_T^{\alpha,\beta}(X)$.

The starting point of the next statement is the description for each time of the right hand side of a parabolic equation as a $<$ paracontrolled distribution whenever this makes sense. The statement provides as an outcome a description of the solution of the equation as a $\prec$ paracontrolled function. This can be read as a kind of Schauder-type estimate in the setting of paracontrolled calculus. See Section 5 of [11] for a proof.

9 – *Proposition. Pick a positive regularity exponent* b. *For* $\pi \in C_T C^{\alpha-2}$ *let* $\Pi \in \mathscr{C}_T^\alpha$ *be the solution of the equation*

$$(\partial_t - \Delta)\Pi = \pi$$

with null initial condition at time 0. *Then for every* $w', w^\# \in \mathscr{C}_T^\alpha$ *such that*

$$\sup_{t\in(0,T]} t^{\beta/2}\big\|w_t^\#\big\|_{(\alpha-2)+\beta} < \infty \tag{3.7}$$

and $u_0 \in C^\alpha$, *the solution* u *to the equation*

$$(\partial_t - \Delta)u = w' < \pi + w^\#, \quad u(0) = u_0, \tag{3.8}$$

belongs to $\mathcal{D}_T^{\alpha,\beta}(\Pi)$ *and* $u' = w'$. *We further have the estimate*

$$\|(u', u^\sharp)\|_{\mathcal{D}_T^{\alpha,\beta}(\Pi)} \lesssim \|u_0\|_{C^\alpha} + T^{(\alpha-\beta)/2}\Big(\|w'\|_{\mathscr{C}_T^\alpha}\big(1 + \|\pi\|_{C_T C^{\alpha-2}}\big) + \sup_{t\in(0,T]} t^{\beta/2}\big\|w_t^\#\big\|_{C^{(\alpha-2)+\beta}}\Big).$$

For different $w_i', w_i^\#$ *satisfying condition* (3.7), *initial conditions* $u_{i,0}$ *and noises* $\pi_i \in C_T C^{\alpha-2}$, *for* $i \in \{1,2\}$, *setting*

$$m' := \max_{i\in\{1,2\}}\Big\{1, \|w_i'\|_{\mathscr{C}_T^\alpha}, \|\pi_i\|_{C_T C^{\alpha-2}}\Big\}$$

and denoting by u_1, u_2 *the corresponding solutions to equation* (3.8) *with corresponding paracontrolled decomposition* $\mathbf{u}_1, \mathbf{u}_2$, *we have*

$$d_{\mathcal{D}_T^{\alpha,\beta}}(\mathbf{u}_1, \mathbf{u}_2) \lesssim \|u_{1,0} - u_{2,0}\|_{C^\alpha} + P(m')\,T^{(\alpha-\beta)/2}\Big(\|w_1' - w_2'\|_{\mathscr{C}_T^\alpha} + \|\pi_1 - \pi_2\|_{C_T C^{\alpha-2}} + \sup_{t\in(0,T]} t^{\beta/2}\big\|w_1^\#(t) - w_2^\#(t)\big\|_{C^{(\alpha-2)+\beta}}\Big),$$

for some quadratic polynomial P.

3.3 – Long range mean field equations. As a direct application of the results of Section 3.1 we treat in this section a particular case of mean field singular stochastic PDE where the function f in (1.3) has a simple structure. Let a function $F \in C_b^3(\mathbf{R}^2, \mathbf{R})$ and a C_b^2 kernel $k(z, z')$ on the torus T^2 be given, together with a constant $\beta \in (2/3, \alpha)$. For $a \in C^\alpha$ and $\mu \in \mathcal{P}_p(C^\alpha)$ we set in this section

$$f(a, \mu)(z) = \int_{C^\alpha} \int_{\mathsf{T}^2} F\big(a(z), b(z')\big) k(z, z')\, dz' \mu(db). \tag{3.9}$$

This is a linear function of its measure argument. The setting and results of Section 3.1 are sufficient to deal with the mean field equation

$$(\partial_t - \Delta) u = f(u, \mathcal{L}(u_t))\zeta + g(u, \mathcal{L}(u_t)), \tag{3.10}$$

when f has the form (3.9) and g satisfies the following Lipschitz condition.

Assumption (A_g) – *One has $\|g\big(a_1, \mu_1\big) - g\big(a_2, \mu_2\big)\|_{C^{(\alpha-2)+\beta}} \lesssim \|a_1 - a_2\|_{C^\alpha} + \mathcal{W}_{p,C^\alpha}\big(\mu_1, \mu_2\big)$.*

We first deal with the paracontrolled structure of $f(a, \mu)$. Fix $t > 0$ and some reference function $X_t \in C^\alpha$.

10 – *Proposition. For $a \in \mathcal{D}^\beta(X_t)$ and $\mu \in \mathcal{P}_p(C^\alpha)$ one has*

$$f(a, \mu) = f(a, \mu)' < X_t + f(a, \mu)^{\#}$$

with

$$f(a, \mu)'(z) = \int_{C^\alpha} \int_{\mathsf{T}^2} \partial_1 F\big(a(z), b(z')\big) k(z, z')\, dz' \mu(db),$$

and

$$\|f(a, \mu)^{\#}\|_{C^{\alpha+\beta}} \lesssim \big(1 + \|X_t\|_{C^\alpha}^2\big)\Big(1 + \|a'\|_{C^\beta} + \|a^{\#}\|_{C^\alpha}\Big)\Big(1 + \|a'\|_{C^\beta} + \|a^{\#}\|_{C^{\alpha+\beta}}\Big).$$

Furthermore, for $X_t^i \in C^\alpha$ and $a_i \in \mathcal{D}^\beta(X_t^i), \mu_i \in \mathcal{P}_p(C^\alpha)$, for $1 \leq i \leq 2$, one has

$$\|f(a_1, \mu_1)^{\#} - f(a_2, \mu_2)^{\#}\|_{C^{\alpha+\beta}} \lesssim d_{\mathcal{D}^\beta}\big(a_1, a_2\big) + \mathcal{W}_{p,C^\alpha}\big(\mu_1, \mu_2\big) + \|X_t^1 - X_t^2\|_{C^\alpha}, \tag{3.11}$$

for an implicit constant that is a polynomial of degree 3 on

$$\max_{i=1,2} \Big\{1, \|a_i\|_{\mathcal{D}^\beta(X^i)}, \mathcal{W}_{p,C^\alpha}(\mu_i, \delta_0), \|X_t^i\|_{C^\alpha}\Big\}.$$

Proof – We paralinearize with respect to the z variable, with z' in the role of a parameter in the paraproducts below. We use the shorthand notations

$$k_{z'}(z) := k(z, z'), \quad F_{b(z')}(w) := F(w, b(z')).$$

With these notations one has

$$\begin{aligned} F(a, b(z')) &= \partial_1 F(a, b(z')) < a + F_{b(z')}(a)^\sharp \\ &= \big\{\partial_1 F(a, b(z'))a'\big\} < X_t + \partial_1 F(a, b(z')) < a^{\#} \\ &\quad + \partial_1 F\big(a, b(z')\big) < (a' < X_t) - \big(\partial_1 F\big(a, b(z')\big)a' < X_t\big) + F_{b(z')}(a)^\sharp \end{aligned}$$

and

$$\begin{aligned} f(a, \mu) = &\left\{a' \int_{\mathsf{T}^2 \times C^\alpha} \partial_1 F\big(a, b(z')\big) k_{z'}\, dz' \mu(db)\right\} < X_t \\ &+ \int_{\mathsf{T}^2 \times C^\alpha} \Big(\big(\partial_1 F\big(a, b(z')\big)a' < X_t\big)k_{z'} - \big\{k_{z'}\partial_1 F\big(a, b(z')\big)a'\big\} < X_t\Big) dz' \mu(db) \end{aligned}$$

$$
\begin{aligned}
&+ \int_{\mathsf{T}^2\times C^\alpha} k_{z'}\Big\{\partial_1 F(a,b(z')) < (a' < X_t) - \big\{\partial_1 F(a,b(z'))a'\big\} < X_t\Big\}\, dz'\mu(db) \\
&+ \int_{\mathsf{T}^2\times C^\alpha} k_{z'} F_{b(z')}(a)^\sharp\, dz'\mu(db) + \int_{C^\alpha}\int_{\mathsf{T}^2} \big(\partial_1 F(a,b) < a^{\#}\big) k_{z'}\, dz'\mu(db) \\
=: &\left\{a' \int_{\mathsf{T}^2\times C^\alpha} \partial_1 F\big(a,b(z')\big) k_{z'}\, dz'\mu(db)\right\} < X_t + f(a,b)^{\#}.
\end{aligned}
$$

We estimate each term separately to show that the remainder is regular, using commutator type estimates when needed. First, since $k_{z'}$ is C_b^2 and $\alpha+\beta<2$ we have from (3.1) the continuity estimate

$$
\begin{aligned}
\Big\| \big(\big\{\partial_1 F\big(a,b(z')\big)a'\big\} < X_t\big)k_{z'} &- \big\{k_{z'}\partial_1 F\big(a,b(z')\big)a'\big\} < X_t \Big\|_{C^{\alpha+\beta}} \\
&\lesssim \|k_{z'}\|_{C^{2\alpha}} \|\partial_1 F\big(a,b(z')\big)a'\|_{C^\beta} \|X_t\|_{C^\alpha} \\
&\lesssim \|k\|_{C_b^2} \big(1+\|a\|_{C^\alpha}\big)\|a'\|_{C^\beta}\|X_t\|_{C^\alpha} \\
&\lesssim \big(1+\|X_t\|_{C^\alpha}^2\big)\Big(1+\|a'\|_{C^\beta}^2+\|a^{\#}\|_{C^\alpha}^2\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
\Big\|\partial_1 F(a,b(z')) < (a' < X_t) &- \big\{\partial_1 F(a,b(z'))a'\big\} < X_t\Big\|_{C^{\alpha+\beta}} \\
&\lesssim \|\partial_1 F(a,b(z'))\|_{C^\beta}\|a'\|_{C^\beta}\|X_t\|_{C^\alpha} \\
&\lesssim \big(1+\|a\|_{C^\alpha}\big)\|a'\|_{C^\beta}\|X_t\|_{C^\alpha} \\
&\lesssim \big(1+\|X_t\|_{C^\alpha}^2\big)\Big(1+\|a'\|_{C^\beta}^2+\|a^{\#}\|_{C^\alpha}^2\Big)
\end{aligned}
$$

and

$$
\|k_{z'}F_{b(z')}(a)^\sharp\|_{C^{\alpha+\beta}} \lesssim \|F_{b(z')}\|_{C_b^3}\big(1+\|a\|_{C^\alpha}^2\big) \lesssim \big(1+\|X_t\|_{C^\alpha}^2\big)\Big(1+\|a'\|_{C^\beta}^2+\|a^{\#}\|_{C^\alpha}^2\Big)
$$

and

$$
\begin{aligned}
\|\big(\partial_1 F(a,b(z')) < a^{\#}\big)k_{z'}\|_{C^{\alpha+\beta}} &\lesssim \big(1+\|a\|_{C^\alpha}\big)\|a^{\#}\|_{C^{\alpha+\beta}} \\
&\lesssim \big(1+\|X_t\|_{C^\alpha}\big)\Big(1+\|a'\|_{C^\beta}+\|a^{\#}\|_{C^\alpha}\Big)\|a^{\#}\|_{C^{\alpha+\beta}}.
\end{aligned}
$$

Integrating over z' and summing we get

$$
\|f(a,\mu)^{\#}\|_{C^{\alpha+\beta}} \lesssim \big(1+\|X_t\|_{C^\alpha}^2\big)\Big(1+\|a'\|_{C^\beta}+\|a^{\#}\|_{C^\alpha}\Big)\Big(1+\|a'\|_{C^\beta}+\|a^{\#}\|_{C^{\alpha+\beta}}\Big).
$$

We leave the proof of the estimate (3.11) to the reader as it is similar to what is above. ▷

For $\widehat{\zeta}\in\mathfrak{N}$ we write $Z := \mathscr{L}^{-1}(\zeta)$, so $Z\in\mathscr{C}_T^\alpha$. We emphasize below the fact that u is paracontrolled in the product of $f(u_t,\mu_t)$ with λ_t by writing $f(\mathbf{u}_t,\mu_t)\lambda_t$.

11 – *Proposition. Assume Assumption* $(\boldsymbol{A}_g)$ *holds and fix* $0<T_0<\infty$. *For every initial condition* $u_0\in C^\alpha$, *for every enhanced noise* $\widehat{\zeta}\in\mathfrak{N}$ *and any* $\mu\in\mathcal{P}_p(\mathscr{C}_{T_0}^\alpha)$ *there exists a positive time horizon* $T\le T_0$ *and a unique solution to the equation*

$$
(\partial_t-\Delta)u = f(\mathbf{u}_t,\mu_t)\lambda_t + g(u_t,\mu_t) \tag{3.12}
$$

in $\mathcal{D}_T^{\alpha,\beta}(Z)$. *This solution is a locally Lipschitz function of* $u_0\in C^\alpha,\mu\in\mathcal{P}_p(\mathscr{C}_T^\alpha)$ *and* $\widehat{\zeta}\in\mathfrak{N}$.

Proof – Rewrite equation (3.12) as the fixed point equation

$$
u_t = P_t u_0 + \int_0^t P_{t-s}\big(f(\mathbf{u}_s,\mu_s)\zeta_s + g(u_s,\mu_s)\big)\, ds.
$$

We get from Proposition 10 and Proposition 8 that $f(\mathbf{u}_s,\mu_s)\zeta_s+g(u_s,\mu_s)$ is for each s an element of $\mathcal{D}^\alpha(\zeta_s)$ with Gubinelli derivative $f(u_s,\mu_s)$ and remainder $(f(\mathbf{u}_s,\mu_s)\zeta_s)^{\#}+g(u_s,\mu_s)$. With Proposition 9 in mind we check that $f(\mathbf{u},\mu)\in\mathscr{C}_{T_0}^\alpha$ and $(f(u_s,\mu_s)\zeta_s)^{\#}+g(u_s,\mu_s)$ satisfies

(3.7). Take $\mathbf{u}\in\mathcal{D}_T^{\alpha,\beta}(Z)$. First one has for $(s,x),(t,y)\in[0,T_0]\times\mathsf{T}^2$

$$\begin{aligned}\big|f(u_t,\mu_t)(y)-f(u_s,\mu_s)(x)\big| &= \Big|\int_{\mathsf{T}^2\times\mathscr{C}_T^\alpha} F\big(u_t(y),v_t(z)\big)k(y,z)-F\big(u_s(x),v_s(z)\big)k(X,L)\,dz\mu(dv)\Big|\\ &\leq \int_{\mathsf{T}^2\times\mathscr{C}_T^\alpha}\bigg(\big|F\big(u_t(y),v_t(z)\big)\big(k(y,z)-k(X,L)\big)\big|\\ &\qquad + \big|F\big(u_t(y),v_t(y)\big)-F\big(u_s(x),v_s(x)\big)\big|\,|k(X,L)|\bigg)\,dz\mu(dv)\end{aligned}$$

$$\begin{aligned}&\lesssim \int_{\mathsf{T}^2\times\mathscr{C}_T^\alpha}\Big(|x-y|+\big(\|u\|_{\mathscr{C}_T^\alpha}+\|v\|_{\mathscr{C}_T^\alpha}\big)\big(|x-y|^\alpha+|t-s|^{\alpha/2}\big)\Big)\,dz\mu(dv)\\ &\lesssim \big(1+\|u\|_{\mathscr{C}_T^\alpha}+\mathcal{W}_{p,\mathscr{C}_T^\alpha}(v,\delta_{\mathbf{0}})\big)\big(|x-y|^\alpha+|t-s|^{\alpha/2}\big),\end{aligned}$$

so we have the norm estimate

$$\|f(u,\mu)\|_{\mathscr{C}_{T_0}^\alpha}\lesssim\big(1+\|Z\|_{\mathscr{C}_{T_0}^\alpha}\big)\Big(1+\|\mathbf{u}\|_{\mathcal{D}_{T_0}^{\alpha,\beta}}+\mathcal{W}_{p,\mathscr{C}_{T_0}^\alpha}(\mu,\delta_{\mathbf{0}})\Big).$$

Second, one gets for $0<T\leq T_0$

$$\sup_{t\in(0,T]} t^{\beta/2}\big\|\big(f(\mathbf{u}_t,\mu_t)\lambda_t\big)^{\#}+g(u_t,\mu_t)\big\|_{\alpha+\beta-2}\lesssim\big(1+\|\widehat{\zeta}\,\|_{\mathfrak{N}}^3\big)\Big(1+\|\mathbf{u}\|^2_{\mathcal{D}_T^{\alpha,\beta}}+\mathcal{W}_{p,\mathscr{C}_T^\alpha}(\mu,\delta_{\mathbf{0}})\Big). \tag{3.13}$$

from Proposition 10 and Proposition 8. It follows from Proposition 9 that the map

$$\Phi_{\widehat{\zeta},u_0,\mu}:\mathcal{D}_T^{\alpha,\beta}(Z)\to\mathcal{D}_T^{\alpha,\beta}(Z)$$

which associates to $\mathbf{u}\in\mathcal{D}_T^{\alpha,\beta}(Z)$ the solution w of the equation

$$\mathscr{L}w=f(\mathbf{u},\mu)\zeta+g(u,\mu),$$

with initial condition $w_0=u_0$, is well-defined and satisfies the estimate

$$\big\|\Phi_{\widehat{\zeta},u_0,\mu}(\mathbf{u})\big\|_{\mathcal{D}_T^{\alpha,\beta}}\lesssim\|u_0\|_{C^\alpha}+T^{\frac{\alpha-\beta}{2}}\Big(1+\big\|\widehat{\zeta}\,\big\|^3_{\mathfrak{N}}\Big)\Big(1+\|\mathbf{u}\|^2_{\mathcal{D}_T^{\alpha,\beta}(X)}+\mathcal{W}_{p,\mathscr{C}_T^\alpha}(\mu,\delta_{\mathbf{0}})\Big). \tag{3.14}$$

One can then find

$$M=M\Big(\|u_0\|_\alpha\vee\|\widehat{\zeta}\,\|_{\mathfrak{N}}\vee\mathcal{W}_{p,\mathscr{C}_T^\alpha}(\mu,\delta_{\mathbf{0}})\Big)$$

and

$$T=T\Big(\|u_0\|_\alpha\vee\|\widehat{\zeta}\|_{\mathfrak{N}}\vee\mathcal{W}_{p,\mathscr{C}_T^\alpha}(\mu,\delta_{\mathbf{0}})\Big)$$

such that the map $\Phi_{\widehat{\zeta},u_0,\mu}$ sends the ball $\Big\{\mathbf{u}\in\mathcal{D}_T^{\alpha,\beta}(Z)\,;\,\|\mathbf{u}\|_{\mathcal{D}_T^{\alpha,\beta}}\leq M\Big\}$ into itself. One can choose M as an increasing function of its arguments and T as a decreasing function of its arguments.
Given $\widehat{\zeta}_1,\widehat{\zeta}_2$ in $\mathfrak{N}$, two initial conditions u_{01},u_{02} in C^α and μ_1,μ_2 in $\mathcal{P}_p(\mathscr{C}_T^\alpha)$, set

$$M'=M\Big(\max_{i=1,2}\Big\{\|u_{0i}\|_{C^\alpha}\vee\|\widehat{\zeta}_i\|_{\mathfrak{N}}\vee\mathcal{W}_{p,\mathscr{C}_T^\alpha}(\mu_i,\delta_{\mathbf{0}})\Big\}\Big).$$

For $\|\mathbf{u}\|_{\mathcal{D}_T^{\alpha,\beta}}\leq M'$, Proposition 9 tells us that

$$\begin{aligned}&\mathrm{d}_{\mathcal{D}_T^{\alpha,\beta}}\big(\Phi_{\widehat{\zeta}_1,u_{01},\mu_1}(\mathbf{u}_1),\Phi_{\widehat{\zeta}_2,u_{02},\mu_2}(\mathbf{u}_2)\big)\\ &\qquad\lesssim\|u_{01}-u_{02}\|_{C^\alpha}+T^{(\alpha-\beta)/2}\Big\{\mathrm{d}_{\mathcal{D}_T^{\alpha,\beta}}(\mathbf{u}_1,\mathbf{u}_2)+\|\widehat{\zeta}_1-\widehat{\zeta}_2\|_{\mathfrak{N}}+\mathcal{W}_{p,\mathscr{C}_T^\alpha}(\mu^1,\mu^2)\Big\}.\end{aligned}$$

So choosing T small ensures that the map $\Phi_{\widehat{\zeta},u_0,\mu}$ has a unique fixed point $\mathbf{u}=(u',u^\sharp)$ which depends in a locally Lipschitz way on $u_0\in C^\alpha,\mu\in\mathcal{P}_p(\mathscr{C}_T^\alpha)$ and $\widehat{\zeta}\in\mathfrak{N}$. ▷

Before we can consider the case where ζ is random and formulate a fixed point equation to get $\mu_t=\mathcal{L}(u_t)$ we need a setting where the local solution to equation (3.12) can be turned into a fixed horizon solution. The following statement is a first step to do that. It gives an explosion criterion. It is a small variation on a similar result in Theorem 5.4 of [11].

12 – Lemma. *For every $R > 0$, the solution* $\mathbf{u}$ *to equation* (3.12) *is defined up to the time*

$$T^* = \inf\big\{t \geq 0, \quad \|u(t)\|_{L^\infty} \geq R\big\}.$$

Proof – The existence time T from Proposition 11 is a decreasing function

$$T = T\big(\|u_0\|_{C^\alpha}, \|\widehat{\zeta}\|_{\mathfrak{N}}, \mathcal{W}_{p,\mathscr{C}_T^\alpha}(\mu, \delta_{\mathbf{0}})\big)$$

of its arguments. One fixes here $\widehat{\zeta}$ and μ and consider T as a function of $\|u_0\|_{C^\alpha}$. We obtain below a constant bound for $\|u\|_{C^\alpha}$ that is valid as long as $\|u\|_{L^\infty} \leq R$. As $\|u\|_{C_T C^\alpha} \lesssim_{\widehat{\zeta}} \|\mathbf{u}\|_{\mathcal{D}_T^{\alpha,\beta}}$ we actually prove that

$$\|u\|_{\mathcal{D}_T^{\alpha,\beta}} \lesssim_{\mu,\widehat{\zeta}} 1 + \|u\|_{C_T L^\infty}^2.$$

This is done as follows. Since $u'_t = f(u_t, v_t)$, we have

$$\|u'\|_{\mathscr{C}_T^\beta} \lesssim_\mu 1 + \|u\|_{\mathscr{C}_T^\beta}.$$

Yet since $u = u' \prec X + u^\#$ where u' appears as an L^∞ contribution we have

$$\|u'\|_{\mathscr{C}_T^\beta} \lesssim_{\mu,\widehat{\zeta},R} 1 + \|u^\#\|_{\mathscr{C}_T^\beta}.$$

We now use the fact that

$$(\partial_t - \Delta) u^\# = \Phi^\# \tag{3.15}$$

where

$$\Phi^\# = \big(f(\mathbf{u},\mu)\zeta - f(u,\mu) \prec \zeta\big) + g(u,\mu).$$

The refined paralinearization lemma C.1 from [11] ensures that

$$\begin{aligned}
\big\|f\big(u' \prec X + u^\#, \mu\big) - f'\big(u' \prec X + u^\#, \mu\big) \prec \big(u' \prec X + u^\#\big)\big\|_{C^{\alpha+\beta}} \\
\lesssim_\mu \big(1 + \|u' \prec X\|_{C^\alpha}^2 + \|u^\#\|_{L^\infty}^2\big)\big(1 + \|u^\#\|_{C^{\alpha+\beta}}\big) \\
\lesssim_{\widehat{\zeta},\mu} \big(1 + \|u\|_{L^\infty}^2\big)\big(1 + \|u^\#\|_{C^{\alpha+\beta}}\big),
\end{aligned}$$

so using the continuity relation (3.3) and the estimate (3.6) from Definition 7 we obtain

$$\begin{aligned}
\|\Phi^\#\|_{C^{\alpha+\beta-2}} &\lesssim_{\widehat{\zeta},\mu} \Big(1 + \|u\|_{C_T L^\infty}^2\Big)\Big(1 + \|u\|_{\mathscr{C}_T^\alpha} + \|u^\#\|_{C^{\alpha+\beta}}\Big) \\
&\lesssim_{\widehat{\zeta},\mu} \Big(1 + \|u\|_{C_T L^\infty}^2\Big)\Big(1 + \|u^\#\|_{\mathscr{C}_T^\alpha} + \|u^\#\|_{C^{\alpha+\beta}}\Big),
\end{aligned}$$

where the constant is a polynomial in $\|\widehat{\zeta}\|_{\mathfrak{N}}$ of degree 3. Schauder estimates – Lemma 5.3 of [11], ensure that

$$\sup_{0<t<T} t^{\beta/2}\|u^\#\|_{C^{\alpha+\beta}} \lesssim_{u_0} 1 + \sup_{0<t<T} t^{\beta/2}\|\Phi^\#\|_{C^{\alpha+\beta-2}}, \tag{3.16}$$

and

$$\|u^\#\|_{\mathscr{C}_T^\alpha} \lesssim_{u_0} 1 + \sup_{0<t<T} t^{\beta/2}\|\Phi^\#\|_{C^{\alpha+\beta-2}}, \tag{3.17}$$

so we have

$$\sup_{0<t\leq T} t^{\beta/2}\|\Phi^\#\|_{\alpha+\beta-2} \lesssim_{u_0,\mu,\widehat{\zeta}} \Big(1 + \|u\|_{C_T L^\infty}^2\Big)\Big(1 + \sup_{0<t\leq T} t^{\beta/2}\|\Phi^\#\|_{C^{\alpha+\beta-2}}\Big). \tag{3.18}$$

