A tourist's guide to regularity structures and singular stochastic PDEs

I. BAILLEUL & M. HOSHINO

Abstract. We give a short essentially self-contained treatment of the fundamental analytic and algebraic features of regularity structures and its applications to the study of singular stochastic PDEs.

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1 – Introduction

The class of singular stochastic partial differential equations (PDEs) is charactarised by the appearance in their formulation of ill-defined products due to the presence in the equation of distributions with low regularity, typically realisations of random distributions. Here are three typical examples.

- The 2 or 3-dimensional parabolic Anderson model equation (PAM)
$$(\partial_t - \Delta_x)u = u\xi, \tag{1.1}$$
with ξ a space white noise. It represents the evolution of a Brownian particle in a 2 or 3-dimensional white noise environment in the torus. (The operator Δ_x stands here for the 2 or 3-dimensional Laplacian.)
- The scalar Φ^4_3 equation from quantum field theory
$$(\partial_t - \Delta_x)u = -u^3 + \zeta, \tag{1.2}$$
with ζ a 3-dimensional spacetime white noise and Δ_x the 3-dimensional Laplacian in the torus or the Euclidean space. Its invariant measure is the scalar Φ^4_3 measure from quantum field theory.
- The generalized (KPZ) equation
$$(\partial_t - \partial_x^2)u = f(u)\zeta + g(u)|\partial_x u|^2, \tag{1.3}$$
with ζ a 1-dimensional spacetime white noise. In a more sophisticated form, it provides amongst others a description of the random motion of a rubber on a Riemannian manifold under a random perturbation of the mean curvature flow motion.

A d-dimensional space white noise has Hölder regularity $-d/2-\kappa$, and a d-dimensional spacetime white noise has Hölder regularity $-d/2-1-\kappa$ under the parabolic scaling, almost surely for every positive κ. Whereas one expects from the heat operator that its inverse regularizes a distribution by 2, this is not sufficient to make sense of any of the products $u\xi, u^3, f(u)\zeta, |\partial_x u|^2, g(u)|\partial_x u|^2$ above, as the product of two Hölder distributions is well-defined if and only if the sum of their regularity exponents is positive. Why then bother about such equations? It happens that they appear as scaling limits of a number of microscopic nonlinear random dynamics where the strength of the nonlinearity and the randomness balance each other. Many microscopic random systems exhibit this feature as you will see from reading Corwin & Shen's nice review [24] on singular stochastic PDEs.

A typical statement about a singular stochastic PDE takes the following informal form, stated here in restricted generality. Consider a subcritical singular stochastic PDE

$$(\partial_t - \Delta_x)u = f(u, \partial u)\zeta + g(u, \partial u) =: F(u, \partial u; \zeta) \tag{1.4}$$

driven by a possibly multi-dimensional irregular random noise ζ that is almost surely of spacetime regularity $\alpha - 2$, for a deterministic constant $\alpha \in \mathbb{R}$. (The notion of 'subcriticality' will be properly defined later in the text.) We talk of the sufficiently regular function F as a 'nonlinearity' – even though a particular F could depend linearly or affinely of one or all of its arguments. Denote by $\mathfrak{F}$ the space of nonlinearities that are affine functions of the noise argument. Denote also by ζ_ε a regularized version of the noise, obtained for instance by convolution with a deterministic smooth kernel. Write

$$\textsc{Sol}(\zeta_\varepsilon; F)$$

for the solution to the well-posed parabolic equation

$$(\partial_t - \Delta_x)u = F(u, \partial u; \zeta_\varepsilon)$$

started at time 0 from a given (regular enough) fixed initial condition.

Meta-theorem 1. (What it means to be a solution) – *The following three points hold true.*

- *One can associate to each subcritical singular stochastic PDE a finite dimensional unbounded Lie group called the renormalization group. Denote by k its generic elements.*
- *This group acts explicitly on the right on the nonlinearity space $\mathfrak{F}$*
$$\big(k, F\big) \mapsto F^{(k)} \in \mathfrak{F}. \tag{1.5}$$
- *There exists diverging elements k_ε of the renormalization group such that the solutions*
$$\textsc{Sol}\Big(\zeta_\varepsilon; (F^{(k_\varepsilon)})^{(k)}\Big)$$
to the well-posed stochastic PDE
$$(\partial_t - \Delta_x)u = \big(F^{(k_\varepsilon)}\big)^{(k)}(u, \partial u; \zeta_\varepsilon)$$
with given initial condition, converge in probability in an appropriate function/distribution space, for any element k of the renormalization group, as ε goes to 0.

A solution to a singular stochastic PDE is not a single function or distribution, but rather a family *of functions/distributions indexed by the renormalization group.*

We stick to the tradition and talk about any of the above limit functions/distributions as a solution to equation (1.4). We talk of the family of solutions. To have a picture in mind, consider the family of maps

$$S_\varepsilon(x) = \big(x - 1/\varepsilon\big)^2, \tag{1.6}$$

on $\mathbb{R}$. It explodes in every fixed interval as ε goes to 0, but remains finite, and converges, in a moving window $S_\varepsilon(x + 1/\varepsilon)$, where it is equal to x^2. It also converges in the other moving window $S_\varepsilon(x + 1/\varepsilon + 1)$, where it is equal to $(x + 1)^2$. No given moving window is a priori better than another. In this parallel, the function $\textsc{Sol}(\zeta_\varepsilon; \cdot)$ plays the role of S_ε, with the infinite dimensional nonlinearity space $\mathfrak{F}$ in the role of the state space $\mathbb{R}$. The role of the translations $x \mapsto x + 1/\varepsilon$, is played by the group action (1.5) of k_ε on the space of nonlinearities $\mathfrak{F}$. The explicit action of k_ε on the space of nonlinearities gives formulas of the form

$$F^{(k_\varepsilon)}(u, \partial u; \xi) = f(u, \partial u)\xi + g(u, \partial u) + h_\varepsilon(u, \partial u),$$

so $\textsc{Sol}\big(\zeta_\varepsilon; F^{(k_\varepsilon)}\big)$ is the solution to the equation

$$(\partial_t - \Delta_x)u = f(u, \partial u)\zeta_\varepsilon + g(u, \partial u) + h_\varepsilon(u, \partial u), \tag{1.7}$$

with given initial condition, for an explicit function h_ε built from f, g and their derivatives. We talk of the function h_ε as a counterterm; it diverges as ε goes to 0. Note that the convergence involved in Meta-theorem 1 is in the probabilistic sense, not almost surely.

In a robust solution theory for differential equations a solution to a differential equation ends up being a continuous function of the parameters in the equation. In the case of equation (1.4) the parameters are the functions f, g, the noise ζ and the initial condition of the equation. While it is unreasonable and wrong to expect that the solutions from Meta-theorem 1 are continuous functions of each realization of the noise, they happen to be continuous functions of a measurable functional of the latter build by probabilistic means. We talk about that functional of the noise as an *enhanced noise*.

Meta-theorem 2 (Continuity of a solution with respect to the enhanced noise) – *For ζ in a class of random noises including space or spacetime white noises, there is a measurable functional $\mathsf{\Pi}$ of the noise taking values in a metric space and such that any individual solution of equation* (1.4) *is a continuous function of $\mathsf{\Pi}$.*

This is a fundamental point to be compared with the fact that in the stochastic calculus approach to stochastic (partial) differential equations, the solutions to the equations are only measurable functionals of the noise. In a setting where both approaches can be used and coincide, the regularity structures point of view provides a factorization of the measurable solution map of stochastic calculus under the form of the composition of a measurable function $\mathsf{\Pi}$ of the noise with a continuous function of $\mathsf{\Pi}$. A number of probabilistic statements about $\mathsf{\Pi}$ are then automatically transfered to the solution of the equation, by continuity. The support of the law of the random variable $\mathsf{\Pi}$ determines for instance the support of the law of the solution to the equation. Large deviation results for a family of random $\mathsf{\Pi}$'s is also automatically transported by continuity into a large deviation result for the corresponding family of solutions of the equation.

The functional $\mathsf{\Pi}$ is built as a limit *in probability* of elementary functionals of the noise. This is in the end the reason why the convergence result in Meta-theorem 1 holds in probability.

How is it that one can prove such statements? The point is that solutions of singular stochastic PDEs are not expected to be any kind of Hölder functions or distributions. Rather, under an assumption on the equation captured by the notion of subcriticality, we expect possible solutions to be described locally in terms of a *finite* number of equation-dependent reference functions/distributions that are polynomials functionals of the noise. The theory of regularity structures provides a complete description of the local structure of these possible solutions, in terms of their local expansion properties with respect to the preceding polynomial functions of the noise. It actually turns the problem upside down by reformulating singular stochastic PDEs as equations in spaces of functions/distributions characterised by their local behaviour. The product problem is isolated along the way in the problem of making sense of the reference functions/distributions that are used as ingredients in the local description of a possible solution. This is the very point where the fact that the noise is random plays a crucial role. It allows indeed to build the reference functions/distributions not as functions of the noise but rather as random variables jointly defined with the noise on a common probability space. This realizes a wonderful decoupling of probability and analysis. To the former the task of building the enhanced noise: Functions or distribution-valued random variables that play the role of ill-defined products of equation-dependent quantities involving the noise *only*. This is done by a limiting procedure called *renormalization*, after similar procedures used in quantum field theory to tackle similar problems. To the latter the task of solving uniquely an equation in a side space built from the enhanced noise, regardless of any multiplication problem.

The fundamentals of the theory of regularity structures were built gradually by M. Hairer and his co-authors in four groundbreaking works [44, 16, 20, 13]. In paper [44], M. Hairer sets the analytic framework of regularity structures and provides an ad hoc study of the renormalization problem for the parabolic Anderson model equation (1.1) and scalar Φ^4_3 equation (1.2). The algebra involved in the renormalization process of a large class of singular stochastic PDEs was unveiled in Bruned, Hairer and Zambotti's work [16]. The proof that the renormalization algorithm provided in [16] converges was done by Chandra & Hairer in [20]. Last, the fact that the renormalization can be 'implemented' at the level of the equation was proved in Bruned, Chandra, Chevyrev and Hairer's work [13], giving a wonderful analogue of the equivalence of the "substraction scheme" versus "counterterms" approaches to renormalization problems in quantum field theory. Altogether, these four works provide a black box for the local well-posedness theory of subcritical singular stochastic PDEs. This work gives an essentially self-contained short treatment of the fundamental analytic and algebraic features of regularity structures and its applications to the study of singular stochastic PDEs that contains the essential points of the works [44, 16, 13]. It is intended for readers who already have an idea of the subject and who wish to understand in depth the mechanics at work. We hope nonetheless that even a newcomer to the field may grasp the matter by following the road taken here. Regularity structures and the fundamental tools are developed in generality within a highly abstract setting. No trees are in particular involved in the analysis before we actually construct an example of regularity structure adapted to the study of the generalized (KPZ) equation (1.3) in Section 9. When it comes to applying these tools to singular stochastic PDEs, we trade generality for the concrete example of the generalized (KPZ) equation (1.3), that involves all the difficulties of the most general case. We do not treat about Chandra & Hairer's work [20] constructing the measurable functional Π of the noise involved in Meta-theorem 2 using Bruned, Hairer and Zambotti's renormalization process [16].

We stress here that Hairer's approach to singular stochastic PDEs is somewhat orthogonal to the purely probabilistic approaches of stochastic PDEs pioneered by Pardoux, Walsh or da Prato & Zabczyk, using martingale technics. No knowledge of these approaches is needed to understand what follows.

It is our aim here to give a concise self-contained version of what seems to us to be the most important features of the 437 pages of the works [44, 16, 13]. A number of comments about different statements, concepts, other works, are deferred to Appendix D so as to keep focused in the main body of the text. The reader is invited to read this section at any point along her/his reading. We expect that the reader will see from the present work the simplicity that governs the architecture of the theory. The climb may be hard but the view after the walk is stunning.

Besides the original articles [44, 16, 20, 13], Hairer's lectures notes [45, 47], the book [33] by Friz and Hairer, and Chandra & Weber's article [21], provide other accessible accounts of part of the material presented here. The work [24] of Corwin and Shen provides a nice non-technical overview of the context in which singular stochastic PDEs arise.

We will introduce the different pieces of the puzzle one after the other to arrive at a clear understanding of the mathematical form of the above meta-theorems. This will be done along the following lines. We will first set the scene to talk of the local behaviour of functions/distributions

$$f(\cdot) \sim \sum_{\tau} f_\tau(x)(\mathsf{\Pi}^{\mathsf{g}}_x\tau)(\cdot), \tag{1.8}$$

near each spacetime point x, giving a generalisation of the notion of jet, in terms of reference functions/distributions $(\mathsf{\Pi}^{\mathsf{g}}_x\tau)(\cdot)$. This will involve the setting of *(concrete) regularity structures* $\mathscr{T} = \big((T^+,\Delta^+),(T,\Delta)\big)$, and *models* $\mathsf{M} = (\mathsf{g},\mathsf{\Pi})$, from which the reference functions/distributions $(\mathsf{\Pi}^{\mathsf{g}}_x\tau)(\cdot)$ are built. In the same way as a family of functions $(f_1,\dots,f_n)$ on $\mathbb{R}^d$ needs to satisfy a quantitative consistency condition for a function f satisfying

$$f(\cdot) \sim \sum_{k\in\mathbb{N}^d} f_k(x)(\cdot - x)^k, \qquad \text{near all } x,$$

to exist, a collection of functions $(f_\tau)_\tau$ needs to satisfy a quantitative consistency condition for a distribution f satisfying (1.8) to exist. This condition will involve the notion of *modelled distribution* and *reconstruction operator* R^{M}, with a notion of consistency that will depend on the component g of the model M. At that stage, given a regularity structure and a model on it, we will have a convenient way of representing a class of functions/distributions on the state space – not all of them. Given a (system of) singular stochastic PDE(s), a good choice of concrete regularity structure will allow to represent the set of functions/distributions that appear in a naive analysis of the equation via a Picard iteration. Unlike what happens in the study of controlled ordinary differential equations driven by an ℓ-dimension control, there is no universal concrete regularity structure for the set of all singular stochastic PDEs. One associates to each (system of) subcritical singular stochastic PDE(s) a specific regularity structure. The regularity structure associated with equation (1.4) is built from a noise symbol and operators $(\mathcal{I}_n)_{n\in\mathbb{N}^{d+1}}$ that play the role in the regularity structure of the operators $\partial^n(\partial_t-\Delta_x)^{-1}$, involved in the Picard fixed point formulation of the equation. (The letter $\partial=(\partial_t,\partial_x)$ stands here for the time/space derivative operator.) One proceeds then by formulating the equation as a fixed point problem in the space of consistent jets of distributions/functions, encoded in the notion of modelled distribution. This will require to introduce a tweaked version $\mathcal{K}^{\mathsf{M}}$ of the operator $\mathcal{I}$, as the latter does not produce consistent jets from consistent jets – the notion of consistency depends on g whereas the operator $\mathcal{I}$ does not. The equation on the jet space will happen then to have a unique solution in small time under proper mild conditions; this solution will be a continuous function of all the parameters in the equations, the model in particular. Along the way, we will turn the initial analytical multiplication problem into the problem of defining models with appropriate properties – the so called *admissible models*. It is straightforward to construct what is called the canonical lift of a regularized version ζ_ε of the noise ζ as an admissible model M^ε; this can be done for any smooth noise. In those terms, the solution u_ε to a well-posed (system of) singular stochastic PDE(s) with smooth noise ζ_ε can be written as the reconstruction

