3. Rough paths

Guide for this section

Hölder p-rough paths, which control the rough differential equations

$$dx_t = \mathrm{F}(x_t)\mathbf{X}(dt), \quad d\varphi_t = \mathrm{F}^{\otimes}\mathbf{X}(dt),$$

and play the role of the control h in the model classical ordinary differential equation

$$dx_t = V_i(x_t)\, dh_t^i = \mathrm{F}(x_t)\, dh_t$$

are defined in section 3.1.2. As $\mathbb{R}^\ell$-valued paths, they are not regular enough for the formula

$$\mu_{ts}(x) = x + X_{ts}^i V_i(x)$$

to define an approximate flow, as in the classical Euler scheme studied in exercice 1. The missing bit of information needed to stabilize the situation is a substitute of the non-existing iterated integrals $\int_s^t X_r^j dX_r^k$, and higher order iterated integrals, which provide a partial description of what happens to X during any time interval (s,t). A (Hölder) p-rough path is a *multi-level object* whose higher order parts provide precisely that information. We saw in the introduction that iterated integrals appear naturally in Taylor-Euler expansions of solutions to ordinary differential equations; they provide higher order numerical schemes like Milstein' second order scheme. It is an important fact that p-rough paths take values in a very special kind of algebraic structure, whose basic features are explained in section 3.1.1. A Hölder p-rough path will then appear as a kind of $\frac{1}{p}$-Hölder path in that space. We shall then study in section 3.2 the space of p-rough path for itself.

3.1. **Definition of a p-rough path.** Iterated integrals, as they appear for instance in the form $\int_s^t \int_s^y dh_r^j\, dh_u^k$ or $\int_s^t \int_s^y \int_s^r (\cdots)$ are multi-indexed quantities. A useful formalism to with such object is provided by the notion of tensor produc. We first start our investigation by recalling some elementary facts about that notion. Eventually, all what will be used for practical computations on rough differential equations will be a product operation very similar to the product operation on polynomials. This abstract setting however greatly clarifies the meaning of these computations.

3.1.1. *An algebraic prelude: tensor algebra over $\mathbb{R}^\ell$ and free nilpotent Lie group.* Let first recall what the algebraic tensor product $\mathrm{U} \otimes \mathrm{V}$ of any two Banach spaces U and V is. Denote by V' the set of all continuous linear forms on V. Given $u \in \mathrm{U}$ and $v \in \mathrm{V}$, we define a continuous linear map on V' setting

$$(u \otimes v)(v') = (v', v)\, u,$$

for any $v' \in\in \mathrm{V}'$. The algebraic tensor product $\mathrm{U} \otimes \mathrm{V}$ is the set of all finite linear combinations of such maps. Elementary elements $u \otimes v)$ are 1-dimensional rank maps. Note that an element of $\mathrm{U} \otimes \mathrm{V}$ can have several different decompositions as a sum of elementary elements; this has no consequences as they all define the same map from V' to U.

As an example, $(\mathbb{R}^\ell)\otimes(\mathbb{R}^\ell)'$ is the set of all linear maps from $\mathbb{R}^\ell$ to itself that is $\mathrm{L}(\mathbb{R}^\ell)$. We keep that interpretation for $(\mathbb{R}^\ell)\otimes(\mathbb{R}^\ell)$ as $\mathbb{R}^\ell$ and $(\mathbb{R}^\ell)'$ are canonically identified. To see which element of $\mathrm{L}(\mathbb{R}^\ell)$ corresponds to $u\otimes v$, it suffices to look at the image of the j^{th} vector ϵ_j of the canonical basis by the map; it gives the j^{th} column of the matrix of $u\otimes v$ in the canonical basis. We have

$$(u\otimes v)\big(\epsilon_j\big)=(v,\epsilon_j)\,u.$$

For $N\in\mathbb{N}\cup\{\infty\}$, write $T_\ell^{(N)}$ for the direct sum $\bigoplus_{r=0}^{N}\big(\mathbb{R}^\ell\big)^{\otimes r}$, with the convention that $\big(\mathbb{R}^\ell\big)^{\otimes 0}$ stands for $\mathbb{R}$. Denote by $\mathbf{a}=\overset{N}{\underset{r=0}{\oplus}}a^r$ and $\mathbf{b}=\overset{N}{\underset{r=0}{\oplus}}b^r$ two generic elements of $T_\ell^{(N)}$. The vector space $T_\ell^{(N)}$ is an algebra for the operations

$$\begin{aligned}
&\mathbf{a}+\mathbf{b}=\overset{N}{\underset{r=0}{\oplus}}(a^r+b^r),\\
&\mathbf{ab}=\overset{N}{\underset{r=0}{\oplus}}c^r,\quad\text{with }\ c^r=\sum_{k=0}^{r}a^k\otimes b^{r-k}\ \in(\mathbb{R}^\ell)^{\otimes r}
\end{aligned}\tag{3.1}$$

