LARGE DEVIATION PRINCIPLE FOR BRIDGES OF DEGENERATE DIFFUSION PROCESSES

I. BAILLEUL

ABSTRACT. We prove that bridges of subelliptic diffusions on a compact manifold, with different ends, satisfy a large deviation principle in the space of Hölder continuous functions, with a good rate function, when the travel time tends to 0. This leads to the identification of the deterministic first order asymptotics of the distribution of the bridge under generic conditions on the endpoints of the bridge.

1. INTRODUCTION

Let M be a compact, connect and oriented m-dimensonal smooth manifold, and $V_1, \ldots, V_\ell$ be smooth vector fields on M, whose Lie algebra has maximal rank everywhere. Given another vector field V on M, set

$$(1.1) \qquad \mathcal{L} = \frac{1}{2} \sum_{i=1}^{\ell} V_i^2 + V.$$

The semi-group associated with $\mathcal{L}$ has a smooth positive fundamental solution $p_t(z, z')$ with respect to any smooth volume measure VOL on M. Given x, y in M denote by $\Omega^{x,y}$ the set of continuous paths $\omega : [0, 1] \to M$ with $X_0(\omega) = x$ and $X_1(\omega) = y$, where $X_\bullet$ stands for the canonical coordinate process on $\Omega^{x,y}$. For $\epsilon > 0$, we define uniquely a probability measure $\mathbb{P}_\epsilon^{x,y}$ on $\Omega^{x,y}$ defining $\mathbb{P}_\epsilon^{x,y}\big(X_{t_1} \in A_1, \ldots, X_{t_k} \in A_k\big)$ for all $k \geqslant 1$, $0 < t_1 < \cdots < t_k < 1$ and any Borel sets $A_1, \ldots, A_k$ of M, by the formula

$$(1.2) \quad \frac{1}{p_\epsilon(x,y)} \int \left(\prod_{j=1}^{k} \big(p_{t_j\epsilon - t_{j-1}\epsilon}(x_{j-1}, x_j) \mathbf{1}_{A_j}(x_j)\big) \right) p_{\epsilon - t_k \epsilon}(x_k, y) \, \text{VOL}(dx_1) \cdots \text{VOL}(dx_k)$$

where $t_0 = 0$ and $x_0 = x$. This formula describes the law of the diffusion process associated with $\epsilon \mathcal{L}$, conditionned on having position x at time 0 and position y at time 1. By Whitney's embedding theorem, there is no loss of generality in supposing that M is a submanifold of an ambient Euclidean space $\big(\mathbb{R}^d, \|\cdot\|_d\big)$.

Write H_0^1 for the set of $\mathbb{R}^\ell$-valued paths h over the time interval $[0,1]$, with starting point 0; its H^1-norm is denoted by $\|h\|$. Given $h \in H_0^1$, we define a path γ^h by solving the differential equation

$$(1.3) \qquad \dot{\gamma}_t^h = \sum_{i=1}^{\ell} V_i\big(\gamma_t^h\big) \dot{h}_t^i,$$

for $0 \leqslant t \leqslant 1$, given any specified starting point. The Lie bracket condition ensures that one defines a metric topology identical to the manifold topology setting for any pair of

points (a, b) in M

$$d(a, b) = \inf \int_0^1 |\dot{h}_s|_\ell ds$$

where the infimum is over the non-empty set of H_0^1-controls h such that $\gamma_0^h = a$ and $\gamma_1^h = b$. It is called the sub-Riemannian distance associated with $\mathcal{L}$. The notation $|\cdot|_\ell$ stands here for the Euclidean norm on $\mathbb{R}^\ell$. We define an $[0, \infty]$-valued function J on $\Omega^{x,y}$ setting

$$(1.4) \qquad\qquad J(\gamma) = \frac{1}{2}\Big(\inf\big\{\|h\|^2 \,;\, \gamma^h = \gamma\big\} - d(x, y)^2\Big),$$

with the convention $\inf \emptyset = \infty$. The above infimum is called the **energy of the path** γ, classically denoted by $2I(\gamma)$. Given any $0 < \alpha < 1$, denote by $\|x\|_\alpha$ the α-Hölder norm of a path x from $[0, 1]$ to the ambient space $\mathbb{R}^d$. Write $\mathcal{C}_{x,y}^\alpha\big([0, 1], M\big)$ for the set of all M-valued paths with finite α-Hölder norm, with endpoints x and y; it is equipped with the topology associated with $\|\cdot\|_\alpha$.