$$u_\varepsilon=\mathbf{R}^{\mathsf{M}^\varepsilon}(\boldsymbol{u}_\varepsilon) \tag{1.9}$$

of a consistent jet $\boldsymbol{u}_\varepsilon$ obtained as the fixed point

$$\boldsymbol{u}_\varepsilon=\Phi\big(\mathsf{M}^\varepsilon,\boldsymbol{u}_\varepsilon\big) \tag{1.10}$$

of a map Φ that depends continuously on its model argument. So does the reconstruction map $\mathbf{R}^{\mathsf{M}^\varepsilon}$. However the model M^ε does not converge in the appropriate space as the positive regularization parameter ε goes to 0, so a solution to the (system of) singular stochastic PDE(s) under study cannot be defined as the limit of the u_ε as ε goes to 0. The situation is similar to what happens to the function S_ε from (1.6). One has to look at M^ε in a moving window to obtain a finite limit. The *renormalization group* will provide us precisely with this possibility, and will provide a family of renormalized canonical models ${}^{k_\varepsilon}\mathsf{M}^\varepsilon$. To make the final step from here to the meta-theorems, we will see that this action of the renormalization group on models has a dual action on the space $\mathfrak{F}$ of nonlinearities. The ${}^{k_\varepsilon}\mathsf{M}^\varepsilon$-reconstruction u_ε of the unique ${}^{k_\varepsilon}\mathsf{M}^\varepsilon$-dependent fixed point equation in the space of jets will happen to solve a 'renormalized' version of the singular equation (1.4), with additional ε-dependent terms diverging as the regularization parameter ε tends to 0, as in (1.7). The continuity of both the solution of the fixed point equation and the reconstruction map, as functions of the underlying model, will ensure the convergence of u_ε to a limit u for converging renormalized models ${}^{k_\varepsilon}\mathsf{M}^\varepsilon$ with limit M. The limit function u will satisfy a system

$$u=\mathbf{R}^{\mathsf{M}}(\boldsymbol{u}),\quad \boldsymbol{u}=\Phi\big(\mathsf{M},\boldsymbol{u}\big),$$

similar to the system of equations (1.9) and (1.10) satisfied by u_ε. It is in this sense that u will deserve to be called a solution of the singular stochastic PDE under study. Think of u as a function/distribution defined from its 'Taylor' jet $\boldsymbol{u}$, with the latter solution of a fixed point problem.

A word about algebra. It is one of the features of the theory of regularity structures that algebra plays an important role, unlike what one usually encounters in the analytic study of PDEs. This is partly due to the choice of description of the objects involved in the analysis, in terms of "jets-like" quantities. Elementary consistency requirements directly bring algebra into play, under the form of Hopf algebras and actions of the latter on vector spaces. This is what regularity structures are. The appearance of algebra in the study of singular stochastic PDEs is also due to the fact that the renormalization algorithm used to define the random variables that play the role of a number of ill-defined polynomial functionals of the noise is conveniently encoded in an algebraic structure that we call *renormalization structure*; it differs from a regularity structure. These two points involve Hopf algebras. A last piece of algebra is also needed under the form of pre-Lie algebras. This is an algebraic structure that behaves as the differentiation operation $(f,g)\mapsto g'f$, derivative of g in the direction of f. Using an algebraic language sheds a gentle light on the meaning of the renormalization process at the level of the equation. Pre-Lie algebras are the ingredient we need to understand how to build h_ε in Meta-theorem 1 and equation (1.7).

The analysis or probability-oriented reader should not be frightened by the perspective of working with algebraic tools; we will hardly need anything more than a few definitions and elementary facts that are direct consequences of the latter; everything else is proved. We refer the reader to Manchon's lecture notes [59], or the first four chapters of Sweedler's book [65], for accessible references on Hopf algebras, and to Foissy's work [31] for basics on pre-Lie algebras; all we need is elementary and recalled below. Appendix B contains in any case all the results from algebra that we use without proving them, with precise pointers to the litterature.

The work has been organized as follows. Basics on regularity structures are introduced in Section 2, under the form of concrete regularity structures. The reconstruction theorem, that ensures that a consistent jet describes a distribution in the state space is proved there, in Theorem 4. This allows to formulate in Section 4 a singular PDE as an equation in a space of modelled distributions over a regularity structure associated with the singular PDE. A fixed point argument is used in Section 4 to prove a local in time well-posedness result in a space of modelled distributions. Despite their possible differences, the regularity structures built for the study of different subcritical elliptic or parabolic singular PDEs all involve the construction of the counterpart of a (or several) regularizing convolution operator(s) and the proof of its (/their) continuity properties in spaces of modelled distributions. This is done in Section 3. Section 5 sets the scene of renormalization structures. They encode the renormalization algorithm used to build the random variables whose realisations play the role of a finite number of reference functions/distributions. The renormalization algorithm is described in Section 7. The dual action of the renormalization operation on the genuine singular PDE is clarified by the introduction of pre-Lie structures; this is done in Section 6. Nothing so far requires a deep understanding of how the regularity or the renormalization structure associated with a given singular stochastic PDE are built. It suffices to assume that they satisfy a small number of simple assumptions to run the analysis. Section 9 is dedicated to constructing explicitly such structures in the concrete example of the generalized (KPZ) equation. A summary of notations is given in Appendix A. Appendix B contains a number of elementary facts from algebra that we use. Precise pointers to the proofs of these facts are given. Appendix C contains the proof of technical results that were not given in the body of the text to keep concentrated on the essential features of the method. A number of comments about the notions, the statements or the litterature are collected in Appendix D. The reader is invited to read them at any time.

Notations – *We use a number of greek letters with different meanings. As a rule, α, β, γ stand for real numbers, while $\tau, \sigma, \mu, \nu, \eta, \varphi, \psi$ stand for elements of regularity or renormalization structures.*

- *Given two statements $\mathfrak{a}$ and $\mathfrak{a}^+$, we agree to write $\mathfrak{a}^{(+)}$ to mean both the statement $\mathfrak{a}$ and the statement $\mathfrak{a}^+$.*
- *Denote by $e_i = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0) \in \mathbb{R}^d$ the i-th basis vector of $\mathbb{R}^d$.*

All the notations introduced along the way are gathered in Appendix A, with pointers to the section where they are introduced.

2 – Basics on regularity structures

This section introduces the main actors of the play: Regularity structures in Section 2.2, models, modelled distributions and their reconstructions in Section 2.3, and the fundamental operations of product and derivatives in a regularity structure setting in Section 2.4.

Hairer's theory of regularity structures builds on Gubinelli's approach [38] to T. Lyons' theory of rough paths and rough differential equations [58]. This is a theory of controlled ordinary differential equations

$$dz_t = V(z_t)d\mathbf{X}_t, \quad z_t \in \mathbb{R}^k,$$

driven by irregular controls $\mathbf{X}$. Gubinelli's notion of 'path controlled by a rough path $\mathbf{X}$' gives a Taylor-like description of a path around each time s in terms of 'monomials' given by the different components of the increments of the rough path $\mathbf{X}$ between the running time and time s. This notion of controlled path turns out to be stable by nonlinear maps and by a map defining the integral against the reference rough path $\mathbf{X}$. These facts allow to formulate controlled ordinary differential equations driven by a rough path as an integral equation in a space of controlled paths and to prove local well-posedness of the equation under regularity assumptions on the vector fields involved in the dynamics by fixed point arguments.

Hairer chooses a similar angle to build his theory of singular stochastic PDEs. He provides a setting whose rough path analogue not only gives a pointwise description of a potential solution path z, but also a 'local' description of the $\mathbb{R}^k$-valued distribution $V(z_t)d\mathbf{X}_t$ on a time interval, around each time s, sticking to a differential formulation of the problem. We start this section by explaining in Section 2.1 in simple terms why this strategy of local expansion devices automatically brings algebra into play, independently of any problem of dynamical nature.

2.1 Algebra as the mechanics of local expansion devices

Regularity structures are the backbone of expansion devices for the local description of functions and distributions in (an open set of) a Euclidean space, say $\mathbb{R}^d$. The usual notion of local description of a function near a point $x \in \mathbb{R}^d$ involves Taylor expansion and amounts to comparing a function to a polynomial centered at x

$$f(\cdot) \simeq \sum_n f_n(x)\,(\cdot - x)^n, \quad \text{near } x. \tag{2.1}$$

The sum over n is finite, the approximation quantified and we end up describing the class of Hölder functions with positive regularity exponents. One gets a local description of f near another point x' writing

$$f(\cdot) \simeq \sum_{\ell \leqslant n} f_n(x) \binom{n}{\ell} (\cdot - x')^\ell (x'-x)^{n-\ell} \simeq \sum_\ell \left(\sum_{n;\ell\leqslant n} f_n(x) \binom{n}{\ell} (x'-x)^{n-\ell} \right) (\cdot - x')^\ell. \tag{2.2}$$

A more general local description device involves an $\mathbb{R}^d$-indexed collection of functions or distributions $(\Pi_x\tau)(\cdot)$, with labels τ in a finite set $\mathcal{B}$. Consider the real vector space T spanned freely by $\mathcal{B}$. Functions or distributions are locally described as

$$f(\cdot) \simeq \sum_\tau f_\tau(x)(\Pi_x\tau)(\cdot), \quad \text{near each } x \in \mathbb{R}^d.$$

This implicitly assumes that the coefficients $f_\tau(x)$ are function of x. One has $\mathcal{B} = \mathbb{N}^d$ and $(\Pi_x k)(\cdot) = (\cdot - x)^k$, in the usual, Taylor, polynomial setting. Like in the former setting, in a general local description device the reference objects

$$(\Pi_{x'}\tau)(\cdot) = \big(\Pi_x(\Gamma_{xx'}\tau)\big)(\cdot) \tag{2.3}$$

at a different base point x' are linear combinations of the $\Pi_x\sigma$, for a linear map

$$\Gamma_{xx'} : T \to T,$$

and one can switch back and forth between local descriptions at different points. The same thing is encoded in equation (2.2), in the Taylor polynomial setting. The linear maps $\Gamma_{xx'}$ are thus invertible and one has a group action of an $\mathbb{R}^d \times \mathbb{R}^d$-indexed group on the local description structure T.

Whereas one uses the same polynomial-type local description for the f_n as for f itself in the usual Hölder setting C^a, there is no reason in a more general local description device to use the same reference objects for f and for its local coefficients. This is in particular the case if the $(\Pi_x\tau)(\cdot)$ are meant to describe distributions, among others, while it makes sense to use functions only as reference objects to describe the functions f_τ. A simple setting consists in having all the f_τ locally described by a possibly different finite collection $\mathcal{B}^+$ of labels μ, in terms of reference functions $\mathsf{g}_{yx}(\mu)$, with

$$f_\tau(y) \simeq \sum_{\mu\in\mathcal{B}^+} f_{\tau\mu}(x)\mathsf{g}_{yx}(\mu), \quad \text{near } x. \tag{2.4}$$

Note that we take care not to write $(\Pi_x\tau)(y)$ as the $\Pi_x\tau$ may be distributions. It would be consistent to write $\mathsf{g}_x(\mu(y))$ but we stick to the established and convenient notation $\mathsf{g}_{yx}(\mu)$. With this notation, one has both

$$f(\cdot) \simeq \sum_{\tau\in\mathcal{B}} f_\tau(x)(\Pi_x\tau)(\cdot) \simeq \sum_{\tau\in\mathcal{B},\,\mu\in\mathcal{B}^+} f_{\tau\mu}(y)\mathsf{g}_{xy}(\mu)(\Pi_x\tau)(\cdot) \tag{2.5}$$

and

$$f(\cdot) \simeq \sum_{\sigma\in\mathcal{B}} f_\sigma(y)(\Pi_y\sigma)(\cdot).$$

Consistency dictates that the two expressions coincide, giving in particular the fact that the coefficients $f_{\tau\mu}(y)$ are linear combinations of the $f_\sigma(y)$. Write T^+ for the vector space spanned freely by $\mathcal{B}^+$. Re-indexing identity (2.5) and using the notation σ/τ for the μ corresponding to $\tau\mu \simeq \sigma$, one then has

$$f(\cdot) \simeq \sum_{\sigma\in\mathcal{B},\tau\in\mathcal{B}} f_\sigma(y)\, \mathsf{g}_{xy}(\sigma/\tau)(\Pi_x\tau)(\cdot). \tag{2.6}$$

The transition map $\Gamma_{xy} : T \to T$, from (2.3) is thus given in terms of the splitting map

$$\Delta : T \to T \otimes T^+, \quad \Delta\sigma = \sum_\tau \tau \otimes (\sigma/\tau)$$

that appears in the above decomposition, with

$$\Pi_y\sigma = \sum_{\tau\in\mathcal{B}} \mathsf{g}_{xy}(\sigma/\tau)\Pi_x\tau$$

so the transition maps

$$\Gamma_{xy}\sigma = \sum_{\tau\in\mathcal{B}} \mathsf{g}_{xy}(\sigma/\tau)\tau$$

for the reference distributions $\Pi_x\tau$ involve the same ingredients $\mathsf{g}_{yx}(\cdot)$ as those that appear in the local expansion of the coefficients f_τ of f. If one further expands $f_\sigma(y)$ in (2.6) around another reference point z, one gets

$$\begin{aligned} f(\cdot) &\simeq \sum_{\tau,\sigma,\nu\in\mathcal{B}} f_\nu(z)\, \mathsf{g}_{yz}(\nu/\sigma)\mathsf{g}_{xy}(\sigma/\tau)(\Pi_x\tau)(\cdot) \\ &\simeq \sum_{\nu\in\mathcal{B}} f_\nu(z)(\Pi_z\nu)(\cdot) \simeq \sum_{\tau,\nu\in\mathcal{B}} f_\nu(z)\, \mathsf{g}_{xz}(\nu/\tau)(\Pi_x\tau)(\cdot). \end{aligned} \tag{2.7}$$

Here again, *consistency* requires that the two expressions coincide, giving the identity

$$\sum_{\sigma\in\mathcal{B}} \mathsf{g}_{yz}(\nu/\sigma)\mathsf{g}_{xy}(\sigma/\tau) = \mathsf{g}_{xz}(\nu/\tau)$$

in terms of another splitting map

$$\Delta^+ : T^+ \to T^+ \otimes T^+$$

satisfying

$$\Delta^+(\nu/\tau) = \sum_\sigma \sigma/\tau \otimes \nu/\sigma,$$

and, by construction, the identity

$$(\Delta\otimes\mathrm{Id})\Delta = (\mathrm{Id}\otimes\Delta^+)\Delta \tag{2.8}$$

encoded in identity (2.7). Developing $f_\nu(z)$ in (2.7) in terms of another reference point leads by consistency to the identity

$$(\mathrm{Id}\otimes\Delta^+)\Delta^+ = (\Delta^+\otimes\mathrm{Id})\Delta^+.$$

If we insist that the family of reference functions $\mathsf{g}_{yx}(\mu), \mu \in \mathcal{B}^+$, be sufficiently rich to describe locally an *algebra* of functions, it happens to be convenient to assume that the linear span T^+ of $\mathcal{B}^+$ has an algebra structure and the maps g_{yx} on T^+ are characters of the algebra, that is multiplicative maps. Building on the example of the polynomials, it is also natural to assume that T^+ has a grading structure. Using the assumed invertibility of the transition maps Γ_{xy}, an elementary fact from algebra then leads directly to the Hopf algebra structure that appears below in the definition of a concrete regularity structure. (The curious reader can see Proposition 45 in Appendix B. We do not need to understand the details of its simple proof now.)