The coefficient in front of the sup term in the right hand side does not allow a priori to absorb that term in the left hand side. We follow [11] and use a scaling argument to isolate the $\Phi^\#$ terms. Let

$$(\Lambda^\lambda u)(t,x) := u(\lambda^2 t, \lambda x)$$

and

$$\mathsf{T}_\lambda^2 = \big(\mathbf{R}/(2\pi\lambda^{-1}\mathbf{Z})\big)^2.$$

We have

$$(\partial_t - \Delta) \circ \Lambda^\lambda = \lambda^2 \Lambda^\lambda \circ (\partial_t - \Delta)$$

and

$$\zeta^\lambda := \lambda^{2-\alpha}\Lambda^\lambda\zeta, \quad \|\zeta^\lambda\|_{\alpha-2} \simeq \|\zeta\|_{C^{\alpha-2}},$$

a deterministic estimate, and

$$u^\lambda := \Lambda^\lambda u$$

is a solution of the equation

$$(\partial_t - \Delta)u^\lambda = \lambda^\alpha f(\mathbf{u}^\lambda, \mu^\lambda)\zeta^\lambda + \lambda^2 g(u^\lambda, \mu^\lambda).$$

We now rewrite (3.18) for the rescaled equation, that is replacing f with $\lambda^\alpha f$ and g with $\lambda^2 g$, the bound for $\Phi^\#$ becomes for $\lambda \leq 1$

$$\begin{aligned}\|\Phi^{\#,\lambda}\|_{C^{\alpha+\beta-}} &\lesssim (\lambda^\alpha + \lambda^2)\big(1 + \|u^\lambda\|^2_{C_T L^\infty}\big)\big(1 + \|u^{\#,\lambda}\|_{\mathscr{C}^\alpha_T} + \|u^{\#,\lambda}\|_{C^{\alpha+\beta}}\big) \\ &\lesssim \lambda^\alpha\big(1 + \|u^\lambda\|^2_{C_T L^\infty}\big)\big(1 + \|u^{\#,\lambda}\|_{\mathscr{C}^\alpha_T} + \|u^{\#,\lambda}\|_{C^{\alpha+\beta}}\big),\end{aligned}$$

so

$$\sup_{0\leq t\leq T/\lambda^2} t^{\beta/2}\|\Phi^{\#,\lambda}\|_{C^{\alpha+\beta-2}} \lesssim \lambda^\alpha\Big(1 + \|u\|^2_{C_T L^\infty}\Big)\Big(1 + \sup_{0\leq t\leq T/\lambda^2} t^{\beta/2}\|\Phi^{\#,\lambda}\|_{C^{\alpha+\beta-2}}\Big),$$

and choosing λ small enough we finally get after inverse scaling

$$\sup_{0\leq t\leq T} t^{\beta/2}\|\Phi^\#\|_{C^{\alpha+\beta-2}} \lesssim_{u_0,\widehat{\zeta},\mu} 1 + \|u\|^2_{C_T L^\infty}.$$

In the end we obtain from the estimates (3.16) and (3.17) the bound

$$\|u^\#\|_{\mathscr{C}^\alpha_T} + \sup_{0\leq t\leq T} t^{\beta/2}\|u^\#\|_{C^{\alpha+\beta}} \lesssim_{u_0,\widehat{\zeta},\mu} 1 + \|u\|^2_{C_T L^\infty}.$$

▷

We are thus looking now for a condition on f that ensures a good control of the L^∞ norm of the solution to equation (3.12). We follow Proposition 3.28 of Cannizzaro, Friz & Gassiat's work [7] and introduce the following assumption to control the L^∞ norm of the solution u to (4.5).

Assumption (B) – *There exists a positive constant C_0 such that*

$$f(\pm C_0, \mu) = 0, \quad g(\pm C_0, \mu) = 0$$

for all $\mu \in \mathcal{P}_p(C^\alpha)$.

Examples of such functions can be constructed from functions F such that $F(\cdot, \mu)$ is compactly supported with a support independent of μ. Alternatively one can think of functions of the form $F(c, \mu) = F_1(c)F_2(\mu)$ with separate variables, with $F_1(\pm C_0) = 0$. We now specialize the result of Proposition 11 to the case where $\widehat{\zeta}$ is the random enhancement $\widehat{\xi}$ of a random noise ξ provided by Theorem 6. We emphasize that point by writing $\mathbf{u}_{\widehat{\xi}}$ for the solution to equation (3.12) in that case. Given $\varepsilon_k > 0$ set

$$c_k := \mathbb{E}[X^{\varepsilon_k} \odot \xi_t^{\varepsilon_k}].$$

13 – Lemma. *There is a sequence $\varepsilon_k > 0$ converging to 0 such that one has*

$$u = \lim_{n+\infty} u^{\varepsilon_k}$$

where u^{ε_k} stands for the well-defined solution in $\big[0, T(\|u_0\|_{C^\alpha})\big]$ of the equation

$$(\partial_t - \Delta)u^{\varepsilon_k} = f(u^{\varepsilon_k}(t), \mu_t)\zeta_t^{\varepsilon_k} - c_k f(u^{\varepsilon_k}(t), \mu_t) f(u^{\varepsilon_k}(t), \mu_t)' + g(u^{\varepsilon_k}(t), \mu_t) \tag{3.19}$$

Proof – The enhanced noise $\widehat{\zeta}$ is the limit in $\mathfrak{N}$ of the sequence of enhanced smooth noises $\widehat{\zeta}^{\varepsilon_k} := (\zeta^{\varepsilon_k}, (X \odot \zeta)^{\varepsilon_k})$ where

$$(X \odot \zeta)^{\varepsilon_k} := \zeta^{\varepsilon_k} \odot X^{\varepsilon_k} - c_k.$$

It follows from Proposition 11 that the function u is the limit in C^α of the sequence $\widetilde{u}^{\varepsilon_k}$ where $\widetilde{\mathbf{u}}_n$ is the solution to equation (3.12) with noise $\widehat{\zeta}^{\varepsilon_k}$. We have

$$f(\widetilde{\mathbf{u}}_n,\mu)\zeta^{\varepsilon_k}+g(\widetilde{u}^{\varepsilon_k},\mu)=f(\widetilde{u}^{\varepsilon_k},\mu)<\zeta^{\varepsilon_k}+\zeta^{\varepsilon_k}<f(\widetilde{u}^{\varepsilon_k},\mu)+f(\widetilde{u}^{\varepsilon_k},\mu)^{\#}\odot\zeta^{\varepsilon_k}$$
$$+\mathsf{C}\big(f(\widetilde{u}^{\varepsilon_k},\mu)',X_n,\zeta^{\varepsilon_k}\big)+f(\widetilde{u}^{\varepsilon_k},\mu)\big(X\odot\zeta\big)^{\varepsilon_k}+g(\widetilde{u}^{\varepsilon_k},\mu)$$
$$=f(\widetilde{u}^{\varepsilon_k},\mu)\zeta^{\varepsilon_k}-c_k(ff')(\widetilde{u}^{\varepsilon_k},\mu)+g(\widetilde{u}^{\varepsilon_k},\mu),$$

so $\widetilde{u}^{\varepsilon_k}$ is a solution of the equation

$$(\partial_t-\Delta)\widetilde{u}^{\varepsilon_k}=f(\widetilde{\mathbf{u}}_n(t),\mu_t)\zeta_t^{\varepsilon_k}-c_k(ff')(\widetilde{u}^{\varepsilon_k}(t),\mu_t)+g(\widetilde{u}^{\varepsilon_k}(t),\mu_t),$$

and one has indeed $u^{\varepsilon_k}=\widetilde{u}^{\varepsilon_k}$. ▷

*14 – Proposition. Under the assumptions **(A_g-B)**, if $\|u_0\|_{L^\infty}\leq C_0$ then $\mathbf{u}_{\widehat{\xi}}$ is defined globally in time.*

Proof – For every $n\in\mathbf{N}$ the constant function C_0 is a sub-solution and $-C_0$ is a super-solution of renormalized regularized equation (3.19). It follows from the classical comparison principle that one has

$$|u^{\varepsilon_k}(t,x)|\leq C_0$$

for all $t\leq T$ and $x\in\mathsf{T}^2$. The local Lipschitz continuity of $\mathbf{u}_{\widehat{\xi}}$ as a function of $\widehat{\xi}$ and the convergence in $\mathfrak{N}$ of $\widehat{\xi}_t^{\varepsilon_k}$ ensure that u^{ε_k} is converging to u in C_TL^∞. It follows that we have $\|u(t)\|_{L^\infty}\leq C_0$ for all $0\leq t\leq T$. The result of the statement follows from the explosion criterion of Lemma 12. ▷

15 – Proposition. If $\|u_0\|_{L^\infty}\leq C_0$ the random variable $\|\mathbf{u}\|_{\mathcal{D}_{\mathbf{T}}^{\alpha,\beta}}(\omega)$ has moments of any order.

Proof – Following what was done in the proof of the Proposition 22 we have an estimate

$$\|\mathbf{u}\|_{\mathcal{D}_T^{\alpha,\beta}}\lesssim_{\widehat{\zeta},u_0,\mu}1+\|u\|^2_{C_TL^\infty}\lesssim_{\widehat{\xi},u_0,\mu}1+C_0$$

with an implicit multiplicative constant that is polynomial function in $\|\widehat{\xi}\|_{\mathfrak{N}}$ of degree 3. ▷

*16 – Theorem. Fix $T_0>0$. Suppose that f and g satisfy assumptions **(A_g-B)** and pick $1\leq p<\infty$. There exists a positive time $T\leq T_0$ with the following property. For every $u_0\in C^\alpha$ there exists a unique solution to the mean field equation (3.10) in $L^p\big(\Omega,\mathscr{C}^\alpha_{T_0}\big)$. It is a locally Lipschitz continuous function of the initial condition u_0 and the enhanced noise $\widehat{\xi}\in L^{12p}(\Omega,\mathfrak{N})$. Furthermore u is the limit in $L^p\big(\Omega,\mathscr{C}^\alpha_{T_0}\big)$ of the solutions u^ε of the renormalized equations*

$$(\partial_t-\Delta)u^\varepsilon=f\big(u^\varepsilon,\mathcal{L}(u^\varepsilon(t))\big)\zeta^\varepsilon-c_\varepsilon(t)(\partial_1ff)\big(u^\varepsilon,\mathcal{L}(u^\varepsilon(t))\big)+g\big(u^\varepsilon,\mathcal{L}(u^\varepsilon(t))\big).$$

Proof – Pick $0<T\leq T_0$. Write $\mathbf{u}^\mu_{\widehat{\xi},u_0}$ for the solution to equation (3.12). We define from Proposition 11 a map $\Psi_{\widehat{\xi},u_0}$ from $L^p(\Omega,\mathscr{C}^\alpha_T)$ into itself setting

$$\Psi_{\widehat{\xi},u_0}(\mu)=\mathbf{u}^\mu_{\widehat{\xi},u_0}.$$

One has from the estimate (3.14)

$$\begin{aligned}\big\|\mathbf{u}^\mu_{\widehat{\xi},u_0}\big\|_{\mathcal{D}_T^{\alpha,\beta}}&\lesssim\|u_0\|_{C^\alpha}+T^{(\alpha-\beta)/2}\Big(1+\big\|\widehat{\xi}\big\|^3_{\mathfrak{N}}\Big)\Big(1+\|\mathbf{u}^\mu_{\widehat{\xi},u_0}\|^2_{\mathcal{D}_T^{\alpha,\beta}(X)}+\mathcal{W}_{p,\mathscr{C}^\alpha_T}\big(\mu,\delta_{\mathbf{0}}\big)\Big)\\&\lesssim\|u_0\|_{C^\alpha}+T^{(\alpha-\beta)/2}\Big(1+\big\|\widehat{\xi}\big\|^3_{\mathfrak{N}}\Big)\Big\{1+\|\mathbf{u}^\mu_{\widehat{\xi},u_0}\|^{1/2}_{\mathcal{D}_T^{\alpha,\beta}(X)}\Big(1+\|u_0\|_{C^\alpha}+\mathcal{W}_{p,\mathscr{C}^\alpha_T}\big(\mu,\delta_{\mathbf{0}}\big)\Big)^{3/2}\\&\qquad+\mathcal{W}_{p,\mathscr{C}^\alpha_T}\big(\mu,\delta_{\mathbf{0}}\big)\Big\}.\end{aligned}$$

Integrating and using Cauchy-Schwarz inequality we get for $\mathbb{E}\big[\|\mathbf{u}^\mu_{\widehat{\xi},u_0}\|^{2p}_{\mathcal{D}^{\alpha,\beta}_T}\big]^2$ the upper bound

$$\|u_0\|^{4p}_{C^\alpha} + T^{4p\delta}\Big(1+\mathbb{E}\Big[\big\|\widehat{\xi}\,\big\|^{12p}_{\mathfrak{N}}\Big]\Big)\Big\{\mathbb{E}\big[\|\mathbf{u}^\mu_{\widehat{\xi},u_0}\|^{2p}_{\mathcal{D}^{\alpha,\beta}_T}\big]\Big(1+\|u_0\|_{C^\alpha}+\mathcal{W}_{p,\mathscr{C}^\alpha_T}\big(\mu,\delta_{\mathbf{0}}\big)\Big)^{6p} + \mathcal{W}_{p,\mathscr{C}^\alpha_T}\big(\mu,\delta_{\mathbf{0}}\big)^{4p}\Big\}.$$

So for $T=T\big(\mathcal{W}_{p,\mathscr{C}^\alpha_T}(\mu,\delta_{\mathbf{0}})\big)$ sufficiently small we have

$$\mathbb{E}\big[\|\mathbf{u}^\mu_{\widehat{\xi},u_0}\|^{2p}_{\mathcal{D}^{\alpha,\beta}_T}\big]^{\frac{1}{2p}} \lesssim \|u_0\|_{C^\alpha} + T^\delta\Big(1+\mathbb{E}\big[\|\widehat{\xi}\|^{12p}_{\mathfrak{N}}\big]^{\frac{1}{4p}}\Big)\,\mathcal{W}_{p,\mathscr{C}^\alpha_T}\big(\mu,\delta_{\mathbf{0}}\big).$$

We have

$$\|u\|_{\mathscr{C}^\alpha_T} \lesssim (1+\|X\|_{\mathscr{C}^\alpha_T})\|\mathbf{u}^\mu_{\widehat{\xi},u_0}\|_{\mathcal{D}^{\alpha,\beta}_T(X)},$$

so we have from Cauchy-Schwarz inequality

$$\mathbb{E}\big[\|\mathbf{u}^\mu_{\widehat{\xi},u_0}\|^p_{\mathscr{C}^\alpha_T}\big]^{\frac{1}{p}} \lesssim \mathbb{E}\big[\|\mathbf{u}^\mu_{\widehat{\xi},u_0}\|^{2p}_{\mathcal{D}^{\alpha,\beta}_T}\big]^{\frac{1}{2p}}\Big(1+\mathbb{E}\big[\|\widehat{\xi}\|^{2p}_{\mathfrak{N}}\big]^{\frac{1}{2p}}\Big) \tag{3.20}$$

$$\lesssim \Big(1+\|u_0\|_{C^\alpha}\Big)\Big(1+\mathbb{E}\big[\|\widehat{\xi}\|^{12p}_{\mathfrak{N}}\big]^{\frac{1}{3p}}\Big)\Big(1+T^\delta\mathcal{W}_{p,\mathscr{C}^\alpha_T}\big(\mu,\delta_{\mathbf{0}}\big)\Big). \tag{3.21}$$

Pick $A>0$. For M sufficiently big and $T=T(M,A)$ even smaller, for every $u_0\in C^\alpha$ with $\|u_0\|_{C^\alpha}\le A$, the map $\Psi_{\widehat{\xi},u_0}$ sends the ball

$$\Big\{\mu\in L^p(\Omega,\mathscr{C}^\alpha_T)\,;\,\mathcal{W}_{p,\mathscr{C}^\alpha_T}(\mu,\delta_{\mathbf{0}})\le M\Big\}$$

into itself. Now pick μ_1,μ_2 in $L^p(\Omega,\mathscr{C}^\alpha_T)$, two initial conditions u_{01},u_{02} in C^α and $\widehat{\xi}_1,\widehat{\xi}_2$ in $L^{12p}(\Omega,\mathfrak{N})$ such that one has

$$\mathbb{E}\big[\|\widehat{\xi}_i\|^{8p}_{\mathfrak{N}}\big]\vee\|u_{0i}\|_{C^\alpha}\le A,\qquad \mathcal{W}_{p,\mathscr{C}^\alpha_T}\big(\mu_i,\delta_{\mathbf{0}}\big)\le M,$$

for $1\le i\le 2$. Write $\mathbf{u}_i$ for $\Phi_{\widehat{\xi}_i,u_{0i}}(\mu_i)$ and define the random variable

$$R:=\|\widehat{\xi}_1\|_{\mathfrak{N}}+\|\widehat{\xi}_2\|_{\mathfrak{N}}.$$

We have from the Schauder estimates of Proposition 9

$$\begin{aligned}\mathrm{d}_{\mathcal{D}^{\alpha,\beta}_T}\big(\mathbf{u}_1,\mathbf{u}_2\big) &\lesssim_R \|u_{01}-u_{02}\|_{C^\alpha}+T^\delta\Big\{\|\widehat{\xi}_1-\widehat{\xi}_2\|_{\mathfrak{N}}+\mathrm{d}_{\mathcal{D}^{\alpha,\beta}_T}(\mathbf{u}_1,\mathbf{u}_2)+\mathcal{W}_{p,\mathscr{C}^\alpha_T}\big(\mu^1,\mu^2\big)\Big\}\\ &\lesssim_R \|u_{01}-u_{02}\|_{C^\alpha}+T^\delta\Big\{\|\widehat{\xi}_1-\widehat{\xi}_2\|_{\mathfrak{N}}+\mathrm{d}_{\mathcal{D}^{\alpha,\beta}_T}(\mathbf{u}_1,\mathbf{u}_2)^{\frac{1}{2}}+\mathcal{W}_{p,\mathscr{C}^\alpha_T}\big(\mu^1,\mu^2\big)\Big\},\end{aligned}$$

for some implicit positive multiplicative constant that is a polynomial of R, which is of degree 5, combining Proposition 9, Proposition 10 and Proposition 8. Integrating and using Cauchy-Schwarz inequality we obtain the estimate

$$\begin{aligned}\mathbb{E}\big[\mathrm{d}_{\mathcal{D}^{\alpha,\beta}_T}\big(\mathbf{u}_1,\mathbf{u}_2\big)^{2p}\big]^2 \lesssim& \|u_{01}-u_{02}\|^{4p}_{C^\alpha}+\mathbb{E}\big[\|\widehat{\xi}_1-\widehat{\xi}_2\|^{4p}_{\mathfrak{N}}\big]\\ &+T^{4p\delta}\Big\{\mathbb{E}\big[\mathrm{d}_{\mathcal{D}^{\alpha,\beta}_T}\big(\mathbf{u}_1,\mathbf{u}_2\big)^{2p}\big]+\mathcal{W}_{p,\mathscr{C}^\alpha_T}(\mu_1,\mu_2)^{4p}\Big\},\end{aligned}$$

so taking $T>0$ deterministic, small enough, independently of u_{0i} and $\widehat{\xi}_i$, ensures that we have

$$\mathbb{E}\big[\mathrm{d}_{\mathcal{D}^{\alpha,\beta}_T}\big(\mathbf{u}_1,\mathbf{u}_2\big)^{2p}\big]^2 \lesssim \|u_{01}-u_{02}\|^{4p}_{C^\alpha}+\mathbb{E}\big[\|\widehat{\xi}_1-\widehat{\xi}_2\|^{4p}\big]+T^{4p\delta}\mathcal{W}_{p,\mathscr{C}^\alpha_T}(\mu_1,\mu_2)^{4p}.$$

We have moreover

$$\|u_1-u_2\|_{\mathscr{C}^\alpha_T}\lesssim\big(1+\|X_1\|_{\mathscr{C}^\alpha_T}\big)\,\mathrm{d}_{\mathcal{D}^{\alpha,\beta}_T}(\mathbf{u}_1,\mathbf{u}_2)+\|X_1-X_2\|_{\mathscr{C}^\alpha_T}\|\mathbf{u}_2\|_{\mathcal{D}^{\alpha,\beta}_T(X_2)},$$

so we obtain from Cauchy-Schwarz inequality that

$$\mathbb{E}\big[\|u_1-u_2\|^p_{\mathscr{C}^\alpha_T}\big]^2\lesssim\big(1+\mathbb{E}[\|X_1\|^{2p}_{\mathscr{C}^\alpha_T}]\big)\,\mathbb{E}\big[\mathrm{d}_{\mathcal{D}^{\alpha,\beta}_T}(\mathbf{u}_1,\mathbf{u}_2)^{2p}\big]+\mathbb{E}\big[\|X_1-X_2\|^{2p}_{\mathscr{C}^\alpha_T}\big]\,\mathbb{E}\big[\|\mathbf{u}_2\|^{2p}_{\mathcal{D}^{\alpha,\beta}_T(X_2)}\big],$$

hence

$$\mathcal{W}_{p,\mathscr{C}_T^\alpha}(\Psi(\mu_1),\Psi(\mu_2)) \lesssim \|u_{01}-u_{02}\|_{C^\alpha}^{4p} + \mathbb{E}\big[\|\widehat{\xi}_1-\widehat{\xi}_2\|_{\mathfrak{N}}^{4p}\big] + T^\delta \mathcal{W}_{p,\mathscr{C}_T^\alpha}(\mu_1,\mu_2).$$

We conclude that equation (1.3) has a unique local solution $\mathbf{u}$ in $\mathcal{P}_p(\mathscr{C}_T^\alpha)$, and that the law $\mathcal{L}(\mathbf{u}) \in \mathcal{P}_p(\mathcal{D}_T^{\alpha,\beta}(X))$ of $\mathbf{u}$ depends continuously on $\widehat{\xi} \in L^{12p}(\Omega,\mathfrak{N})$ and on $u_0 \in C^\alpha$. ▷

We remark that the integrability exponent $12p$ in the condition $\widehat{\xi} \in L^{12p}(\Omega,\mathfrak{N})$ in Proposition 15 comes from both the nonlinearity and the use of the Cauchy-Schwarz inequality when passing from $\mathcal{D}_T^{\alpha,\beta}$ to $\mathscr{C}_T^\alpha$. In the next section we obtain a better exponent $8p$ as the last step is skipped, working directly in $\mathcal{D}_T^{\alpha,\beta}$. For the class of Gaussian noises of Theorem 6 we have $\widehat{\xi} \in L^q(\Omega,\mathfrak{N})$ for all $1 \leq q < \infty$.

4 – Mean field type singular SPDEs

We deal in this section with a large family of mean field type singular SPDEs (1.3). The enhancement of the noise needed to make sense of (1.3) is specific to the mean field setting and described in Section 4.1. The paracontrolled structure needed to make sense of (1.3) is described in Section 4.2. This structure is proved to be stable by a certain solution map to a fixed point equation (4.5) similar to (1.3) where the measure argument is frozen and has a particular structure. The proper statement and proof of item *(a)* of Theorem 1 is done in Section 4.3.

4.1 – Mean field enhancement of the noise. We work here as above with the class of random Gaussian noises specified in Theorem 6. The random field ξ is initially defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. We extend it canonically as a random variable defined on the probability space $\big(\Omega^2,\mathcal{F}^{\otimes 2},\mathbb{P}^{\otimes 2}\big)$ setting

$$\xi(\omega,\varpi) = \xi(\omega).$$

We also define

$$\overline{\xi}(\omega,\varpi) := \xi(\varpi);$$

this is under $\mathbb{P}^{\otimes 2}$ an independent copy of ξ. For a distribution Λ on T^2 and a positive regularization parameter ε set

$$\Lambda^\varepsilon := \Lambda \circ e^{\varepsilon\Delta} \in C^\infty.$$

Recall T_0 stands for the time horizon that we use in our definition of the space of enhanced noises $\mathfrak{N}$ – the interval $[0,T_0]$ is our maximal interval of time. Pick $1 \leq p < \infty$. We define on $\big(\Omega^2,\mathcal{F}^{\otimes 2},\mathbb{P}^{\otimes 2}\big)$ the random variable

$$\overline{X} := \mathscr{L}^{-1}(\overline{\xi}).$$

and denote by

$$\xi \odot \overline{X} \in L^{8p}(\mathbb{P}^{\otimes 2}),$$

the limit of the $\xi^\varepsilon(\omega) \odot \mathscr{L}^{-1}(\overline{\xi}^\varepsilon(\varpi))$ as $\varepsilon > 0$ goes to 0. We have

$$\big\|\big(\xi \odot \overline{X}\big)(\omega,\cdot)\big\|_{L^{8p}(\Omega, C_{T_0}C^{2\alpha-2})} < \infty$$

and

$$\big\|\big(\xi \odot \overline{X}\big)(\cdot,\varpi)\big\|_{L^{8p}(\Omega, C_{T_0}C^{2\alpha-2})} < \infty$$

for $\mathbb{P}$-almost every $\omega \in \Omega$ and $\varpi \in \Omega$. We will use the notation $\overline{\mathbb{E}}$ to denote the expectation operator with respect to ϖ on the product probability space.