THEOREM   **1** (Large deviation principle for bridges of degenerate diffusion processes).

  (i) *Given any $\frac{1}{3} < \alpha < \frac{1}{2}$, the probabilities $\mathbb{P}_\epsilon^{x,y}$ are supported on $\mathcal{C}_{x,y}^\alpha\big([0, 1], M\big)$.*
  (ii) *The family $\big(\mathbb{P}_\epsilon^{x,y}\big)_{0<\epsilon<1}$ satisfies a large deviation principle in $\mathcal{C}_{x,y}^\alpha\big([0, 1], M\big)$, with good rate function J.*

REMARKS **2.**   (1) *The above definition of the space $\mathcal{C}_{x,y}^\alpha\big([0, 1], M\big)$ rests on the ambiant Euclidean metric. It is straigtforward to see that it coincides with the set of M-valued paths which are α-Hölder for any choice of Riemannian metric on M, so $\mathcal{C}_{x,y}^\alpha\big([0, 1], M\big)$ is intrinsically defined.*

  (2) *Inahama proved in* [1] *a similar result under a stronger ellipticity condition. His analysis rests on the dynamic description of the diffusion associated with $\mathcal{L}$, given by the stochastic differential equation $dx_t = V(x_t)dt + \sum_{i=1}^\ell V_i(x_t) \circ dB_t^i$, or rather on its rough path counterpart. By using quasi-sure analysis, he is able to lift the measures $\mathbb{P}_\epsilon^{x,y}$ to some measures $\mathbf{P}_\epsilon^{x,y}$ on the space of geometric rough paths, which requires the quasi-sure existence of the Brownian rough path. The large deviation principle for $\mathbb{P}_\epsilon^{x,y}$ is then obtained as a consequence of a subtle large deviation principle for $\mathbf{P}_\epsilon^{x,y}$, as the Ito-Lyons map is continuous. Our proof is more analytic, in that its essential ingredients are the heat kernel estimates of Léandre and Sanchez-Calle. We also use the machinery of rough paths as a convenient tool for proving the exponential tightness of the family of probability measures $(\mathbb{P}_\epsilon^{x,y})_{0<\epsilon\leqslant 1}$ on $\mathcal{C}_{x,y}^\alpha\big([0, 1], M\big)$.*

  (3) *We shall see in section 3 that the large deviation principle stated in theorem 1 leads directly to the identification of the first order asymptotics of $\mathbb{P}_\epsilon^{x,y}$ under some mild conditions on (x, y), in the sense that $\mathbb{P}_\epsilon^{x,y}$ converges weakly to a Dirac mass on some particular path γ from x to y. It is natural in that setting to push further the analysis and try and get a second order asymptotics. This is done in the forthcoming work* [2] *where it is proved that the fluctuation process around the deterministic limit γ is a Gaussian process whose covariance involves the (non-constant rank) sub-Riemannian geometry associated with the operator $\mathcal{L}$. This requires that the pair (x, y) lies outside some intrinsic cutlocus associated with $\mathcal{L}$.*

2. Proof of the large deviation principle

The proof of theorem 1 follows the pattern of proof devised by Hsu in [3] to prove a similar large deviation principle in a Riemannian setting where $\mathcal{L}$ is the Laplacian of some Riemannian metric on M. Our reasoning relies crucially on Léandre's logarithmic estimate [4], [5]

$$(2.1) \qquad \lim_{\epsilon \searrow 0} \epsilon \log p_\epsilon(z, z') = -\frac{d^2(z, z')}{2},$$

which holds uniformly with respect to $(z, z') \in M^2$, as well as on Sanchez-Calle's estimate

$$(2.2) \qquad p_t(z, z') \leqslant c\, t^{-m},$$

which holds for some positive constant c and all $z, z' \in M$ and $t > 0$, see [6].