Note that the dimension d of the state space plays no role in this discussion.

We choose to record the essential features of this discussion in the definition of a 'concrete' regularity structure given below; this is a special form of the more general notion of regularity structure from Hairer' seminal work [44]. The reader should keep in mind that the entire algebraic setting can be understood at a basic level from the above consistency requirements on a given local description device. We invite the reader to look at Appendix B for definitions and basics on bialgebras, Hopf algebras, and comodules, and read this appendix in the light of the preceeding discussion.

2.2 Regularity structures

We define in this section a particular form of regularity structure that turns out to be sufficient for the study of (systems of) singular stochastic PDE(s). A number of notations are fixed here. The following definition is to be read in the light of the discussion of Section 2.1 and will be best understoof by recall first from Appendix B the definition of a Hopf algebra and the meaning of connectedness in this setting.

Definition – *A* **concrete regularity structure** $\mathscr{T} = (T^+, T)$ *is the pair of graded vector spaces*

$$T^+ = \bigoplus_{\alpha \in A^+} T^+_\alpha, \qquad T = \bigoplus_{\beta \in A} T_\beta$$

such that the following holds.

- *The vector spaces* T^+_α *and* T_β *are finite dimensional.*
- *The vector space* T^+ *is a connected graded bialgebra with unit* $\mathbf{1}_+$*, counit* $\mathbf{1}'_+$*, coproduct* $\Delta^+ : T^+ \to T^+ \otimes T^+$*, and grading* $A^+ \subset [0, \infty)$*.*
- *The index set* A *for* T *is a locally finite subset of* $\mathbb{R}$ *bounded below. The vector space* T *is a right comodule over* T^+*, that is* T *is equipped with a splitting map* $\Delta : T \to T \otimes T^+$ *which satisfies*
$$(\Delta \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \Delta^+)\Delta, \quad \textit{and} \quad (\mathrm{Id} \otimes \mathbf{1}'_+)\Delta = \mathrm{Id}. \tag{2.9}$$
Moreover, for any $\beta \in A$
$$\Delta T_\beta \subset \bigoplus_{\alpha \geqslant 0} T_{\beta - \alpha} \otimes T^+_\alpha. \tag{2.10}$$

We denote by

$$\mathscr{T} := \big((T^+, \Delta^+), (T, \Delta)\big)$$

a concrete regularity structure.

The different elements that appear in the definition of a concrete regularity structure will acquire later a concrete meaning. The elements of T and T^+ will index expansion devices for the study of a given (system of) singular stochastic PDE(s) – remember each equation has its own regularity structure. We saw in Section 2.1 the meaning of the splitting maps Δ and Δ^+ and their intertwining relation (2.9) in terms of expansion devices. Recall from the definition of the graded bialgebra given in Appendix B that $T^+_0 = \langle \mathbf{1}_+ \rangle$ and $T^+_\alpha T^+_\beta \subset T^+_{\alpha+\beta}$, for any $\alpha, \beta \in A^+$. By Proposition 45, the bialgebra T^+ is indeed a Hopf algebra; we denote by S_+ its antipode. Denoting by $\mathcal{M} : T^+ \otimes T^+ \to T^+$ the multiplication operator $\mathcal{M}(a \otimes b) := ab$, and by $\mathbf{1}'_+$ the counit of T^+ – think of it as a dual vector to the vector $\mathbf{1}_+$, the antipode S_+ is characterized by the identity

$$\mathcal{M}(\mathrm{Id} \otimes S_+)\Delta^+\tau = \mathcal{M}(S_+ \otimes \mathrm{Id})\Delta^+\tau = \mathbf{1}'_+(\tau)\mathbf{1}_+.$$

Moreover, its coproduct Δ^+ satisfies $\Delta^+\mathbf{1}_+ = \mathbf{1}_+ \otimes \mathbf{1}_+$, and

$$\Delta^+\tau \in \left\{ \tau \otimes \mathbf{1}_+ + \mathbf{1}_+ \otimes \tau + \sum_{0 < \beta < \alpha} T^+_\beta \otimes T^+_{\alpha - \beta} \right\}, \tag{2.11}$$

for any $\tau \in T^+_\alpha$ with $\alpha > 0$. Similarly, it is straightforward from (2.9) and (2.10) to check that

$$\Delta\tau \in \left\{ \tau \otimes \mathbf{1}_+ + \sum_{\beta < \alpha} T_\beta \otimes T^+_{\alpha - \beta} \right\} \tag{2.12}$$

for any $\tau \in T_\alpha$. This identity will later imply for a set $\Pi_x\tau$ of reference functions/distributions that $\Pi_{x'}\tau \simeq \Pi_x\tau$, up to terms of smaller 'homogeneity' $\beta < \alpha$.

Note that we do not assume any relation between the linear spaces T^+_α and T_β at that stage. Note also that the homogeneity function $|\cdot|$ takes values in $\mathbb{R}$, and that the parameter β in (2.12) can be non-positive, unlike in (2.11). For an arbitrary element τ in T, set

$$\tau = \sum_{\beta \in A} \tau_\beta \in \bigoplus_{\beta \in A} T_\beta.$$

We use a similar notation for elements of T^+. An element τ of $T_\alpha^{(+)}$ is said to be *homogeneous* and is assigned **homogeneity** $|\tau| := \alpha$. The homogeneous spaces T_β and T_α^+ being finite dimensional, all norms on them are equivalent; we use a generic notation $\|\cdot\|_\beta$ or $\|\cdot\|_\alpha$ for norms on these spaces. For simplicity, we write

$$\|\tau\|_\alpha := \|\tau_\alpha\|_\alpha. \tag{2.13}$$

Notations. • *Let $\mathcal{B}_\alpha^+$ and $\mathcal{B}_\beta$ be bases of T_α^+ and T_β, respectively. Set*

$$\mathcal{B}^+ := \bigcup_{\alpha\in A^+} \mathcal{B}_\alpha^+, \quad \mathcal{B} := \bigcup_{\beta\in A} \mathcal{B}_\beta.$$

• *Recall from the end of Section 1 our convention about statements of the form $\mathfrak{s}^{(+)}$. Given $\sigma, \tau \in \mathcal{B}^{(+)}$, we use the notation $\sigma \leqslant^{(+)} \tau$ to mean that $\sigma = \tau$ or $|\sigma| < |\tau|$; we write $\tau/^{(+)}\sigma$ for the element of T^+ defined by the expansion*

$$\Delta^{(+)}\tau = \sum_{\sigma\in\mathcal{B}^{(+)},\, \sigma\leqslant^{(+)}\tau} \sigma \otimes (\tau/^{(+)}\sigma).$$

Write $\sigma <^{(+)} \tau$ to mean further that σ is different from τ. The notations $\tau/^{(+)}\sigma$ and $\sigma <^{(+)} \tau$ are only used for τ and σ in $\mathcal{B}^{(+)}$.

Interpreting the splitting maps Δ and Δ^+ as chopping elements into pieces, keep in mind that $\tau/^{(+)}\sigma$ can be a sum of elements of T^+, in case σ appears 'at different places' as an element of τ. This will be particularly clear in formula (2.15) below for the polynomial regularity structure, where the binomial coefficient $\binom{n}{\ell}$ will account for the number of X^ℓ inside X^n, for $0 \leqslant \ell \leqslant n$. Note that for $\sigma <^{(+)} \tau$ in $\mathcal{B}^{(+)}$, we have

$$\begin{aligned} \Delta^+(\tau/^{(+)}\sigma) &= \sum_{\sigma\leqslant^{(+)}\eta\leqslant^{+}\tau} (\eta/^{(+)}\sigma) \otimes (\tau/^{+}\eta) \\ &= (\tau/^{(+)}\sigma) \otimes \mathbf{1}_+ + \mathbf{1}_+ \otimes (\tau/^{(+)}\sigma) + \sum_{\sigma<^{(+)}\eta<^{(+)}\tau} (\eta/^{(+)}\sigma) \otimes (\tau/^{(+)}\eta). \end{aligned} \tag{2.14}$$

These two identities are direct consequences of the *co-associativity properties*

$$(\Delta^{(+)} \otimes \mathrm{Id})\Delta^{(+)} = (\mathrm{Id} \otimes \Delta^+)\Delta^{(+)},$$

of the coproduct $\Delta^{(+)}$, obtained by identifying the corresponding terms in the left and right hand sides. In the setting of singular stochastic PDEs where the elements of T are (decorated) trees, τ/σ will be a product of trees, and each of these trees will eventually be involved in the action of re-centering the corresponding analytic objects to a given running point, while leaving the trunk tree σ untouched. The definition of a model given in Section 2.3 illustrates exactly this picture. Here are two examples of regularity structures.

– Let symbols $X_1, \dots, X_d$ be given. For $n \in \mathbb{N}^d$, set $X^n := X_1^{n_1} \cdots X_d^{n_d}$; this is an element of the free commutative algebra with unit $\mathbf{1}(=: X^0)$ generated by the X_i. We can see that $T_X := \mathrm{span}\{X^n; n \in \mathbb{N}^d\}$ is a bialgebra with the coproduct

$$\Delta_{\mathrm{pol}} X^n := \sum_{\ell\leqslant n} \binom{n}{\ell} X^\ell \otimes X^{n-\ell}, \tag{2.15}$$

with $\ell \leqslant n$ if $\ell_i \leqslant n_i$ for all $1 \leqslant i \leqslant d$. Let $\mathfrak{s} = (\mathfrak{s}_i) \in (\mathbb{N}\backslash\{0\})^d$ an integer-valued fixed vector, called a **scaling**. This vector accounts for the natural scaling properties in the different directions of $\mathbb{R}^d$ for the problem at hand. If for instance $\mathbb{R}^d$ does not stand for the isotropic Euclidean space but rather for a non-isotropic space with topology $\mathbb{R}^d$, as a Lie group, different directions will naturally have different homogeneities, depending on the geometry of the space. We define the scaled degree of $n \in \mathbb{N}^d$ by

$$|n|_{\mathfrak{s}} = \sum_{i=1}^d \mathfrak{s}_i n_i.$$

Then the definition $T_\alpha = \mathrm{span}\{X^n; |n|_{\mathfrak{s}} = \alpha\}$ gives a grading for the bialgebra T_X. Since $T_0 = \mathrm{span}\{\mathbf{1}\}$, the space T_X is a connected graded bialgebra. Thus it is indeed a Hopf algbera; the antipode is actually given by $S_+ X^n = (-X)^n$. The **polynomial regularity structure** is given by

$$\mathscr{T}_X := \big((T_X, \Delta_{\mathrm{pol}}), (T_X, \Delta_{\mathrm{pol}})\big).$$

– To have another picture in mind, think of T and T^+ as sets of possibly labelled rooted trees, with T^+ consisting only of trees with positive tree homogeneities – a homogeneity is assigned to each labelled tree. This notion of homogeneity induces the decomposition (2.13) of T into linear spaces spanned by trees with equal homogeneities; a similar decomposition holds for T^+. The coproduct $\Delta^+\tau$ is typically a sum over subtrees σ of τ with the same root as τ, and τ/σ is the quotient tree obtained from τ by identifying σ with the root; this quotient tree is better seen as a product of trees. One understands the splitting $\Delta\tau$ of an element $\tau \in T$ in similar terms. See Section 9 for constructions of regularity structures of this sort associated with singular stochastic PDEs. For such regularity structures, the minimum regularity of the elements of T is given by the minimum regularity of the noises in the equation. One can leave aside trees by the time we arrive at Section 9 and work in the abstract setting of this section throughout.

A **character** g on the Hopf algebra T^+ is a linear map $g : T^+ \to \mathbb{R}$, such that $g(\mathbf{1}_+) = 1$ and $g(\tau_1\tau_2) = g(\tau_1)g(\tau_2)$, for any $\tau_1, \tau_2 \in T^+$. The antipode S_+ of the Hopf algebra T^+ turns the set of characters of the algebra T^+ into a group G^+ for the *convolution law* $*$ defined by

$$(g_1 * g_2)\tau := (g_1 \otimes g_2)\Delta^+\tau, \quad \tau \in T^+.$$

Think of the usual convolution product $(f * g)(x) = \int f(y)g(x-y)dy$, where one first splits x into y and $x-y$, then apply f and g to each piece, before taking the product and summing over all possible splittings. The convolution inverse of a character g of T^+ is $g \circ S_+$. One associates to a character g of T^+ the map

$$\widehat{g} := (\mathrm{Id} \otimes g)\Delta : T \to T,$$

from T to itself. For $g \in G^+$, the map $\widehat{g}$ is denoted by Γ_g in Hairer's work [44]; we prefer the former Fourier-like notation, which is consistent with the fact that the 'hat' map defines a linear representation of G^+ into $L(T)$. We have indeed

$$\widehat{g_1 * g_2} = \widehat{g_1} \circ \widehat{g_2}$$

for any $g_1, g_2 \in G^+$, as a direct consequence of the comodule property (2.8). Also, for any $\tau \in T_\beta$,

$$\Big(\widehat{g}(\tau) - \tau\Big) \in \bigoplus_{\beta' < \beta} T_{\beta'},$$

as a consequence of the structural identity (2.12). Similarly, one defines the action of G^+ on T^+ by

$$\widehat{g}^+ := (\mathrm{Id} \otimes g)\Delta^+ : T^+ \to T^+.$$

for $g \in G^+$.