17 – Definition. The ***mean field enhancement of the random noise*** ξ *is the random variable*

$$\widehat{\xi}^+(\omega,\varpi) := \Big(\xi(\omega), (\xi\odot X)(\omega), \overline{\xi}(\varpi), \big(\xi\odot\overline{X}\big)(\omega,\varpi)\Big) \in \mathfrak{N}^2,$$

defined on $\big(\Omega^2, \mathcal{F}^2, \mathbb{P}^{\otimes 2}\big)$. *We define on* $(\Omega, \mathcal{F}, \mathbb{P})$ *the random variable*

$$\begin{aligned}(\!|\widehat{\xi}^{+}|\!)_\omega := \|\xi(\omega)\|_{C_{T_0}C^{\alpha-2}} + \big\|\xi^{(2)}(\omega)\big\|_{C_{T_0}C^{2\alpha-2}} \\ + \overline{\mathbb{E}}\big[\|\overline{\xi}(\omega,\cdot)\|^4_{C_{T_0}C^{\alpha-2}}\big]^{\frac14} + \overline{\mathbb{E}}\big[\|(\xi\odot\overline{X})(\omega,\cdot)\|^4_{C_{T_0}C^{2\alpha-2}}\big]^{\frac14}.\end{aligned} \tag{4.1}$$

This is an element of $L^{8p}(\Omega,\mathbf{R})$ *– it actually has moments of any finite order.*

4.2 – Paracontrolled structure for mean field singular SPDEs. The appropriate notion of paracontrolled structure for the study of a large class of mean field singular SPDEs is captured by the following definition.

18 – *Definition. Pick an* L^2 *random variable* $\Lambda : \Omega \to C^\alpha$. *A* C^α***-valued random variable*** v *on* Ω *is said to be* ω***-paracontrolled by*** Λ *if there are some random variables*

$$\delta_z v : \Omega \to C^\beta$$

and

$$\delta_\mu v : \Omega \to L^{\frac43}\big(\Omega, C^\beta\big)$$

and

$$v^\sharp : \Omega \to C^{\alpha+\beta}$$

such that one has

$$v(\omega) = (\delta_z v)(\omega) < \Lambda(\omega) + \overline{\mathbb{E}}\big[(\delta_\mu v)(\omega,\cdot) < \overline{\Lambda}(\cdot)\big] + v^\sharp(\omega) \tag{4.2}$$

for $\mathbb{P}$*-almost all* $\omega\in\Omega$, *and*

$$\|\delta_z v\|_{L^2(\Omega)} + \|\delta_\mu v\|_{L^2(\Omega)} + \|v^\sharp\|_{L^2(\Omega)} < \infty.$$

We simply say that v *is paracontrolled by* Λ. We first check that the datum of a mean field enhancement $\widehat{\xi}^{+}$ of the random noise ξ comes with a natural definition of the product of ξ by a random function $v \in C_TC^\alpha$ with the property that v_t is paracontrolled by X_t for each $0 < t \le T$. To emphasize the fact that we use the paracontrolled structure of v to make sense of that product we write

$$\mathbf{v}_t\xi_t,$$

using a bold letter $\mathbf{v}$. Set then

$$(\mathbf{v}_t\xi_t)(\omega) := v_t(\omega) < \xi_t(\omega) + (\mathbf{v}_t\xi_t)^\sharp(\omega)$$

where

$$\begin{aligned}(\mathbf{v}_t\xi_t)^\sharp(\omega) := \xi_t(\omega) < v_t(\omega) + v_t^{\#}(\omega)\odot\xi_t(\omega) \\ + \mathsf{C}\big((\delta_z v)(\omega), X(\omega), \xi_t(\omega)\big) + \overline{\mathbb{E}}\Big[\mathsf{C}\big((\delta_\mu v)(\omega,\cdot), \overline{X}(\cdot), \xi_t(\omega)\big)\Big] \\ + (\delta_z v)(\omega)\xi_t^{(2)}(\omega) + \overline{\mathbb{E}}\Big[(\delta_\mu v)(\omega,\cdot)\,\big(\xi\odot\overline{X}\big)(\omega,\cdot)\Big].\end{aligned}$$

The proof of the next statement comes from standard continuity estimates on paraproducts and correctors and from Hölder inequality in the expectation $\overline{\mathbb{E}}$; it is left to the reader.

19 – *Proposition. One has* $\mathbb{P}$*-almost surely* $\mathbf{v}\xi \in C_TC^{\alpha-2}$ *and*

$$\|(\mathbf{v}_t\xi_t)^{\#}(\omega)\|_{C^{\alpha+\beta-2}} \lesssim \big(1 + (\!|\widehat{\xi}^{+}|\!)^2_\omega\big)\bigg(\|(\delta_z v)(\omega)\|_{C^\beta} + \overline{\mathbb{E}}\big[\|\delta_\mu v\|^{\frac43}_{C^\beta}\big]^{\frac34} + \|v^{\#}(\omega)\|_{C^{\alpha+\beta}}\bigg).$$

Furthermore, for two enhanced noises $\widehat{\xi}^{1+}, \widehat{\xi}^{2+}$ *in our class, and with* $v^i \in C_TC^\alpha$ *with* v^i_t *paracontrolled by* X^i_t, *for integers* $1 \le i \le 2$, *for each* $0 < t \le T$, *one has*

$$\begin{aligned}&\|(\mathbf{v}^1_t\xi^1_t)^{\#}(\omega) - (\mathbf{v}^2_t\xi^2_t)^{\#}(\omega)\|_{C^{\alpha+\beta-2}} \\ &\lesssim (\star)_{12}(\omega)\bigg(\|\delta_z v^1 - \delta_z v^2\|_{C^\beta} + \overline{\mathbb{E}}\big[\|\delta_\mu v^1 - \delta_\mu v^2\|^{\frac43}_{C^\beta}\big]^{\frac34} + \|v^{1\#} - v^{2\#}\|_{C^{\alpha+\beta}} + (\!|\widehat{\xi}^{+1} - \widehat{\xi}^{+2}|\!)_\omega\bigg),\end{aligned}$$

where

$$(\star)_{12}(\omega) = P\Big(\max_{i\in\{1,2\}}\Big\{(\!|\widehat{\xi}^{+i}|\!)_\omega, \|\delta_z v^i\|_{C^\alpha}, \overline{\mathbb{E}}\big[\|\delta_\mu v^i\|_{C^\alpha}^{\frac{4}{3}}\big]^{\frac{3}{4}}, \|v^{\#i}\|_{C^{\alpha+\beta}}\Big\}\Big),$$

for some quadratic polynomial P.

For a noise $\xi \in C_T C^{\alpha-2}$ in our class of noises we set

$$X := \mathscr{L}^{-1}(\xi) \in \mathscr{C}_T^\alpha.$$

Fix $t > 0$. We prove now that the class of random functions on T^2 paracontrolled by X_t is stable by a certain family of nonlinear functions $f : C^\alpha \times \mathcal{W}_p(C^\alpha) \to C^\alpha$. This comes under the form of a paralinearization formula. Our primary goal is to give a useful description of the random variable $f(v_t, \mathcal{L}(v_t))$ when v_t is paracontrolled by X_t. For that purpose it will be useful to lift any function $f : C^\alpha \times \mathcal{W}_p(C^\alpha) \to C^\alpha$ into a real valued function on $C^\alpha \times L^p(\Omega, \overline{\mathbb{P}}; C^\alpha)$ setting, with a slight abuse of notation,

$$f(v, A) := f\big(v, \mathcal{L}(A)\big),$$

for $A \in L^p(\Omega, \overline{\mathbb{P}}; C_T^\alpha)$. We assume in this work that f depends polynomially on its measure argument

$$f(u, \mu)(z) = \int F\big(u(z), v_1(z), \dots, v_m(z)\big)\, \mu^{\otimes m}(dv_1 \dots dv_m) \tag{4.3}$$

for some integer $m \geq 1$, for a function $F : \mathbf{R}^{m+1} \to \mathbf{R}$ of class C_b^3 – or is a linear combination of such monomials. With $m = 1$, and compared to the long range interaction (3.9) studied in Section 3.3, this function corresponds to a pointwise singular Dirac kernel

$$k(z, z') = \delta_z(z').$$

It will be useful to work on the probability space $(\Omega^{m+1}, \mathcal{F}^{\otimes(m+1)}, \mathbb{P}^{\otimes(m+1)})$ and write

$$(\omega, \omega_1, \dots, \omega_m)$$

for an element of Ω^{m+1}. We set $\overline{\mathbb{E}}^i$ for the expectation operator with respect to the variable ω_i and for $I = (i_1, \dots, i_k)$ a subset of the integer interval $[\![1, m]\!]$ we write $\overline{\mathbb{E}}^I$ for the expectation operator with respect to the variables $(\omega_{i_1}, \dots, \omega_{i_k})$. In those terms, and for $A \in L^p(\Omega, \overline{\mathbb{P}}; C_T^\alpha)$ and $\mu = \mathcal{L}(A)$, one has

$$f(v, \mu)(z) = f(v, A)(z) = \overline{\mathbb{E}}^{[\![1,m]\!]}\Big[F\big(v(z), A(\omega_1)(z), \dots, A(\omega_m)(z)\big)\Big].$$

As $F \in C_b^3 \subset C_b^1$ one has

$$\|F(v, A(\omega_1), \dots, A(\omega_m)\|_{C^\alpha} \lesssim 1 + \|v\|_{C^\alpha} + \sum_{j=1}^m \|A(\omega_j)\|_{C^\alpha},$$

and as $A \in C^\alpha$ is integrable the function $f(v, A)$ on T^2 is indeed an element of C^α. For $i \in [\![1, m]\!]$ we set

$$\partial_i f(v, A)(z) := \overline{\mathbb{E}}^{[\![1,m]\!]}\Big[(\partial_i F)\big(v(z), A(\omega_1)(z), \cdots, A(\omega_m)(z)\big)\Big].$$

20 – *Proposition. Fix $t > 0$ and assume we are given two $L^{8p}(\Omega, \mathcal{D}^\alpha(X_t))$ random variables $(h', h^\sharp)$ and $(k', k^\sharp)$ with corresponding C^α functions h, k on T^2. Then $f(h, k)$ is paracontrolled by X_t in the sense of Definition 18, with*

$$(\delta_z f)(h, k)(\omega) = (\partial_1 f)\big(h(\omega), k\big)h'(\omega)$$

and

$$(\delta_\mu f)(h, k)(\omega, \varpi)$$
$$= \sum_{j=1}^m \overline{\mathbb{E}}^{[\![1,m]\!]\setminus\{j\}}\Big[(\partial_{j+1}F)\Big(h(\omega), k(\omega_1), \cdots, k(\omega_{j-1}), k(\varpi), k(\omega_{j+1}), \cdots, k(\omega_m)\Big)\Big]k'(\varpi),$$

and

$$\|f(h(\omega),k)^{\#}\|_{C^{\alpha+\beta}} \lesssim \Big(1+\|X_t(\omega)\|_{C^\alpha}^2+\overline{\mathbb{E}}\big[\|\overline{X}_t\|_{C^\alpha}^4\big]^{\frac{1}{2}}\Big)$$
$$\times\Big(1+\|h'(\omega)\|_{C^\beta}+\|h^{\#}(\omega)\|_{C^\alpha}+\overline{\mathbb{E}}\big[\|k'\|_{C^\beta}^4\big]^{\frac14}+\overline{\mathbb{E}}\big[\|k^{\#}\|_{C^\alpha}^4\big]^{\frac14}\Big)$$
$$\times\Big(1+\|h'(\omega)\|_{C^\beta}+\|h^{\#}(\omega)\|_{C^{\alpha+\beta}}+\overline{\mathbb{E}}\big[\|k'\|_{C^\beta}^4\big]^{\frac14}+\overline{\mathbb{E}}\big[\|k^{\#}\|_{C^{\alpha+\beta}}^4\big]^{\frac14}\Big).$$

Moreover for $\widehat{\xi}^{+i}\in L^{8p}(\Omega^2,\mathfrak{R}^2)$ *and* h *and* k *in* $L^{8p}(\Omega,\mathcal{D}^\alpha(X_t))$*, for* $1\leq i\leq 2$*, we have*

$$\|f\big(h^1(\omega),k^1\big)^{\#}-f\big(h^2(\omega),k^2\big)^{\#}\|_{C^{\alpha+\beta}}\lesssim(\star)_{12}(\omega)\times$$
$$\Big\{\|X_t^1(\omega)-X_t^2(\omega)\|_{C^\alpha}+\overline{\mathbb{E}}\big[\big\|\overline{X}_t^1-\overline{X}_t^2\big\|_{C^\alpha}^4\big]^{\frac14}+d_{\mathcal{D}^\beta}\big(h^1(\omega),h^2(\omega)\big)+\overline{\mathbb{E}}\big[d_{\mathcal{D}^\beta}\big(k^1,k^2\big)^4\big]^{\frac14}\Big\},\tag{4.4}$$

where

$$(\star)_{12}(\omega)=P\Big(\max_{i\in\{1,2\}}\Big\{\|X_t^i(\omega)\|_{C^\alpha},\,\overline{\mathbb{E}}\big[\|X_t^i\|_{C^\alpha}^4\big]^{\frac14},\,\|h^i(\omega)\|_{\mathcal{D}^\alpha},\,\overline{\mathbb{E}}\big[\|k^i\|_{\mathcal{D}^\alpha}^4\big]^{\frac14}\Big\}\Big),$$

for some polynomial P.

Proof – One has from paralinearisation

$$\begin{aligned}
&F\big(h(\omega),k(\omega_1),\dots,k(\omega_m)\big)\\
&\quad=\partial_1F\big(h(\omega),k(\omega_1),\dots,k(\omega_m)\big)<h(\omega)+\sum_{j=1}^m\partial_{j+1}F\big(h(\omega),k(\omega_1),\dots,k(\omega_m)\big)<k(\omega_j)\\
&\qquad+R_F\big(h(\omega),k(\omega_1),\dots,k(\omega_m)\big)\\
&\quad=\big(\partial_1F\big(h(\omega),k(\omega_1),\dots,k(\omega_m)\big)h'(\omega)\big)<X_t(\omega)\\
&\qquad+\sum_{j=1}^m\Big(\partial_{j+1}F\big(h(\omega),k(\omega_1),\dots,k(\omega_m)\big)k'(\omega_j)\Big)<\overline{X}_t(\omega_j)+R_F+R_0+\sum_{j=1}^mR_j
\end{aligned}$$

where $R_F=R_F\big(h(\omega),k(\omega_1),\cdots,k(\omega_m)\big)\in C^{\alpha+\beta}$ and

$$\begin{aligned}
R_0=&\Big\{\partial_1F\big(h(\omega),k(\omega_1),\dots,k(\omega_m)\big)<\big(h'(\omega)<X_t(\omega)\big)\\
&\qquad-\big(\partial_1F\big(h(\omega),k(\omega_1),\dots,k(\omega_m)\big)h'\big)<X_t(\omega)\Big\}\\
&+\partial_1F\big(h(\omega),k(\omega_1),\cdots,k(\omega_m)\big)<h^{\#}(\omega),\\
R_j=&\Big\{\partial_{j+1}F\big(h(\omega),k(\omega_1),\cdots,k(\omega_m)\big)<\big(k'(\omega_j)<\overline{X}_t(\omega_j)\big)\\
&\qquad-\Big(\partial_{j+1}F\big(h(\omega),k(\omega_1),\cdots,k(\omega_m)\big)k'(\omega_j)\Big)<\overline{X}_t(\omega_j)\Big\}\\
&+\partial_{j+1}F\big(h(\omega),k(\omega_1),\cdots,k(\omega_m)\big)<k^{\#}(\omega_j).
\end{aligned}$$

From classical results in paradifferential calculus we have

$$\|R_F\|_{C^{\alpha+\beta}}\lesssim\|F\|_{C^2}\Big(1+\|h(\omega)\|_{C^\alpha}^2+\sum_{j=1}^m\|k(\omega_j)\|_{C^\alpha}^2\Big)$$

$$\lesssim\Big(1+\|X_t(\omega)\|_{C^\alpha}^2+\sum_{j=1}^m\|\overline{X}(\omega_j)\|_{C^\alpha}^2\Big)$$
$$\times\Big(1+\|h'(\omega)\|_{C^\beta}^2+\|h^{\#}(\omega)\|_{C^\alpha}^2+\sum_{j=1}^m\|k'(\omega_j)\|_{C^\beta}^2+\|k^{\#}(\omega_j)\|_{C^\alpha}^2\Big),$$

and

$$\|R_0\|_{C^{\alpha+\beta}} \lesssim \|\partial_1 F\big(h(\omega),k(\omega_1),\cdots,k(\omega_m)\big)\|_{C^\alpha}\Big(\|h'(\omega)\|_{C^\beta}\|X_t(\omega)\|_{C^\alpha}+\|h^{\#}(\omega)\|_{C^{\alpha+\beta}}\Big)$$

$$\lesssim \Big(1+\|h(\omega)\|_{C^\alpha}+\sum_{j=1}^m\|k(\omega_j)\|_{C^\alpha}\Big)\Big(\|h'\|_{C^\beta}\|X_t(\omega)\|_{C^\alpha}+\|h^{\#}(\omega)\|_{C^{\alpha+\beta}}\Big)$$

$$\lesssim \Big(1+\|X_t(\omega)\|^2_{C^\alpha}+\sum_{j=1}^m\|\overline{X}(\omega_j)\|^2_{C^\alpha}\Big)$$

$$\times\Big(1+\|h'(\omega)\|_{C^\beta}+\|h^{\#}(\omega)\|_{C^\alpha}+\sum_{j=1}^m\|k'(\omega_j)\|_{C^\beta}+\|k^{\#}(\omega_j)\|_{C^\alpha}\Big)$$

$$\times\Big(1+\|h'(\omega)\|_{C^\beta}+\|h^{\#}(\omega)\|_{C^{\alpha+\beta}}+\sum_{j=1}^m\|k'(\omega_j)\|_{C^\beta}+\|k^{\#}(\omega_j)\|_{C^\alpha}\Big),$$

and, for $1\le i\le m$, we have for $\|R_i\|_{C^{\alpha+\beta}}$ the upper bound

$$\Big(1+\|X_t(\omega)\|^2_{C^\alpha}+\sum_{j=1}^m\|\overline{X}(\omega_j)\|^2_{C^\alpha}\Big)$$

$$\times\Big\{1+\|h'(\omega)\|_{C^\beta}+\|h^{\#}(\omega)\|_{C^\alpha}+\sum_{j=1}^m\|k'(\omega_j)\|_{C^\beta}+\|k^{\#}(\omega_j)\|_{C^\alpha}\Big\}$$

$$\times\Big\{1+\|h'(\omega)\|_{C^\beta}+\|h^{\#}(\omega)\|_{C^\alpha}+\|k^{\#}(\omega_i)\|_{C^{\alpha+\beta}}+\sum_{j=1}^m\|k'(\omega_j)\|_{C^\beta}+\|k^{\#}(\omega_j)\|_{C^\alpha}\Big\}.$$

So we have for $\big\|R_F+\sum_{j=0}^m R_j\big\|_{C^{\alpha+\beta}}$ the bound

$$\Big(1+\|X_t(\omega)\|^2_{C^\alpha}+\sum_{j=1}^m\|\overline{X}_t(\omega_j)\|^2_{C^\alpha}\Big)\Big(1+\|h'\|_{C^\beta}+\|h^{\#}\|_{C^\alpha}+\sum_{j=1}^m\|k'(\omega_j)\|_{C^\beta}+\|k^{\#}(\omega_j)\|_{C^\alpha}\Big)$$

$$\times\Big(1+\|h'\|_{C^\beta}+\|h^{\#}\|_{C^{\alpha+\beta}}+\sum_{j=1}^m\|k'(\omega_j)\|_{C^\beta}+\|k^{\#}(\omega_j)\|_{C^{\alpha+\beta}}\Big).$$

Taking the $\overline{\mathbb{E}}^{[\![1,m]\!]}$ expectation one gets

$$f(h(\omega),k)=\big(\partial_1 f(h(\omega),k)h'(\omega)\big)<X_t(\omega)$$

$$+\overline{\mathbb{E}}^{[\![1,m]\!]}\Bigg[\sum_{j=1}^m\Big(\big(\partial_{j+1}F\big)(h(\omega),k(\omega_1),\dots,k(\omega_m))k'(\omega_j)\Big)<\overline{X}_t(\omega_j)\Bigg]+f(h(\omega),k)^{\#}$$

$$=\big(\partial_1 f(h(\omega),k)h(\omega)'\big)<X_t(\omega)+\sum_{j=1}^m$$

$$\overline{\mathbb{E}}\Bigg[\overline{\mathbb{E}}^{[\![1,m]\!]\setminus\{j\}}\Big[\big(\partial_{j+1}F\big)\Big(h(\omega),k(\omega_1),\cdots,k(\omega_{j-1}),k(\varpi),k(\omega_{j+1}),\dots,k(\omega_m)\Big)k'(\varpi)\Big]<\overline{X}_t(\varpi)\Bigg]$$

$$+f(h(\omega),k)^{\#},$$

with

$$
\begin{aligned}
&\|f(h(\omega),k)^{\#}\|_{C^{\alpha+\beta}} \\
&\qquad \lesssim \Big(1+\|X_t(\omega)\|_{C^\alpha}^2+\overline{\mathbb{E}}\big[\|\overline{X}_t\|_{C^\alpha}^4\big]^{\frac12}\Big) \\
&\qquad\quad \times\bigg(1+\|h'(\omega)\|_{C^\beta}+\|h^{\#}(\omega)\|_{C^\alpha}+\overline{\mathbb{E}}\big[\|k'\|_{C^\beta}^4\big]^{\frac14}+\overline{\mathbb{E}}\big[\|k^{\#}\|_{C^\alpha}^4\big]^{\frac14}\bigg) \\
&\qquad\quad \times\bigg(1+\|h'(\omega)\|_{C^\beta}+\|h^{\#}(\omega)\|_{C^{\alpha+\beta}}+\overline{\mathbb{E}}\big[\|k'\|_{C^\beta}^4\big]^{\frac14}+\overline{\mathbb{E}}\big[\|k^{\#}\|_{C^{\alpha+\beta}}^4\big]^{\frac14}\bigg).
\end{aligned}
$$

One proves (4.4) in a similar way. ▷

We fix $4 \leq p < \infty$ and assume from now on that the following Lipschitz condition holds true.

Assumption $(\boldsymbol{A}_f)$ – *There exists a constant L such that for every a_1, a_2 in C^α and b_1, b_2 in $L^p(\Omega; C^\alpha)$ we have*

$$
\|f(a_1,b_1)-f(a_2,b_2)\|_{C^\alpha} \leq L\Big(\|a_1-a_2\|_{C^\alpha}+\overline{\mathbb{E}}\big[\|b_1-b_2\|_{C^\alpha}^p\big]^{\frac1p}\Big).
$$

We proceed as usual in two steps to prove the well-posed character of equation (1.3). We freeze the measure argument in a first step and show that the corresponding equation is well-posed. This is what Proposition 21 below is about. This gives a solution u^μ that depends on the measure argument μ. Another fixed point argument is done in a second step to find a measure such that the law of u^μ coincides with μ. In order to proceed in this way we need to make sure that the fixed measure dynamics is defined on a fixed interval, not on a small interval, as is typically given by fixed point arguments. Assumption ***(B)*** guarantees the long time existence.

Recall from (3.4) the definition of the maps L_c, for $c \in C([0,T_0],\mathbf{R})$, and the existence of functions $c_k \in C([0,T_0],\mathbf{R})$ such that the random variables $\mathsf{L}_{c_k}(\xi_t^{\varepsilon_k})$ are converging in $L^{8p}(\Omega,\mathbf{R})$ to the random variable $\xi \odot \mathscr{L}^{-1}(\xi)$. We emphasize below in the product (4.5) of $f(u,v)$ by ξ the fact that u is seen therein as a paracontrolled function by using the bold notation $\mathbf{u}$.