It is called the **(truncated) tensor algebra of** $\mathbb{R}^\ell$ (if N is finite). Note the similarity between these rules and the analogue rules for addition and product of polynomials.

The exponential map $\exp: T_\ell^{(\infty)}\to T_\ell^{(\infty)}$ and the logarithm map $\log: T_\ell^{(\infty)}\to T_\ell^{(\infty)}$ are defined by the usual series

$$\exp(\mathbf{a})=\sum_{n\geqslant 0}\frac{\mathbf{a}^n}{n!},\qquad \log(\mathbf{b})=\sum_{n\geqslant 1}\frac{(-1)^n}{n}(1-\mathbf{b})^n,\tag{3.2}$$

with the convention $bfa^0=1\in\mathbb{R}\subset T_\ell^{(\infty)}$. Denote by $\pi_N: T_\ell^{(\infty)}\to T_\ell^{(N)}$ the natural projection. We also denote by exp and log the restrictions to $T_\ell^{(N)}$ of the maps $\pi_N\circ\exp$ and $\pi_N\circ\log$ respectively. Denote by $T_\ell^{(N),1}$, resp. $T_\ell^{(N),0}$, the elements $a_0\oplus\cdots\oplus c_N$ of $T_\ell^{(N)}$ such that $a_0=0$, resp. $a_0=1$. All the elements of $T_\ell^{(N),1}$ are invertible, and $\exp: T_\ell^{(N),0}\to T_\ell^{(N),1}$ and $\log: T_\ell^{(N),1}\to T_\ell^{(N),0}$ are smooth reciprocal bijections.

The set $T_\ell^{(N),1}$ is naturally equipped with a norm defined by the formula

$$\|\mathbf{a}\|:=\sum_{i=1}^{\ell}\big\|a^i\big\|_{\mathrm{Eucl}}^{\frac{1}{i}},$$

where $\big\|a^i\big\|_{\mathrm{Eucl}}$ stands for the Euclidean norm of $a^i\in(\mathbb{R}^\ell)^{\otimes i}$ identified with an element of $\mathbb{R}^{\ell^i}$ by looking at its coordinates in the canonical basis. The choice of power $\frac{1}{i}$ comes from the fact that $T_\ell^{(N),1}$ is naturally equipped with a *dilation* operation

$$\delta_\lambda(\mathbf{a})=\big(1,\lambda a^1,\dots,\lambda^N a^N\big),$$

so the norm $\|\cdot\|$ is homogeneous with respect to this dilation, in the sense that one has

$$\left\|\delta_\lambda(\mathbf{a})\right\| = |\lambda|\|\mathbf{a}\|$$

for all $\lambda \in \mathbb{R}$, and all $\mathbf{a} \in T_\ell^{(N),1}$.

The formula $[\mathbf{a}, \mathbf{b}] = \mathbf{ab} - \mathbf{ba}$, defines a Lie bracket on $T_\ell^{(N)}$. Define inductively $F = F^1 = \mathbb{R}^\ell$, considered as a subset of $T_\ell^{(\infty)}$, and $F^{n+1} = [F, F^n] \subset T_\ell^{(\infty)}$.