2.3 Models and modelled distributions

The preceding section contains the algebraic backbone of regularity structures. Its analytic flesh is introduced in this section on models and modelled distributions. This analytic setting depends on which (system of) singular stochastic PDE(s) one studies. We will not use the same function spaces to analyse a class of equations involving only the heat operator $(\partial_t - \Delta_x)$ and a system of two equations involving $(\partial_t - \Delta_x)$ for one and an operator with a different scaling for the other, like $(\partial_t + (-\Delta_x)^a)$, or simply $(-\Delta_x)^a$, with $0 < a \neq 1$, for the other. *We choose to concentrate in the present work on parabolic equations involving the heat operator only.* We will thus work throughout with the parabolic space $\mathbb{R} \times \mathbb{R}^d$, with generic point $x = (x_0, x') = (x_i)_{i=0}^d$, equipped with the distance function

$$d(x,y) = d\big((x_0,x'),(y_0,y')\big) = \sqrt{|x_0 - y_0|} + |x' - y'|.$$

The Hölder spaces introduced in the first paragraph of this section will play a prominent role. They are used in the second paragraph to define models over a given regularity structure. Models give us the reference functions/distributions $(\Pi_x\tau)(\cdot)$ and $\mathsf{g}_{yx}(\mu)$ that we will use in our expansion devices to describe potential solutions of given singular stochastic PDEs. Expansion devices associate to each spacetime point a distribution meant to give the local description of a globally defined distribution. There is however no reason that such a globally defined distribution exists if no condition on its local 'jets' is imposed. The appropriate consistency condition is encoded in the definition of a modelled distribution. Under this consistency condition, it is a fundamental fact that all these local descriptions can be patched together to define a unique globally defined distribution locally that is close to its local description, everywhere. This is what the reconstruction theorem does for us. We end this section with a paragraph on the special properties of modelled distributions representing functions.

§ Function spaces

Set
$$\mathfrak{s} := (2, 1, \dots, 1) \in \mathbb{N} \times \mathbb{N}^d,$$
and define, for any multi-index $n = (n_0, n_1, \dots, n_d) \in \mathbb{N} \times \mathbb{N}^d$, the scaled degree of n by
$$|n|_{\mathfrak{s}} := 2n_0 + n_1 + \cdots + n_d.$$
We define a non-positive elliptic operator on $\mathbb{R} \times \mathbb{R}^d$
$$\mathcal{G} := \partial_{x_0}^2 - \Delta_{x'}^2$$
and denote by
$$P_t := e^{t\mathcal{G}}$$
its heat semigroup, and by $p_t(x, y)$ its kernel with respect to Lebesgue measure. It is a symmetric function of (x, y) that satisfies the scaling property
$$p_t(x, y) = t^{-(d+2)/4} p\left(\left(t^{-\mathfrak{s}_j/4}(x_j - y_j)\right)_{0 \leqslant j \leqslant d}\right)$$
for a Schwartz function $p \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$. The estimate
$$\int \left|\partial_x^n p_t(x, y)\right| d^a(x, y)\, dy \lesssim t^{\frac{a - |n|_{\mathfrak{s}}}{4}}, \tag{2.16}$$
holds as a consquence for any multiindex $n \in \mathbb{N} \times \mathbb{N}^d$ and any positive exponent a. For a fixed positive integer $N \in \mathbb{N}$, we define operators $Q_t^{(N)}$ and $P_t^{(N)}$ setting
$$Q_t^{(N)} := (-t\mathcal{G})^N e^{t\mathcal{G}}, \quad P_t^{(N)} := \int_t^\infty Q_s^{(N)} \frac{ds}{s}.$$
This implies that $P_t^{(N)} = p(-t\mathcal{G})e^{t\mathcal{G}}$, for a polynomial p of degree N, with constant coefficient 1. One has in particular $P_t^{(1)} = P_t$, and
$$P_t^{(N)} = \int_t^1 Q_s^{(N)} \frac{ds}{s} + P_1^{(N)}. \tag{2.17}$$
(Those who know a little about Littlewood-Paley decomposition will recognise in $Q_t^{(N)}$ the counterpart of the Littlewood-Paley projectors Δ_i and in the integral with respect to the measure ds/s the counterpart of the uniform measure on the integers; the integral operator associated with $P_1^{(N)}$ plays the role of Δ_{-1}; this is an infinitely smoothing operator.)

Definition – *Fix $N \geqslant 1$ and pick a real number $\alpha < 4N$. We define the α-**Hölder space** $\mathcal{C}^\alpha(\mathbb{R} \times \mathbb{R}^d)$ as the set of distributions on $\mathbb{R} \times \mathbb{R}^d$ with finite $\mathcal{C}^\alpha$-norm defined by*
$$\|\Lambda\|_{\mathcal{C}^\alpha} := \left\|P_1^{(N)}(\Lambda)\right\|_{L^\infty(\mathbb{R} \times \mathbb{R}^d)} + \sup_{0 < t \leqslant 1} t^{-\frac{\alpha}{4}} \left\|Q_t^{(N)}(\Lambda)\right\|_{L^\infty(\mathbb{R} \times \mathbb{R}^d)}. \tag{2.18}$$

The constraint on α comes from the fact that all polynomials of scaled degree no greater than $4N$ are in the kernel of the operator $Q_t^{(N)}$. The above definition of the Hölder spaces depends on N, for the range of regularity exponents considered; write momentarily $\mathcal{C}_N^\alpha(\mathbb{R} \times \mathbb{R}^d)$. We remark that if $\alpha/4 < N < N'$ are given then on can prove that the N and N'-dependent norms are equivalent on $\mathcal{C}_N^\alpha(\mathbb{R}\times\mathbb{R}^d)$ – this is a classical fact, worked out e.g. in Appendix A of [3]. In the sequel, the exponent N is fixed once and for all to a large enough value depending on the problem at hand, so we do not record it in the notations for the Hölder spaces. More generally, one can define Hölder spaces using other elliptic operators than $\mathcal{G}$; the spaces will be identical and the different norms equivalent. We will use this remark only in the proof of Proposition 11 on the classical Schauder estimates. One can also show that for a positive non-integer regularity exponent a the space $\mathcal{C}^a(\mathbb{R}\times\mathbb{R}^d)$ coincides with the usual space of a-Hölder functions, for the parabolic distance d, with equivalent norms. See e.g. the proof of Proposition 2.5 in [3].

Note that, if $\alpha<0$, then the equivalence

$$\|\Lambda\|_{\mathcal{C}^\alpha}\simeq\sup_{0<t\leqslant 1}t^{-\frac{\alpha}{4}}\|P_t(\Lambda)\|_{L^\infty(\mathbb{R}\times\mathbb{R}^d)} \tag{2.19}$$

holds. The left hand side is bounded by the right one, because $P_1^{(1)}=P_1$ and $Q_t^{(1)}=\varphi(t\mathcal{G})P_{t/2}$, with a uniformly bounded operator $\varphi(t\mathcal{G})$. The other direction follows from identity (2.17) relating the operators P and $Q^{(1)}$.

§ **Models**

Recall from the introduction of Section 2 the intuitive motivation for introducing regularity structures. Whereas the algebra involved in the use of local description devices is captured by the notion of regularity structure, the actual family of functions and distributions involved in these local descriptions is captured by the notion of model over a regularity structure.

Definition 1. *A* **model over a regularity structure** $\mathscr{T}$ *is a pair* $(\mathsf{\Pi},\mathsf{g})$ *of maps*

$$\mathsf{g}:\mathbb{R}\times\mathbb{R}^d\to G^+,\qquad \mathsf{\Pi}:T\to\mathcal{S}'(\mathbb{R}\times\mathbb{R}^d)$$

with the following properties.

- *Set* $\mathsf{g}_{yx}:=\mathsf{g}_y*\mathsf{g}_x^{-1}$, *for each* $x,y\in\mathbb{R}\times\mathbb{R}^d$. *For each exponent* $\gamma\in\mathbb{R}$, *one has*

$$\|\mathsf{g}\|_\gamma:=\sup_{\tau\in\mathcal{B}^+,|\tau|<\gamma}\ \sup_{x,y\in\mathbb{R}\times\mathbb{R}^d}\frac{|\mathsf{g}_{yx}(\tau)|}{d(y,x)^{|\tau|}}<\infty. \tag{2.20}$$

- *The map* $\mathsf{\Pi}$ *is linear. Set*

$$\mathsf{\Pi}_x^{\mathsf{g}}:=(\mathsf{\Pi}\otimes\mathsf{g}_x^{-1})\Delta$$

for each $x\in\mathbb{R}\times\mathbb{R}^d$. *For each exponent* $\gamma\in\mathbb{R}$, *one has*

$$\|\mathsf{\Pi}^{\mathsf{g}}\|_\gamma:=\sup_{\sigma\in\mathcal{B},|\sigma|<\gamma}\ \sup_{x\in\mathbb{R}\times\mathbb{R}^d,\,0<t\leqslant 1}t^{-\frac{|\sigma|}{4}}\big|\big\langle\mathsf{\Pi}_x^{\mathsf{g}}\sigma,p_t(x,\cdot)\big\rangle\big|<\infty. \tag{2.21}$$

In the class of problems we consider, it is sufficient in each problem to fix $\gamma\in\mathbb{R}$ to a large enough value; we omit as a consequence this parameter from the notations, unless necessary. In Hairer's original work [44], the notations Π_x and Γ_{yx} are used instead of $\mathsf{\Pi}_x^{\mathsf{g}}$ and $\widehat{\mathsf{g}_{yx}}$, respectively. We emphasize the dependence of Π_x on g using our notation. We stress that $\mathsf{\Pi}\tau$ is only an element of $\mathcal{S}'(\mathbb{R}\times\mathbb{R}^d)$. Think of $\mathsf{\Pi}$ as an interpretation operator for the symbols τ, with τ encoding the structure of the analytic object $\mathsf{\Pi}\tau$. One can think of $\mathsf{\Pi}_x^{\mathsf{g}}\tau=(\mathsf{\Pi}\otimes\mathsf{g}_x^{-1})\Delta\tau$, as $\mathsf{\Pi}\tau$ 'fully recentered' at x, to give it a concrete meaning. The splitting map Δ identifies the different sets of internal pieces of τ that can be 'recentered' to the point x by the action of the map g_x^{-1}, with the full recentering operation on $\mathsf{\Pi}\tau$ being the result of all these recentering operations. Condition (2.21) conveys the idea that $\mathsf{\Pi}_x^{\mathsf{g}}\tau$ behaves *at point* x like an element of $\mathcal{C}^{|\tau|}(\mathbb{R}\times\mathbb{R}^d)$, as a result of this full recentering operation. We will see in Section 9 concrete examples of recentering operations that can be understood as replacing a function by its Taylor remainder of a certain degree.

Emphasize that g acts on T^+, while $\mathsf{\Pi}$ acts on T, and *note that* g *plays on* T^+ *the same role as* $\mathsf{\Pi}$ *on* T: For $\tau\in T^+$ and $\sigma\in T$, one has

$$\mathsf{g}_{yx}(\tau)=\Big(\mathsf{g}_y(\cdot)\otimes\mathsf{g}_x^{-1}\Big)\Delta^+\tau,\quad (\mathsf{\Pi}_x^{\mathsf{g}}\sigma)(y)=\Big(\mathsf{\Pi}(\cdot)(y)\otimes\mathsf{g}_x^{-1}\Big)\Delta\sigma, \tag{2.22}$$

in a distributional sense for the latter. Note also the fundamental relation

$$\mathsf{\Pi}_y^{\mathsf{g}}=\mathsf{\Pi}_x^{\mathsf{g}}\circ\widehat{\mathsf{g}_{xy}}, \tag{2.23}$$

for all $x, y \in \mathbb{R}\times\mathbb{R}^d$; it comes directly from the comodule property (2.8). The following consequence of the bound (2.21) will be useful in the next section.

Proposition 2. *One has*

$$\sup_{x\in\mathbb{R}\times\mathbb{R}^d,\,0<t\leqslant 1} t^{-\frac{|\tau|-|n|_{\mathfrak{s}}}{4}} \big|\big\langle \mathsf{\Pi}^{\mathsf{g}}_x\tau, \partial^n_x p_t(x,\cdot)\big\rangle\big| < \infty,$$

for any model $(\mathsf{\Pi},\mathsf{g})$ *on* $\mathscr{T}$, $\tau\in\mathcal{B}$, *and* $n\in\mathbb{N}\times\mathbb{N}^d$.