Write Ω^x for the set of continuous paths $\omega : [0, 1] \to M$ started from x; we equip Ω^x and $\Omega^{x,y}$ with the metric of uniform convergence inherited from the ambient space. Fix $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$.

a) Exponential tightness of the family of probability measures $(\mathbb{P}_\epsilon^{x,y})_{0 < \epsilon \leqslant 1}$ on $\mathcal{C}_{x,y}^\alpha\big([0, 1], M\big)$. Given $n = n(N) \geqslant 7$ and $K = K(N)$, to be fixed later as functions of some parameter N, we define a compact subset C_N of $\Omega^{x,y}$ setting

$$C_N = \left\{ \omega \in \Omega^{x,y} \,;\; \sup_{0 < t-s \leqslant \frac{1}{n}} \frac{|\omega_t - \omega_s|_d}{|t-s|^\alpha} \leqslant K \right\}.$$

The above supremum is over the set of all times $s, t \in [0, 1]$. We first work on the time interval $[0, 2/3]$ to avoid the difficulties coming from the singularities of the drift at time 1, in the classical dynamical description of the bridge as the solution to a stochastic differential equation.

$$(\star) := \mathbb{P}_\epsilon^{x,y} \left(\sup_{s,t \in [0,2/3],\, 0 < t-s \leqslant \frac{1}{n}} \frac{|\omega_t - \omega_s|_d}{|t-s|^\alpha} > K \right) \leqslant \frac{n}{2} \sup_{0 \leqslant r \leqslant 2/3} \mathbb{P}_\epsilon^{x,y} \left(\sup_{r \leqslant s < t \leqslant r+2/n} \frac{|\omega_t - \omega_s|_d}{|t-s|^\alpha} > K \right)$$

Using (1.2) and the Markov property provides the upper bound

$$
\begin{aligned}
(2.3) \qquad (\star) &\leqslant \sup_{0 \leqslant r \leqslant \frac{2\epsilon}{3}} \mathbb{E}^x \left[\frac{p_{\epsilon - r - \frac{2\epsilon}{n}}\big(\omega_{\epsilon(r + \frac{2\epsilon}{n})}, y\big)}{p_\epsilon(x, y)} \,;\; \sup_{r \leqslant s < t \leqslant r + \frac{2\epsilon}{n}} \frac{|\omega_t - \omega_s|_d}{|t-s|^\alpha} \geqslant K \right] \\
&\leqslant \frac{c\epsilon^{-m}}{p_\epsilon(x, y)} \sup_{z \in M} \mathbb{P}^z \left(\sup_{0 \leqslant s < t \leqslant \frac{2\epsilon}{n}} \frac{|\omega_t - \omega_s|_d}{|t-s|^\alpha} \geqslant K \right).
\end{aligned}
$$

By Lyons' universal limit theorem, as stated for instance under the form given in theorem 11 in [7], there exists universal controls on the oscillation of solutions of stochastic differential equations in terms of the oscillations of Brownian motion and its Lévy area. More precisely, there exists positive constants a_i, b_i, depending only on the vector fields V, V_i, such that

$$(2.4)$$

$$\sup_{z \in M} \mathbb{P}^z \left(\sup_{0 \leqslant s < t \leqslant \frac{2\epsilon}{n}} \frac{|\omega_t - \omega_s|}{|t-s|^\alpha} \geqslant K \right) \leqslant a_1 \left\{ \mathbf{P}\left(\big\| \mathbf{B}_{[0,(2\epsilon)/n]} \big\| \geqslant b_1 K \right) + \mathbf{P}\left(\big\| \mathbf{B}_{[0,(2\epsilon)/n]} \big\|^3 \geqslant K \wedge \frac{n}{3} \right) \right\},$$

$$\leqslant a_2 \mathbf{P}\left(\big\| \mathbf{B}_{[0,(2\epsilon)/n]} \big\| \geqslant b_2 (K \wedge n)^{1/3} \right)$$