Definition 1.
- *The Lie algebra $\mathfrak{g}_\ell^N$ generated by the $F^1, \dots, F^N$ in $T_\ell^{(N)}$ is called the N-**step free nilpotent Lie algebra**.*
- *As a consequence of Baker-Campbell-Hausdorf-Dynkin formula, the subset $\exp\left(\mathfrak{g}_\ell^N\right)$ of $T_\ell^{(N),1}$ is a group for the multiplication operation. It is called the N-**step nilpotent Lie group** on $\mathbb{R}^\ell$ and denoted by $G_\ell^{(N)}$.*

As all finite dimensional Lie groups, the N-step nilpotent Lie group is equipped with a natural (sub-Riemannian) distance inherited from its manifold structure. Its definition rests on the fact that the element $\mathbf{a}u$ of $T_\ell^{(N)}$ is for any $\mathbf{a} \in G_\ell^{(N)}$ and $u \in \mathbb{R}^\ell \subset T_\ell^{(N)}$ a tangent vector to $G_\ell^{(N)}$ at point $\mathbf{a}$ (as u is tangent to $G_\ell^{(N)}$ at the identity and tnagent vectors are transported by left translation in the group). So the ordinary differential equation

$$d\mathbf{a}_t = \mathbf{a}_t \, \dot{h}_t$$

makes sense for any $\mathbb{R}^\ell$-valued smooth control h, and defines a path in $G_\ell^{(N)}$ started from the identity. We define the size $|\mathbf{a}|$ of $\mathbf{a}$ by the formula

$$|\mathbf{a}| = \inf \int_0^1 \left|\dot{h}_t\right| dt,$$

where the infimum is over the set of all smooth controls h such that $\mathbf{a}_1 = \mathbf{a}$. This set is non-empty as $\mathbf{a} \in \exp\left(\mathfrak{g}^N\right)$ can be written as $\mathbf{a}_1$ for some piecewise $\mathcal{C}^1$ control, as a consequence of a theorem of sub-Riemannian geometry due to Chow; see for instance the textbook [11] for a nice account of that theorem. The distance between any two points $\mathbf{a}$ and $\mathbf{b}$ of $G_\ell^{(N)}$ is then defined as $\left|\mathbf{a}^{-1}\mathbf{b}\right|$. It is homogeneous in the sense that if $\mathbf{a} = \exp(u)$, with $u \in \mathbb{R}^\ell \subset T_\ell^{(N)}$, then $\left|\exp(\lambda u)\right| = |\lambda||\mathbf{a}|$, for all $\lambda \in \mathbb{R}$ and all $u \in \mathbb{R}^\ell \subset T_\ell^{(N)}$.

This way of defining a distance is intrinsic to $G_\ell^{(N)}$ and classical in geometry. From an extrinsic point of view, one can also consider $G_\ell^{(N)}$ as a subset of $T_\ell^{(N)}$ and use the ambiant metric to define the distance between any two points $\mathbf{a}$ and $\mathbf{b}$ of $G_\ell^{(N)}$ as $\left\|\mathbf{a}^{-1}\mathbf{b}\right\|$. It can be proved (this is elementary, see e.g. proposition 10 in Appendix A of [12], pp. 76-77) that the two norms $|\cdot|$ and $\|\cdot\|$ on $G_\ell^{(N)}$ are equivalent, so one can equivalently work with one or the other, depending on the context. This will be useful in defining the Brownian rough path for example.

3.1.2. *Definition of a p-rough path.* The relevance of the algebraic framework provided by the N-step nilpotent Lie group for the study of smooth paths was first noted by Chen in his seminal work [13]. Indeed, for any $\mathbb{R}^\ell$-valued smooth path $(x_s)_{s\geqslant 0}$, the family of iterated integrals

$$\mathfrak{X}^N_{ts} := \left(1, x_t - x_s, \int_s^t \int_s^{s_1} dx_{s_2} \otimes dx_{s_1}, \dots, \int_{s\leqslant s_1 \leqslant \cdots \leqslant s_N \leqslant t} dx_{s_1} \otimes \cdots \otimes dx_{s_N}\right)$$

defines for all $0 \leqslant s \leqslant t$ an element of $T^{(N),1}_\ell$ with the property that if $x_\bullet$ is scaled into $\lambda x_\bullet$ then $\mathfrak{X}^N$ becomes $\delta_\lambda \mathfrak{X}^N$. We actually have $\mathfrak{X}^N_{ts} \in G^{(N)}_\ell$. To see that, notice that, as a function of t, the function $\mathfrak{X}^N_{ts}$ satisfies the differential equation

$$d\mathfrak{X}^N_{ts} = \mathfrak{X}^N_{ts}\, dx_t,$$

in $T^{(N)}_\ell$ driven by the $\mathbb{R}^\ell$-valued smooth contro x, so it defines a $G^{(N)}_\ell$-valued path as an integral curve of a field of tangent vectors. The above differential equation also makes it clear the we have the following *Chen relations*