Proof – By the semigroup property,

$$\partial^n_x p_t(x,y) = \int \partial^n_x p_{\frac{t}{2}}(x,z)p_{\frac{t}{2}}(z,y)dz.$$

We need to apply the distribution $\mathsf{\Pi}^{\mathsf{g}}_x\tau$ to the kernel $p_{\frac{t}{2}}(z,\cdot)$. Using relation (2.23) to write $\mathsf{\Pi}^{\mathsf{g}}_x\tau$ in terms of $\mathsf{\Pi}^{\mathsf{g}}_z\tau$, one has

$$\big|\big\langle \mathsf{\Pi}^{\mathsf{g}}_x\tau, p_{\frac{t}{2}}(z,\cdot)\big\rangle\big| = \Big|\sum_{\sigma\leqslant\tau}\mathsf{g}_{zx}(\tau/\sigma)\big\langle \mathsf{\Pi}^{\mathsf{g}}_z\sigma, p_{\frac{t}{2}}(z,\cdot)\big\rangle\Big| \lesssim \sum_{\sigma\leqslant\tau} d(z,x)^{|\tau|-|\sigma|}\,t^{\frac{|\sigma|}{4}}$$

from the bound (2.21) in the definition of a model. Using the bound (2.16) on the moments of the heat kernel, we then have

$$\begin{aligned}\Big|\big\langle \mathsf{\Pi}^{\mathsf{g}}_x\tau, \partial^n_x p_t(x,\cdot)\big\rangle\Big| &\lesssim \sum_{\sigma\leqslant\tau} t^{\frac{|\sigma|}{4}}\int \big|\partial^n_x p_{\frac{t}{2}}(x,z)\big|\, d(z,x)^{|\tau|-|\sigma|}dz \\ &\lesssim \sum_{\sigma\leqslant\tau} t^{\frac{|\sigma|}{4}}\, t^{\frac{|\tau|-|\sigma|-|n|_{\mathfrak{s}}}{4}} \lesssim t^{\frac{|\tau|-|n|_{\mathfrak{s}}}{4}}.\end{aligned}$$

$\rhd$

If ever all the $\mathsf{\Pi}\tau$ happen to be a continuous function, then it follows from the bound on $\big\langle \mathsf{\Pi}^{\mathsf{g}}_x\tau, p_t(x,\cdot)\big\rangle$, and the fact that $p_t(x,\cdot)$ is converging to a Dirac mass at x, that the function $\mathsf{\Pi}^{\mathsf{g}}_x\tau$ satisfies $(\mathsf{\Pi}^{\mathsf{g}}_x\tau)(x) = 0$, for all $\tau\in T$ such that $|\tau|>0$. This will be the case of the smooth (possibly renormalized) models from Section 6. We close this paragraph by defining a pseudo-distance on the space of models over a given regularity structure setting for each $\gamma\in\mathbb{R}$

$$\mathsf{d}_\gamma(\mathsf{M},\mathsf{M}') := \sup_{\substack{\tau\in\mathcal{B}^+,|\tau|<\gamma\\ \sigma\in\mathcal{B},|\sigma|<\gamma}}\ \sup_{x,y\in\mathbb{R}\times\mathbb{R}^d}\left(\frac{\big|\mathsf{g}_{yx}(\tau)-\mathsf{g}'_{yx}(\tau)\big|}{d(y,x)^{|\tau|}} + \sup_{0<t\leqslant 1} t^{-\frac{|\sigma|}{4}}\Big|\big\langle \mathsf{\Pi}^{\mathsf{g}}_x\sigma - \mathsf{\Pi}'^{\mathsf{g}'}_x\sigma, p_t(x,\cdot)\big\rangle\Big|\right). \tag{2.24}$$

§ Modelled distributions and their reconstruction

Think of a T-valued function $\boldsymbol{f}$ on $\mathbb{R}\times\mathbb{R}^d$ as the data needed to associate with each spacetime point $x\in\mathbb{R}\times\mathbb{R}^d$ the local description $\mathsf{\Pi}^{\mathsf{g}}_x\boldsymbol{f}(x)$ of a possibly globally defined distribution close to $\mathsf{\Pi}^{\mathsf{g}}_x\boldsymbol{f}(x)$ near each x. There is no reason that such a globally defined object exists if one does not impose relations between the different components of $\boldsymbol{f}$. This is what the next definition does. For $\gamma\in\mathbb{R}$, set

$$T_{<\gamma} := \bigoplus_{\beta<\gamma} T_\beta, \qquad T^+_{<\gamma} := \bigoplus_{\alpha<\gamma} T^+_\alpha.$$

Recall from (2.13) the meaning of the notation $\|h\|_\alpha$, for $\alpha\in A$ and $h\in T$.

Definition 3. *Let* $\mathsf{g}:\mathbb{R}\times\mathbb{R}^d\to G^+$ *satisfy* (2.20)*. Fix a regularity exponent* $\gamma\in\mathbb{R}$*. One defines the space* $\mathcal{D}^\gamma(T,\mathsf{g})$ *of* **distributions modelled on the regularity structure $\mathscr{T}$, with transition** g, *as the space of functions* $\boldsymbol{f}:\mathbb{R}\times\mathbb{R}^d\to T_{<\gamma}$ *such that*

$$[\![\boldsymbol{f}]\!]_{\mathcal{D}^\gamma} := \max_{\beta<\gamma}\ \sup_{x\in\mathbb{R}\times\mathbb{R}^d}\big\|\boldsymbol{f}(x)\big\|_\beta < \infty,$$

$$\|\boldsymbol{f}\|_{\mathcal{D}^\gamma} := \max_{\beta<\gamma}\ \sup_{x,y\in\mathbb{R}\times\mathbb{R}^d}\frac{\big\|\boldsymbol{f}(y)-\widehat{\mathsf{g}_{yx}}\boldsymbol{f}(x)\big\|_\beta}{d(y,x)^{\gamma-\beta}} < \infty.$$

Set $|\!|\!|\boldsymbol{f}|\!|\!|_{\mathcal{D}^\gamma} := [\![\boldsymbol{f}]\!]_{\mathcal{D}^\gamma} + \|\boldsymbol{f}\|_{\mathcal{D}^\gamma}$.

For a basis element $\sigma\in\mathcal{B}\subset T$, and an arbitrary element h in T, denote by h_σ its component on σ in the basis $\mathcal{B}$. For a modelled distribution $\boldsymbol{f}(\cdot) = \sum_{\sigma\in\mathcal{B}} f_\sigma(\cdot)\,\sigma$ in $\mathcal{D}^\gamma(T,\mathsf{g})$, and $\sigma_0\in\mathcal{B}$, we have

$$\begin{aligned}\big(\boldsymbol{f}(y)-\widehat{\mathsf{g}_{yx}}\boldsymbol{f}(x)\big)_{\sigma_0} &= f_{\sigma_0}(y)-\sum_{\tau\geqslant\sigma_0}\mathsf{g}_{yx}(\tau/\sigma_0)\,f_\tau(x)\\ &= f_{\sigma_0}(y)-f_{\sigma_0}(x)-\sum_{\tau>\sigma_0}\mathsf{g}_{yx}(\tau/\sigma_0)\,f_\tau(x).\end{aligned}\tag{2.25}$$

Note that we use $\widehat{\mathsf{g}_{yx}}$ to compare $\boldsymbol{f}(x)$ and $\boldsymbol{f}(y)$ in the definition of a modelled distribution while we use $\widehat{\mathsf{g}_{xy}}$ to relate the interpretation operators Π_y^{g} and Π_x^{g} in (2.23).

Examples • *The archetype of a modelled distribution is given by the lift*

$$\boldsymbol{f}(x):=\sum_{|n|_{\mathfrak{s}}<\gamma}\frac{f^{(n)}(x)}{n!}X^n,\tag{2.26}$$

in the polynomial regularity structure of a γ-Hölder real valued function f on $\mathbb{R}\times\mathbb{R}^d$ with a positive regularity exponent γ. The identities (2.25) become in that case the Taylor expansions

$$f^{(n)}(y)-f^{(n)}(x)-\sum_{|\ell|_{\mathfrak{s}}<\gamma-|n|_{\mathfrak{s}}}\frac{1}{\ell!}f^{(n+\ell)}(x)(y-x)^\ell=O\Big(d(y,x)^{\gamma-|n|_{\mathfrak{s}}}\Big)\tag{2.27}$$

satisfied by each $f^{(n)}$. Note here that the classical and elementary fact that a family $(f_n)_{n\in\mathbb{N}\times\mathbb{N}^d}, |n|_{\mathfrak{s}}<\gamma$ of functions on $\mathbb{R}\times\mathbb{R}^d$ is γ-Hölder for the parabolic metric iff the functions f_n satisfy (2.27) with f_n in the role of $f^{(n)}$. So the notion of modelled distribution with values in the polynomial regularity structure captures exactly the classical notion of regularity.

• *Given a basis element $\tau\in\mathcal{B}$, set*

$$\boldsymbol{h}^\tau(x):=\sum_{\sigma<\tau}\mathsf{g}_x(\tau/\sigma)\sigma.\tag{2.28}$$

It follows from identity (2.10) in the definition of a concrete regularity structure that $\boldsymbol{h}^\tau$ takes values in $T_{<|\tau|}$. It follows also from identity (2.14) giving $\Delta^+(\tau/\sigma)$, that

$$\begin{aligned}\widehat{\mathsf{g}_{yx}}\big(\boldsymbol{h}^\tau(x)\big) &= \sum_{\eta\leqslant\sigma<\tau}\mathsf{g}_{yx}(\sigma/\eta)\mathsf{g}_x(\tau/\sigma)\eta=\sum_{\eta<\tau}\Big(\mathsf{g}_y(\tau/\eta)-\mathsf{g}_{yx}(\tau/\eta)\Big)\eta\\ &= \boldsymbol{h}^\tau(y)-\sum_{\eta<\tau}\mathsf{g}_{yx}(\tau/\eta)\eta.\end{aligned}$$

The size estimate $\big|\mathsf{g}_{yx}(\tau/\eta)\big|\lesssim d(y,x)^{|\tau|-|\eta|}$ required from the g-component of a model, then shows that $\boldsymbol{h}^\tau$ is a modelled distribution in $\mathcal{D}^{|\tau|}(T_{<|\tau|},\mathsf{g})$.

• *If $\boldsymbol{f}(\cdot)=\sum_{\sigma\in\mathcal{B}}f_\sigma(\cdot)\,\sigma$, is an element of $\mathcal{D}^\gamma(T,\mathsf{g})$, then, for each $\tau\in\mathcal{B}$, the T^+-valued function*

$$\boldsymbol{f}/\tau(\cdot):=\sum_{\sigma\geqslant\tau}f_\sigma(\cdot)\,\sigma/\tau.$$

is an element of $\mathcal{D}^{\gamma-|\tau|}(T^+,\mathsf{g})$.

The next statement says that the consistency condition encoded in the notion of modelled distribution $\boldsymbol{f}$ ensures the existence of a globally defined object close to $\Pi_x^{\mathsf{g}}\boldsymbol{f}(x)$ near each $x\in\mathbb{R}\times\mathbb{R}^d$, and gives condition for uniqueness. Recall A stands for the index set in the grading of T and set

$$\beta_0:=\min A.$$

Theorem 4. (Reconstruction theorem) *Let $\mathscr{T}$ be a concrete regularity structure and $\mathsf{M}=(\Pi,\mathsf{g})$ be a model over $\mathscr{T}$. Fix a regularity exponent $\gamma\in\mathbb{R}\backslash\{0\}$. There exists a linear continuous operator*

$$\mathsf{R}^{\mathsf{M}}:\mathcal{D}^\gamma(T,\mathsf{g})\to\mathcal{C}^{\beta_0\wedge 0}(\mathbb{R}\times\mathbb{R}^d)$$

satisfying the property

$$\Big|\Big\langle\mathsf{R}^{\mathsf{M}}\boldsymbol{f}-\Pi_x^{\mathsf{g}}\boldsymbol{f}(x),p_t(x,\cdot)\Big\rangle\Big|\lesssim\|\Pi^{\mathsf{g}}\|\,\|\boldsymbol{f}\|_{\mathcal{D}^\gamma}\,t^{\frac{\gamma}{4}},\tag{2.29}$$

uniformly in $\boldsymbol{f}\in\mathcal{D}^\gamma(T,\mathsf{g}),x\in\mathbb{R}\times\mathbb{R}^d$ and $0<t\leqslant 1$. Such an operator is unique if the exponent γ is positive.

A distribution $\mathsf{R}^{\mathsf{M}}\boldsymbol{f}$ satisfying identity (2.29) is called a **reconstruction of the modelled distribution $\boldsymbol{f}$**. When $\gamma=0$, the existence of a reconstruction is not ensured by (2.29) in general.

See Example 5.5 in [18]. We will see as a particular case of Corollary 6 that the lift (2.26) of a γ-Hölder function f in the polynomial regularity structure has indeed f as a reconstruction.

Notice from the definition of Π^{g}_x that the constraint $\big|\big\langle \Pi^{\mathsf{g}}_x\tau, p_t(x,\cdot)\big\rangle\big| \lesssim t^{|\tau|/4}$, that needs to be satisfied by a model, is equivalent to the estimate

$$\Big|\Big\langle \Pi\tau - \Pi^{\mathsf{g}}_x \boldsymbol{h}^\tau(x), p_t(x,\cdot)\Big\rangle\Big| = \Big|\Big\langle \Pi\tau - \sum_{\sigma<\tau} \mathsf{g}_x(\tau/\sigma)\Pi^{\mathsf{g}}_x\sigma, p_t(x,\cdot)\Big\rangle\Big| \lesssim t^{|\tau|/4}, \tag{2.30}$$

says that $\Pi\tau$ is a/the reconstruction of the modelled distribution $\boldsymbol{h}^\tau$ from (2.28), depending on whether $|\tau| \leqslant 0$ or $|\tau| > 0$. Since, for $|\tau| < 0$, the difference $(*)$ of two reconstructions of $\boldsymbol{h}^\tau$ satisfy

$$\big|\big\langle (*), p_t(x,\cdot)\big\rangle\big| \lesssim t^{|\tau|/4}$$

for all $x \in \mathbb{R}\times\mathbb{R}^d$, this difference is a $\mathcal{C}^{|\tau|}(\mathbb{R}\times\mathbb{R}^d)$ distribution from identity (2.19). So the estimate (2.30) shows in particular that we could require from scratch that the Π map of a model of $\mathscr{T}$ takes values in $\mathcal{C}^{\beta_0\wedge 0}(\mathbb{R}\times\mathbb{R}^d)$ rather than $\mathcal{D}'(\mathbb{R}\times\mathbb{R}^d)$. The case $|\tau| = 0$ does not cause any problem as we assume that the only element of T of null homogeneity is $\mathbf{1}$.

We will only work with $\mathcal{D}^\gamma(T,\mathsf{g})$-spaces with positive regularity exponents γ in our study of singular stochastic PDEs. We only give a proof of the reconstruction theorem in that setting, following Otto & Weber's nice approach [61]. See Friz and Hairer's lecture notes [33] for another treatment along these lines. See Hairer's original work [44] or the references given in Appendix D for a proof of Theorem 4 when $\gamma \leqslant 0$.