21 – *Proposition. Fix $0 < T_0 < \infty$. Assume the assumptions **$(\boldsymbol{A}_f$-$\boldsymbol{A}_g$-$\boldsymbol{B})$** hold true. For every $\mathbf{v} \in L^p\big(\Omega, \mathcal{D}_{T_0}^{\alpha,\beta}(X)\big)$ and $u_0 \in C^\alpha$ there exists a positive random time*

$$
T = T\big(\llparenthesis\widehat{\xi}^+\rrparenthesis_\omega, \mathbf{v}, u_0\big) \leq T_0
$$

and a unique solution in $\mathbf{u}_{\widehat{\xi}^+,u_0,\mathbf{v}} \in \mathcal{D}_T^{\alpha,\beta}(X)$ to the equation

$$
(\partial_t - \Delta)u = f(\mathbf{u},\mathbf{v})\,\xi + g(u,v), \tag{4.5}
$$

where $\mathbf{u}$ is $\omega-$paracontrolled by X with null δ_μ derivative. This random solution $\mathbf{u}_{\widehat{\xi}^+,u_0,\mathbf{v}}(\omega)$ satisfies the local Lipschitz continuity property

$$
d_{\mathcal{D}_T^{\alpha,\beta}}\big(\mathbf{u}_{\widehat{\xi}_1^+,u_0,\mathbf{v}_1}(\omega), \mathbf{u}_{\widehat{\xi}_2^+,u_0,\mathbf{v}_2}(\omega)\big) \lesssim_\omega \|u_{01}-u_{02}\|_{C^\alpha} + \overline{\mathbb{E}}\big[\|\mathbf{v}_1-\mathbf{v}_2\|_{L^p(\Omega,\mathcal{D}_T^{\alpha,\beta})}\big] + \llparenthesis\widehat{\xi}_1^+-\widehat{\xi}_2^+\rrparenthesis_\omega. \tag{4.6}
$$

The random function $u(\omega) \in \mathscr{C}_T^\alpha$ associated with $\mathbf{u}_{\widehat{\xi}^+,u_0,\mathbf{v}}$ is the limit in probability of the solutions u^{ε_k} of the equations

$$
(\partial_t - \Delta)u^{\varepsilon_k} = f(u^{\varepsilon_k},v)\,\xi_t^{\varepsilon_k} + g(u^{\varepsilon_k},v) - c_k(t)(f\partial_1 f)(u^{\varepsilon_k},v), \tag{4.7}
$$

with initial condition u_0.

We should more properly write $u(\omega), u'(\omega), u^\sharp(\omega)$ rather than just $u, u', u^\sharp$. Also the randomness in $\mathbf{u}_{\widehat{\xi}^+,u_0,\mathbf{v}}(\omega)$ only occurs via $\widehat{\xi}^+(\omega)$.

Proof – Rewrite equation (4.5) as the fixed point equation

$$
u_t = P_t u_0 + \int_0^t P_{t-s}\big(f(\mathbf{u}_s,\mathbf{v}_s)\xi_t + g(u_s,v_s)\big)\,ds.
$$

We get from Lemma 19 and Lemma 20 that $f(\mathbf{u}_s,\mathbf{v}_s)\xi_t + g(u_s,v_s)$ is for each s an element of $\mathcal{D}^\alpha(\xi_s)$ with Gubinelli derivative $f(\mathbf{u}_s,\mathbf{v}_s)$ and remainder $\big(f(\mathbf{u}_s,\mathbf{v}_s)\xi\big)^{\#} + g(u_s,v_s)$. With

Proposition 9 in mind we check that $f(u,v)\in\mathscr{C}_T^\alpha$ and $\big(f(\mathbf{u}_s,\mathbf{v}_s)\xi\big)^{\#}+g(u_s,v_s)$ satisfies (3.7). Recall from (4.1) the definition of the mixed pathwise/averaged random variable $(\!|\widehat{\xi}^+|\!)_\omega$. Take $\mathbf{u}\in\mathcal{D}_T^{\alpha,\beta}(X)$. First one has

$$\begin{aligned}\|f(u,v)\|_{\mathscr{C}_T^\alpha} &\lesssim 1+\|u\|_{\mathscr{C}_T^\alpha}+\overline{\mathbb{E}}\big[\|v\|_{\mathscr{C}_T^\alpha}\big]\\ &\lesssim \Big(1+\|X\|_{\mathscr{C}_T^\alpha}+\overline{\mathbb{E}}\big[\|\overline{X}(\omega)\|^2_{\mathscr{C}_T^\alpha}\big]^{\frac12}\Big)\Big(1+\|u\|_{\mathcal{D}_T^{\alpha,\beta}}+\overline{\mathbb{E}}\big[\|v\|^2_{\mathcal{D}_T^{\alpha,\beta}}\big]^{\frac12}\Big)\\ &\lesssim \big(1+(\!|\widehat{\xi}^+|\!)_\omega\big)\Big(1+\|\mathbf{u}\|^2_{\mathcal{D}_T^{\alpha,\beta}}+\overline{\mathbb{E}}\big[\|v\|^4_{\mathcal{D}_T^{\alpha,\beta}}\big]^{\frac14}\Big)\end{aligned}$$

Second, combining the estimates from Lemmas 19 and 20 one gets at some fixed time t the estimates

$$\|(f(\mathbf{u},\mathbf{v})\xi)^{\#}\|_{C^{\alpha+\beta-2}}\lesssim\big(1+(\!|\widehat{\xi}^+|\!)^2_\omega\big)\Big(\|\delta_z f(u,v)\|_{C^\beta}+\overline{\mathbb{E}}\big[\|\delta_\mu f(u,v)\|^{\frac43}_{C^\beta}\big]^{\frac34}+\|f(\mathbf{u},\mathbf{v})^{\#}\|_{C^{\alpha+\beta}}\Big)$$

$$\begin{aligned}&\lesssim\big(1+(\!|\widehat{\xi}^+|\!)^2_\omega\big)\Big\{\Big(1+\|u\|_{C^\alpha}+\overline{\mathbb{E}}\big[\|v\|_{C^\alpha}\big]\Big)\|u'\|_{C^\beta}\\ &\qquad+\Big(1+\|u\|_{C^\alpha}+\overline{\mathbb{E}}\big[\|v\|^2_{C^\alpha}\big]^{\frac12}\Big)\overline{\mathbb{E}}\big[\|v'\|^4_{C^\beta}\big]^{\frac14}+\|f(\mathbf{u},\mathbf{v})^{\#}\|_{C^{\alpha+\beta}}\Big\}\\ &\lesssim\big(1+(\!|\widehat{\xi}^+|\!)^4_\omega\big)\Big(1+\|u'\|_{C^\beta}+\|u^{\#}\|_{C^\alpha}+\overline{\mathbb{E}}\big[\|v'\|^4_{C^\beta}\big]^{\frac14}+\overline{\mathbb{E}}\big[\|v^{\#}\|^4_{C^\alpha}\big]^{\frac14}\Big)\\ &\qquad\times\Big(1+\|u'\|_{C^\beta}+\|u^{\#}\|_{C^{\alpha+\beta}}+\overline{\mathbb{E}}\big[\|v'\|^4_{C^\beta}\big]^{\frac14}+\overline{\mathbb{E}}\big[\|v^{\#}\|^4_{C^{\alpha+\beta}}\big]^{\frac14}\Big),\end{aligned}$$

so

$$\sup_{t\in(0,T]}t^{\beta/2}\|(f(\mathbf{u}_t,\mathbf{v}_t)\xi_t)^{\#}\|_{C^{\alpha+\beta-2}}\lesssim\big(1+(\!|\widehat{\xi}^+|\!)^4_\omega\big)\Big(1+\|\mathbf{u}\|^2_{\mathcal{D}_T^{\alpha,\beta}}+\overline{\mathbb{E}}\big[\|\mathbf{v}\|^4_{\mathcal{D}_T^{\alpha,\beta}}\big]^{\frac12}\Big).$$

We have also

$$\begin{aligned}\sup_{t\in(0,T]}t^{\beta/2}\|g(u_t,v_t)\|_{C^{\alpha+\beta-2}}&\lesssim\sup_{t\in(0,T]}t^{\beta/2}\Big(1+\|u_t\|_{C^\alpha}+\overline{\mathbb{E}}\big[\|v_t\|^2_{C^\alpha}\big]^{\frac12}\Big)\\ &\lesssim\big(1+(\!|\widehat{\xi}^+|\!)_\omega\big)\Big(1+\|\mathbf{u}\|^2_{\mathcal{D}_T^{\alpha,\beta}}+\overline{\mathbb{E}}\big[\|\mathbf{v}\|^4_{\mathcal{D}_T^{\alpha,\beta}}\big]^{\frac12}\Big),\end{aligned}$$

so we have in the end the pathwise estimate

$$\sup_{t\in(0,T]}t^{\beta/2}\|(f(\mathbf{u}_t,\mathbf{v}_t)\xi_t)^{\#}+g(u_t,v_t)\|_{C^{\alpha+\beta-2}}\lesssim\big(1+(\!|\widehat{\xi}^+|\!)^4_\omega\big)\Big(1+\|\mathbf{u}\|^2_{\mathcal{D}_T^{\alpha,\beta}}+\overline{\mathbb{E}}\big[\|\mathbf{v}\|^4_{\mathcal{D}_T^{\alpha,\beta}}\big]^{1/2}\Big).$$

It follows from Proposition 9 that the map

$$\Phi_{\widehat{\xi}^+,u_0,\mathbf{v}}:\mathcal{D}_T^{\alpha,\beta}(X(\omega))\to\mathcal{D}_T^{\alpha,\beta}(X(\omega))$$

which associates to $\mathbf{u}\in\mathcal{D}_T^{\alpha,\beta}(X(\omega))$ the solution w of the equation

$$(\partial_t-\Delta)w=f(\mathbf{u},\mathbf{v})\xi+g(u,v)$$

with initial condition $w_0=u_0$, is well-defined and satisfies the bound

$$\big\|\Phi_{\widehat{\xi}^+,u_0,\mathbf{v}}(\mathbf{u})\big\|_{\mathcal{D}_T^{\alpha,\beta}}\lesssim\|u_0\|_{C^\alpha}+T^{(\alpha-\beta)/2}\big(1+(\!|\widehat{\xi}^+|\!)^4_\omega\big)\Big(1+\|\mathbf{u}\|^2_{\mathcal{D}_T^{\alpha,\beta}}+\overline{\mathbb{E}}\big[\|\mathbf{v}\|^4_{\mathcal{D}_T^{\alpha,\beta}}\big]^{1/2}\Big).$$

Recall $4\le p<\infty$. One can then find some random positive constants

$$M=M\Big(\|u_0\|_\alpha\vee\overline{\mathbb{E}}\big[\|\mathbf{v}\|^p_{\mathcal{D}_T^{\alpha,\beta}}\big]\vee(\!|\widehat{\xi}^+|\!)_\omega\Big)$$

and

$$T=T\Big(\|u_0\|_\alpha\vee\overline{\mathbb{E}}\big[\|\mathbf{v}\|^p_{\mathcal{D}_T^{\alpha,\beta}}\big]\vee(\!|\widehat{\xi}^+|\!)_\omega\Big)$$

so that the map $\Phi_{\widehat{\xi}^+,u_0,\mathbf{v}}$ sends the ball

$$\Big\{\mathbf{u}\in\mathcal{D}_T^{\alpha,\beta}(X(\omega))\,;\,\|\mathbf{u}\|_{\mathcal{D}_T^{\alpha,\beta}}\leq M\Big\}$$

into itself. Now, given $\widehat{\xi}_1^+,\widehat{\xi}_2^+$ in $L^{8p}(\Omega^2,\mathfrak{N}^2)$, two initial conditions u_{01},u_{02} in C^α and $\mathbf{v}_1,\mathbf{v}_2$ in $L^p\big(\Omega,\mathcal{D}_{T_0}^{\alpha,\beta}(X(\omega))\big)$, we define a random constant

$$M'_\omega=M\Big(\max_{i=1,2}\Big\{\|u_{0i}\|_{C^\alpha}\vee\overline{\mathbb{E}}\big[\|\mathbf{v}_i\|^p_{\mathcal{D}_{T_0}^{\alpha,\beta}}\big]\vee(\!|\widehat{\xi}_i^+|\!)_\omega\Big\}\Big).$$

For $\|\mathbf{u}\|_{\mathcal{D}_T^{\alpha,\beta}}\leq M'_\omega$, Proposition 9 tells us that

$$\begin{aligned}&\mathrm{d}_{\mathcal{D}_T^{\alpha,\beta}}\big(\Phi_{\widehat{\xi}_1^+,u_{01},\mathbf{v}_1}(\mathbf{u}_1),\Phi_{\widehat{\xi}_2^+,u_{02},\mathbf{v}_2}(\mathbf{u}_2)\big)\\&\lesssim_{M'_\omega}\|u_{01}-u_{02}\|_{C^\alpha}+T^{(\alpha-\beta)/2}\Big\{\mathrm{d}_{\mathcal{D}_T^{\alpha,\beta}}\big(\mathbf{u}_1,\mathbf{u}_2\big)+\overline{\mathbb{E}}\big[\|\mathbf{v}_1-\mathbf{v}_2\|_{L^p(\Omega,\mathcal{D}_T^{\alpha,\beta})}\big]+(\!|\widehat{\xi}_1^+-\widehat{\xi}_2^+|\!)_\omega\Big\}.\end{aligned}$$

So choosing

$$T\Big(\max_{i=1,2}\Big\{\|u_{0i}\|_{C^\alpha}\vee\overline{\mathbb{E}}\big[\|\mathbf{v}_i\|_{L^p(\Omega,\mathcal{D}_T^{\alpha,\beta})}\big]\vee(\!|\widehat{\xi}_i^+|\!)_\omega\Big\}\Big)$$

small enough ensures that the map $\Phi_{\widehat{\xi}^+,u_0,\mu}$ has a unique fixed point $\mathbf{u}_{\widehat{\xi}^+,u_0,\mu}(\omega)$ which satisfies the local Lipschitz property

$$\mathrm{d}_{\mathcal{D}_T^{\alpha,\beta}}\big(\mathbf{u}_{\widehat{\xi}_1^+,u_0,\mathbf{v}_1}(\omega),\mathbf{u}_{\widehat{\xi}_2^+,u_0,\mathbf{v}_2}(\omega)\big)\lesssim_{M'_\omega}\|u_{01}-u_{02}\|_{C^\alpha}+\overline{\mathbb{E}}\big[\|\mathbf{v}_1-\mathbf{v}_2\|_{L^p(\Omega,\mathcal{D}_T^{\alpha,\beta})}\big]+(\!|\widehat{\xi}_1^+-\widehat{\xi}_2^+|\!)_\omega.$$

Recall that $(\xi,X\odot\xi)\in\mathfrak{N}$ is the limit in any $L^q(\Omega,\mathbb{P})$ space, $1\leq q<\infty$, of the sequence of enhanced noises

$$\big(\xi_t^{\varepsilon_k},\xi_t^{\varepsilon_k}\odot X_n-c_k\big)=:(\xi_t^{\varepsilon_k},(\xi\odot X)_n)$$

for some diverging function c_k, and that $\xi\odot\overline{X}$ is the limit in $L^q(\Omega^2,\mathbb{P}^{\otimes 2})$ of $\xi^{\varepsilon_k}\odot\overline{X}^{\varepsilon_k}$. We then have

$$\begin{aligned}f(\mathbf{u}_n,\mathbf{v})\xi_t^{\varepsilon_k}+g(u^{\varepsilon_k},v)&=f(u^{\varepsilon_k},v)<\xi_t^{\varepsilon_k}+\xi_t^{\varepsilon_k}<f(u^{\varepsilon_k},v)+f(u^{\varepsilon_k},v)^{\#}\odot\xi_t^{\varepsilon_k}\\&\quad+\mathsf{C}\big(\delta_zf(u^{\varepsilon_k},v),X_n,\xi_t^{\varepsilon_k}\big)+\overline{\mathbb{E}}\Big[\mathsf{C}\big(\delta_\mu f(u^{\varepsilon_k},v),\xi_t\overline{X}^{\varepsilon_k},\xi_t^{\varepsilon_k}\big)\Big]\\&\quad+\delta_zf(u^{\varepsilon_k},v)\big(X\odot\xi\big)^{\varepsilon_k}+\overline{\mathbb{E}}\Big[\delta_\mu f(u^{\varepsilon_k},v)\big(\xi_t^{\varepsilon_k}\odot\xi_t\overline{X}^{\varepsilon_k}\big)\Big]\\&\quad+g(u^{\varepsilon_k},v)\\&=f(u^{\varepsilon_k},v)\xi_t^{\varepsilon_k}-c_k(f\partial_1f)(u^{\varepsilon_k})+g(u^{\varepsilon_k},v),\end{aligned}$$

so the function u^{ε_k} is a solution of the renormalized equation

$$(\partial_t-\Delta)u^{\varepsilon_k}=f(u^{\varepsilon_k},v)\xi_t^{\varepsilon_k}-c_k(f\partial_1f)(u^{\varepsilon_k})+g(u^{\varepsilon_k},v).$$

As we know that the solution $\mathbf{u}_{\widehat{\xi}^+,u_0,\mathbf{v}}\in\mathcal{D}_T^{\alpha,\beta}(X)$ is a continuous function of $\widehat{\xi}^+\in\mathfrak{N}^2$, and since $\widehat{\xi}_n^+$ converges to $\widehat{\xi}^+$ in probability, we see that $\mathbf{u}_{\widehat{\xi}^+,u_0,\mathbf{v}}$ is the limit in probability in $\mathcal{D}_T^{\alpha,\beta}$ of the sequence $(u^{\varepsilon_k},f(u^{\varepsilon_k},v))\in\mathcal{D}_T^{\alpha,\beta}(X_n)$. ▷

The following statement is the analogue of Lemma 12 in the present setting.

22 – Lemma. For every $R>0$, the solution $\mathbf{u}_{\widehat{\xi}^+,u_0,\mathbf{v}}(\omega)$ to equation (4.5) is defined up to the time

$$T^*=\inf\big\{t\geq 0,\quad\|u(t)\|_{L^\infty}\geq R\big\}.$$

Proof – The proof is a direct adaptation of the proof of Lemma 12. We give the details for the interested reader. To lighten the notations we write $\mathbf{u}$ for $\mathbf{u}_{\widehat{\xi}^+,u_0,\mathbf{v}}(\omega)$. Recall that the local well-posedness time from the Picard iteration argument for $\mathbf{u}$ reads as a decreasing function

$$T=T\big(u_0,\widehat{\xi}^+,\overline{\mathbb{E}}[\|\mathbf{v}\|^p_{\mathcal{D}_T^{\alpha,\beta}}]\big).$$

If we fix $\widehat{\xi}^+$ and v, one ends up with a function $T=T\big(\|u_0\|_{C^\alpha}\big)$, so that it is sufficient to obtain a bound for $\|u\|_{C_TC^\alpha}$ that depends only on the constant R. As $\|u\|_{C_TC^\alpha}\lesssim_{\widehat{\xi}^+}\|\mathbf{u}\|_{\mathcal{D}_T^{\alpha,\beta}}$

we actually show that

$$\|u\|_{\mathcal{D}_T^{\alpha,\beta}} \lesssim_{,\widehat{\xi}^+} 1 + \|u\|^2_{C_T L^\infty} + \mathbb{E}\big[\|v\|^4_{C_T L^\infty}\big]^{1/2}.$$

We proceed as follows. Since $u'_t = f(u_t, v_t)$, we have

$$\|u'\|_{\mathscr{C}_T^\beta} \lesssim 1 + \|u\|_{\mathscr{C}_T^\beta} + \|v\|_{\mathscr{C}_T^\beta}.$$

Yet since $u = u' \prec X + u^\#$ where u' appears as an L^∞ contribution we have

$$\|u'\|_{\mathscr{C}_T^\beta} \lesssim_{,\widehat{\xi}^+,R} 1 + \|u^\#\|_{\mathscr{C}_T^\beta} + \|v^\#\|_{\mathscr{C}_T^\beta}.$$

We now use the fact that

$$(\partial_t - \Delta)u^\# = \Phi^\# \tag{4.8}$$

where

$$\Phi^\# = \big(f(\mathbf{u},\mathbf{v})\xi - f(u,v) \prec \xi\big) + g(u,v).$$

The refined paralinearization lemma C.1 from [11] ensures here that

$$\begin{aligned}
&\big\|F\big(u' \prec X + u^\#, v' \prec \overline{X} + v^\#\big) - \\
&\qquad \nabla f\big(u' \prec X + u^\#, v \prec \overline{X} + v^\#\big) \prec \big(u' \prec X + u^\#, v' \prec \overline{X} + v^\#\big)\big\|_{\alpha+\beta} \\
&\lesssim \big(1 + \|u' \prec X\|^2_{C^\alpha} + \|v' \prec \overline{X}\|^2_{C^\alpha} + \|u^\#\|^2_{L^\infty} + \|v^\#\|^2_{L^\infty}\big)\big(1 + \|u^\#\|_{C^{\alpha+\beta}} + \|v^\#\|_{C^{\alpha+\beta}}\big) \\
&\lesssim \big(1 + \|X\|^2_{C^\beta} + \|\overline{X}\|^2_{C^\alpha}\big)\big(1 + \|u\|^2_{L^\infty} + \|v\|^2_{L^\infty}\big)\big(1 + \|u^\#\|_{C^{\alpha+\beta}} + \|v^\#\|_{C^{\alpha+\beta}}\big),
\end{aligned}$$

so that using continuity relation 3.3 and estimate from Definition 7

$$\begin{aligned}
\|\Phi^\#\|_{C^{\alpha+\beta-2}} &\lesssim \overline{\mathbb{E}}\Big[\big(1 + \|\widehat{\xi}^+\|^3\big)\big(1 + \|u\|^2_{C_T L^\infty} + \|v\|^2_{C_T L^\infty}\big) \\
&\qquad \times \big(1 + \|u\|_{\mathscr{C}_T^\alpha} + \|v\|_{\mathscr{C}_T^\alpha} + \|u^\#\|_{C^{\alpha+\beta}} + \|v^\#\|_{C^{\alpha+\beta}}\big)\Big] \\
&\lesssim \Big(1 + (\!|\widehat{\xi}^+|\!)^3_\omega\Big)\Big(1 + \|u\|^2_{C_T L^\infty} + \mathbb{E}\big[\|v\|^4_{C_T L^\infty}\big]^{1/2}\Big) \\
&\qquad \times \Big(\|u^\#\|_{\mathscr{C}_T^\alpha} + \|u^\#\|_{C^{\alpha+\beta}} + \mathbb{E}\big[\|v^\#\|^4_{\mathscr{C}_T^\alpha}\big]^{1/4} + \mathbb{E}\big[\|v^\#\|^4_{C^{\alpha+\beta}}\big]^{1/4}\Big)
\end{aligned}$$

The Schauder estimates from Lemma 5.3 of [11] ensure that

$$\sup_{0<t<T} t^{\beta/2}\|u^\#\|_{C^{\alpha+\beta}} \lesssim_{u_0} 1 + \sup_{0<t<T} t^{\beta/2}\|\Phi^\#\|_{C^{\alpha+\beta-2}}, \tag{4.9}$$

and

$$\|u^\#\|_{\mathscr{C}_T^\alpha} \lesssim_{u_0} 1 + \sup_{0<t<T} t^{\beta/2}\|\Phi^\#\|_{C^{\alpha+\beta-2}}, \tag{4.10}$$

so we have

$$\sup_{0<t\le T} t^{\beta/2}\|\Phi^\#\|_{\alpha+\beta-2} \lesssim \Big(1 + \|u\|^2_{C_T L^\infty} + \mathbb{E}\big[\|v\|^4_{C_T L^\infty}\big]^{1/2}\Big) \tag{4.11}$$

$$\times \Big(1 + \sup_{0<t\le T} t^{\beta/2}\|\Phi^\#\|_{C^{\alpha+\beta-2}} + \overline{\mathbb{E}}\big[\|v^\#\|^4_{\mathscr{C}_T^\alpha}\big]^{1/4} + \overline{\mathbb{E}}\big[\|v^\#\|^4_{C^{\alpha+\beta}}\big]^{1/4}\Big). \tag{4.12}$$

We use again a scaling argument to isolate the $\Phi^\#$ terms. Let

$$(\Lambda^\lambda u)(t,x) := u(\lambda^2 t, \lambda x)$$

and

$$\mathsf{T}^2_\lambda = \big(\mathbf{R}/(2\pi\lambda^{-1}\mathbf{Z})\big)^2.$$

We have

$$(\partial_t - \Delta)\circ\Lambda^\lambda = \lambda^2\Lambda^\lambda\circ(\partial_t - \Delta)$$

and

$$\xi^\lambda := \lambda^{2-\alpha}\Lambda^\lambda\xi, \quad \|\xi^\lambda\|_{\alpha-2} \simeq \|\xi\|_{C^{\alpha-2}},$$

a deterministic estimate, and

$$u^\lambda := \Lambda^\lambda u$$

is a solution of the equation

$$(\partial_t - \Delta)u^\lambda = \lambda^\alpha f(\mathbf{u}^\lambda, \mathbf{v}^\lambda)\xi^\lambda + g(u^\lambda, v^\lambda).$$

It follows from the estimate (4.11) that we have

$$\sup_{0\le t\le T/\lambda^2} t^{\beta/2}\|\Phi^{\#,\lambda}\|_{C^{\alpha+\beta-2}} \lesssim_{\widehat{\xi}^+} \lambda^\alpha\Big(1 + \|u\|^2_{C_TL^\infty} + \mathbb{E}\big[\|v\|^2_{C_TL^\infty}\big]^{1/2}\Big)$$
$$\times\Big(1 + \sup_{0\le t\le T/\lambda^2} t^{\beta/2}\|\Phi^{\#,\lambda}\|_{C^{\alpha+\beta-2}} + \mathbb{E}\big[\|v^\#\|^4_{\mathscr{C}^\alpha_T}\big]^{1/4} + \mathbb{E}\big[\|v^\#\|^4_{C^{\alpha+\beta}}\big]^{1/4}\Big),$$

so choosing λ small enough we finally get

$$\sup_{0\le t\le T} t^{\beta/2}\|\Phi^{\#}\|_{C^{\alpha+\beta-2}} \lesssim_{\widehat{\xi}^+} \Big(1 + \|u\|^2_{C_TL^\infty} + \mathbb{E}\big[\|v\|^2_{C_TL^\infty}\big]^{1/2}\Big)$$
$$\times\Big(1 + \mathbb{E}\big[\|v^\#\|^4_{\mathscr{C}^\alpha_T}\big]^{1/4} + \mathbb{E}\big[\|v^\#\|^4_{C^{\alpha+\beta}}\big]^{1/4}\Big).$$

In the end we obtain from Proposition 4.9 and Proposition 4.10 the estimate

$$\|u^\#\|_{\mathscr{C}^\alpha_T} + \sup_{0\le t\le T} t^{\beta/2}\|u^\#\|_{C^{\alpha+\beta}} \lesssim_{\widehat{\xi}^+} \Big(1 + \|u\|^2_{C_TL^\infty} + \mathbb{E}\big[\|v\|^2_{C_TL^\infty}\big]^{1/2}\Big)$$
$$\times\Big(1 + \mathbb{E}\big[\|v^\#\|^4_{\mathscr{C}^\alpha_T}\big]^{1/4} + \mathbb{E}\big[\|v^\#\|^4_{C^{\alpha+\beta}}\big]^{1/4}\Big),$$

▷

*23 – Proposition. Under assumptions **(A_f-A_g-B)**, if $\|u_0\|_{L^\infty} \le C_0$ then $\mathbf{u}_{\widehat{\xi}^+,u_0,\mathbf{v}}$ is defined globally in time and $\|\mathbf{u}\|_{\mathcal{D}^{\alpha,\beta}_{\mathbf{T}}}(\omega)$ has moments of order p.*

Proof – The global in time existence is a direct consequence of the explosion criterion of Lemma 14 and the maximum principle applied to the solution u^{ε_k} of the renormalized equation (4.7). Following what is done in the proof of the Proposition 22 we have an estimate of the form

$$\|u\|_{\mathcal{D}^{\alpha,\beta}_T} \lesssim_{\widehat{\xi}^+,u_0} 1 + \|u\|^2_{C_TL^\infty} \lesssim_{\widehat{\xi}^+,u_0} 1 + C_0$$

with an implicit multiplicative constant that is polynomial function in $(\!|\widehat{\xi}^+|\!)_\omega$ of degree 3. ▷

4.3 – Solving equation (1.3). The proof of well-posedness of equation (1.3) requires a second fixed point which is the object of the next statement. We fix as above $4 \le p < \infty$.