where $\mathbf{B}_{[0,(2\epsilon)/n]}$ is the Brownian $\frac{1}{\alpha}$-rough path on the time interval $\left[0, \frac{2\epsilon}{n}\right]$, defined on some probablity space $(\mathbf{\Omega}, \mathcal{F}, \mathbf{P})$, and $\left\|\mathbf{B}_{[0,(2\epsilon)/n]}\right\|$ stands for the homogeneous rough path norm of $\mathbf{B}_{[0,(2\epsilon)/n]}$; see for instance chapter 10.1 of [8]. It follows from the equality in law $\left\|\mathbf{B}_{[0,(2\epsilon)/n]}\right\| = \sqrt{\frac{2\epsilon}{n}}\left\|\mathbf{B}_{[0,1]}\right\|$, the Gaussian character of $\left\|\mathbf{B}_{[0,1]}\right\|$ under $\mathbf{P}$, and Léandre's estimate (2.1) for $p_\epsilon(x,y)$, that

$$\epsilon \log \mathbb{P}_\epsilon^{x,y}\left(\sup_{s,t\in[0,2/3],\, 0<t-s\leqslant\frac{1}{n}} \frac{|\omega_t - \omega_s|}{|t-s|^\alpha} > K\right) \leqslant \frac{d^2(x,y)}{2} + o_\epsilon(1) - \frac{n(K\wedge n)^{2/3}}{2}b_2^2,$$

so we have

$$(2.5) \qquad \overline{\lim_{\epsilon\searrow 0}} \, \epsilon \log \mathbb{P}_\epsilon^{x,y}\left(\sup_{s,t\in[0,2/3],\, 0<t-s\leqslant\frac{1}{n}} \frac{|\omega_t - \omega_s|}{|t-s|^\alpha} > K\right) \leqslant -N$$

by choosing $n = n(N)$ and $K = K(N)$ big enough.

To get a similar estimate when working on the whole time interval $[0,1]$, remark that since M is compact and the operator $\mathcal{L}$ is hypoelliptic, it has a smooth positive invariant measure. If we use this measure as our reference measure VOL, then $\widehat{p}_t(z,z') = p_t(z',z)$ is the heat kernel of another operator $\widehat{\mathcal{L}}$ which satisfies the same conditions as $\mathcal{L}$. Write $\widehat{\mathbb{P}}_\epsilon^{z,z'}$ for the law of the associated bridge. So the class of measures $\left(\mathbb{P}_\epsilon^{z,z'}\right)_{z\neq z'\in M}$ constructed from hypoelliptic operators $\mathcal{L}$ as in (1.1), satisfying the Lie bracket assumption, is preserved under time reversal. Applying inequality (2.5) to the measure $\widehat{\mathbb{P}}_\epsilon^{y,x}$ on $\Omega^{y,x}$ obtained by time-reversal of $\mathbb{P}_\epsilon^{x,y}$, we conclude with (2.5) that

$$\overline{\lim_{\epsilon\searrow 0}} \, \epsilon \log \mathbb{P}_\epsilon^{x,y}(C_N^c) \leqslant -N.$$

So the family $\left(\mathbb{P}_\epsilon^{x,y}\right)_{0<\epsilon<1}$ of probabilities on $\mathcal{C}_\alpha^{x,y}\big([0,1],M\big)$ is exponentially tight, which proves in particular point 1. As the inclusion of $\mathcal{C}_\alpha^{x,y}\big([0,1],M\big)$ into $\big(\Omega^{x,y}, \|\cdot\|_\infty\big)$ is continuous, it suffices, by the inverse contraction principle, to prove that $\left(\mathbb{P}_\epsilon^{x,y}\right)_{0<\epsilon<1}$ satisfies a large deviation principle in $\big(\Omega^{x,y}, \|\cdot\|_\infty\big)$, with good rate function J, to prove point 2 of the theorem, in so far as J is also a good rate function on $\mathcal{C}_\alpha^{x,y}\big([0,1],M\big)$. We follow closely Hsu's work [3] to prove that fact.

b) Large deviation upper bound for $(\mathbb{P}_\epsilon^{x,y})_{0<\epsilon\leqslant 1}$. We first prove the upper bound for a compact subset C of $\Omega^{x,y}$. For $0 < a < 1$, set

$$C^a = \big\{\omega \in \Omega^{x,y} \,;\, \exists\, \rho \in C \text{ such that } \omega_s = \rho_s, \text{ for } 0 \leqslant s \leqslant 1-a\big\}$$

and

$$C_*^a = \big\{\omega \in \Omega^x \,;\, \exists\, \rho \in C \text{ such that } \omega_s = \rho_{(1-a)s}, \text{ for all } 0 \leqslant s \leqslant 1\big\}.$$