Proof – *Existence* – We construct explicitly a reconstruction operator. Note first that since

$$\begin{aligned}\Big(\Pi^{\mathsf{g}}_y \boldsymbol{f}(y) - \Pi^{\mathsf{g}}_x \boldsymbol{f}(x)\Big)(\cdot) &= \Big(\Pi^{\mathsf{g}}_y\big(\boldsymbol{f}(y) - \widehat{\mathsf{g}_{yx}}\boldsymbol{f}(x)\big)\Big)(\cdot) \\ &= \sum_{\tau\in\mathcal{B}} \big(\boldsymbol{f}(y) - \widehat{\mathsf{g}_{yx}}\boldsymbol{f}(x)\big)_\tau \big(\Pi^{\mathsf{g}}_y\tau\big)(\cdot)\end{aligned}$$

one has

$$\big|\big\langle \Pi^{\mathsf{g}}_y \boldsymbol{f}(y) - \Pi^{\mathsf{g}}_x \boldsymbol{f}(x), p_t(y,\cdot)\big\rangle\big| \lesssim \sum_{\tau\in\mathcal{B}, |\tau|<\gamma} d(y,x)^{\gamma-|\tau|} t^{\frac{|\tau|}{4}},$$

from the bounds on models and modelled distributions. For $0 < s \leqslant t \leqslant 1$ and $x \in \mathbb{R}\times\mathbb{R}^d$, set

$$\mathsf{I}^t_s(x) := \int p_{t-s}(x,y)\big\langle \Pi^{\mathsf{g}}_y \boldsymbol{f}(y), p_s(y,\cdot)\big\rangle\, dy.$$

We will obtain the distribution $\mathsf{R}^{\mathsf{M}}\boldsymbol{f}$ from I^t_s under the form $\lim_{t\downarrow 0}\lim_{s\downarrow 0}\mathsf{I}^t_s$, with the limits taken in that order, with s sent to 0 first, and then t send to 0. First, from the bounds on modelled distributions, we have

$$\mathsf{I}^t_t(x) = \big\langle \Pi^{\mathsf{g}}_x \boldsymbol{f}(x), p_t(x,\cdot)\big\rangle \quad\text{and}\quad \big|\mathsf{I}^t_t(x)\big| \leqslant \sum_{\tau\in\mathcal{B}} |f_\tau(x)|\,\big|\big\langle \Pi^{\mathsf{g}}_x\tau, p_t(x,\cdot)\big\rangle\big| \lesssim t^{\frac{\beta_0}{4}},$$

and moreover, for $0 < s' < s < t \leqslant 1$, we have from the semigroup property of the kernel p the x-uniform estimate

$$\begin{aligned}\big|\mathsf{I}^t_{s'}(x) - \mathsf{I}^t_s(x)\big| &= \left|\int p_{t-s}(x,z)p_{s-s'}(z,y)\big\langle \Pi^{\mathsf{g}}_y \boldsymbol{f}(y) - \Pi^{\mathsf{g}}_z \boldsymbol{f}(z), p_{s'}(y,\cdot)\big\rangle\, dzdy\right| \\ &\leqslant \sum_{\tau\in\mathcal{B}, |\tau|<\gamma} \int p_{t-s}(x,z)p_{s-s'}(z,y)\, d(y,z)^{\gamma-|\tau|}(s')^{\frac{|\tau|}{4}}\, dzdy \\ &\lesssim \sum_{\tau\in\mathcal{B}, |\tau|<\gamma} (s-s')^{\frac{\gamma-|\tau|}{4}}(s')^{\frac{|\tau|}{4}}.\end{aligned}$$

For $s' \in [s/2, s)$, this implies

$$\big|\mathsf{I}^t_{s'}(x) - \mathsf{I}^t_s(x)\big| \lesssim s^{\frac{\gamma}{4}}. \tag{2.31}$$

For $s' \in (0, s/2)$, by taking $n \in \mathbb{N}$ such that $s' \in \big[s/2^{n+1}, s/2^n\big)$, we have

$$\begin{aligned}\big|\mathsf{I}^t_{s'}(x) - \mathsf{I}^t_s(x)\big| &\leqslant \sum_{m=0}^{n-1} \big|\mathsf{I}^t_{s/2^m}(x) - \mathsf{I}^t_{s/2^{m+1}}(x)\big| + \big|\mathsf{I}^t_{s/2^n}(x) - \mathsf{I}^t_{s'}(x)\big| \\ &\lesssim \sum_{m=0}^{n-1} (s/2^m)^{\frac{\gamma}{4}} + (s/2^n)^{\frac{\gamma}{4}} \lesssim s^{\frac{\gamma}{4}}.\end{aligned}$$

Thus the bound (2.31) holds uniformly over $0 < s' < s$. Hence the (locally in t) uniform limit

$$\mathsf{I}_0^t(x) := \lim_{s\to 0} \mathsf{I}_s^t(x)$$

exists, since γ is positive. As the identity $P_{t'}\mathsf{I}_0^t = \mathsf{I}_0^{t+t'}$ follows from the semigroup property, we see that $\{\mathsf{I}_0^t\}_{0<t\leqslant 1}$ is bounded in the space $\mathcal{C}^{\beta_0}(\mathbb{R}\times\mathbb{R}^d)$. (Note that all of the above estimates on I_s^t holds over $0 < s \leqslant t \leqslant 2$, since the bounds on $\Pi_x^{\mathsf{g}}\tau$ can be extended to $0 < t \leqslant 2$ by a similar argument to Proposition 2.) Hence $\{\mathsf{I}_0^t\}_{0<t\leqslant 1}$ has a subsequence $\{\mathsf{I}_0^{t_n}\}$ converging in $\mathcal{C}^{\beta_0-\varepsilon}(\mathbb{R}\times\mathbb{R}^d)$ as t_n goes to 0, for any $\varepsilon > 0$. Denote its limit by $\mathbf{R}^{\mathsf{M}}\boldsymbol{f}$. Since

$$\big\langle \mathbf{R}^{\mathsf{M}}\boldsymbol{f}, p_t(x,\cdot)\big\rangle = \lim_{t_n\to 0}\big\langle \mathsf{I}_0^{t_n}, p_t(x,\cdot)\big\rangle = \lim_{t_n\to 0} \mathsf{I}_0^{t+t_n}(x) = \mathsf{I}_0^t(x),$$

we have actually $\mathbf{R}^{\mathsf{M}}\boldsymbol{f} \in \mathcal{C}^{\beta_0}(\mathbb{R}\times\mathbb{R}^d)$ from the x-uniform bound $|\mathsf{I}_0^t(x)| \lesssim t^{\beta_0/4}$. Letting $s = t$ and sending s' to 0 in (2.31), we can check that $\mathbf{R}^{\mathsf{M}}\boldsymbol{f}$ satisfies the bound (2.29).

Uniqueness – To prove uniqueness of the reconstruction operator on $\mathcal{D}^\gamma(T,\mathsf{g})$ when the regularity exponent γ is positive, we start from the identity

$$\big|\big\langle \mathbf{R}^{\mathsf{M}}\boldsymbol{f} - (\mathbf{R}^{\mathsf{M}})'\boldsymbol{f}, p_t(x,\cdot)\big\rangle\big| \lesssim t^{\gamma/4},$$

satisfied uniformly in $x \in \mathbb{R}\times\mathbb{R}^d$ by any other reconstruction operator $(\mathbf{R}^{\mathsf{M}})'$. As for any Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$ the convolutions $\int\varphi(x)p_t(x,z)dx$, converge to φ in the smooth topology, one has from the symmetry of the kernels p_t and the fact that γ is positive

$$\big\langle \mathbf{R}^{\mathsf{M}}\boldsymbol{f} - \big(\mathbf{R}^{\mathsf{M}}\big)'\boldsymbol{f}, \varphi\big\rangle = \lim_{t\to 0}\int \Big\langle \mathbf{R}^{\mathsf{M}}\boldsymbol{f} - (\mathbf{R}^{\mathsf{M}})'\boldsymbol{f}, p_t(x,\cdot)\Big\rangle\varphi(x)dx = \lim_{t\to 0} O\big(t^{\gamma/4}\big) = 0.$$

$\rhd$

One can use Proposition 2 to improve estimate (2.29) under the form

$$\Big|\Big\langle \mathbf{R}^{\mathsf{M}}\boldsymbol{f} - \Pi_x^{\mathsf{g}}\boldsymbol{f}(x), \partial_x^n p_t(x,\cdot)\Big\rangle\Big| \lesssim t^{\frac{\gamma-|n|_{\mathfrak{s}}}{4}}, \tag{2.32}$$

uniformly in $x \in \mathbb{R}\times\mathbb{R}^d$, for each $n \in \mathbb{N}\times\mathbb{N}^d$. It is important that the reconstruction operator $\mathbf{R}^{\mathsf{M}}$ is a local operator. The following fact implies that $\big\langle \mathbf{R}^{\mathsf{M}}\boldsymbol{f}, \varphi\big\rangle$ depends only on the restriction of $\boldsymbol{f}$ to the support of φ. This fact is used to define the reconstructions of modelled distributions which are given in $(0,t)\times\mathbb{R}^d$ with $t \in (0,\infty]$, not in $\mathbb{R}\times\mathbb{R}^d$. SeeTheorem 20 and Section 4.3.

Corollary 5. *Pick γ positive. If $\boldsymbol{f} \in \mathcal{D}^\gamma(T,\mathsf{g})$ is null on an open set $U \subset \mathbb{R}\times\mathbb{R}^d$, then $\mathbf{R}^{\mathsf{M}}\boldsymbol{f} = 0$ on U.*

Proof – Since $\boldsymbol{f}$ and $\Pi_x^{\mathsf{g}}\boldsymbol{f}(x)$ are null on U, it follows from estimate (2.29) that

$$\big|\big\langle \mathbf{R}^{\mathsf{M}}\boldsymbol{f}, p_t(x,\cdot)\big\rangle\big| \lesssim t^{\gamma/4},$$

for all $x \in U$. For a smooth function φ with compact support in U, one can use the uniform convergence of $\int\varphi(x)p_t(x,z)dx$ to φ as $t > 0$ goes to 0, to get

$$\big\langle \mathbf{R}^{\mathsf{M}}\boldsymbol{f}, \varphi\big\rangle = \lim_{t\to 0}\int \varphi(x)\big\langle \mathbf{R}^{\mathsf{M}}\boldsymbol{f}, p_t(x,\cdot)\big\rangle dx = 0.$$

$\rhd$

The following fact is an immediate consequence of uniqueness in the reconstruction theorem; it is used in Section 4 and implies in particular that the the lift (2.26) of a γ-Hölder function f in the polynomial regularity structure has indeed f as a reconstruction.

Corollary 6. *Pick γ positive and $\boldsymbol{f} \in \mathcal{D}^\gamma(T,\mathsf{g})$. If the model (Π,g) takes values in the space of smooth functions on $\mathbb{R}\times\mathbb{R}^d$, then the mapping $x \mapsto \big(\Pi_x^{\mathsf{g}}\boldsymbol{f}(x)\big)(x)$ is itself a continuous function and*

$$\big(\mathbf{R}^{\mathsf{M}}\boldsymbol{f}\big)(x) = \Big(\Pi_x^{\mathsf{g}}\boldsymbol{f}(x)\Big)(x). \tag{2.33}$$

Remark – *One may wonder for which class of T-valued functions the reconstruction theorem holds. Caravenna and Zambotti proved in [18] that a slightly modified notion of modelled distribution is actually necessary and sufficient for that. So the notion of modelled distribution captures (almost) exactly the notion of 'jet' of a distribution.*

Throughout we will work with regularity structures satisfying the following assumption saying that T and T^+ contain the polynomial regularity structure.

Assumption (A1) – *The concrete regularity structure* $\big((T^+,\Delta^+),(T,\Delta)\big)$ *contains the polynomial regularity structure in the following sense.*

(*a*) *One has* $\mathcal{B}^+_\alpha = \big\{X_+^n\,;\,|n|_\mathfrak{s}=\alpha\big\}$*, for a symbol* $X_+\in T^+$ *and any integer* $\alpha\in\mathbb{N}$*, and*

$$\Delta^+ X_+^n = \sum_{\ell\leqslant n}\binom{n}{\ell} X_+^\ell\otimes X_+^{n-\ell}.$$

(*b*) *One has* $\mathcal{B}_\alpha = \big\{X^n\,;\,|n|_\mathfrak{s}=\alpha\big\}$*, for a symbol* $X\in T$ *and any integer* $\alpha\in\mathbb{N}$*, and*

$$\Delta X^n = \sum_{\ell\leqslant n}\binom{n}{\ell} X^\ell\otimes X_+^{n-\ell}.$$

The notation X_+^n allows to distinguish the elements in $\mathcal{B}^+$ and $\mathcal{B}$. Set

$$\mathcal{B}^+_X := \big\{X_+^n\,;\,n\in\mathbb{N}\times\mathbb{N}^d\big\},\qquad \mathcal{B}_X := \big\{X^n\,;\,n\in\mathbb{N}\times\mathbb{N}^d\big\},$$

and write

$$\mathbf{1}_+ = X_+^0,\qquad \mathbf{1}=X^0.$$

Note the use of X_+ in the formula for ΔX^n. The space

$$T_X^+ := \mathrm{span}(\mathcal{B}_X^+)$$

with Δ^+ is isomorphic to a polynomial regularity structure, while the space

$$T_X := \mathrm{span}(\mathcal{B}_X)$$

with Δ is a right comodule over T_X^+. These polynomials are the only elements of T or T^+ with integer homogeneities. One defines a *canonical model* over the polynomial regularity structure

$$\mathscr{T}_X := \big((T_X^+,\Delta^+),(T_X,\Delta)\big)$$

setting for all $x,y\in\mathbb{R}\times\mathbb{R}^d$

$$(\mathsf{\Pi} X^n)(y) := y^n,\quad \mathsf{g}_x(X_+^n) := x^n.$$

We see that $\mathsf{g}_{yx}(X_+^n) = (y-x)^n$ and $(\mathsf{\Pi}^\mathsf{g}_x X^n)(y) = (y-x)^n$, so $(\mathsf{\Pi},\mathsf{g})$ is indeed a model over $\mathscr{T}_X$.