$$\mathfrak{X}^N_{ts} = \mathfrak{X}^N_{us}\, \mathfrak{X}^N_{tu},$$

for all $0 \leqslant s \leqslant u \leqslant t$; they imply in particular the identity

$$\mathfrak{X}^N_{ts} = \left(\mathfrak{X}^N_{s0}\right)^{-1} \mathfrak{X}^N_{t0},$$

which is here nothing but the "flow" property for ordinary differential equation solutions. Rough paths and weak geometric rough paths are somehow an abstract version of this family of iterated integrals.

DEFINITION **2.** *Let $2 \leqslant p$. A* **Hölder p-rough path on** $[0,T]$ *is a $T^{([p]),1}_\ell$-valued path $\mathbf{X} : t \in [0,T] \mapsto 1 \oplus X^1_t \oplus X^2_t \oplus \cdots \oplus X^{[p]}_t$ such that*

$$\left\|X^i\right\|_{\frac{i}{p}} := \sup_{0\leqslant s<t\leqslant T} \frac{|X^i_{ts}|}{|t-s|^{\frac{i}{p}}} < \infty, \tag{3.3}$$

for all $i = 1 \dots [p]$, where we set $\mathbf{X}_{ts} = \mathbf{X}_s^{-1}\mathbf{X}_t$. We define the norm of $\mathbf{X}$ to be

$$\|\mathbf{X}\| := \max_{i=1\dots[p]} \left\|X^i\right\|_{\frac{i}{p}}, \tag{3.4}$$

and a distance $d(\mathbf{X},\mathbf{Y}) = \|\mathbf{X}-\mathbf{Y}\|$ on the set of Hölder p-rough path. A **Hölder weak geometric p-rough path on** $[0,T]$ *is a $G^{([p])}_\ell$-valued p-rough path.*

So a (weak geometric) Hölder p-rough path is in a way nothing but a $T^{(N),1}_\ell$ (or $G^{(N)}_\ell$)-valued $\frac{1}{p}$-Hölder continuous path, for the $\|\cdot\|$-norm introduced above and the use of $\mathbf{X}_s^{-1}\mathbf{X}_t$ in place of the usual $\mathbf{X}_t - \mathbf{X}_s$. Note that the Chen relation

$$\mathbf{X}_{ts} = \mathbf{X}_{us}\mathbf{X}_{tu}$$

is granted by the definition of $\mathbf{X}_{ts} = \mathbf{X}_s^{-1}\mathbf{X}_t$.

For $2 \leqslant p < 3$, Chen's relation is equivalent to

- **(i)** $X^1_{ts} = X^1_{tu} + X^1_{us}$,
- **(ii)** $X^2_{ts} = X^2_{tu} + X^1_{us} \otimes X^1_{tu} + X^2_{us}$.

Condition **(i)** means that $X^1_{ts} = X^1_{t0} - X^1_{s0}$ represents the increment of the $\mathbb{R}^d$-valued path $\big(X^1_{r0}\big)_{0\leqslant r\leqslant T}$. Condition **(ii)** is nothing but the analogue of the elementary property $\int_s^t\int_s^r = \int_s^u\int_s^r + \int_u^t\int_s^u + \int_u^t\int_u^r$, satisfied by any reasonable notion of integral on $\mathbb{R}$ that satisfies the Chasles relation

$$\int_s^t = \int_s^u + \int_u^t$$

This remark justifies thinking of the $\big(\mathbb{R}^\ell\otimes\mathbb{R}^\ell\big)$-part of a rough path as a kind of iterated integral of X^1 against itself, although this hypothetical iterated integral does not make sense in itself for lack of an integration operation for a general Hölder path in $\mathbb{R}^\ell$. In that setting, a p-rough path $\mathbf{X}$ is a weak geometric p-rough path iff the symmetric part of X^2_{ts} is $\frac{1}{2}X^1_{ts}\otimes X^1_{ts}$, for all $0\leqslant s\leqslant t\leqslant T$.

Note that the space of Hölder p-rough paths is not a vector space; this prevents the use of the classical Banach space calculus.