The set C^a is closed in both Ω^x and $\Omega^{x,y}$, and $C \subset C^a$. Using (1.2) and the Markov property, we get as in (2.3) the inequality

$$\mathbb{P}_\epsilon^{x,y}(C) \leqslant \mathbb{P}_\epsilon^{x,y}(C^a) \leqslant \mathbb{E}_\epsilon^x\left[\frac{p_{a\epsilon}(\omega_1,y)}{p_\epsilon(x,y)}\mathbf{1}_{\omega\in C_*^a}\right]$$

$$\leqslant \frac{c\epsilon^{-m}}{p_\epsilon(x,y)}\mathbb{P}_\epsilon^x(C_*^a).$$

As C_*^a is closed in Ω^x, we have by the classical Freidlin-Wentzell large deviation principle for $\mathbb{P}_\epsilon^x$

$$\limsup_{\epsilon \searrow 0} \epsilon \log \mathbb{P}_\epsilon^{x,y}(C) \leqslant \frac{d^2(x,y)}{2} - \frac{1}{1-a} \inf_{\omega \in C_*^a} I(\omega).$$

Using the lower semicontinuity of I on Ω^x, it is straightforward to use the compacity of C to see that $\limsup_{a \searrow 0} \inf_{\omega \in C_*^a} I(\omega) \geqslant \inf_{\omega \in C} I(\omega)$, as done in [3], p.112. This proves the upper bound

$$\limsup_{\epsilon \searrow 0} \epsilon \log \mathbb{P}_\epsilon^{x,y}(C) \leqslant - \inf_C J,$$

for a compact set C; it is classical that the exponential tightness proved in point a) implies in that case the upper bound for any closed set.

c) Large deviation lower bound for $(\mathbb{P}_\epsilon^{x,y})_{0<\epsilon\leqslant 1}$**.** We use the notation $\|f\|_{[a,b]}$ to denote the uniform norm of some function f defined on some time interval $[a,b]$. Given an open set U in $\Omega^{x,y}$, we aim at proving that we have

(2.6)
$$\liminf_{\epsilon \searrow 0} \epsilon \log \mathbb{P}_\epsilon^{x,y}(U) \geqslant -J(\gamma)$$

for any $\gamma \in U$ with finite energy $I(\gamma)$. Pick such a path $\gamma \in U$ and $b > 0$ small enough for the ball in Ω^x with center γ and radius b to be included in U. Set for $0 < a < 1$

$$U^{a,b} = \big\{\omega \in \Omega^{x,y}\,;\, \|\omega - \gamma\|_{[0,1-a]} < b\big\}, \quad F^{a,b} = \big\{\omega \in \Omega^{x,y}\,;\, \|\omega - \gamma\|_{[1-a,1]} \geqslant b\big\}$$

and $U_*^{a,b} = \big\{\omega_* \in \Omega^x\,;\, \exists\, \omega \in U \text{ such that } \omega_*(s) = \omega_{(1-a)s}, \text{ for all } 0 \leqslant s \leqslant 1\big\}$. We have $U^{a,b} \subset \big(U \cup F^{a,b}\big)$, so $\mathbb{P}_\epsilon^{x,y}(U) \geqslant \mathbb{P}_\epsilon^{x,y}(U^{a,b}) - \mathbb{P}_\epsilon^{x,y}(F^{a,b})$. We prove (2.6) by showing that $\liminf_{\epsilon \searrow 0} \epsilon \log \mathbb{P}_\epsilon^{x,y}(U^{a,b}) \geqslant -J(\gamma)$, and $\liminf_{\epsilon \searrow 0} \epsilon \log \mathbb{P}_\epsilon^{x,y}(F^{a,b}) = -\infty$.