Assumption (A2) – *Under Assumption* **(A1)***, we only consider models* $(\mathsf{\Pi},\mathsf{g})$ *whose restriction to* $\mathscr{T}_X$ *is the canonical model.*

We only work from now on with regularity structures satisfying assumptions **(A1)** and **(A2)**. It is useful, to deal with sub-regularity structures of a given regularity structure, to introduce the following notion. A linear subspace V of T is called a **subcomodule** if, defining $V_\alpha := V\cap T_\alpha$, the pair $\big((T^+,\Delta^+),(V,\Delta)\big)$ is a regularity structure, that is if $\Delta V\subset V\otimes T^+$. A subcomodule V is said to be **function-like**, if V satisfies assumptions **(A1)** and **(A2)** and if $V_\beta=0$ whenever $\beta<0$. Given a subcomodule V, set

$$\alpha_0(V) := \min\Big\{\alpha\in A\,;\,V_\alpha\neq 0,\,\alpha\notin\mathbb{N}\Big\}.$$

Corollary 7. *Let V be a function-like comodule. For a positive regularity exponent $\gamma\geqslant\alpha_0(V)$ and $\boldsymbol{f}\in\mathcal{D}^\gamma(V,\mathsf{g})$, one has $\mathsf{R}^\mathsf{M}\boldsymbol{f}\in\mathcal{C}^{\alpha_0(V)}(\mathbb{R}\times\mathbb{R}^d)$ and for all $x\in\mathbb{R}\times\mathbb{R}^d$*

$$\big(\mathsf{R}^\mathsf{M}\boldsymbol{f}\big)(x) = f_\mathbf{1}(x).$$

Proof – We write in the proof $a^{>0}$ to denote an element of the form a^b, with a positive constant b whose value is irrelevant and may change from place to place.
The uniqueness part of the proof of the reconstruction theorem, Theorem 4, makes it clear that the reconstruction $\mathsf{R}^\mathsf{M}\boldsymbol{f}$ of $\boldsymbol{f}\in\mathcal{D}^\gamma(T,\mathsf{g})$, with $\gamma>0$, is characterized by the estimate

$$\Big|\big\langle \mathsf{R}^\mathsf{M}\boldsymbol{f} - \mathsf{\Pi}^\mathsf{g}_x\boldsymbol{f}(x), p_t(x,\cdot)\big\rangle\Big| \lesssim t^{\gamma'},$$

whatever positive exponent γ' appears in the upper bound. Set $\overline{\boldsymbol{f}} := \boldsymbol{f} - f_{\mathbf{1}}\mathbf{1}$. It is elementary to see that $f_{\mathbf{1}}$ is a usual $\alpha_0(V)$-Hölder function, from the fact that $\boldsymbol{f} \in \mathcal{D}^\gamma(V, \mathsf{g})$. Since

$$\Pi^{\mathsf{g}}_x\big(\boldsymbol{f}(x)\big)(\cdot) - f_{\mathbf{1}}(\cdot) = f_{\mathbf{1}}(x) - f_{\mathbf{1}}(\cdot) + \Pi^{\mathsf{g}}_x\big(\overline{\boldsymbol{f}}(x)\big)(\cdot),$$

the result follows from the fact that one has

$$\big|\big\langle f_{\mathbf{1}}(x) - f_{\mathbf{1}}(\cdot), p_t(x,\cdot)\big\rangle\big| \lesssim \int d(x,y)^{>0}\,|p_t(x,y)|dy \lesssim t^{>0},$$

from (2.16), and

$$\big|\big\langle \Pi^{\mathsf{g}}_x\big(\overline{\boldsymbol{f}}(x)\big)(\cdot), p_t(x,\cdot)\big\rangle\big| \lesssim t^{>0},$$

from the bound of a model and the fact that $\boldsymbol{f}$ takes values in $\bigoplus_{\beta \geqslant \alpha_0(V)\wedge 1} V_\beta$. ▷

2.4 Products and derivatives

Other regularity structures than the polynomial regularity structure can be used to 'model' functions. In good cases, they come equipped with a bilinear operation that plays the role plaid by multiplication in the usual setting, and allows to define the image of a modelled distribution by a nonlinear map. This is what this section is about.

Let V, W be subcomodules of T and set

$$V_\alpha := V \cap T_\alpha, \qquad W_\alpha := W \cap T_\alpha.$$

Definition – *A* **product** *on $V \times W$ is a continuous bilinear map $\star : V \times W \to T$, such that $V_\alpha \star W_\beta \subset T_{\alpha+\beta}$, for all $\alpha, \beta \in A$. The product is said to be* **regular** *if*

$$\Delta(\tau \star \sigma) = (\Delta\tau)(\Delta\sigma)$$

for all $\tau \in V$ and $\sigma \in W$. In the right hand side, the product $(V \otimes T^+) \times (W \otimes T^+) \to T \otimes T^+$ is canonically defined from $\star$ and the product of T^+ setting $(\tau \otimes \mu)(\sigma \otimes \nu) = (\tau \star \sigma) \otimes (\mu\nu)$.

The regularity structures used in the study of singular PDEs have elements that are decorated rooted trees. The product is given as a tree product in that setting, and such a product is regular in the above sense. You will find the details in Section 9. A regular product $\star$ satisfies

$$\widehat{g}(\tau \star \sigma) = \widehat{g}(\tau) \star \widehat{g}(\sigma), \tag{2.34}$$

for any character g on T^+. For regularity structures containing the polynomial regularity structure, one asks the following consistency assumption.

Assumption (A3) – *Under Assumption* **(A1)***, the product between T_X and T is always defined and satisfies*

$$\mathbf{1} \star \tau = \tau \star \mathbf{1} = \tau, \textit{ for all } \tau \in T, \quad X^k \star X^\ell = X^{k+\ell}, \textit{ for all } k, \ell \in \mathbb{N} \times \mathbb{N}^d.$$

Assumptions **(A1)**, **(A2)**, and **(A3)** are jointly called assumption **(A)**. The proof of the next statement is elementary and left to the reader. See the proof of Theorem 4.6 in [44] if needed. For $\alpha \leqslant 0 < \gamma$, denote by $\mathcal{D}^\gamma_\alpha$ the space of modelled distributions of the form

$$\boldsymbol{f} = \sum_{\alpha \leqslant |\tau| < \gamma} f_\tau \tau$$

and write

$$\mathcal{Q}_{<\gamma} : T \to T_{<\gamma}$$

for the canonical projection.

Proposition 8. *Let $\star : V \times W \to T$ be a regular product. Given $\boldsymbol{f}_1 \in \mathcal{D}^{\gamma_1}_{\alpha_1}(V, \mathsf{g})$ and $\boldsymbol{f}_2 \in \mathcal{D}^{\gamma_2}_{\alpha_2}(W, \mathsf{g})$, set $\gamma = (\gamma_1 + \alpha_2) \wedge (\gamma_2 + \alpha_1)$. Then one has*

$$\mathcal{Q}_{<\gamma}(\boldsymbol{f}_1 \star \boldsymbol{f}_2) \in \mathcal{D}^\gamma_{\alpha_1+\alpha_2}(T, \mathsf{g}).$$

The mapping $(\boldsymbol{f}_1, \boldsymbol{f}_2) \mapsto \mathcal{Q}_{<\gamma}(\boldsymbol{f}_1 \star \boldsymbol{f}_2)$ is continuous.

Let V be a *function-like comodule* of T equipped with an associative product

$$\star : V \times V \to V.$$

Then $\star$ is naturally extended to the multilinear map from V^n to V, for any $n \geqslant 1$. For any $\boldsymbol{f} \in \mathcal{D}^\gamma(V, \mathrm{g})$ with $\gamma > 0$ and a smooth function $F : \mathbb{R} \to \mathbb{R}$, we define

$$F^\star(\boldsymbol{f}) := \mathcal{Q}_{<\gamma}\left(\sum_{n=0}^{\infty} \frac{F^{(n)}(f_{\mathbf{1}})}{n!}\,\overline{\boldsymbol{f}}^{\star n}\right), \quad \overline{\boldsymbol{f}} := \boldsymbol{f} - f_{\mathbf{1}}\mathbf{1}.$$

The sum contains only finitely many terms since the sector V is function-like, so $\overline{\boldsymbol{f}} \in \mathcal{D}^\gamma_{\alpha_1}(V, \mathrm{g})$ for an $\alpha_1 > 0$ and $\overline{\boldsymbol{f}}^{\star n} \in \mathcal{D}^\gamma_{n\alpha_1}(V, \mathrm{g})$, so $\mathcal{Q}_{<\gamma}\big(\overline{\boldsymbol{f}}^{\star n}\big) = 0$, for $n\alpha_1 \geqslant \gamma$. The proof of the next proposition is elementary and left to the reader; see Theorem 4.15 in [44] for a proof.

Proposition 9. *Pick a positive regularity exponent γ. For any $\boldsymbol{f} \in \mathcal{D}^\gamma(V, \mathrm{g})$ and a smooth function F, one has $F^\star(\boldsymbol{f}) \in \mathcal{D}^\gamma(V, \mathrm{g})$. Moreover, the mapping $\boldsymbol{f} \mapsto F^\star(\boldsymbol{f})$ is locally Lipschitz continuous.*

Definition – *A* **derivative** *is a continuous linear map $D : T \to T$, such that $DT_\alpha \subset T_{\alpha-1}$ for all $\alpha \in A$, and*

$$\Delta(D\tau) = (D \otimes \mathrm{Id})\Delta\tau$$

for any $\tau \in T$ – by an abuse of notation, we mean $T_{\alpha-1} = \{0\}$ if $\alpha - 1 \notin A$.

The assumption on D implies

$$\widehat{g}(D\tau) = D\widehat{g}(\tau)$$

for any character g on T^+. From this property, it is straightforward to show the following statement.

Proposition 10. *The mapping $\mathcal{D}^\gamma(T, \mathrm{g}) \ni \boldsymbol{f} \mapsto D\boldsymbol{f} \in \mathcal{D}^{\gamma-1}(T, \mathrm{g})$ is continuous. Moreover, if $\mathsf{\Pi} \circ D = \mathscr{D} \circ \mathsf{\Pi}$, holds for a first order differential operator $\mathscr{D}$ then*

$$\mathsf{R}^{\mathsf{M}}(D\boldsymbol{f}) = \mathscr{D}\Big(\mathsf{R}^{\mathsf{M}}\boldsymbol{f}\Big)$$

for any $\boldsymbol{f} \in \mathcal{D}^\gamma(T, \mathrm{g})$ with $\gamma > 1$.

3 – Regularity structures built from integration operators

The regularity structures used for the study of singular stochastic PDEs have a particular structure that comes from the fixed point formulation of the (system of) PDE(s) under study. We concentrate here on the case where only one second order differential operator is involved, typically $\partial_t - \Delta_x$. (See Section 9 and Appendix D for comments on the general case.) We work then with regularity structures equipped with an operator that plays the role of the convolution operator $(\partial_t - \Delta_x)^{-1}$, involved in the fixed point formulation of the equation under study. This operator is called an abstract integration map; it is introduced in Section 3.2. One can associate to any given model $(\mathsf{\Pi}, \mathrm{g})$ on $\mathscr{T}$ a notion of admissible $\mathsf{\Pi}$-maps; this is done in Section 3.3. They essentially intertwin the abstract integration map with the convolution operator $(\partial_t - \Delta_x)^{-1}$. They are used in Section 3.4 to lift the latter into a map sending continuously $\mathcal{D}^\gamma(T, \mathrm{g})$ into $\mathcal{D}^{\gamma+2}(T, \mathrm{g})$, for any positive non-integer γ, which the abstract integration map fails to do. Section 3.5 is dedicated to constructing admissible models. We set the scene in Section 3.1.

We will restrict our study to regularity structures for which the minimum homogeneity of its elements satisfies

$$\beta_0 = \min A > -2. \tag{3.1}$$

This condition ensures that elements of the form $\mathcal{I}(\tau)$ that appear in the expansion of solutions to the regularity structure lift of the considered class of singular stochastic PDE are of positive homogeneity. (So if $\mathcal{I}$ were increasing the homogeneity of all symbols by a we would require $\beta_0 > -a$.) While the generalized (KPZ) equation satisfies for instance this assumption, not all singular stochastic PDEs satisfy it. This is for instance the case of the Φ^4_2, Φ^4_3 or sine-Gordon equations. This kind of equations can nonetheless by studied within the setting of regularity structures by writing their solutions as the sum of an explicit functional of the noise and a remainder term that solves an equation that can be formulated in a regularity structure satisfying condition (3.1) – the so-called da Prato-Debussch trick, after similar operation was used in their work [25]. We consider as an example the case of the Φ^4_3 equation

$$(\partial_t - \Delta_x)u = -u^3 + \zeta,$$

set on the 3-dimensional torus, with ζ a spacetime white noise of Hölder regularity $(-5/2)^-$. We decompose a priori the solution u into $u = X + v$, where

$$\begin{aligned}(\partial_t - \Delta_x)X &= \zeta,\\ (\partial_t - \Delta_x)v &= -v^3 - 3v^2X - 3vX^2 - X^3.\end{aligned}$$

The polynomial functions $(X^n)_{1\leqslant n\leqslant 3}$ can directly be defined as elements of $\mathcal{C}^{(-n/2)^-}$ by probabilistic means. The above equation for v can then be formulated in a regularity structure with three noise symbols for X, X^2 and X^3, in which $\beta_0 = (-3/2)^- > -2$. The interested reader will find more details on this matter for a general class of singular stochastic PDEs in Section 5 of Bruned, Chandra, Chevyrev and Hairer's work [13].

3.1 Operators on $\mathbb{R} \times \mathbb{R}^d$

We will be interested in (systems of) singular stochastic PDEs that involve possibly two types of differential operators. The derivatives ∂_i in the directions of the canonical basis of $\mathbb{R} \times \mathbb{R}^d$, and the second order differential operators

$$\mathbf{L} := \partial_{x_0} - \Delta_{x'} + 1$$

Denote by $\mathbf{L}^{-1}$ the resolution operator associating to a Schwartz function $v \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$ the solution $u \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$ to the equation

$$\mathbf{L}u = v.$$

The strictly positiveness of $-\Delta+1$ ensures uniqueness of a solution $u \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$ to the preceding equation. (This is the reason why we work with $\mathbf{L}$ rather than the heat operator.) The operator $\mathbf{L}^{-1}$ can be represented using a variant of the elliptic operators $\mathcal{G}$ introduced in Section 2.3. Indeed, we have

$$\begin{aligned}\big(\partial_{x_0} - \Delta_{x'} + 1\big)^{-1} &= -\big(\partial_{x_0} + \Delta_{x'} - 1\big)\,\big(-\partial_{x_0}^2 + (\Delta_{x'} - 1)^2\big)^{-1}\\ &= -\int_0^\infty \big(\partial_{x_0} + \Delta_{x'} - 1\big)\, e^{r(\partial_{x_0}^2 - (\Delta_{x'}-1)^2)} dr.\end{aligned}$$

Write

$$\mathbf{L}^{-1} = -\int_0^\infty \big(\partial_{x_0} + \Delta_{x'} - 1\big)\, e^{r(\partial_{x_0}^2 - (\Delta_{x'}-1)^2)}\, dr =: \int_0^\infty K_r\, dr.$$

We thus use the inhomogeneous operator

$$\widetilde{\mathcal{G}} := \partial_{x_0}^2 - (\Delta_{x'} - 1)^2$$

instead of the operator $\mathcal{G}$ considered in Section 2.3. One has $\widetilde{\mathcal{G}} = \mathcal{G} + 2\Delta_{x'} - 1$, so the smooth kernel of $e^{r\widetilde{\mathcal{G}}}$ is up to the multiplicative constant e^{-r} the space convolution of the kernel p_r of $e^{r\mathcal{G}}$ with the classical heat kernel at time $2r$. It follows from this fact that Proposition 2 holds for the kernel of $e^{r\widetilde{\mathcal{G}}}$. Hence the kernel $q_r(x, y)$ of the operator K_r satisfies the x-uniform bounds

$$\big|\big\langle \Pi_x^{\mathsf{g}}\tau, \partial_x^n q_r(x,\cdot)\big\rangle\big| \lesssim r^{\frac{|\tau| - |n|_{\mathfrak{s}} - 2}{4}} \tag{3.2}$$

for any $\tau \in \mathcal{B}$, $r \in (0, 1]$, and $n \in \mathbb{N} \times \mathbb{N}^d$, with the exponent $|n|_{\mathfrak{s}} + 2$ coming from the derivative operators ∂_x^n and $(\partial_{x_0} + \Delta_{x'} - 1)$ applied to $e^{r\widetilde{\mathcal{G}}}$. It is convenient, for technical purposes, to decompose $\mathbf{L}^{-1}$ under the form

$$\mathbf{L}^{-1} = \mathbf{K} + \mathbf{K}',$$

with

$$\mathbf{K} := \int_0^1 K_r\, dr - (e-1)K_1, \quad \mathbf{K}' := \int_1^\infty K_r\, dr + (e-1)K_1.$$

It is elementary to see that the operators $\mathbf{K}'$ maps $\mathcal{C}^\gamma(\mathbb{R}\times\mathbb{R}^d)$ into $\mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}^d)$, for any regularity exponent $\gamma\in\mathbb{R}$. We concentrate on the operator $\mathbf{K}$ in the remainder of this section. Denote by

$$K(x,y) := \int_0^1 q_r(x,y)\,dr - (e-1)q_1(x,y) \tag{3.3}$$

its kernel. Since K_1 is a smoothing operator, the second term in the definition of the kernel K does not matter in the arguments of this section. The compensation of K ensures that

$$\int_{\mathbb{R}\times\mathbb{R}^d} K(x,y)dy = 0, \qquad \int_{\mathbb{R}\times\mathbb{R}^d} y_i\,K(x,y)dy = 0, \qquad (|e_i|_{\mathfrak{s}}=1), \tag{3.4}$$

which follows from $\int_{\mathbb{R}\times\mathbb{R}^d} q_r(x,y)dy = e^{-r}$. These identities are not used in this section, but they are important in Section 7; see Assumption **(E)**.