It is clear that considering the iterated integrals of any given smooth path defines a p-rough path above it, for any $p\geqslant 2$. This *lift is not unique*, as if we are given a p-rough path $\mathbf{X} = \big(X^1, X^2\big)$, with $2\leqslant p<3$ say, and any $\frac{2}{p}$-Hölder continuous $(\mathbb{R}^\ell)^{\otimes 2}$-valued path $\big(M_t\big)_{0\leqslant t\leqslant 1}$, we define a new rough path setting $\mathbb{M}_{ts} = M_t - M_s$, and

$$\mathbf{X}'_{ts} = \big(X^1_{ts}, X^2_{ts} + \mathbb{M}_{ts}\big)$$

for all $0\leqslant s\leqslant t\leqslant 1$. Relations **(i)** and **(ii)** above are indeed easily checked.

Last, note that a Hölder p-rough path is also a Hölder q-rough path for any $p<q<[p]+1$.

3.2. **The metric space of p-rough paths.** The distance d defined in definition 2 is actually not a distance since only the increments $\mathbf{X}_{ts} - \mathbf{Y_{ts}}$ are taken into account. We define a proper metric on the set of all Hölder p-rough paths setting

$$\overline{d}(\mathbf{X},\mathbf{Y}) = \big|X^1_0 - Y^1_0\big| + d(\mathbf{X},\mathbf{Y}).$$

Proposition **3.** *The metric $\overline{d}$ turns the set of all Hölder p-rough paths into a (non-separable) complete metric space.*

Proof – Given a Cauchy sequence of Hölder p-rough paths ${}^{(n)}\mathbf{X}$, there is no loss of generality in supposing that their first level starts from the same point in $\mathbb{R}^\ell$. It follows from the uniform Hölder bounds for $\Big\|{}^{(n)}X^i_{ts} - {}^{(m)}X^i_{ts}\Big\|_{\frac{i}{p}}$, and (an easily proved version of) Ascoli-Arzela theorem (for 2-parameter maps) that ${}^{(n)}X$ converges uniformly to some Hölder p-rough path $\mathbf{X}$. To prove the convergence of ${}^{(n)}X$ to $\mathbf{X}$ in d-distance, it suffices to send m to infinity in the inequality

$$\Big|{}^{(n)}X^i_{ts} - {}^{(m)}X^i_{ts}\Big| \leqslant \epsilon\,|t-s|^{\frac{i}{p}},$$

which holds for all n,m bigger than some N_ϵ, uniformly with respect to $0\leqslant s\leqslant t\leqslant 1$.

An uncountable family of $\mathbb{R}^\ell$-valued $\frac{1}{p}$-Hölder continuous functions at pairwise $\frac{1}{p}$-Hölder distance bounded below by a positive constant is constructed in example

5.28 of [3]. As the set of all first levels of the set of Hölder p-rough paths is a subset of the set of $\mathbb{R}^\ell$-valued $\frac{1}{p}$-Hölder paths, this examples implies the non-separability of set of all Hölder p-rough paths. $\triangleright$

The following *interpolation result* will be useful in several places to prove rough paths convergence results at a cheap price.

PROPOSITION **4.** *Assume ${}^{(n)}\mathbf{X}$ is a sequence of Hölder p-rough paths with uniform bounds*

$$\sup_n \left\| {}^{(n)}X \right\| \leqslant C < \infty, \tag{3.5}$$

which converge pointwise, in the sense that ${}^{(n)}\mathbf{X}_{ts}$ converges to some $\mathbf{X}_{ts}$ for each $0 \leqslant s \leqslant t \leqslant 1$. Then the limit object $\mathbf{X}$ is a Hölder p-rough path, and ${}^{(n)}\mathbf{X}$ converges to $\mathbf{X}$ as a Hölder q-rough path, for any $p < q \leqslant [p]+1$.