Given $\lambda > 0$, write $B_\lambda(y)$ for the sub-Riemannian open ball in M, with center y and radius λ. Using the Markov property as above, we have

$$\mathbb{P}_\epsilon^{x,y}(U^{a,b}) = \mathbb{E}_\epsilon^{x,y}\Big[\mathbb{P}_{\epsilon(1-a)}^{x,X_{1-a}}(U_*^{a,b})\Big] \geqslant \int \mathbb{P}_{\epsilon(1-a)}^{x,z}(U_*^{a,b}) \frac{p_{\epsilon(1-a)}(x,z)p_{\epsilon a}(z,y)}{p_\epsilon(x,y)} \mathbf{1}_{z \in B_\lambda(y)}\, dz$$

$$\geqslant \frac{\min\limits_{z \in B_\lambda(y)} p_{\epsilon a}(z,y)}{p_\epsilon(x,y)} \int \mathbb{P}_{\epsilon(1-a)}^{x,z}(U_*^{a,b}) \mathbf{1}_{z \in B_\lambda(y)} p_{\epsilon(1-a)}(x,z)\, dz$$

$$\geqslant \frac{\min\limits_{z \in B_\lambda(y)} p_{\epsilon a}(z,y)}{p_\epsilon(x,y)} \mathbb{P}_{\epsilon(1-a)}^{x}\big(U_*^{a,b} \cap \{\omega_1 \in B_\lambda(y)\}\big)$$

Define $\gamma_a(s) = \gamma_{(1-a)s}$ for all $0 \leqslant s \leqslant 1$. As γ has finite energy, one can pick some control $h \in H_0^1$ such that $\gamma^h = \gamma$; we have $d(\gamma_a(1),y) \leqslant \int_{1-a}^1 |\dot{h}_s|_\ell\, ds \leqslant \sqrt{a \int_{1-a}^1 |\dot{h}_s|_\ell^2\, ds}$. The choice of $\lambda = \lambda(a) = 2\sqrt{a \int_{1-a}^1 |\dot{h}_s|_\ell^2\, ds}$ ensures that the open set $U_*^{a,b} \cap \{\omega_1 \in B_\lambda(y)\}$ contains γ_a, so it is nonempty; also, $\frac{\lambda(a)^2}{a} \to 0$ as a tends to 0. Using the classical Freidlin-Wentzell large deviation theory and the uniform character of Léandre's estimate (2.1), the above lower bound for $\mathbb{P}_\epsilon^{x,y}(U^{a,b})$ gives

$$\liminf_{\epsilon \searrow 0} \epsilon \log \mathbb{P}_\epsilon^{x,y}(U^{a,b}) \geqslant \frac{-I(\gamma_a)}{1-a} + \frac{d(x,y)^2}{2} - \frac{\lambda(a)^2}{2a},$$

from which the inequality $\liminf_{\epsilon \searrow 0} \epsilon \log \mathbb{P}_\epsilon^{x,y}(U^{a,b}) \geqslant -J(\gamma)$ follows, since $I(\gamma_a) \to I(\gamma)$ and $\frac{\lambda(a)^2}{a} \to 0$ as a tends to 0.

We now deal with the term $\mathbb{P}_\epsilon^{x,y}(F^{a,b})$. Set $\overline{\gamma}_s = \gamma_{1-s}$, for $0 \leqslant s \leqslant 1$, and choose a small enough to have $\|\overline{\gamma} - y\|_{[0,a]} \leqslant \frac{b}{2}$. We use the same time reversal trick and notations as above to estimate $\mathbb{P}_\epsilon^{x,y}(F^{a,b})$. Write

$$\mathbb{P}_\epsilon^{x,y}(F^{a,b}) = \widehat{\mathbb{P}}_\epsilon^{y,x}\left(\|\omega - \overline{\gamma}\|_{[0,a]} \geqslant b\right) \leqslant \widehat{\mathbb{P}}_\epsilon^{y,x}\left(\|\omega - y\|_{[0,a]} \geqslant \frac{b}{2}\right)$$
$$\leqslant \frac{c\epsilon^{-m}}{p_\epsilon(y,x)} \widehat{\mathbb{P}}_\epsilon^y\left(\|\omega - y\|_{[0,a]} \geqslant \frac{b}{2}\right).$$

Léandre's estimate (2.1) and the classical large deviation results for $\widehat{\mathbb{P}}_\epsilon^y$ give the existence of a positive constant c such that we have

$$\liminf_{\epsilon \searrow 0} \epsilon \log \mathbb{P}_\epsilon^{x,y}(F^{a,b}) \leqslant \frac{d(x,y)^2}{2} - \frac{c}{a};$$

this upper bound tends to $-\infty$ as a tends to 0. Points a), b) and c) all together prove theorem 1.