*24 – Theorem. We assume that the assumptions **(A_f-A_g-B)** hold true. There exists a positive deterministic positive time $T \le T_0$ with the following property.*

- *For every $u_0 \in C^\alpha$ such that $\|u_0\|_{L^\infty} \le C_0$ there exists a unique solution $\mathbf{u} = (u', u^\sharp)$ to the mean field equation (1.3) in $L^p(\Omega, \mathcal{D}^{\alpha,\beta}_T(X))$. The law $\mathcal{L}(\mathbf{u}) \in \mathcal{P}_p(\mathcal{D}^{\alpha,\beta}_T(X))$ of $\mathbf{u}$ depends continuously on $\widehat{\xi}^+ \in L^{8p}(\Omega^2, \mathfrak{N}^2)$ and $u_0 \in C^\alpha$.*
- *Write $u = u' < X + u^\sharp$. The function $u \in \mathscr{C}^\alpha_T$ is the limit in probability of the family of solutions of the renormalized equations*

$$(\partial_t - \Delta)u^{\varepsilon_k} = f\big(u^{\varepsilon_k}, \mathcal{L}(u^{\varepsilon_k}(t))\big)\xi^{\varepsilon_k}_t - c_k(t)(f\partial_1 f)\big(u^{\varepsilon_k}, \mathcal{L}(u^{\varepsilon_k}(t))\big) + g\big(u^{\varepsilon_k}, \mathcal{L}(u^{\varepsilon_k}(t))\big).$$

Proof – Write here $\mathbf{u}^{\mathbf{v}}_{\widehat{\xi}^+,u_0}$ for $\mathbf{u}_{\widehat{\xi}^+,u_0,\mathbf{v}}$. We define from Proposition 21 a map $\Psi_{\widehat{\xi}^+,u_0}$ from $L^p(\Omega, \mathcal{D}^{\alpha,\beta}_T(X))$ into itself setting

$$\Psi_{\widehat{\xi}^+,u_0}(\mathbf{v}) = \mathbf{u}^{\mathbf{v}}_{\widehat{\xi}^+,u_0}.$$

One has from Proposition 9

$$\|\mathbf{u}^{\mathbf{v}}_{\widehat{\xi}^+,u_0}\|_{\mathcal{D}^{\alpha,\beta}_T} \lesssim \|u_0\|_{C^\alpha} + T^\delta\big(1+(\!|\widehat{\xi}^+|\!)^4_\omega\big)\Big(1+\|\mathbf{u}\|^2_{\mathcal{D}^{\alpha,\beta}_T} + \overline{\mathbb{E}}\big[\|\mathbf{v}\|^4_{\mathcal{D}^{\alpha,\beta}_T}\big]^{1/2}\Big)$$

$$\lesssim \|u_0\|_{C^\alpha} + T^\delta\big(1+(\!|\widehat{\xi}^+|\!)^4_\omega\big)\bigg(\|\mathbf{u}\|^{\frac12}_{\mathcal{D}^{\alpha,\beta}_T}\Big\{1+\|u_0\|_{C^\alpha}+\overline{\mathbb{E}}\big[\|\mathbf{v}\|^4_{\mathcal{D}^{\alpha,\beta}_T}\big]^{\frac12}\Big\}^{\frac32} + \overline{\mathbb{E}}\big[\|\mathbf{v}\|^4_{\mathcal{D}^{\alpha,\beta}_T}\big]^{\frac12}\bigg).$$

Integrating and using Cauchy-Schwarz inequality we get

$$\mathbb{E}\big[\|\mathbf{u}^{\mathbf{v}}_{\widehat{\xi}^+,u_0}\|^p_{\mathcal{D}^{\alpha,\beta}_T}\big]^2 \lesssim \|u_0\|^{2p}_{C^\alpha} + T^{2p\delta}\big(1+\mathbb{E}\big[(\!|\widehat{\xi}^+|\!)^{8p}\big]\big)\bigg\{\mathbb{E}\big[\|\mathbf{u}\|^p_{\mathcal{D}^{\alpha,\beta}_T}\big]\Big(1+\|u_0\|_{C^\alpha}+\overline{\mathbb{E}}\big[\|\mathbf{v}\|^4_{\mathcal{D}^{\alpha,\beta}_T}\big]^{\frac12}\Big)^{3p} + \overline{\mathbb{E}}\big[\|\mathbf{v}\|^p_{\mathcal{D}^{\alpha,\beta}_T}\big]^2\bigg\}.$$

So for $T = T\big(\overline{\mathbb{E}}\big[\|\mathbf{v}\|^p_{\mathcal{D}^{\alpha,\beta}_T}\big]\big)$ sufficiently small we have

$$\mathbb{E}\big[\|\mathbf{u}\|^p_{\mathcal{D}^{\alpha,\beta}_T}\big]^{\frac1p} \lesssim \|u_0\|_{C^\alpha} + T^\delta\Big(1+\mathbb{E}\big[(\!|\widehat{\xi}^+|\!)^{8p}\big]^{\frac{1}{2p}}\Big)\,\mathbb{E}\big[\|\mathbf{v}\|^p_{\mathcal{D}^{\alpha,\beta}_T}\big]^{\frac1p}.$$

Pick

$$A > C_0^2 \vee 2\mathbb{E}\big[(\!|\widehat{\xi}^+|\!)^{8p}\big].$$

For M sufficiently big and $T = T(M,A)$ even smaller, for every $u_0\in C^\alpha$ with $\|u_0\|_{C^\alpha}\le A$ the map $\Psi_{\widehat{\xi}^+,u_0}$ sends the ball

$$\Big\{\mathbf{v}\in L^p(\Omega,\mathcal{D}^{\alpha,\beta}_T(X))\,;\,\|\mathbf{v}\|_{L^p(\Omega,\mathcal{D}^{\alpha,\beta}_T)}\le M\Big\}$$

into itself. Now pick $\mathbf{v}_1,\mathbf{v}_2$ in $L^p(\Omega,\mathcal{D}^{\alpha,\beta}_T(X))$, two initial conditions u_{01},u_{02} in C^α and $\widehat{\xi}^+_1,\widehat{\xi}^+_2$ in $L^{8p}(\Omega^2,\mathfrak{N}^2)$ such that one has

$$\mathbb{E}\big[(\!|\widehat{\xi}^+_i|\!)^{8p}\big]\vee\|u_{0i}\|_{C^\alpha}\le A,\qquad \overline{\mathbb{E}}\big[\|\mathbf{v}_i\|^p_{\mathcal{D}^{\alpha,\beta}_T}\big]\le M,$$

for $1\le i\le 2$. Write $\mathbf{u}_i$ for $\Phi_{\widehat{\xi}^+_i,u_{0i}}(\mathbf{v}_i)$ and define the random variable

$$R_\omega := (\!|\widehat{\xi}^+_1|\!)_\omega + (\!|\widehat{\xi}^+_2|\!)_\omega.$$

We have

$$\mathrm{d}_{\mathcal{D}^{\alpha,\beta}_T}(\mathbf{u}_1,\mathbf{u}_2) \lesssim_{R_\omega} \|u_{01}-u_{02}\|_{C^\alpha} + T^\delta\Big\{(\!|\widehat{\xi}^+_1-\widehat{\xi}^+_2|\!)_\omega + \mathrm{d}_{\mathcal{D}^{\alpha,\beta}_T}(\mathbf{u}_1,\mathbf{u}_2) + \overline{\mathbb{E}}\big[\mathrm{d}_{\mathcal{D}^{\alpha,\beta}_T}(\mathbf{v}^1,\mathbf{v}^2)^4\big]^{\frac14}\Big\}$$

$$\lesssim_{R_\omega} \|u_{01}-u_{02}\|_{C^\alpha} + T^\delta\Big\{(\!|\widehat{\xi}^+_1-\widehat{\xi}^+_2|\!)_\omega + \mathrm{d}_{\mathcal{D}^{\alpha,\beta}_T}(\mathbf{u}_1,\mathbf{u}_2)^{\frac12} + \overline{\mathbb{E}}\big[\mathrm{d}_{\mathcal{D}^{\alpha,\beta}_T}(\mathbf{v}^1,\mathbf{v}^2)^4\big]^{\frac14}\Big\},$$

for some implicit positive multiplicative constant that is a polynomial of R_ω, which is of degree 5 combining Propositions 9 20 and 19. Integrating and using Cauchy-Schwarz inequality we obtain the estimate

$$\mathbb{E}\big[\mathrm{d}_{\mathcal{D}^{\alpha,\beta}_T}(\mathbf{u}_1,\mathbf{u}_2)^p\big]^2 \lesssim \|u_{01}-u_{02}\|^{2p}_{C^\alpha} + \mathbb{E}\big[(\!|\widehat{\xi}^+_1-\widehat{\xi}^+_2|\!)^{2p}\big] + T^{2p\delta}\Big\{\mathbb{E}\big[\mathrm{d}_{\mathcal{D}^{\alpha,\beta}_T}(\mathbf{u}_1,\mathbf{u}_2)^p\big] + \overline{\mathbb{E}}\big[\mathrm{d}_{\mathcal{D}^{\alpha,\beta}_T}(\mathbf{v}_1,\mathbf{v}_2)^4\big]^{\frac p2}\Big\},$$

so taking $T>0$ deterministic, small enough, independently of u_{0i} and $\widehat{\xi}^+_i$, ensures that we have

$$\mathbb{E}\big[\mathrm{d}_{\mathcal{D}^{\alpha,\beta}_T}(\mathbf{u}_1,\mathbf{u}_2)^p\big]^2 \lesssim \|u_{01}-u_{02}\|^{2p}_{C^\alpha} + \mathbb{E}\big[(\!|\widehat{\xi}^+_1-\widehat{\xi}^+_2|\!)^{2p}\big] + T^{2p\delta}\,\overline{\mathbb{E}}\big[\mathrm{d}_{\mathcal{D}^{\alpha,\beta}_T}(\mathbf{v}_1,\mathbf{v}_2)^4\big]^{\frac p2}.$$

As $4 \leq p < \infty$, we conclude that equation (1.3) has a unique local solution $\mathbf{u}$ in $\mathcal{P}_p(\mathcal{D}_T^{\alpha,\beta}(X))$, and that the law $\mathcal{L}(\mathbf{u}) \in \mathcal{P}_p(\mathcal{D}_T^{\alpha,\beta}(X))$ of $\mathbf{u}$ depends continuously on $\widehat{\xi}^+ \in L^{2p}(\Omega^2, \mathfrak{N}^2)$ and on $u_0 \in C^\alpha$. ▷

5 – Propagation of chaos

Let now (ξ^i, u_0^i) be a sequence of independent and identically distributed random variables with common law $\mathcal{L}(\xi, u_0)$, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We fix $\omega \in \Omega$ and an integer $n \geq 1$ and study the dynamics

$$\begin{aligned} (\partial_t - \Delta) u^{i,n}(\omega) &= f\big(u^{i,n}(\omega), \mu_t^n\big)\xi^i(\omega) + g\big(u^{i,n}(\omega), \mu_t^n(\omega)\big), \qquad (1 \leq i \leq n) \\ \mu_t^n(\omega) &:= \frac{1}{n}\sum_{i=1}^n \delta_{u_t^{i,n}(\omega)}, \end{aligned} \tag{5.1}$$

with initial conditions $\big(u_0^1(\omega), \dots, u_0^n(\omega)\big)$. We suppose that f and g satisfy the assumptions ***(A**$_f$**-A**$_g$**-B)***. System (5.1) can either be understood as a multidimensional singular stochastic PDE driven by a multidimensional (enhanced) noise or as a mean field singular stochastic PDE. We prove in paragraph *(a)* that these two interpretations coincide and prove in paragraph *(b)* that we have a propagation of chaos result for (5.1). We write $[\![1, n]\!]$ for the set of integers between 1 and n.

(a) Singular systems of interacting fields – To lighten the notations we consider here the case that the diffusivity f is linear in the measure argument – see (5.2) below. The polynomial case is treated similarly. One can see equation (5.1) as a single multidimensional singular stochastic equation

$$(\partial_t - \Delta)\mathsf{u} = \mathsf{f}(\mathsf{u})\xi^{[1,n]} + \mathsf{g}(\mathsf{u})$$

with unknown $\mathsf{u} = \big(u^{1,n}, \dots, u^{n,n}\big)$ and noise $\xi^{[1,n]} = \big(\xi^1, \dots, \xi^n\big)$, and where f is $(f^1, \dots, f^n)$ with

$$f^i : \big(u^{1,n}, \dots, u^{n,n}\big) \mapsto f\left(u^{i,n}, \frac{1}{n}\sum_{j=1}^n \delta_{u^{j,n}}\right) =: f(u^{i,n}, \mu^n),$$

with a similar definition of g. The noise $\xi^{[1,n]}$ needs to be enhanced to make sense of the equation. The solution will be a tuple of paracontrolled functions

$$u^{i,n} = (u^{i,n})' < X^i + (u^{i,n})^{\#} = f^i(u^{1,n}, \dots, u^{n,n}) < X^i + (u^{i,n})^{\#}$$

so we will have from paralinearisation

$$f^i\big(u^{1,n}, \cdots, u^{n,n}\big) = \sum_{j=1}^n \Big(\partial_j f^i\big(u^{1,n}, \dots, u^{n,n}\big)(u^{j,n})'\Big) < X^j + f^i\big(u^{1,n}, \dots, u^{n,n}\big)^{\#},$$

with

$$\partial_j f^i\big(u^{1,n}, \dots, u^{n,n}\big) = \delta_{i,j}\partial_1 f\big(u^{i,n}, \mu^n\big) + \frac{1}{n}\partial_2 F\big(u^{i,n}, \mu^n\big),$$

since

$$f(u^{i,n}, \mu^n) = \frac{1}{n}\sum_{j=1}^n F\big(u^{i,n}, u^{j,n}\big). \tag{5.2}$$

The singular product in (5.1) then reads

$$\begin{aligned} f\big(u^{i,n}, \mu^n\big)\xi^i &= f\big(u^{i,n}, \mu^n\big) < \xi^i + \xi^i < f\big(u^{i,n}, \mu^n\big) + f\big(u^{i,n}, \mu^n\big)^{\#} \odot \xi^i \\ &\quad + \mathsf{C}\Big(\partial_1 f\big(u^{i,n}, \mu^n\big)(u^{i,n})', X^i, \xi^i\Big) + \frac{1}{n}\sum_{j=1}^n \mathsf{C}\Big(\partial_2 F(u^i, \mu^n)(u^{j,n})', X^j, \xi^i\Big) \\ &\quad + \partial_1 f\big(u^{i,n}, \mu\big)(u^{i,n})'\big(\xi^i \odot X^j\big) + \frac{1}{n}\sum_{j=1}^n \partial_2 F(u^i, \mu)(u^{j,n})'\big(\xi^i \odot X^j\big). \end{aligned} \tag{5.3}$$

Our task is now to prove that (5.1) may also be understood as a mean field singular stochastic PDE with a suitable enhancement of the noise and that the two interpretations coincide. With the notations of Section 2.2, Tanaka's trick gives an interpretation of (5.1) as the mean field type equation

$$(\partial_t - \Delta)u^{i,n}(\omega) = f\Big(u^{i,n}(\omega), u^{U_n(\cdot),n}(\omega)\Big)\xi^i(\omega) + g\Big(u^{i,n}(\omega), u^{U_n(\cdot),n}(\omega)\Big) \tag{5.4}$$

studied in Section 4, but now set on the finite probability space $(\llbracket 1,n\rrbracket, 2^{\llbracket 1,n\rrbracket}, \lambda_n)$, with generic chance element i. The enhanced noise from Definition 17 is then

$$\Big\{\xi^i,\ \ \xi^i\odot X^i,\ \ \xi^j,\ \ \xi^j\odot X^i\Big\}_{1\le i,j\le n},$$

where the index i plays the role of ω and j the role of ϖ. Let us now clarify the meaning of the singular product. We have

$$\delta_z f\big(u^{i,n}, u^{u^{\varepsilon_k}(\cdot)}\big) = \partial_1 f\big(u^{i,n}, u^{U_n(\cdot),n}\big)\big(u^{i,n}\big)',$$

and

$$\delta_\mu f\big(u^{i,n}, u^{U_n(\cdot),n}\big) = \partial_2 F\big(u^{i,n}, v^{U_n(\cdot),n}\big)\big(u^{U_n(\cdot),n}\big)'.$$

In the sense of Section 4.2 the singular product in Equation (5.4) is defined as

$$\begin{aligned} f\big(u^{i,n}, u^{U_n(\cdot),n}\big)\xi^i &= f\big(u^{i,n}, u^{U_n(\cdot),n}\big) < \xi^i + \xi^i < f\big(u^{i,n}, u^{U_n(\cdot),n}\big) + f\big(u^{i,n}, u^{U_n(\cdot),n}\big)^{\#}\odot\xi^i \\ &\quad + \mathsf{C}\Big(\partial_1 f\big(u^{i,n}, u^{U_n(\cdot),n}\big)\big(u^{i,n}\big)', X^i, \xi^i\Big) + \partial_1 f\big(u^{i,n}, u^{U_n(\cdot),n}\big)\big(u^{i,n}\big)'\big(\xi\odot X\big)^i \\ &\quad + \frac{1}{n}\sum_{j=1}^n \mathsf{C}\Big(\partial_2 F\big(u^{i,n}, u^{j,n}\big)\big(u^{j,n}\big)', X^j, \xi^i\Big) \\ &\quad + \frac{1}{n}\sum_{j=1}^n \partial_2 F\big(u^{i,n}, u^{U_n(\cdot),n}\big)\big(u^{U_n(\cdot),n}\big)'\big(\xi^i\odot X^j\big). \end{aligned} \tag{5.5}$$

We conclude from (5.3) and (5.5) that the two formulations coincide as they amount to solving the same classical PDE for the remainders $(u^{i,n})^{\#}$.

(b) Mean field limit – We know from the continuity result of Theorem 24 that the almost sure convergence of

$$\mathcal{W}_p\left(\frac{1}{n}\sum_{i=1}^n \delta_{(\widehat{\xi}^{i,+}, u_0^i)(\omega)}, \mathcal{L}(\widehat{\xi}^{+}, u_0)\right)$$

to 0 granted by the law of large numbers entails the convergence of $\mathcal{W}_{p,C_TC^\alpha}\big(\frac{1}{n}\sum_{i=1}^n \delta_{u^{i,n}}, \mathcal{L}(u)\big)$ to 0, where u is the function associated with the solution $\mathbf{u}$ of the mean field dynamics (1.3). It follows then from Sznitman's Proposition 2.2 in [17] that there is propagation of chaos for the system (5.1) of interacting fields to the mean field limit dynamics (1.3).

25 – *Corollary. For any fixed integer k, the law of $\big(u^{1,n},\dots,u^{k,n}\big)$ converges to $\mathscr{L}(u)^{\otimes k}$ when n tends to ∞.*

Note that Shen, Smith, Zhu & Zhu have also looked at some systems of mean field type singular SPDEs in their works [15, 16] on the linear sigma model. The precise structure of the equations they consider allows them to bypass the use of the sophisticated setting presented in the present work.

6 – Systems with a common noise

We study in this section some systems of interacting fields of the form

$$(\partial_t - \Delta)u^{i,n} = f_1\big(u_t, \mu_t^n\big)\xi^i + f_2\big(u_t,\mu_t^n\big)\lambda + g\big(u_t,\mu_t^n\big), \qquad (1\le i\le n) \tag{6.1}$$

where the ξ^i are independent random noises with values in $C_TC^{\alpha-2}$ and λ is another noise with values in $C_TC^{\alpha-2}$, independent of the ξ^i. To lighten the notations, we we work with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of the form $(\Omega_1\times\Omega_2, \mathcal{F}_1\otimes\mathcal{F}_2, \mathbb{P}_1\otimes\mathbb{P}_2)$, with generic element $\omega=(\omega_1,\omega_2)$. We assume that the random variables $\xi^i(\omega)=\xi^i(\omega_1)$ are measurable with respect to $(\Omega_1,\mathcal{F}_1)$ and $\lambda(\omega)=\lambda(\omega_2)$ is measurable with respect to $(\Omega_2,\mathcal{F}_2)$. For any random variable Y on Ω, the conditional expectation

$$\mathbb{E}^1\big[Y\big](\omega_2) := \int_{\Omega_1} Y(\omega_1,\omega_2)\,\mathbb{P}_1(d\omega_1)$$

The mean field type equation corresponding to Equation (6.1) reads

$$(\partial_t-\Delta_t)u = f_1\big(u_t,\mathcal{L}(u_t|\lambda)\big)\xi_t + f_2\big(u_t,\mathcal{L}(u_t|\lambda)\big)\lambda_t + g\big(u_t,\mathcal{L}(u_t|\lambda)\big). \tag{6.2}$$

As in the previous sections it will turn out to be convenient to view the random variable $\mathcal{L}(U|\lambda)$ as an ω_2-dependent element $v(\omega_2)\in L^p(\Omega_1)$. We work in this section with functions f_1 and f_2 that are linear with respect to their measure argument

$$f_j\big(u,v\big)(z)(\omega_1,\omega_2) = \int_{\Omega_1} F_j\big(u(\omega_1,\omega_2)(z), v(\varpi_1,\omega_2)(z)\big)\,\mathbb{P}(d\varpi_1),$$

with F_j of class C_3^b. There is no difficulty in adapting what follows to the case where the functions depend polynomially on their measure argument. We suppose that the function g satisfies the assumptions ***(A$_g$)*** and ***(B)*** and that both f_1 and f_2 satisfies assumptions ***(A$_f$)*** and ***(B)***. The analysis of Equation (6.2) is very similar to what was done in Section 4. We only describe below the main changes that need to be done and leave some of the details to the interested reader.

6.1 – Paracontrolled structure for mean field singular SPDEs with common noise. We now adapt the definitions of enhanced data and controlled distributions from Section 4 for the purpose of this section. We still let $X=(\partial_t-\Delta)^{-1}\xi$ and set $L=(\partial_t-\Delta)^{-1}\lambda$.