PROOF – (Following the solution of exercice 2.9 in [5]) The fact that $\mathbf{X}$ is a Hölder p-rough path is a direct consequence of the uniform bounds (3.5) and pointwise convergence:

$$\left|X^i_{ts}\right| = \lim_n \left|{}^{(n)}X^i_{ts}\right| \leqslant C|t-s|^{\frac{i}{p}}.$$

Would the convergence of ${}^{(n)}\mathbf{X}$ to $\mathbf{X}$ be uniform, we could find a sequence ϵ_n decreasing to 0, such that, uniformly in s,t,

$$\left|X^i_{ts} - {}^{(n)}X^i_{ts}\right| \leqslant \epsilon_n, \quad \left|X^i_{ts} - {}^{(n)}X^i_{ts}\right| \leqslant 2C|t-s|^{\frac{i}{p}}.$$

Using the geometric interpolation $a\wedge b \leqslant a^{1-\theta}b^\theta$, with $\theta = \frac{p}{q} < 1$, we would have

$$\left|X^i_{ts} - {}^{(n)}X^i_{ts}\right| \leqslant \epsilon_n^{1-\frac{p}{q}}|t-s|^{\frac{i}{p}},$$

which entails the convergence result as a Hölder q-rough path.
We proceed as follow to see that pointwise convergence suffices to get the result. Given a partition π of $[0,1]$ and any $0 \leqslant s \leqslant t \leqslant 1$, denote by $\overline{s}, \overline{t}$ the nearest points in π to s and t respectively. Writing

$$d\Big(\mathbf{X}_{ts}, {}^{(n)}\mathbf{X}_{ts}\Big) \leqslant d\big(\mathbf{X}_{ts}, \mathbf{X}_{\overline{t}\overline{s}}\big) + d\Big(\mathbf{X}_{\overline{t}\overline{s}}, {}^{(n)}\mathbf{X}_{\overline{t}\overline{s}}\Big) + d\Big({}^{(n)}\mathbf{X}_{\overline{t}\overline{s}}, {}^{(n)}\mathbf{X}_{ts}\Big) \tag{3.6}$$

and the fact that

$$\mathbf{X}_{\overline{t}\overline{s}} = \mathbf{X}_{s\overline{s}}\mathbf{X}_{ts}\mathbf{X}_{\overline{t}t}, \quad {}^{(n)}\mathbf{X}_{\overline{t}\overline{s}} = {}^{(n)}\mathbf{X}_{s\overline{s}}{}^{(n)}\mathbf{X}_{ts}{}^{(n)}\mathbf{X}_{\overline{t}t}$$

and the uniform estimate (3.5) to see that the first and third terms in the above upper bound can be made arbitrarily small by choosing a partition with a small enough mesh, uniformly in s,t and n. The second term is dealt with the pointwise convergence assumption as it involves only finitely many points once the partition π has been chosen as above. $\triangleright$

3.3. **Exercices. 7. Lyons' extension theorem** [10]**.** Let n be a positive integer. An *n-truncated multiplicative functional over $\mathbb{R}^\ell$ in the sense of Chen-Lyons* is a $T_\ell^{(n),1}$-valued map $X = \big(X_{st}\big)_{0\leqslant s\leqslant t\leqslant 1}$, with components X_{st}^k, such that we have

$$X_{st} = X_{su}X_{ut}$$

for all $0\leqslant s\leqslant u\leqslant t\leqslant 1$, that is

$$X_{ts}^i = \sum_{k=0}^{i} X_{su}^k X_{ut}^{i-k}$$

for all $0\leqslant i\leqslant n$. A $T_\ell^{(n),1}$-valued map X is an *n-almost-multiplicative functional* if for every we have

$$\Big|X_{ts}^k - \big(X_{us}X_{tu}\big)^k| \leqslant c\,|t-s|^a$$

for all $0\leqslant s\leqslant t\leqslant 1$ and $0\leqslant k\leqslant n$, for some control ω and some constant $a>1$. Prove that if X is an n-truncated multiplicative functional and Y_{ts}^{n+1} is a continuous $(\mathbb{R}^d)^{\otimes n+1}$-valued map defined for $0\leqslant s\leqslant t\leqslant 1$, such that

$$Y = \big(1, X^1, \dots, X^n, Y^{n+1}\big)$$

is an $(n+1)$-truncated almost-multiplicative functional, then there exists a unique X_{ts}^{n+1} such that

$$\Big|X_{ts}^{n+1} - Y_{ts}^{n+1}\Big| \leqslant c_1\,|t-s|^a$$

holds for some positive constant c_1, and

$$Z = \Big(1, X^1, \dots, X^n, X^{n+1}\Big)$$

is an $(n+1)$-multiplicative functional.