3. FIRST ORDER ASYMPTOTICS FOR BRIDGES OF DEGENERATE DIFFUSION PROCESSES

Theorem 1 provides a straightforward mean for investigating the first order asymptotics of $\mathbb{P}_\epsilon^{x,y}$ as ϵ tends to 0, for x and y in generic positions.

THEOREM **3** (First order asymptotics of $\mathbb{P}_\epsilon^{x,y}$). *If there exists a unique path γ with minimal energy from x to y, then $\mathbb{P}_\epsilon^{x,y}$ converges weakly in $(\Omega^{x,y}, \|\cdot\|_\infty)$ to a Dirac mass on γ as ϵ tends to 0.*

PROOF – We follow the proof of lemma 3.1 in [3]. Since the family $(\mathbb{P}_\epsilon^{x,y})_{0 < \epsilon \leqslant 1}$ is tight by point a) in section 2, let $\mathbb{Q}$ be any limit measure. Given $b > 0$, set

$$C_N^b = C_N \cap \{\omega \in \Omega^{x,y} \,;\, \|\omega - \gamma\|_\infty > b\};$$

then $\inf_{\omega \in C_N^b} J(\omega) > 0$. Indeed, since the paths of C_N^b are equicontinuous, if the infimum were null, we could extract from any sequence of paths $(\omega_n)_{n \geqslant 0}$ such that $J(\omega_n)$ converges to 0 a uniformly converging subsequence with limit $\omega \in \overline{C}_N^b$ say. We should then have $J(\omega) = 0$, by the lower semicontinuity of J, that is $\omega = \gamma$, since there is a unique path from x to y with minimal energy, in contradiction with the fact that elements of $\overline{C}_N^b$ satisfy the inequality $\|\omega - \gamma\|_\infty \geqslant b > 0$. As a consequence, the above large deviation upper bound implies

$$\mathbb{Q}(C_N^b) \leqslant \liminf_{\epsilon \searrow} \mathbb{P}_\epsilon^{x,y}(\overline{C}_N^b) = 0;$$

sending N tend to infinity, it follows that

$$\mathbb{Q}(\omega \in \Omega^{x,y} \,;\, \|\omega - \gamma\|_\infty > b) = 0.$$

As this holds for all $b > 0$, we have $\mathbb{Q} = \delta_\gamma$, from which the convergence of $\mathbb{P}_\epsilon^{x,y}$ to δ_γ follows. $\triangleright$

Note that the set of pairs of points $(x,y) \in M^2$ such that x and y are joined by a unique path of minimal energy is dense in M^2.

References

[1] Inahama, Y. Large deviation principle of Freidlin-Wentzell type for pinned diffusion processes. *Preprint*, 2012.

[2] Bailleul, I. and Mesnager, L. and Norris, J. Small time fluctuations for Riemannian and sub-Riemannian diffusions. *Preprint*, 2013.

[3] Hsu, P. Brownian bridges on Riemannian manifolds. *Prob. Theory and Related Fields*, 84(1):103–118, 1990.

[4] Léandre, R. Majoration en temps petit de la densité d'une diffusion dégénérée. *Probab. Theory Related Fields*, 74(2):289–294, 1987.

[5] Léandre, R. Minoration en temps petit de la densité d'une diffusion dégénérée. *J. Funct. Anal.*, 74(2):399–414, 1987.

[6] Sanchez-Calle, A. Fundamental solution and geometry of the sum of squares of vector fields. *Inv. Math.*, 78:143–160, 1984.

[7] Bailleul, I. Flows driven by rough paths. *Submitted*, 2012.

[8] Friz, P. and Victoir, N. Multidimensional stochastic processes as rough paths. *C.U.P.*, 2010.

IRMAR, 263 Avenue du General Leclerc, 35042 RENNES, France

E-mail address: ismael.bailleul@univ-rennes1.fr