Definition – The ***mean field enhancement of the random noise*** (ξ,λ) *is the random variable*

$$\widehat{(\xi,\lambda)}^{+}(\omega,\varpi) := \Big(\xi(\omega),\overline{\xi}(\varpi),\lambda(\omega),(\xi\odot X)(\omega),\big(\xi\odot\overline{X}\big)(\omega,\varpi),$$
$$\big(\xi\odot L\big)(\omega),\big(\lambda\odot X\big)(\omega),\big(\lambda\odot\overline{X}\big)(\omega,\varpi),\big(\lambda\odot L\big)(\omega)\Big)$$

defined on $\big(\Omega^2,\mathcal{F}^2,\mathbb{P}^{\otimes 2}\big)$. *We define on* $(\Omega,\mathcal{F},\mathbb{P})$ *the random variable*

$$\begin{aligned}(\!(\widehat{(\xi,\lambda)}^{+})\!)_\omega := &\,\|\xi(\omega)\|_{C_{T_0}C^{\alpha-2}} + \big\|(\xi\odot X)(\omega)\big\|_{C_{T_0}C^{2\alpha-2}} + \|\lambda(\omega)\|_{C_TC^{\alpha-2}}\\ &+\overline{\mathbb{E}}\big[\|\overline{\xi}(\omega,\cdot)\|^4_{C_{T_0}C^{\alpha-2}}\big]^{\frac14} + \overline{\mathbb{E}}\big[\|(\xi\odot\overline{X})(\omega,\cdot)\|^4_{C_{T_0}C^{2\alpha-2}}\big]^{\frac14}\\ &+\|(\xi\odot L)(\omega)\|_{C_TC^{2\alpha-2}} + \|(\lambda\odot L)(\omega)\|_{C_TC^{2\alpha-2}} + \|(\lambda\odot X)(\omega)\|_{C_{T_0}C^{2\alpha-2}}\\ &+\overline{\mathbb{E}}\big[\|(\lambda\odot\overline{X})(\omega,\cdot)\|^4_{C_{T_0}C^{2\alpha-2}}\big]^{\frac14}\end{aligned}$$

Definition – A ***parabolic function*** u ***on*** $[0,T]\times\mathsf{T}^2$ *is said to be* ***paracontrolled by*** (X,L) *if there exists functions* $u'_X,u'_L\in\mathscr{C}_T^\beta$, *such that*

$$u^{\#} := u - u'_X\prec X - u'_L\prec L\in\mathscr{C}_T^\alpha$$

and

$$\sup_{t\in(0,T]} t^{\beta/2}\big\|u_t^{\#}\big\|_{C^{\beta+\rho}} < +\infty.$$

We denote by $\mathcal{D}_T^{\alpha,\beta}(X,L)$ the space of all such tuples $\mathbf{u}=(u'_X,u'_L,u^\sharp)$; it is equipped with the norm

$$\|\mathbf{u}\|_{\mathcal{D}_T^{\alpha,\beta}} = \big\|(u'_X,u'_L,u^\sharp)\big\|_{\mathcal{D}_T^{\alpha,\beta}} := \big\|u'_X\big\|_{\mathscr{C}_T^\beta} + \big\|u'_L\big\|_{\mathscr{C}_T^\beta} + \big\|u^\#\big\|_{\mathscr{C}_T^\alpha} + \sup_{t\in(0,T]} t^{\beta/2}\big\|u_t^\#\big\|_{C^{\beta+\alpha}}.$$

We will often write u' for the pair (u'_X,u'_L). For two pairs of reference functions (X_1,L_1), (X_2,L_2) in $\mathscr{C}_T^\alpha$ and $\mathbf{u}^1=(u'^1,u^{1\#})\in\mathcal{D}_T^{\alpha,\beta}(X_1,L_1)$ and $\mathbf{u}^2=(u'^2,u^{2\#})\in\mathcal{D}_T^{\alpha,\beta}(X_2,L_2)$ we set

$$d_{\mathcal{D}_T^{\alpha,\beta}}(\mathbf{u}^1,\mathbf{u}^2) := \big\|u'^1-u'^2\big\|_{\mathscr{C}_T^\beta} + \big\|u^{1\#}-u^{2\#}\big\|_{\mathscr{C}_T^\alpha} + \sup_{t\in(0,T]} t^{\beta/2}\big\|u^{1\#}-u^{2\#}\big\|_{\beta+\alpha}.$$

26 – *Definition. Fix $t\geq 0$. A C^α-**valued random variable** w on Ω is said to be ω-**paracontrolled by** (X_t,L_t) if there are some random variables*

$$\delta_z^X w,\ \delta_z^L w : \Omega\to C^\beta$$

and

$$\delta_\mu w : \Omega \to L^{\frac{4}{3}}\big(\Omega_1, C^\beta\big)$$

and

$$w^\sharp : \Omega\to C^{\alpha+\beta}$$

such that one has

$$w(\omega) = (\delta_z^X w)(\omega) < X(\omega) + (\delta_z^L w)(\omega) < L(\omega) + \overline{\mathbb{E}}^1\big[(\delta_\mu w)(\omega,\cdot) < \overline{X}(\cdot)\big] + w^\sharp(\omega). \tag{6.3}$$

We will often write $\delta_z w$ for the pair $(\delta_z^X w,\delta_z^L w)$.

The datum of a mean field enhancement $\widehat{(\xi,\lambda)}^+$ of the random noises ξ,λ comes with a natural definition of the product of ξ_t and λ_t by a random function $v\in C_TC^\alpha$ with the property that v_t is ω-paracontrolled by (X_t,L_t) for each $0<t\leq T$. We set

$$(\mathbf{v}_t\xi_t)(\omega) := v_t(\omega) < \xi_t(\omega) + (\mathbf{v}_t\xi_t)^\sharp(\omega)$$

where

$$\begin{aligned}
(\mathbf{v}_t\xi_t)^\sharp(\omega) := {}& \xi_t(\omega) < v_t(\omega) + v_t^\#(\omega)\odot\xi_t(\omega)\\
&+ \mathsf{C}\big((\delta_z^X v)(\omega),X(\omega),\xi_t(\omega)\big) + \overline{\mathbb{E}}^1\Big[\mathsf{C}\big((\delta_\mu v)(\omega,\cdot),\overline{X}(\cdot),\xi_t(\omega)\big)\Big]\\
&+ (\delta_z^L v)(\omega)(\xi\odot X)_t(\omega) + \overline{\mathbb{E}}^1\Big[(\delta_\mu v)(\omega,\cdot)\,\big(\xi\odot\overline{X}\big)(\omega,\cdot)\Big]\\
&+ \mathsf{C}\big((\delta_z^L v)(\omega),L(\omega),\xi_t(\omega)\big) + (\delta_z^L v)(\omega)\big(L\odot\xi_t\big)(\omega).
\end{aligned}$$

Likewise we set

$$(\mathbf{v}_t\lambda_t)(\omega) := v_t(\omega) < \lambda_t(\omega) + (\mathbf{v}_t\lambda_t)^\sharp(\omega)$$

where

$$\begin{aligned}
(\mathbf{v}_t\lambda_t)^\sharp(\omega) := {}& \lambda_t(\omega) < v_t(\omega) + v_t^\#(\omega)\odot\lambda_t(\omega)\\
&+ \mathsf{C}\big((\delta_z^X v)(\omega),X(\omega),\lambda_t(\omega)\big) + \overline{\mathbb{E}}^1\Big[\mathsf{C}\big((\delta_\mu v)(\omega,\cdot),\overline{X}(\cdot),\lambda_t(\omega)\big)\Big]\\
&+ (\delta_z^X v)(\omega)(\lambda\odot L)_t(\omega) + \overline{\mathbb{E}}^1\Big[(\delta_\mu v)(\omega,\cdot)\,\big(\lambda\odot\overline{X}\big)(\omega,\cdot)\Big]\\
&+ \mathsf{C}\big((\delta_z^L v)(\omega),L(\omega),\lambda_t(\omega)\big) + (\delta_z^L v)(\omega)\big(L\odot\lambda_t\big)(\omega)
\end{aligned}$$

The proofs of the following two propositions are identical to the proofs of Proposition 19Â and Proposition 20 modulo the obvious changes to be made. We first have a continuity statement for the product map.

27 – *Proposition. One has $\mathbb{P}$-almost surely $\mathbf{v}\xi\in C_TC^{\alpha-2}$ and*

$$\|(\mathbf{v}_t\xi_t)^\#(\omega)\|_{C^{\alpha+\beta-2}} \lesssim \big(1+ (\!|\widehat{(\xi,\lambda)}^+|\!)_\omega^2\big)\left(\|(\delta_z v)(\omega)\|_{C^\beta} + \overline{\mathbb{E}}\Big[\|\delta_\mu v\|_{C^\beta}^{\frac{4}{3}}\Big]^{\frac{3}{4}} + \|v^\#(\omega)\|_{C^{\alpha+\beta}}\right).$$

Furthermore, for two enhanced noises $\widehat{(\xi,\lambda)}^{1+}, \widehat{(\xi,\lambda)}^{2+}$ *in our class, and with* $v^i \in C_T C^\alpha$ *with* v^i_t *paracontrolled by* (X^i_t, L^i_t)*, for* $1 \le i \le 2$*, for each* $0 < t \le T$*, one has*

$$\|(\mathbf{v}^1_t\xi^1_t)^{\#}(\omega)-(\mathbf{v}^2_t\xi^2_t)^{\#}(\omega)\|_{C^{\alpha+\beta-2}}\lesssim(\star)_{12}(\omega)\Big(\|\delta^X_z v^1-\delta^X_z v^2\|_{C^\beta}+\|\delta^L_z v^1-\delta^L_z v^2\|_{C^\beta}$$
$$+\overline{\mathbb{E}}^1\big[\|\delta_\mu v^1-\delta_\mu v^2\|^{\frac43}_{C^\beta}\big]^{\frac34}+\|v^{1\#}-v^{2\#}\|_{C^{\alpha+\beta}}+(\!(\widehat{(\xi,\lambda)}^{+1}-\widehat{(\xi,\lambda)}^{+2})\!)_\omega\Big),$$

where

$$(\star)_{12}(\omega)=P\Big(\max_{i\in\{1,2\}}\Big\{(\!(\widehat{(\xi,\lambda)}^{+i})\!)_\omega,\|\delta_z v^i\|_{C^\alpha},\overline{\mathbb{E}}^1\big[\|\delta_\mu v^i\|^{\frac43}_{C^\alpha}\big]^{\frac34},\|v^{\#i}\|_{C^{\alpha+\beta}}\Big\}\Big),$$

for some quadratic polynomial P.

Second we have a stability result for the paracontrolled structure with respect to nonlinearities.

28 – *Proposition. Fix $t>0$ and assume we are given two $L^{8p}(\Omega,\mathcal{D}^\alpha(X_t,L_t))$ random variables $(h'_X,h'_L,h^\sharp)$ and $(k'_X,k'_L,k^\sharp)$ with corresponding C^α functions h,k on T^2. Then for $f\in\{f_1,f_2\}$, the function $f\big(h(\omega),k(\varpi)\big)(\omega)$ with $\omega=(\omega_1,\omega_2)$ and $\varpi=(\varpi_1,\omega_2)$ is paracontrolled by X_t,L_t in the sense of Definition 26, with*

$$(\delta^X_z f)(h,k)(\omega)=(\partial_1 f)\big(h(\omega),k\big)h'_X(\omega)$$
$$(\delta^L_z f)(h,k)(\omega)=(\partial_1 f)\big(h(\omega),k\big)h'_L(\omega)+\overline{\mathbb{E}}^1\Big[\Big((\partial_2 F)(h(\omega),k(\varpi))k'_L(\varpi)\Big)\Big]$$

and

$$(\delta_\mu f)(h,k)(\omega,\varpi)=\overline{\mathbb{E}}^1\Big[(\partial_2 F)\Big(h(\omega),k(\varpi)\Big)\Big]k'_X(\varpi),$$

and

$$\|f(h(\omega),k)^{\#}\|_{C^{\alpha+\beta}}\lesssim\Big(1+\|X_t(\omega)\|^2_{C^\alpha}+\overline{\mathbb{E}}\big[\|\overline{X}_t\|^4_{C^\alpha}\big]^{\frac12}+\|L_t(\omega_2)\|^2_{C^\alpha}\Big)$$
$$\times\Big(1+\|h'(\omega)\|_{C^\beta}+\|h^{\#}(\omega)\|_{C^\alpha}+\overline{\mathbb{E}}^1\big[\|k'\|^4_{C^\beta}\big]^{\frac14}+\overline{\mathbb{E}}^1\big[\|k^{\#}\|^4_{C^\alpha}\big]^{\frac14}\Big)$$
$$\times\Big(1+\|h'(\omega)\|_{C^\beta}+\|h^{\#}(\omega)\|_{C^{\alpha+\beta}}+\overline{\mathbb{E}}^1\big[\|k'\|^4_{C^\beta}\big]^{\frac14}+\overline{\mathbb{E}}^1\big[\|k^{\#}\|^4_{C^{\alpha+\beta}}\big]^{\frac14}\Big).$$

Moreover for $\widehat{(\xi,\lambda)}^{+i}\in L^{8p}(\Omega^2,\mathfrak{N}^2)$ and h and k in $L^{8p}(\Omega,\mathcal{D}^\alpha(X_t))$, for $1\le i\le 2$, we have

$$\|f\big(h^1(\omega),k^1\big)^{\#}-f\big(h^2(\omega),k^2\big)^{\#}\|_{C^{\alpha+\beta}}\lesssim(\star)_{12}(\omega)\Big\{\|X^1_t(\omega)-X^2_t(\omega)\|_{C^\alpha}$$
$$+\|L^1_t(\omega)-L^2_t(\omega)\|_{C^\alpha}+\overline{\mathbb{E}}\big[\|\overline{X}^1_t-\overline{X}^2_t\|^4_{C^\alpha}\big]^{\frac14}+d_{\mathcal{D}^\beta}\big(h^1(\omega),h^2(\omega)\big)+\overline{\mathbb{E}}^1\big[d_{\mathcal{D}^\beta}\big(k^1,k^2\big)^4\big]^{\frac14}\Big\},\tag{6.4}$$

where

$$(\star)_{12}(\omega)=P\Big(\max_{i\in\{1,2\}}\Big\{\|X^i_t(\omega)\|_{C^\alpha},\,\|L^i_t(\omega)\|_{C^\beta},\,\overline{\mathbb{E}}\big[\|X^i_t\|^4_{C^\alpha}\big]^{\frac14},\,\|h^i(\omega)\|_{\mathcal{D}^\alpha},\,\overline{\mathbb{E}}^1\big[\|k^i\|^4_{\mathcal{D}^\alpha}\big]^{\frac14}\Big\}\Big),$$

for some polynomial P.

6.2 – Fixed point arguments. The local well-posedness of Equation (6.2) follows from the same two fixed point arguments as in Section 4. We only give the details of the first statement to convince the reader that our machinery goes through.

29 – *Proposition. Fix $0<T_0<\infty$. Assume the assumptions **($\boldsymbol{A}_f$-$\boldsymbol{A}_g$-$\boldsymbol{B}$)** hold true. For every $\mathbf{v}\in L^p\big(\Omega,\mathcal{D}^{\alpha,\beta}_{T_0}(X,L)\big)$ and $u_0\in C^\alpha$ there exists a positive random time*

$$T=T\big((\!(\widehat{(\xi,\lambda)}^{+})\!)_\omega,\mathbf{v},u_0\big)\le T_0$$

and a unique solution in $\mathbf{u}_{\widehat{(\xi,\lambda)}^+,u_0,v} \in \mathcal{D}_T^{\alpha,\beta}(X,L)$ *to the equation*

$$(\partial_t - \Delta)u = f_1(\mathbf{u},\mathbf{v})\,\xi + f_2(\mathbf{u},\mathbf{v})\,\lambda + g(u,v), \tag{6.5}$$

where $\mathbf{u}$ *is* $\omega-$*paracontrolled by* (X,L) *with null* δ_μ *derivative. This random solution* $\mathbf{u}_{\widehat{(\xi,\lambda)}^+,u_0,v}(\omega)$ *satisfies the local Lipschitz continuity property*

$$d_{\mathcal{D}_T^{\alpha,\beta}}\big(\mathbf{u}_{\widehat{(\xi,\lambda)}_1^+,u_0,v_1}(\omega), \mathbf{u}_{\widehat{(\xi,\lambda)}_2^+,u_0,v_2}(\omega)\big) \lesssim_\omega \|u_{0,1} - u_{0,2}\|_{C^\alpha} \\ + \overline{\mathbb{E}}^1\big[\|\mathbf{v}_1 - \mathbf{v}_2\|_{L^p(\Omega,\mathcal{D}_T^{\alpha,\beta})}\big] + (\!|\widehat{(\xi,\lambda)}_1^+ - \widehat{(\xi,\lambda)}_2^+|\!)_\omega.$$

The random function $u(\omega) \in \mathscr{C}_T^\alpha$ *associated with* $\mathbf{u}_{\widehat{(\xi,\lambda)}^+,u_0,v}$ *is the limit in probability of the solutions* u^{ε_k} *of the equations*

$$(\partial_t - \Delta)u^{\varepsilon_k} = f(u^{\varepsilon_k},v)\,\xi_t^{\varepsilon_k} + g(u^{\varepsilon_k},v) - c_k^X(t)(f_1\partial_1 f_1)(u^{\varepsilon_k},v) - c_k^L(t)(f_2\partial_1 f_2)(u^{\varepsilon_k},v), \tag{6.6}$$

with initial condition u_0*.*

Proof – We rewrite again equation (6.5) as the fixed point equation

$$u_t = P_t u_0 + \int_0^t P_{t-s}\big(f(\mathbf{u}_s,\mathbf{v}_s)\xi_t + f_2(\mathbf{u}_s,\mathbf{v}_s)\lambda_x + g(u_s,v_s)\big)\,ds.$$

We get from Lemma 27 and Lemma 28 that $f_1(\mathbf{u}_s,\mathbf{v}_s)\xi_t + f_2(\mathbf{u}_s,\mathbf{v}_s)\lambda_x + g(u_s,v_s)$ is for each s an element of $\mathcal{D}^\alpha(X_s,L_s)$ with Gubinelli derivative $\big(f_1(\mathbf{u}_s,\mathbf{v}_s), f_2(\mathbf{u}_s,\mathbf{v}_s)\big)$ and remainder $\big(f_1(\mathbf{u}_s,\mathbf{v}_s)\xi\big)^\# + \big(f_2(\mathbf{u}_s,\mathbf{v}_s)\lambda\big)^\# + g(u_s,v_s)$. We check that $f(u,v) \in \mathscr{C}_T^\alpha$ and $\big(f_1(\mathbf{u}_s,\mathbf{v}_s)\xi\big)^\# + \big(f_2(\mathbf{u}_s,\mathbf{v}_s)\lambda\big)^\# + g(u_s,v_s)$ have the regularity required. Take $\mathbf{u} \in \mathcal{D}_T^{\alpha,\beta}(X)$. First one has

$$\begin{aligned}\|f_1(u,v)\|_{\mathscr{C}_T^\alpha} &\lesssim 1 + \|u\|_{\mathscr{C}_T^\alpha} + \overline{\mathbb{E}}^1\big[\|v\|_{\mathscr{C}_T^\alpha}\big] \\ &\lesssim \Big(1 + \|X\|_{\mathscr{C}_T^\alpha} + \overline{\mathbb{E}}^1\big[\|\overline{X}(\omega)\|^2_{\mathscr{C}_T^\alpha}\big]^{\frac12} + \|L\|_{\mathscr{C}_T^\alpha}\Big)\Big(1 + \|u\|_{\mathcal{D}_T^{\alpha,\beta}} + \overline{\mathbb{E}}^1\big[\|v\|^2_{\mathcal{D}_T^{\alpha,\beta}}\big]^{\frac12}\Big) \\ &\lesssim \big(1 + (\!|\widehat{(\xi,\lambda)}^+|\!)_\omega\big)\Big(1 + \|\mathbf{u}\|^2_{\mathcal{D}_T^{\alpha,\beta}} + \overline{\mathbb{E}}^1\big[\|v\|^4_{\mathcal{D}_T^{\alpha,\beta}}\big]^{\frac14}\Big)\end{aligned}$$

Second, combining the estimates from Lemmas 27 and 28 one gets at some fixed time t the estimates

$$\begin{aligned}\|(f_1(\mathbf{u},\mathbf{v})\xi)^\#\|_{C^{\alpha+\beta-2}} &\lesssim \big(1 + (\!|\widehat{(\xi,\lambda)}^+|\!)^2_\omega\big)\Big(\|\delta_z f(u,v)\|_{C^\beta} + \overline{\mathbb{E}}^1\big[\|\delta_\mu f(u,v)\|^{\frac43}_{C^\beta}\big]^{\frac34} + \|f(\mathbf{u},\mathbf{v})^\#\|_{C^{\alpha+\beta}}\Big) \\ &\lesssim \big(1 + (\!|\widehat{(\xi,\lambda)}^+|\!)^2_\omega\big)\bigg\{\Big(1 + \|u\|_{C^\alpha} + \overline{\mathbb{E}}^1\big[\|v\|_{C^\alpha}\big]\Big)\|u'\|_{C^\beta} \\ &\qquad + \Big(1 + \|u\|_{C^\alpha} + \overline{\mathbb{E}}^1\big[\|v\|^2_{C^\alpha}\big]^{\frac12}\Big)\overline{\mathbb{E}}^1\big[\|v'\|^4_{C^\beta}\big]^{\frac14} + \|f(\mathbf{u},\mathbf{v})^\#\|_{C^{\alpha+\beta}}\bigg\} \\ &\lesssim \Big(1 + (\!|\widehat{(\xi,\lambda)}^+|\!)^4_\omega\Big)\Big(1 + \|u'\|_{C^\beta} + \|u^\#\|_{C^\alpha} + \overline{\mathbb{E}}^1\big[\|v'\|^4_{C^\beta}\big]^{\frac14} + \overline{\mathbb{E}}^1\big[\|v^\#\|^4_{C^\alpha}\big]^{\frac14}\Big) \\ &\qquad \times \Big(1 + \|u'\|_{C^\beta} + \|u^\#\|_{C^{\alpha+\beta}} + \overline{\mathbb{E}}^1\big[\|v'\|^4_{C^\beta}\big]^{\frac14} + \overline{\mathbb{E}}^1\big[\|v^\#\|^4_{C^{\alpha+\beta}}\big]^{\frac14}\Big),\end{aligned}$$

so

$$\sup_{t\in(0,T]} t^{\beta/2}\|(f_1(\mathbf{u}_t,\mathbf{v}_t)\xi_t)^\#\|_{C^{\alpha+\beta-2}} \lesssim \big(1 + (\!|\widehat{(\xi,\lambda)}^+|\!)^4_\omega\big)\Big(1 + \|\mathbf{u}\|^2_{\mathcal{D}_T^{\alpha,\beta}} + \overline{\mathbb{E}}^1\big[\|\mathbf{v}\|^4_{\mathcal{D}_T^{\alpha,\beta}}\big]^{\frac12}\Big),$$

idem for $(f_2(\mathbf{u}_t,\mathbf{v}_t)\lambda_t)^\#$. We have also

$$\begin{aligned}\sup_{t\in(0,T]} t^{\beta/2}\|g(u_t,v_t)\|_{C^{\alpha+\beta-2}} &\lesssim \sup_{t\in(0,T]} t^{\beta/2}\Big(1 + \|u_t\|_{C^\alpha} + \overline{\mathbb{E}}^1\big[\|v_t\|^2_{C^\alpha}\big]^{\frac12}\Big) \\ &\lesssim \big(1 + (\!|\widehat{(\xi,\lambda)}^+|\!)_\omega\big)\Big(1 + \|\mathbf{u}\|^2_{\mathcal{D}_T^{\alpha,\beta}} + \overline{\mathbb{E}}^1\big[\|\mathbf{v}\|^4_{\mathcal{D}_T^{\alpha,\beta}}\big]^{\frac12}\Big),\end{aligned}$$

so we end up with the pathwise estimate

$$\sup_{t\in(0,T]} t^{\beta/2}\|(f_1(\mathbf{u}_t,\mathbf{v}_t)\xi_t)^{\#}+(f_2(\mathbf{u}_t,\mathbf{v}_t)\lambda_t)^{\#}+g(u_t,v_t)\|_{C^{\alpha+\beta-2}} \lesssim \Big(1+\llparenthesis\widehat{(\xi,\lambda)}^{+}\rrparenthesis_\omega^4\Big)\Big(1+\|\mathbf{u}\|^2_{\mathcal{D}_T^{\alpha,\beta}}+\overline{\mathbb{E}}^1\big[\|\mathbf{v}\|^4_{\mathcal{D}_T^{\alpha,\beta}}\big]^{1/2}\Big).$$

It follows from Schauder estimates that the map

$$\Phi_{\widehat{(\xi,\lambda)}^{+},u_0,\mathbf{v}}:\mathcal{D}_T^{\alpha,\beta}(X(\omega),L(\omega))\to\mathcal{D}_T^{\alpha,\beta}(X(\omega),L(\omega))$$

which associates to $\mathbf{u}\in\mathcal{D}_T^{\alpha,\beta}(X(\omega),L(\omega))$ the solution w of the equation

$$(\partial_t-\Delta)w=f_1(\mathbf{u},\mathbf{v})\xi+f_2(\mathbf{u},\mathbf{v})\lambda+g(u,v)$$

with initial condition $w_0=u_0$, is well-defined and satisfies the bound

$$\big\|\Phi_{\widehat{(\xi,\lambda)}^{+},u_0,\mathbf{v}}(\mathbf{u})\big\|_{\mathcal{D}_T^{\alpha,\beta}}\lesssim\|u_0\|_{C^\alpha}+T^{(\alpha-\beta)/2}\Big(1+\llparenthesis\widehat{(\xi,\lambda)}^{+}\rrparenthesis_\omega^4\Big)\Big(1+\|\mathbf{u}\|^2_{\mathcal{D}_T^{\alpha,\beta}}+\overline{\mathbb{E}}^1\big[\|\mathbf{v}\|^4_{\mathcal{D}_T^{\alpha,\beta}}\big]^{1/2}\Big).$$