8. Pure area rough path. Let x^n be the $\mathbb{R}^2$-valued path defined in complex notations by the formula

$$x_t^n = \frac{1}{n}\exp\big(2i\pi n^2 t\big),$$

for $0\leqslant t\leqslant 1$. Let $2<p<3$ be given.

a) Show that the natural lift $\mathbf{X}^n = \big(x^n, \mathbb{X}^n\big)$ of x^n to a Hölder p-rough path converges pointwise to the Hölder p-rough path $\mathbf{X} = (X, \mathbb{X})$ with $X=0$ and

$$\mathbb{X}_{ts} = \pi\,(t-s)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

b) Prove the uniform bounds $\sup_n \big\|x^n\big\|_{\frac12} < \infty$ and $\sup_n \big\|\mathbb{X}^n\big\|_{\frac12} < \infty$.

c) Conclude by interpolation that the convergence of $\mathbf{X}^n$ to $\mathbf{X}$ takes place in the space of Hölder p-rough paths.

9. Wide oscillations. Find a widely oscillating piecewise smooth path converging to $(0,0,tI)$ in the space of Hölder p-rough paths, for $3<p<4$. The letter I stands here for the element of $(\mathbb{R}^\ell)^{\otimes 3}$ given in the canonical basis by $I_{ijk} = \delta_{ij}\delta_{jk}$.

10. Lifting α-Hölder paths to rough paths, for $\alpha > \frac12$. Show that using the Young integral defined in exercice 5 one can lift any α-Hölder paths, with $\alpha > \frac12$, into a Hölder p-rough path, for any $p\geqslant 2$.

11. Gubinelli's controlled paths. Let $2 < p < 3$ be given together with a Hölder p-rough path $\mathbf{X} = \big(X, \mathbb{X}\big)$ over $\mathbb{R}^\ell$. A controlled path is somehow a path whose increments look like the increments of the first level of $\mathbf{X}$, in a quantitative way. More precisely, an $\mathbb{R}^d$-valued path $z_\bullet$ is said to be a ***path controlled by*** $\mathbf{X}$ if its increments $Z_{ts} = z_t - z_s$, satisfy

$$Z_{ts} = Z'_s X_{ts} + R_{ts},$$

for all $0 \leqslant s \leqslant t \leqslant 1$, for some $\mathrm{L}\big(\mathbb{R}^\ell, \mathbb{R}^d\big)$-valued $\frac{1}{p}$-Lipschitz map $Z'_\bullet$, and some $\mathbb{R}^d$-valued $\frac{2}{p}$-Lipschitz map R.

a) Prove that one defines a complete metric on the set of $\mathbb{R}^d$-valued paths controlled by $\mathbf{X}$ setting

$$\|z\| := \|Z'\|_{\frac{1}{p}} + \|R\|_{\frac{2}{p}} + \big|z_0\big|,$$

where $\|\cdot\|_\alpha$ stands for the α-Lipscthiz norm.

b) The crucial property of controlled path is that they admit a notural lift into a Hölder p-rough path.

(i) Show that we define an $(\mathbb{R}^d)^{\otimes 2}$-valued almost-additive functional (see exercice 2) setting

$$\mu_{ts} = z_s \otimes \big(z_t - z_s\big) + Z'_s \otimes Z'_s \, \mathbb{X}_{ts},$$

where

$$\big(Z'_s \otimes Z'_s\big)(a \otimes b) = \big(Z'_s(a)\big) \otimes \big(Z'_s(b)\big)$$

for any two elements a, b of $\mathbb{R}^d$.

(ii) Let φ stand for the associated additive functional. Show that (z, φ) is a Hölder p-rough path.

The Banach space structure of the set of $\mathbb{R}^d$-paths controlled by $\mathbf{X}$ is the main motivation for its introduction, in contrast with the non-linear space of Hölder p-rough paths itself, whose study requires non-conventional tools. The use of controlled paths also seems well-motivated from a dynamical point of view, as we expect any solution to a rough differential equation

$$dx_t = \mathrm{F}(x_t)\mathbf{X}(dt)$$

to be locally described by a first order Taylor expansion, in the sense that

$$x_t = x_s + \mathrm{F}(x_s)\mathbf{X}_{ts} + o(\cdot);$$

we even expect a second order Taylor expansion in that setting... We warmly recommand the forthcoming lecture notes [5] for an account of the theory of rough differential equations from Gubinelli's point of view.

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