Recall $4\le p<\infty$. One can then find some random positive constants

$$M=M\Big(\|u_0\|_\alpha\vee\overline{\mathbb{E}}^1\big[\|\mathbf{v}\|^p_{\mathcal{D}_T^{\alpha,\beta}}\big]\vee\llparenthesis\widehat{(\xi,\lambda)}^{+}\rrparenthesis_\omega\Big)$$

and

$$T=T\Big(\|u_0\|_\alpha\vee\overline{\mathbb{E}}^1\big[\|\mathbf{v}\|^p_{\mathcal{D}_T^{\alpha,\beta}}\big]\vee\llparenthesis\widehat{(\xi,\lambda)}^{+}\rrparenthesis_\omega\Big)$$

so that the map $\Phi_{\widehat{(\xi,\lambda)}^{+},u_0,\mathbf{v}}$ sends the ball

$$\Big\{\mathbf{u}\in\mathcal{D}_T^{\alpha,\beta}(X(\omega),L(\omega))\,;\,\|\mathbf{u}\|_{\mathcal{D}_T^{\alpha,\beta}}\le M\Big\}$$

into itself. Now, given $\widehat{(\xi,\lambda)}_1^{+},\widehat{(\xi,\lambda)}_2^{+}$ in $L^{8p}(\Omega^2,\mathfrak{N}^2)$, two initial conditions $u_{0,1},u_{0,2}$ in C^α and $\mathbf{v}_1,\mathbf{v}_2$ in $L^p\big(\Omega,\mathcal{D}_{T_0}^{\alpha,\beta}(X(\omega),L(\omega))\big)$, we define a random constant

$$M'_\omega=M\Big(\max_{i=1,2}\Big\{\|u_{0,i}\|_{C^\alpha}\vee\overline{\mathbb{E}}^1\big[\|\mathbf{v}_i\|^p_{\mathcal{D}_{T_0}^{\alpha,\beta}}\big]\vee\llparenthesis\widehat{(\xi,\lambda)}_i^{+}\rrparenthesis_\omega\Big\}\Big).$$

For $\|\mathbf{u}\|_{\mathcal{D}_T^{\alpha,\beta}}\le M'_\omega$, Schauder estimates tell us that

$$\mathrm{d}_{\mathcal{D}_T^{\alpha,\beta}}\big(\Phi_{\widehat{(\xi,\lambda)}_1^{+},u_{0,1},\mathbf{v}_1}(\mathbf{u}_1),\Phi_{\widehat{(\xi,\lambda)}_2^{+},u_{0,2},\mathbf{v}_2}(\mathbf{u}_2)\big)\lesssim_{M'_\omega}\|u_{0,1}-u_{0,2}\|_{C^\alpha}+T^{(\alpha-\beta)/2}\Big\{\mathrm{d}_{\mathcal{D}_T^{\alpha,\beta}}(\mathbf{u}_1,\mathbf{u}_2)+\overline{\mathbb{E}}^1\big[\|\mathbf{v}_1-\mathbf{v}_2\|_{L^p(\Omega,\mathcal{D}_T^{\alpha,\beta})}\big]+\llparenthesis\widehat{(\xi,\lambda)}_1^{+}-\widehat{(\xi,\lambda)}_2^{+}\rrparenthesis_\omega\Big\}.$$

So choosing

$$T\Big(\max_{i=1,2}\Big\{\|u_{0,i}\|_{C^\alpha}\vee\overline{\mathbb{E}}^1\big[\|\mathbf{v}_i\|_{L^p(\Omega,\mathcal{D}_T^{\alpha,\beta})}\big]\vee\llparenthesis\widehat{(\xi,\lambda)}_i^{+}\rrparenthesis_\omega\Big\}\Big)$$

small enough ensures that the map $\Phi_{\widehat{(\xi,\lambda)}^{+},u_0,\mu}$ has a unique fixed point $\mathbf{u}_{\widehat{(\xi,\lambda)}^{+},u_0,\mu}(\omega)$ which satisfies the local Lipschitz property

$$\mathrm{d}_{\mathcal{D}_T^{\alpha,\beta}}\big(\mathbf{u}_{\widehat{(\xi,\lambda)}_1^{+},u_0,\mathbf{v}_1}(\omega),\mathbf{u}_{\widehat{(\xi,\lambda)}_2^{+},u_0,\mathbf{v}_2}(\omega)\big)\lesssim_{M'_\omega}\|u_{0,1}-u_{0,2}\|_{C^\alpha}+\overline{\mathbb{E}}^1\big[\|\mathbf{v}_1-\mathbf{v}_2\|_{L^p(\Omega,\mathcal{D}_T^{\alpha,\beta})}\big]+\llparenthesis\widehat{(\xi,\lambda)}_1^{+}-\widehat{(\xi,\lambda)}_2^{+}\rrparenthesis_\omega.$$

The enhanced noise $\widehat{(\xi,\lambda)}\in\mathfrak{N}$ is the limit in any $L^q(\Omega,\mathbb{P})$ space, $1\le q<\infty$, of the sequence of enhanced noises

$$\Big(\xi_t^{\varepsilon_k},\,(\xi\odot X)_n,\,(\xi\odot L)^{\varepsilon_k},\,\lambda^{\varepsilon_k},\,(\lambda\odot X)^{\varepsilon_k},\,(\lambda\odot L)^{\varepsilon_k}\Big):=\Big(\xi_t^{\varepsilon_k},\,\xi^{\varepsilon_k}\odot X^{\varepsilon_k}-c_k^X,\,\xi^{\varepsilon_k}\odot L^{\varepsilon_k},\,\lambda^{\varepsilon_k},\,\lambda^{\varepsilon_k}\odot X^{\varepsilon_k},\,\lambda^{\varepsilon_k}\odot L^{\varepsilon_k}-c_k^L\Big)$$

for some diverging functions c_k^X, c_k^L, and that $\xi \odot \overline{X}$, $\lambda \odot \overline{X}$ is the limit in $L^q(\Omega^2, \mathbb{P}^{\otimes 2})$ of $\xi^{\varepsilon_k} \odot \overline{X}^{\varepsilon_k}$, $\lambda^{\varepsilon_k} \odot \overline{X}^{\varepsilon_k}$. We then have

$$\begin{aligned} f_1(\mathbf{u}^{\varepsilon_k}, \mathbf{v})\xi_t^{\varepsilon_k} &= f_1(u^{\varepsilon_k}, v) < \xi_t^{\varepsilon_k} + \xi_t^{\varepsilon_k} < f_1(u^{\varepsilon_k}, v) + f_1(u^{\varepsilon_k}, v)^{\#} \odot \xi_t^{\varepsilon_k} \\ &\quad + \mathsf{C}\big(\delta_z^X f_1(u^{\varepsilon_k}, v), X_n, \xi_t^{\varepsilon_k}\big) + \overline{\mathbb{E}}^1\Big[\mathsf{C}\big(\delta_\mu f_1(u^{\varepsilon_k}, v), \overline{X}^{\varepsilon_k}, \xi_t^{\varepsilon_k}\big)\Big] \\ &\quad + \delta_z^X f_1(u^{\varepsilon_k}, v)\big(X \odot \xi\big)^{\varepsilon_k} + \overline{\mathbb{E}}^1\Big[\delta_\mu f_1(u^{\varepsilon_k}, v)\big(\xi_t^{\varepsilon_k} \odot \overline{X}^{\varepsilon_k}\big)\Big] \\ &\quad + \mathsf{C}\big(\delta_z^L f_1(u^{\varepsilon_k}, v), L^{\varepsilon_k}, \xi_t^{\varepsilon_k}\big) + \delta_z^L f_1(u^{\varepsilon_k}, v)\big(L \odot \xi\big)_n \\ &= f_1(u^{\varepsilon_k}, v)\xi_t^{\varepsilon_k} - c_k^X(f_1\partial_1 f_1)(u^{\varepsilon_k}, v) \end{aligned}$$

and

$$\begin{aligned} f_2(\mathbf{u}^{\varepsilon_k}, \mathbf{v})\lambda_n &= f_2(u^{\varepsilon_k}, v) < \lambda_n + \lambda_n < f_2(u^{\varepsilon_k}, v) + f_2(u^{\varepsilon_k}, v)^{\#} \odot \lambda_n \\ &\quad + \mathsf{C}\big(\delta_z^X f_2(u^{\varepsilon_k}, v), X_n, \lambda_n\big) + \overline{\mathbb{E}}^1\Big[\mathsf{C}\big(\delta_\mu f_2(u^{\varepsilon_k}, v), \overline{X}^{\varepsilon_k}, \lambda_n\big)\Big] \\ &\quad + \delta_z^X f_2(u^{\varepsilon_k}, v)\big(X \odot \lambda\big)_n + \overline{\mathbb{E}}^1\Big[\delta_\mu f_2(u^{\varepsilon_k}, v)\big(\lambda_n \odot \overline{X}^{\varepsilon_k}\big)\Big] \\ &\quad + \mathsf{C}\big(\delta_z^L f_2(u^{\varepsilon_k}, v), L^{\varepsilon_k}, \lambda_n\big) + \delta_z^L f_2(u^{\varepsilon_k}, v)\big(L \odot \lambda\big)_n \\ &= f_2(u^{\varepsilon_k}, v)\lambda^{\varepsilon_k} - c_k^L(f_2\partial_1 f_2)(u^{\varepsilon_k}, v) \end{aligned}$$

so that

$$\begin{aligned} &f_1(\mathbf{u}^{\varepsilon_k}, \mathbf{v})\xi_t^{\varepsilon_k} + f_2(\mathbf{u}^{\varepsilon_k}, \mathbf{v})\lambda^{\varepsilon_k} + g(u^{\varepsilon_k}, v) \\ &= f_1(u^{\varepsilon_k}, v)\xi_t^{\varepsilon_k} + f_2(u^{\varepsilon_k}, v)\lambda_n - c_k^X(f_1\partial_1 f_1)(u^{\varepsilon_k}, v) - c_k^L(f_2\partial_1 f_2)(u^{\varepsilon_k}, v) + g(u^{\varepsilon_k}, v), \end{aligned}$$

and function u^{ε_k} is a solution of the renormalized equation

$$(\partial_t - \Delta)u^{\varepsilon_k} = f(u^{\varepsilon_k}, v)\xi_t^{\varepsilon_k} - c_k^X(f_1\partial_1 f_1)(u^{\varepsilon_k}, v) - c_k^L(f_2\partial_1 f_2)(u^{\varepsilon_k}, v) + g(u^{\varepsilon_k}, v).$$

As we know that the solution $\mathbf{u}_{\widehat{(\xi,\lambda)}^+, u_0, v} \in \mathcal{D}_T^{\alpha,\beta}(X, L)$ is a continuous function of $\widehat{(\xi,\lambda)}^+ \in \mathfrak{N}^2$, and since $\widehat{(\xi,\lambda)}_n^+$ converges to $\widehat{(\xi,\lambda)}^+$ in probability, we see that $\mathbf{u}_{\widehat{(\xi,\lambda)}^+, u_0, v}$ is the limit in probability in $\mathcal{D}_T^{\alpha,\beta}$ of the sequence $(u^{\varepsilon_k}, f_1(u^{\varepsilon_k}, v), f_2(u^{\varepsilon_k}, v)) \in \mathcal{D}_T^{\alpha,\beta}(X^{\varepsilon_k}, L^{\varepsilon_k})$. ▷

The scaling argument used in the proof of Lemma 22 works verbatim here with the obvious changes.

30 – *Lemma. For every $R > 0$, the solution $\mathbf{u}$ to equation* (6.5) *is defined up to the time*

$$T^* = \inf\big\{t \geq 0, \quad \|u(t)\|_{L^\infty} \geq R\big\}.$$

The proof of the second fixed point works exactly as in the proof of Theorem 24. Details are left to the reader.

31 – *Theorem. We assume that the assumptions **(A_f-A_g-B)** hold true. There exists a positive Ω_2-random time $T \leq T_0$ with the following property.*

- *For every $u_0 \in C^\alpha$ such that $\|u_0\|_{L^\infty} \leq C_0$ there exists a unique local solution $\mathbf{u} = (u', u^\sharp)$ to the mean field equation* (6.2) *in $L^p(\Omega, \mathcal{D}_T^{\alpha,\beta}(X, L))$ and it depends continuously on $\widehat{(\xi,\lambda)}^+ \in L^{8p}(\Omega^2, \mathfrak{N}^2)$ and $u_0 \in C^\alpha$.*
- *Write $u = u'_X < X + u'_L < L + u^\sharp$. The function $u \in \mathscr{C}_T^\alpha$ is the limit in probability of the family of solutions of the renormalized equations*

$$\begin{aligned} (\partial_t - \Delta)u^{\varepsilon_k} &= f_1\big(u^{\varepsilon_k}, \mathcal{L}(u^{\varepsilon_k}(t)|\lambda)\big)\xi_t^{\varepsilon_k} - c_k^X(t)(f_1\partial_1 f_1)\big(u^{\varepsilon_k}, \mathcal{L}(u^{\varepsilon_k}(t)|\lambda)\big) \\ &\quad + f_2\big(u^{\varepsilon_k}, \mathcal{L}(u^{\varepsilon_k}(t)|\lambda)\big)\lambda^{\varepsilon_k} - c_k^L(t)(f_2\partial_1 f_2)\big(u^{\varepsilon_k}, \mathcal{L}(u^{\varepsilon_k}(t)|\lambda)\big) + g\big(u^{\varepsilon_k}, \mathcal{L}(u^{\varepsilon_k}(t)|\lambda)\big). \end{aligned}$$

6.3 – Tanaka's trick and conditional propagation of chaos. Let now (ξ^i, u_0^i) be a sequence of independent and identically distributed random variables with common law $\mathcal{L}(\xi, u_0)$ and some independent noise λ, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as in the beginning of this section. We fix $\omega \in \Omega$ and an integer $n \geq 1$ and study the mean field dynamics

$$(\partial_t - \Delta)u^{i,n}(\omega) = f_1\big(u^{i,n}(\omega), \mu_t^n\big)\xi^i(\omega) + f_2\big(u^{i,n}(\omega), \mu_t^n\big)\lambda(\omega) + g\big(u^{i,n}(\omega), \mu_t^n(\omega)\big), \qquad (6.7)$$

with initial conditions $\big(u_0^1(\omega), \dots, u_0^n(\omega)\big)$, where $1 \leq i \leq n$. We suppose that f_1, f_2 and g satisfy the assumptions ***(A**$_f$**-A**$_g$**-B)***. As in Section 5, the system (6.7) can either be understood as a multidimensional singular stochastic PDE driven by a multidimensional (enhanced) noise or as a mean field singular stochastic PDE. We prove again in paragraph *(a)* that these two interpretations coincide and prove in paragraph *(b)* that we have a propagation of chaos result for (6.7).

(a) Singular systems of interacting fields – One can see equation (6.7) as a single multidimensional singular stochastic equation

$$(\partial_t - \Delta)\mathsf{u} = \mathsf{f}_1(\mathsf{u})\xi^{[1,n]} + \mathsf{f}_2(\mathsf{u})\lambda + \mathsf{g}(\mathsf{u})$$

with unknown $\mathsf{u} = \big(u^{1,n}, \dots, u^{n,n}\big)$ and noise $\xi^{[1,n]} = \big(\xi^1, \dots, \xi^n\big)$, and where f_1 is $(f_1^1, \dots, f_1^n)$ and f_2 is $(f_2^1, \dots, f_2^n)$ with

$$f_j^i : \big(u^{1,n}, \dots, u^{n,n}\big) \mapsto f_j\left(u^{i,n}, \frac{1}{n}\sum_{j=1}^n \delta_{u^{j,n}}\right) =: f_j(u^{i,n}, \mu^n),$$

with a similar definition of g. The noise $(\xi^{[1,n]}, \lambda)$ needs to be enhanced to make sense of the equation. The solution will be a tuple of paracontrolled functions

$$\begin{aligned} u^{i,n} &= (u^{i,n})'_X < X^i + (u^{i,n})'_L < L + (u^{i,n})^\# \\ &= f_1^i(u^{1,n}, \dots, u^{n,n}) < X^i + f_2^i(u^{1,n}, \dots, u^{n,n}) < L + (u^{i,n})^\# \end{aligned}$$

so we will have from paralinearisation

$$\begin{aligned} f_k^i\big(u^{1,n}, \cdots, u^{n,n}\big) = &\sum_{j=1}^n \Big(\partial_j f_k^i\big(u^{1,n}, \dots, u^{n,n}\big)(u^{j,n})'_X\Big) < X^j + f_k^i\big(u^{1,n}, \dots, u^{n,n}\big)^\# \\ &+ \sum_{j=1}^n \Big(\partial_j f_k^i\big(u^{1,n}, \dots, u^{n,n}\big)(u^{j,n})'_L\Big) < L \end{aligned}$$

with

$$\partial_j f_k^i\big(u^{1,n}, \dots, u^{n,n}\big) = \delta_{i,j}\partial_1 f_k\big(u^{i,n}, \mu^n\big) + \frac{1}{n}\sum_{j=1}^n \partial_2 F_k\big(u^{i,n}, u^{j,n}\big),$$

since

$$f_k(u^{i,n}, \mu^n) = \frac{1}{n}\sum_{j=1}^n F_k\big(u^{i,n}, u^{j,n}\big).$$

The singular products in (6.7) then reads

$$\begin{aligned} f_1\big(u^{i,n}, \mu^n\big)\xi^i = f_1\big(u^{i,n}, \mu^n\big) < \xi^i + \xi^i < f_1\big(u^{i,n}, \mu^n\big) + f_1\big(u^{i,n}, u^{j,n}\big)^\# \odot \xi^i \\ + \mathsf{C}\Big(\partial_1 f_1\big(u^{i,n}, \mu^n\big)(u^{i,n})'_X, X^i, \xi^i\Big) + \frac{1}{n}\sum_{j=1}^n \mathsf{C}\Big(\partial_2 F_1(u^i, \mu^n)(u^{j,n})'_X, X^j, \xi^i\Big) \\ + \partial_1 f_1\big(u^{i,n}, \mu\big)(u^{i,n})'_X\big(\xi^i \odot X^i\big) + \frac{1}{n}\sum_{j=1}^n \partial_2 F_1(u^i, \mu)(u^{j,n})'_X\big(\xi^i \odot X^j\big) \\ + \mathsf{C}\Big(\partial_1 f_1\big(u^{i,n}, \mu^n\big)(u^{i,n})'_L, L, \xi^i\Big) + \frac{1}{n}\sum_{j=1}^n \mathsf{C}\Big(\partial_2 F_1(u^i, \mu^n)(u^{j,n})'_L, L, \xi^i\Big) \end{aligned}$$

$$+\,\partial_1 f_1\big(u^{i,n},\mu\big)(u^{i,n})'_L\big(\xi^i\odot L\big)+\frac{1}{n}\sum_{j=1}^n\partial_2 F_1(u^i,\mu)(u^{j,n})'_L\big(\xi^i\odot L\big),\tag{6.8}$$

and

$$\begin{aligned}f_2\big(u^{i,n},\mu^n\big)\lambda = {}& f_2\big(u^{i,n},\mu^n\big)<\lambda+\lambda<f_2\big(u^{i,n},\mu^n\big)+f_2\big(u^{i,n},\mu^n\big)^{\#}\odot\lambda\\&+\mathsf{C}\Big(\partial_1 f_2\big(u^{i,n},\mu^n\big)(u^{i,n})'_X,X^i,\lambda\Big)+\frac{1}{n}\sum_{j=1}^n\mathsf{C}\Big(\partial_2 F_2(u^i,\mu^n)(u^{j,n})'_X,X^j,\lambda\Big)\\&+\partial_1 f_2\big(u^{i,n},\mu\big)(u^{i,n})'_X\big(\lambda\odot X^i\big)+\frac{1}{n}\sum_{j=1}^n\partial_2 F_2(u^i,\mu)(u^{j,n})'_X\big(\lambda\odot X^j\big)\\&+\mathsf{C}\Big(\partial_1 f_1\big(u^{i,n},\mu^n\big)(u^{i,n})'_L,L,\lambda\Big)+\frac{1}{n}\sum_{j=1}^n\mathsf{C}\Big(\partial_2 F_1(u^i,\mu^n)(u^{j,n})'_L,L,\lambda\Big)\\&+\partial_1 f_1\big(u^{i,n},\mu^n\big)(u^{i,n})'_L\big(\lambda\odot L\big)+\frac{1}{n}\sum_{j=1}^n\partial_2 F_1(u^i,\mu^n)(u^{j,n})'_L\big(\lambda\odot L\big),\end{aligned}$$

Our task is now to prove that (6.7) may also be understood as a mean field singular stochastic PDE of the type (6.2) with a suitable enhancement of the noise and that the two interpretations coincide. Tanaka's trick gives an interpretation of (6.7) as the mean field equation

$$\begin{aligned}(\partial_t-\Delta)u^{i,n}(\omega)={}&f_1\Big(u^{i,n}(\omega),u^{U_n(\cdot),n}(\omega)\Big)\xi^i(\omega)+f_2\Big(u^{i,n}(\omega),u^{U_n(\cdot),n}(\omega)\Big)\lambda(\omega)\\&+g\Big(u^{i,n}(\omega),u^{U_n(\cdot),n}(\omega)\Big)\end{aligned}\tag{6.9}$$

now set on the finite probability space $(\llbracket 1,n\rrbracket, 2^{\llbracket 1,n\rrbracket},\lambda_n)$, with generic chance element i. The enhanced noise from Definition 17 is then

$$\Big\{\xi^i,\ \xi^i\odot X^i,\ \xi^j,\ \xi^j\odot X^i,\ \lambda,\ \xi^j\odot L,\ \xi^j\odot X^i,\ \lambda\odot X^i,\ \lambda\odot L\Big\}_{1\le i,j\le n},$$

where the index i plays the role of ω and j the role of ϖ. Let us now clarify the meaning of the singular product. We have

$$\begin{aligned}\delta_z^X f_k\big(u^{i,n},u^{U_n(\cdot),n}\big)&=\partial_1 f_k\big(u^{i,n},u^{U_n(\cdot),n}\big)\big(u^{i,n}\big)'_X,\\ \delta_z^L f_k\big(u^{i,n},u^{U_n(\cdot),n}\big)&=\partial_1 f_k\big(u^{i,n},u^{U_n(\cdot),n}\big)\big(u^{i,n}\big)'_L,\end{aligned}$$

and

$$\delta_\mu f_k\big(u^{i,n},u^{U_n(\cdot),n}\big)=\partial_2 F_k\big(u^{i,n},u^{U_n(\cdot),n}\big)\big(u^{U_n(\cdot),n}\big)'_X.$$

The singular products in Equation (6.9) are defined as

$$\begin{aligned}f_1\big(u^{i,n},u^{U_n(\cdot),n}\big)\xi^i={}&f_1\big(u^{i,n},u^{U_n(\cdot),n}\big)<\xi^i+\xi^i<f_1\big(u^{i,n},u^{U_n(\cdot),n}\big)+f_1\big(u^{i,n},u^{U_n(\cdot),n}\big)^{\#}\odot\xi^i\\&+\mathsf{C}\Big(\partial_1 f_1\big(u^{i,n},u^{U_n(\cdot),n}\big)\big(u^{i,n}\big)'_X,X^i,\xi^i\Big)+\partial_1 f_1\big(u^{i,n},u^{U_n(\cdot),n}\big)\big(u^{i,n}\big)'_X\big(\xi\odot X\big)^i\\&+\mathsf{C}\Big(\partial_1 f_1\big(u^{i,n},u^{U_n(\cdot),n}\big)\big(u^{i,n}\big)'_L,L,\xi^i\Big)+\partial_1 f_1\big(u^{i,n},u^{U_n(\cdot),n}\big)\big(u^{i,n}\big)'_L\big(\xi^i\odot L\big)\\&+\frac{1}{n}\sum_{j=1}^n\mathsf{C}\Big(\partial_2 F_1\big(u^{i,n},u^{j,n}\big)\big(u^{j,n}\big)'_X,X^j,\xi^i\Big)\\&+\frac{1}{n}\sum_{j=1}^n\partial_2 F_1\big(u^{i,n},u^{j,n}\big)\big(u^{j,n}\big)'_X\big(\xi^i\odot X^j\big).\end{aligned}\tag{6.10}$$

and

$$f_2\big(u^{i,n},u^{U_n(\cdot),n}\big)\lambda = f_2\big(u^{i,n},u^{U_n(\cdot),n}\big) \prec \lambda + \lambda \prec f_2\big(u^{i,n},u^{U_n(\cdot),n}\big) + f_2\big(u^{i,n},u^{U_n(\cdot),n}\big)^{\#}\odot\lambda$$
$$+\,\mathsf{C}\Big(\partial_1 f_2\big(u^{i,n},u^{U_n(\cdot),n}\big)\big(u^{i,n}\big)'_X, X^i,\lambda\Big) + \partial_1 f_2\big(u^{i,n},u^{U_n(\cdot),n}\big)\big(u^{i,n}\big)'_X\big(\lambda\odot X^i\big)$$
$$+\,\mathsf{C}\Big(\partial_1 f_2\big(u^{i,n},u^{U_n(\cdot),n}\big)\big(u^{i,n}\big)'_L, L,\lambda\Big) + \partial_1 f_2\big(u^{i,n},u^{U_n(\cdot),n}\big)\big(u^{i,n}\big)'_L\big(\lambda\odot L\big)$$
$$+\,\frac{1}{n}\sum_{j=1}^n \mathsf{C}\Big(\partial_2 F_2\big(u^{i,n},u^{j,n}\big)\big(u^{j,n}\big)'_X, X^j,\lambda\Big)$$
$$+\,\frac{1}{n}\sum_{j=1}^n \partial_2 F_2\big(u^{i,n},u^{U_n(\cdot),n}\big)\big(u^{j,n}\big)'_X\big(\lambda\odot X^j\big).$$

We conclude from (6.8) and (6.10) that the two formulations coincide as they amount to solving the same classical PDE for the remainders $(u^{i,n})^{\#}$.

(b) Mean field limit – We know from the continuity result of Theorem 31 that the $\mathbb{P}$-almost sure convergence of

$$\mathcal{W}_p\bigg(\frac{1}{n}\sum_{i=1}^n \delta_{(\widehat{(\xi,\lambda)}^{i,+},u_0^i)(\omega)}, \mathcal{L}\big(\widehat{(\xi,\lambda)}^{+},u_0\big)\bigg)$$

to 0 granted by the law of large numbers entails the convergence of $\mathcal{W}_{p,C_TC^\alpha}\big(\frac{1}{n}\sum_{i=1}^n\delta_{u^{i,n}},\mathcal{L}(u)\big)$ to 0, where u is the function associated with the solution $\mathbf{u}$ of the mean field dynamics (6.2).

32 – *Corollary. For any fixed integer k, the law $\mathscr{L}\big(u^{1,n},\dots,u^{k,n}\big)$ converges $\mathbb{P}_2(d\omega_2)$-almost surely to $\mathscr{L}(u|\lambda)(\omega_2)^{\otimes k}$ when n tends to ∞*

A – Enhancing random noises

We prove Theorem 6 in this section. Recall from (3.5) the definition of the random variable $X\odot\xi$. Write e_k for the function $x\mapsto\exp(i(k,x))$ and $\widehat{\xi}(k)$ for (ξ,e_k). Our noises satisfy the identity

$$\mathbb{E}\big[\widehat{\xi}_t(k)\widehat{\xi}_s(-k')\big] = \mathbf{1}_{k=k'}\,c(t,s)\,\widehat{\eta}(k). \tag{A.1}$$

We denote below by $\textsc{Var}(A)$ the variance of a random variable A.

33 – *Lemma. There exists a positive constant κ such that on has for all $\ell\in\mathbf{N}$, $s,t,a,b\in\mathbf{R}_+$ and $x\in\mathsf{T}^2$, the estimate*

$$\textsc{Var}\Big(\Delta_\ell\big(P_t\xi_s\odot\xi_a\big)(x)\Big) \lesssim \frac{2^{2\ell}2^{2\ell\eta}}{t}\,e^{-\kappa t2^{2\ell}}\big(c(s,s)\,c(a,a)+c(s,a)^2\big)$$

and

$$\textsc{Var}\Big(\Delta_\ell\Big(\big((\mathrm{Id}-P_b)P_t\xi_s\big)\odot\xi_a\Big)(x)\Big) \lesssim b\,\frac{2^{2\ell}2^{2\ell\eta}}{t}\,e^{-\kappa t2^{2\ell-1}}\big(c(s,s)\,c(a,a)+c(s,a)^2\big).$$

Proof – The proof follows closely the proof of Lemma 5.2 in [11]. We have

$$\begin{aligned}\Delta_\ell\big(P_t\xi_s\odot\xi_a\big)(x) &= (2\pi)^{-2}\sum_{k\in\mathbf{Z}^2} e^{i(k,x)}\rho_\ell(k)\mathcal{F}\big(P_t\xi_s\odot\xi_a\big)(k)\\ &= (2\pi)^{-4}\sum_{k_1,k_2\in\mathbf{Z}^2}\sum_{|i-j|\le1}\rho_\ell(k_1+k_2)\rho_i(k_1)\,e^{-t|k_1|^2}\,\widehat{\xi}_s(k_1)\,\rho_j(k_2)\,\widehat{\xi}_a(k_2)\,e_{k_1+k_2}(x),\end{aligned}$$

then $\textsc{Var}\Big(\Delta_\ell(P_t\xi_s\odot\xi_a)(x)\Big)$ is equal to

$$\begin{aligned}&(2\pi)^{-8}\sum_{k_1,k_2,k_1',k_2'}\sum_{|i-j|\le1}\sum_{|i'-j'|\le1}\rho_\ell(k_1+k_2)\,\rho_i(k_1)\,e^{-t|k_1|^2}\rho_j(k_2)\\ &\quad\times\rho_\ell(k_1'+k_2')\,\rho_{i'}(k_1')\,e^{-t|k_1'|^2}\,\rho_{j'}(k_2')\,\textsc{Cov}\Big(\widehat{\xi}_s(k_1)\widehat{\xi}_a(k_2),\widehat{\xi}_s(k_1')\widehat{\xi}_a(k_2')\Big)e_{k_1+k_2+k_1'+k_2'}(x).\end{aligned}$$

Using Wick theorem and the identity A.1 one gets

$$\begin{aligned}
&\mathrm{Cov}\Big(\widehat{\xi}_s(k_1)\widehat{\xi}_a(k_2),\widehat{\xi}_s(k_1')\widehat{\xi}_a(k_2')\Big)\\
&= \mathbb{E}\Big[\widehat{\xi}_s(k_1)\widehat{\xi}_a(k_2)\widehat{\xi}_s(k_1')\widehat{\xi}_a(k_2')\Big] - \mathbb{E}\Big[\widehat{\xi}_s(k_1)\widehat{\xi}_a(k_2)\Big]\mathbb{E}\Big[\widehat{\xi}_s(k_1')\widehat{\xi}_a(k_2')\Big]\\
&= \mathbb{E}\Big[\widehat{\xi}_s(k_1)\widehat{\xi}_s(k_1')\Big]\mathbb{E}\Big[\widehat{\xi}_a(k_2)\widehat{\xi}_a(k_2')\Big] + \mathbb{E}\Big[\widehat{\xi}_s(k_1)\widehat{\xi}_a(k_2')\Big]\mathbb{E}\Big[\widehat{\xi}_s(k_1')\widehat{\xi}_a(k_2)\Big]\\
&= (2\pi)^4\,\widehat{\eta}(k_1)\,\widehat{\eta}(k_2)\Big(\mathbf{1}_{k_1=-k_1',k_2=-k_2'}\,c(s,s)\,c(a,a) + \mathbf{1}_{k_1=-k_2',k_2=-k_1'}\,c(s,a)^2\Big),
\end{aligned}$$

consequently

$$\begin{aligned}
\mathrm{Var}\Big(\Delta_\ell(P_t\xi_s\circ\xi_a)(x)\Big) = \sum_{k_1,k_2}\sum_{|i-j|\le 1}\sum_{|i'-j'|\le 1}&(2\pi)^4\,\widehat{\eta}(k_1)\,\widehat{\eta}(k_2)\,\rho_\ell(k_1+k_2)^2\,\rho_i(k_1)\,\rho_j(k_2)\\
&\times\Big(c(s,s)\,c(a,a)\,\rho_{i'}(k_1)\,\rho_{j'}(k_2)e^{-2t|k_1|^2} + c(s,a)^2\rho_{i'}(k_2)\rho_{j'}(k_1)\,e^{-t|k_1|^2-t|k_2|^2}\Big).
\end{aligned}$$

The factors $\rho_i(k_1)\,\rho_{i'}(k_1)$ and $\rho_i(k_1)\,\rho_{j'}(k_1)$ ensure that one can restrict the sum on i and i' to couples (i,i') such that $\frac{1}{\mu}|i|\le|i'|\le\mu|i|$ for some constant μ, which will be denoted by $i\sim i'$. Likewise the factor $\rho_\ell(k_1+k_2)$ enables us to restrict the sum to $|i|\ge\frac{1}{\mu'}l$ for some μ'. There exists some $\kappa_0>0$ such that $e^{-2t|k|^2}\lesssim e^{-t\kappa_0 2^{2i}}$ for $k\in\mathrm{supp}(\rho_i)$, so that for some $\kappa>0$

$$\begin{aligned}
&\mathrm{Var}\Big(\Delta_\ell(P_t\xi_s\circ\xi_a)(x)\Big)\\
&\quad\lesssim\big(c(s,s)\,c(a,a)+c(s,a)^2\big)\sum_{i,i',j,j'}\mathbf{1}_{\ell\lesssim i}\mathbf{1}_{i\sim i'\sim j\sim j'}\sum_{k_1,k_2}\mathbf{1}_{\mathrm{supp}(\rho_\ell)}(k_1+k_2)\\
&\qquad\times\mathbf{1}_{\mathrm{supp}(\rho_i)}(k_1)\mathbf{1}_{\mathrm{supp}(\rho_j)}(k_2)2^{2i\eta}e^{-2t\kappa 2^{2i}}\\
&\quad\lesssim\big(c(s,s)\,c(a,a)+c(s,a)^2\big)\sum_{i,l\lesssim i}2^{2i}2^{2\ell}2^{2i\eta}e^{-2t\kappa 2^{2i}}\\
&\quad\lesssim\big(c(s,s)\,c(a,a)+c(s,a)^2\big)\frac{2^{2\ell}2^{2\ell\eta}}{t}e^{-2t\kappa 2^{2\ell}},
\end{aligned}$$

hence the first estimate. For the second estimate we notice that the $e^{-t|k_1|^2}$ is replaced by $(1-e^{-b|k_1|^2})\,e^{-t|k_1^2|}$ and that

$$(1-e^{-b|k_1|^2})\,e^{-t|k_1^2|}\le b|k_1|^2e^{-t|k_1|^2}\lesssim ve^{-t|k_1|^2/2}$$

The remainder of the proof is the same as for the first estimate. ▷

We can now prove Theorem 6. We will estimate $\mathbb{E}\big[\big\|\big(X\odot\xi\big)(t)-\big(X\odot\xi\big)(s)\big\|^{2p}_{B^{2\alpha-2}_{2p,2p}}\big]$ in order to use Kolmogorov continuity criterion and Besov embedding. For $0<s\le t$, write

$$\begin{aligned}
&\int_0^t P_{t-a}(\xi_a)\odot\xi_t\,da-\int_0^s P_{s-a}(\xi_a)\odot\xi_s\,da\\
&\quad=\int_0^s\big((P_{t-s}-\mathrm{Id})P_{s-a}(\xi_a)\big)\odot\xi_t\,da+\int_0^s P_{s-a}(\xi_a)\odot(\xi_t-\xi_s)\,da+\int_s^t P_{t-a}(\xi_a)\odot\xi_t\,da\\
&\quad=:\int_0^s\widetilde{A}_1(a)\,da+\int_0^s\widetilde{A}_2(a)\,da+\int_s^t\widetilde{A}_3(a)\,da,
\end{aligned}$$

and set

$$A_i:=\widetilde{A}_i-\mathbb{E}\big[\widetilde{A}_i\big]\qquad(i\in[\![1,3]\!]).$$

The quantity $\mathbb{E}\big[\big\|\big(X\odot\xi\big)(t)-\big(X\odot\xi\big)(s)\big\|^{2p}_{B^{2\alpha-2}_{2p,2p}}\big]$ is equal to

$$
\begin{aligned}
&\sum_{\ell\geq -1} 2^{2p\ell(2\alpha-2)} \int_{\mathsf{T}^2} \mathbb{E}\Big[\Big|\Delta_\ell\Big(\big(X\odot\xi\big)(t)-\big(X\odot\xi\big)(s)\Big)\Big|^{2p}\Big]\\
&= \sum_{\ell\geq -1} 2^{2p\ell(2\alpha-2)} \int_{\mathsf{T}^2} \mathbb{E}\Big[\Big|\int_0^s \Delta_\ell A_1(a)\,da + \int_0^s \Delta_\ell A_2(a)\,da + \int_s^t \Delta_\ell A_3(a)\,da\Big|^{2p}\Big].
\end{aligned}
$$

From gaussian hypercontractivity we have

$$
\begin{aligned}
&\mathbb{E}\Big[\Big|\int_0^s \Delta_\ell A_1(a)\,da + \int_0^s \Delta_\ell A_2(a)\,da + \int_s^t \Delta_\ell A_3(a)\,da\Big|^{2p}\Big]\\
&\leq \mathbb{E}\Big[\int_0^s \big|\Delta_\ell A_1(a)\big|\,da + \int_0^s \big|\Delta_\ell A_2(a)\big|\,da + \int_s^t \big|\Delta_\ell A_3(a)\big|\,da\Big]^{2p}\\
&\lesssim \Big(\int_0^s \mathbb{E}\big[\big|\Delta_\ell A_1(a)\big|^2\big]^{1/2}\,da\Big)^{2p} + \Big(\int_0^s \mathbb{E}\big[\big|\Delta_\ell A_2(a)\big|^2\big]^{1/2}\,da\Big)^{2p} + \Big(\int_s^t \mathbb{E}\big[\big|\Delta_\ell A_3(a)\big|^2\big]^{1/2}\,da\Big)^{2p}
\end{aligned}
$$

So that the bounds for $\mathbb{E}\big[\|\big(X\odot\xi\big)(t)-\big(X\odot\xi\big)(s)\|^{2p}_{B^{2\alpha-2}_{2p,2p}}\big]^{1/(2p)}$ becomes

$$
\Big(\sum_{\ell\geq -1} 2^{2p\ell(2\alpha-2)} \int_{\mathsf{T}^2} \mathbb{E}\Big[\Big|\int_0^s \Delta_\ell A_1(a)\,da + \int_0^s \Delta_\ell A_2(a)\,da + \int_s^t \Delta_\ell A_3(a)\,da\Big|^{2p}\Big]\Big)^{\frac{1}{2p}}
$$

$$
\begin{aligned}
&\lesssim \sum_{\ell\geq -1} 2^{\ell(2\alpha-2)}\Big(\int_0^s \mathbb{E}\big[\big|\Delta_\ell A_1(a)\big|^2\big]^{\frac12}\,da + \int_0^s \mathbb{E}\big[\big|\Delta_\ell A_2(a)\big|^2\big]^{\frac12}\,da + \int_s^t \mathbb{E}\big[\big|\Delta_\ell A_3(a)\big|^2\big]^{\frac12}\,da\Big)\\
&=: S_1+S_2+S_3
\end{aligned}
$$

We now use Lemma 33 to estimate the $\mathbb{E}\big[\big|\Delta_\ell A_i(a)\big|^2\big]$. First we have

$$
\begin{aligned}
\mathbb{E}\big[\big|\Delta_\ell A_1(a)\big|^2\big] &= \mathrm{Var}\Big(\Delta_\ell\Big(\big((P_{t-s}-\mathrm{Id})P_{s-a}\xi_a\big)\odot\xi_t\Big)\Big)\\
&\lesssim (t-s)\frac{2^{2\ell}2^{2\ell\eta}}{s-a}\, e^{-\kappa(s-a)2^{2\ell}}\big(c(s,s)\,c(a,a)+c(s,a)^2\big)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\big[\big|\Delta_\ell A_2(a)\big|^2\big] &= \mathrm{Var}\Big(\Delta_\ell\Big(\big(P_{s-a}\xi_a\big)\odot\big(\xi_t-\xi_s\big)\Big)\Big)\\
&\lesssim \frac{2^{2\ell}2^{2\ell\eta}}{s-a}\, e^{-\kappa(s-a)2^{2\ell}}\Big(c(a,a)\big(c(t,t)+c(s,s)-2c(s,t)\big)\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\big[\big|\Delta_\ell A_3(a)\big|^2\big] &= \mathrm{Var}\Big(\Delta_\ell\Big(\big(P_{t-a}\xi_a\big)\odot\xi_t\Big)\Big)\\
&\lesssim \frac{2^{2\ell}2^{2\ell\eta}}{t-a} e^{-\kappa(t-a)2^{2\ell}}\big(c(t,t)c(a,a)+c(t,a)^2\big).
\end{aligned}
$$

So, writing c_{st} for $c(t,t)+c(s,s)-2c(s,t)$, we get

$$
\begin{aligned}
\int_0^s \mathbb{E}\Big[\big|\Delta_\ell A_1(a)\big|^2\Big]^{1/2}\,da &\lesssim (t-s)^{\frac12}2^{\ell}2^{2\ell\eta}\int_0^s e^{-\kappa(s-a)2^{2\ell-1}}\,\frac{da}{(s-a)^{1/2}}\\
\int_0^s \mathbb{E}\Big[\big|\Delta_\ell A_2(a)\big|^2\Big]^{1/2}\,da &\lesssim c_{st}^{\frac12}\,2^{\ell}2^{2\ell\eta}\int_0^s e^{-\kappa(s-a)2^{2\ell-1}}\,\frac{da}{(s-a)^{1/2}}\\
\int_s^t \mathbb{E}\Big[\big|\Delta_\ell A_3(a)\big|^2\Big]^{1/2}\,da &\lesssim 2^{\ell}2^{2\ell\eta}\int_s^t e^{-\kappa(t-a)2^{2\ell-1}}\,\frac{da}{(t-a)^{1/2}}.
\end{aligned}
$$

We have

$$S_1 \lesssim (t-s)^{\frac{1}{2}} \sum_{\ell \geq -1} 2^{\ell(2\alpha+2\eta-1)} \int_0^s e^{-\kappa(s-a)2^{2\ell-1}} \frac{da}{(s-a)^{1/2}}$$
$$\lesssim (t-s)^{\frac{1}{2}} \int_0^s \int_{-1}^{+\infty} 2^{x(2\alpha+2\eta-1)} e^{-\kappa(s-a)2^{2x-1}} \frac{dx\, da}{(s-a)^{1/2}}$$
$$\lesssim (t-s)^{\frac{1}{2}} \int_0^s \int_0^{+\infty} (s-a)^{-\alpha-\eta} y^{2\alpha+2\eta-2} e^{-\kappa y^2/2}\, dy\, da$$

and similarly

$$S_2 \lesssim c_{st}^{\frac{1}{2}} \sum_{\ell \geq -1} 2^{\ell(2\alpha-1)} \int_0^s e^{-\kappa(s-a)2^{2\ell-1}} \frac{da}{(s-a)^{1/2}}$$
$$\lesssim c_{st}^{\frac{1}{2}} \int_0^s \int_0^{+\infty} (s-a)^{-\alpha-\eta} y^{2\alpha+2\eta-2} e^{-\kappa y^2/2}\, dy\, da$$

and

$$S_3 \lesssim \sum_{\ell \geq -1} 2^{\ell(2\alpha-1)} \int_s^t e^{-\kappa(t-a)2^{2\ell-1}} \frac{da}{(t-a)^{1/2}}$$
$$\lesssim \int_s^t \int_0^{+\infty} (t-a)^{-\alpha-\eta} y^{2\alpha+2\eta-2} e^{-\kappa y^2/2} \mathrm{d}y\, da.$$

Finally we see that

$$\mathbb{E}\Big[\big\|\big(X \odot \xi\big)(t) - \big(X \odot \xi\big)(s)\big\|^{2p}_{B^{2\alpha-2}_{2p,2p}}\Big] \lesssim \Big((t-s)^{1/2} + c_{st}^{1/2} + (t-s)^{1-\alpha-\eta}\Big)^{2p}$$
$$\lesssim |t-s|^{2pm},$$

with

$$m := \min\big\{1/2,\ \ \delta/2,\ \ 1-\alpha-\eta\big\}.$$

From Kolmogorov continuity criterion and Besov embedding, for every $\alpha < 1$ and $1 \leq p < \infty$ the process $X \odot \xi$ is almost surely an element of $C_T^{m-1/p} C^{2\alpha-2-1/p}(\mathsf{T}^2)$.

The mollifier approximation result in the statement of Theorem 6 comes from the same arguments and calculations writing

$$(X \odot \xi)(t) - \Big((X^\varepsilon \odot \xi^\varepsilon)(t) - \mathbb{E}\big[(X^\varepsilon \odot \xi^\varepsilon)(t)\big]\Big)$$
$$= \int_0^t \Big(P_{t-a}(\xi_a - \xi_a^\varepsilon) \odot \xi_t - \mathbb{E}\big[P_{t-a}(\xi_a - \xi_a^\varepsilon) \odot \xi_t\big]\Big) da$$
$$+ \int_0^t \Big(P_{t-a}\xi_a^\varepsilon \odot (\xi_t - \xi_t^\varepsilon) - \mathbb{E}\big[P_{t-a}\xi_a^\varepsilon \odot (\xi_t - \xi_t^\varepsilon)\big]\Big) da.$$

If φ is the fourier transform of the mollifier, we have

$$\widehat{\xi^\varepsilon}(k) = \varphi(k\varepsilon)\, \widehat{\xi}(k),$$

and the same calculations as in the proof of Lemma 33 give

$$\textsc{Var}\Big(\Delta_\ell(P_{t-a}(\xi_a - \xi_a^\varepsilon) \odot \xi_t)(x)\Big)$$
$$\lesssim \sum_{i,i',j,j'} \mathbf{1}_{l \lesssim i} \mathbf{1}_{i \sim i' \sim j \sim j'} \sum_{k_1,k_2} (1-\varphi(k_1\varepsilon)) \mathbf{1}_{\mathrm{supp}(\rho_\ell)}(k_1+k_2) \mathbf{1}_{\mathrm{supp}(\rho_i)}(k_1) \mathbf{1}_{\mathrm{supp}(\rho_j)}(k_2) 2^{2i\eta} e^{-2t\kappa 2^{2i}}$$
$$\lesssim \sum_i \mathbf{1}_{\ell \lesssim i} \sum_{k_1,k_2} (1-\varphi(k_1\varepsilon)) \mathbf{1}_{\mathrm{supp}(\rho_\ell)}(k_1+k_2) \mathbf{1}_{\mathrm{supp}(\rho_i)}(k_1) \mathbf{1}_{\mathrm{supp}(\rho_j)}(k_2) 2^{2i\eta} e^{-2t\kappa 2^{2i}}.$$

For some integer $N = N(\varepsilon)$, one can decompose the last sum as

$$\sum_{i\leq N} \mathbf{1}_{\ell\lesssim i} \sum_{k_1,k_2} (1-\varphi(k_1\varepsilon))\mathbf{1}_{\mathrm{supp}(\rho_\ell)}(k_1+k_2)\mathbf{1}_{\mathrm{supp}(\rho_i)}(k_1)\mathbf{1}_{\mathrm{supp}(\rho_j)}(k_2)2^{2i\eta}e^{-2t\kappa 2^{2i}}$$
$$+\sum_{i>N} \mathbf{1}_{\ell\lesssim i} \sum_{k_1,k_2} (1-\varphi(k_1\varepsilon))\mathbf{1}_{\mathrm{supp}(\rho_\ell)}(k_1+k_2)\mathbf{1}_{\mathrm{supp}(\rho_i)}(k_1)\mathbf{1}_{\mathrm{supp}(\rho_j)}(k_2)2^{2i\eta}e^{-2t\kappa 2^{2i}}$$
$$\lesssim \sup_{|x|\leq N} \big(1-\varphi(x\varepsilon)\big)\frac{2^{2\ell}2^{2\ell\eta}}{t-a}\,e^{-\kappa(t-a)2^{2\ell}} + \frac{2^{2N}2^{2N\eta}}{t-a}\,e^{-\kappa(t-a)2^{2N}}$$

So that choosing $N(\varepsilon)$ such that $N(\varepsilon)\to\infty$ and $\varepsilon N(\varepsilon)\to 0$ as ε goes to zero, one gets

$$\textsc{Var}\Big(\Delta_\ell\Big(P_{t-a}(\xi_a-\xi_a^\varepsilon)\odot\xi_t(x)\Big)\Big) \lesssim \psi_\ell(\varepsilon)\,\frac{2^{2\ell(1+\eta)}}{t-a}\,e^{-\kappa(t-a)2^{2\ell}},$$

where $0\leq\psi_\ell(\varepsilon)\leq 1$ tends to 0 as $\varepsilon>0$ goes to 0. Likewise one has

$$\textsc{Var}\Big(\Delta_\ell\Big(P_{t-a}\xi_a^\varepsilon\odot(\xi_t-\xi_t^\varepsilon)(x)\Big)\Big) \lesssim \psi_\ell(\varepsilon)\,\frac{2^{2\ell(1+\eta)}}{t-a}\,e^{-\kappa(t-a)2^{2\ell}}.$$

The same calculations as above give for

$$\mathbb{E}\Big[\big\|(X\odot\xi)(t)-\big((X^\varepsilon\odot\xi^\varepsilon)(t)-\mathbb{E}\big[(X^\varepsilon\odot\xi^\varepsilon)(t)\big]\big)\big\|^{2p}_{B^{2\alpha-2}_{2p,2p}}\Big]$$

the bound

$$\sum_{\ell\geq 0}\psi_\ell(\varepsilon)\,2^{\ell(2\alpha+2\eta-2)}\int_0^t \mathbb{E}\big[\big|\Delta_\ell A_3(a)\big|^2\big]^{\frac12}da$$

The result follows from dominated convergence argument as the series

$$\sum_{\ell\geq 0}2^{\ell(2\alpha+2\eta-2)}\int_0^t \mathbb{E}\big[\big|\Delta_\ell A_3(a)\big|^2\big]^{\frac12}da$$

is seen to be convergent.

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