Proposition 11. (Schauder estimates for $\mathbf{L}^{-1}$) *The operators $\mathbf{L}^{-1}$ and $\mathbf{K}$ are continuous operators from $\mathcal{C}^\gamma(\mathbb{R}\times\mathbb{R}^d)$ into $\mathcal{C}^{\gamma+2}(\mathbb{R}\times\mathbb{R}^d)$, for all non-integer regularity exponents $\gamma\in\mathbb{R}$.*

Proof – It is sufficient to show the estimate for $\widetilde{\mathbf{K}} = \int_0^1 K_r\,dr$. Note that $K_t = -e^{t\widetilde{\mathcal{G}}}(\partial_{x_0}+\Delta_{x'}-1)$, as $\widetilde{\mathcal{G}}$ and $(\partial_{x_0}+\Delta_{x'}-1)$ commute. We use the freedom on the choice of the elliptic operator used to define the Hölder spaces, while giving equivalent norms, to work with the norm associated with the operator $\widetilde{\mathcal{G}}$ rather than the operator $\mathcal{G}$. We emphasize that fact by writing $\widetilde{\mathcal{Q}}_s^{(N)}$ the operators built from $\widetilde{\mathcal{G}}$ in the same way as $\mathcal{Q}_s^{(N)}$ is built from $\mathcal{G}$. Given a distribution $\Lambda\in\mathcal{C}^\gamma(\mathbb{R}\times\mathbb{R}^d)$, with $\gamma\in\mathbb{R}$ non-integer, we read on the identity

$$\widetilde{\mathcal{Q}}_s^{(N)}\big(\widetilde{\mathbf{K}}(\Lambda)\big) = \int_0^1 \widetilde{\mathcal{Q}}_s^{(N)}\big(K_r(\Lambda)\big)dr = -\int_0^1 \Big(\frac{s}{s+r}\Big)^N \widetilde{\mathcal{Q}}_{s+r}^{(N)}\Big((\partial_{x_0}+\Delta_{x'}-1)\Lambda\Big)\,dr,$$

the estimate

$$\Big\|Q_s^{(N)}\big(\widetilde{\mathbf{K}}(\Lambda)\big)\Big\|_\infty \lesssim s^{\frac{\gamma-2}{4}+1} + O(s^N).$$

The result follows for all $\gamma+2<4N$. The equivalence of the different Hölder norms corresponding to different choices of N gives the conclusion. ▷

(We refer the reader to the second edition of Friz & Hairer's lecture notes [33] for a particularly nice proof of the classical Schauder estimates using different tools.) For a regularity structure $\mathscr{T}$ for which $\beta_0 = \min A > -2$, and a model $(\mathsf{\Pi},\mathsf{g})$ on it, Schauder estimates imply in particular that all the distributions $\mathbf{K}(\mathsf{\Pi}\tau)(\cdot)$, hence all the distributions $\mathbf{K}(\mathsf{\Pi}_x^{\mathsf{g}}\tau)$, are actually defined pointwise, for any $x\in\mathbb{R}\times\mathbb{R}^d$, making sense of $\mathbf{K}(\mathsf{\Pi}_x^{\mathsf{g}}\tau)(x)$, or even $\partial^n\mathbf{K}(\mathsf{\Pi}_x^{\mathsf{g}}\tau)(x)$, for $|n|_{\mathfrak{s}}<\beta_0+2$. The following lemma allows to take profit from the fact that $\mathsf{\Pi}_x^{\mathsf{g}}\tau$ behaves near x "as" an element of $\mathcal{C}^{|\tau|}(\mathbb{R}\times\mathbb{R}^d)$, to give meaning to $\partial^n\mathbf{K}(\mathsf{\Pi}_x^{\mathsf{g}}\tau)(x)$, for all multiindices n such that $|n|_{\mathfrak{s}}<|\tau|+2$.

Lemma 12. *Assume $\beta_0>-2$. Given $\tau\in\mathcal{B}$ and $n\in\mathbb{N}\times\mathbb{N}^d$, the integral*

$$\big(\partial^n\mathbf{K}(\mathsf{\Pi}_x^{\mathsf{g}}\tau)\big)(x) := \big\langle\mathsf{\Pi}_x^{\mathsf{g}}\tau, \partial_x^n K(x,\cdot)\big\rangle := \int_0^1 \big\langle\mathsf{\Pi}_x^{\mathsf{g}}\tau, \partial_x^n q_r(x,\cdot)\big\rangle\,dr - (e-1)\big\langle\mathsf{\Pi}_x^{\mathsf{g}}\tau, \partial_x^n q_1(x,\cdot)\big\rangle \tag{3.5}$$

converges for all $x\in\mathbb{R}\times\mathbb{R}^d$, provided $|n|_{\mathfrak{s}}<|\tau|+2$.

Proof – It follows from (3.2) that the first term of the right hand side of (3.5) is integrable over $t\in(0,1)$ if

$$|\tau|-|n|_{\mathfrak{s}} > -2.$$

▷

The $\mathbb{R}\times\mathbb{R}^d$-indexed distributions $\mathsf{R}^{\mathsf{M}}\boldsymbol{f}-\mathsf{\Pi}_x^{\mathsf{g}}\boldsymbol{f}(x)$ satisfy a similar bound to (3.2) for any modelled distribution $\boldsymbol{f}\in\mathcal{D}^\gamma(T,\mathsf{g})$. We can then define properly $\partial^n\mathbf{K}\big(\mathsf{R}^{\mathsf{M}}\boldsymbol{f}-\mathsf{\Pi}_x^{\mathsf{g}}\boldsymbol{f}(x)\big)(x)$, for all multiindices n such that $|n|_{\mathfrak{s}}<\gamma+2$, as in the preceding lemma.

3.2 Regularity structures with abstract integration operators

Recall assumption **(A)** essentially says that we consider regularity structures containing the canonical polynomial structure and models that behave naturally on the latter. The regularity structures that are used for the study of singular stochastic PDEs have a special structure that is described in details in Section 9. Presently, we only need to know that they satisfy in addition to assumption **(A)** the following set of assumptions. Recall we denote by $\{e_0, e_1, \dots, e_d\}$ the canonical basis of $\mathbb{N}\times\mathbb{N}^d$.

Assumption (B1) – (a) *The basis $\mathcal{B}^+$ of T^+ is a commutative monoid with unit $\mathbf{1}_+$, freely generated by the symbols*

$$\{X_+^{e_i}\}_{0\leqslant i\leqslant d} \cup \big\{\mathcal{I}_n^+\tau\big\}_{\tau\in\mathcal{B},\, n\in\mathbb{N}\times\mathbb{N}^d,\, |\tau|+2-|n|_{\mathfrak{s}}>0},$$

Each element has homogeneity

$$\big|X_+^{e_i}\big| := \mathfrak{s}_i, \quad \big|\mathcal{I}_n^+\tau\big| := |\tau| + 2 - |n|_{\mathfrak{s}}.$$

The operators Δ and Δ^+ are related by the intertwining relations

$$\Delta^+(\mathcal{I}_n^+\tau) = (\mathcal{I}_n^+\otimes\mathrm{Id})\Delta\tau + \sum_{\ell\in\mathbb{N}\times\mathbb{N}^d,\, |\ell|_{\mathfrak{s}}<|\tau|+2-|n|_{\mathfrak{s}}} \frac{X_+^\ell}{\ell!}\otimes\mathcal{I}_{n+\ell}^+\tau, \tag{3.6}$$

for any $\tau\in\mathcal{B}$.

(b) *For $n\in\{e_1,\dots,e_d\}$, there are operators $\mathcal{I}_n : T\to T$, with*

$$\big|\mathcal{I}_n\tau\big| := |\tau| + 1, \quad \tau\in\mathcal{B}.$$

One has for any $\tau\in\mathcal{B}$

$$\Delta(\mathcal{I}_n\tau) = (\mathcal{I}_n\otimes\mathrm{Id})\Delta\tau + \sum_{\ell\in\mathbb{N}\times\mathbb{N}^d,\, |\ell|_{\mathfrak{s}}<|\tau|+2-|n|_{\mathfrak{s}}} \frac{X^\ell}{\ell!}\otimes\mathcal{I}_{n+\ell}^+\tau. \tag{3.7}$$

Recall from the definition of a model the meaning of the splitting maps Δ and Δ^+ as backbone of 'recentering' operations. Assumption **(B1)** identifies in T a subset of elements built from operators $\mathcal{I}_n$. Identity (3.7) specifies the action of the recentering operations on these elements: Up to a Taylor-type term, the recentered version of $\mathcal{I}_n\tau$ is obtained by applying $\mathcal{I}_n$ to the recentered version of τ. A similar remark holds for T^+, with the difference that T^+ is entirely constructed from the $\mathcal{I}_n^+$ operators and the polynomials, and has no other elements. Note that identity (3.7) identifies $\mathcal{I}_{n+\ell}^+\tau$ as $\ell!(\mathcal{I}_n\tau/X^\ell)$. We let the readers check that identity (3.6) ensures the comodule property

$$(\Delta\otimes\mathrm{Id})\Delta(\mathcal{I}_n\tau) = (\mathrm{Id}\otimes\Delta^+)\Delta(\mathcal{I}_n\tau)$$

on elements of T of the form $\mathcal{I}_n\tau$. The operator $\mathcal{I}_n$ is an abstract version of the convolution operator $\partial^n\mathbf{K}$. The restriction $|n|_{\mathfrak{s}}\leqslant 1$ on n means that we only consider $\mathbf{K}$ or $\partial_i\mathbf{K}$; this is sufficient for the study of all (systems of) singular stochastic PDEs whose solutions are functions involving second order differential operators satisfying the above classical Schauder estimate.

Recall we denote by $\mathcal{M}_+$ the multiplication operator in the algebra T^+ and recall from Appendix B the defining property (B.1) of the antipode S_+ on T^+. As an example of computation using identity (3.6) we check that the antipode S_+ on T^+ satisfies the inductive relation

$$S_+\big(\mathcal{I}^+\tau\big) = -\sum_{\ell\in\mathbb{N}\times\mathbb{N}^d} \frac{(-X)^\ell}{\ell!}\,\mathcal{M}_+\big(\mathcal{I}_{k+\ell}^+\otimes S_+\big)\Delta\tau. \tag{3.8}$$

Together with the relation $S_+(X_+) = -X_+$, such a formula defines indeed a unique algebra morphism. Since it is clear the $\mathcal{M}_+\big(\mathrm{Id}\otimes S_+\big)\Delta^+X_+^k = 0$ for all $k\in\mathbb{N}\times\mathbb{N}^d$, it suffices to see that

$$\mathcal{M}_+\big(\mathrm{Id}\otimes S_+\big)\Delta^+\big(\mathcal{I}_k^+\tau\big) = 0$$

for all $k\in\mathbb{N}\times\mathbb{N}^d$ and $\tau\in T$. This relation follows from (3.8) and (3.6) writing

$$\begin{aligned}\mathcal{M}_+\big(\mathrm{Id}\otimes S_+\big)\Delta^+\big(\mathcal{I}_k^+\tau\big) &= \mathcal{M}_+\big(\mathcal{I}_k^+\otimes S_+\big)\Delta\tau + \sum_{\ell\in\mathbb{N}\times\mathbb{N}^d}\frac{X^\ell}{\ell!}\,S_+\big(\mathcal{I}_{k+\ell}^+\tau\big) \\ &= \Big\{\mathcal{M}_+\big(\mathcal{I}_k^+\otimes S_+\big) - \sum_{\ell,j\in\mathbb{N}\times\mathbb{N}^d}\frac{X^\ell}{\ell!}\,\frac{(-X)^j}{j!}\,\mathcal{M}_+\big(\mathcal{I}_{k+\ell+j}^+\otimes S_+\big)\Big\}\Delta\tau\end{aligned}$$

$$= \Big\{\mathcal{M}_+\big(\mathcal{I}_k^+ \otimes S_+\big) - \sum_{n\in\mathbb{N}\times\mathbb{N}^d} \frac{(X-X)^n}{n!}\,\mathcal{M}_+\big(\mathcal{I}_{k+n}^+ \otimes S_+\big)\Big\}\Delta\tau$$
$$= 0.$$

Write

$$\mathcal{I} := \mathcal{I}_0,$$

and remark that the image by the operator $\mathcal{I}$ of a modelled distribution is not a modelled distribution. The next two sections are dedicated to constructing a model-dependent map $\mathcal{K}^{\mathsf{M}}$ that maps continuously all $\mathcal{D}^\gamma(T,\mathsf{g})$ into $\mathcal{D}^{\gamma+2}(T,\mathsf{g})$ when γ is a positive non-integer real number – the analogue of part of Schauder estimates, and is intertwined to the convolution operator $\mathbf{K}$

$$\mathbf{K}\circ \mathsf{R}^{\mathsf{M}} = \mathsf{R}^{\mathsf{M}} \circ \mathcal{K}^{\mathsf{M}}$$

via the reconstruction operator R^{M} associated with M. We say that $\mathcal{K}^{\mathsf{M}}$ is a *lift of* $\mathbf{K}$. The construction of this operator requires the introduction of the notion of admissible model.

3.3 Admissible models

In this section we consider only the operator $\mathcal{I} = \mathcal{I}_0 : T \to T$. The following notion plays a key role in the proof of the existence of a lift of the convolution operator $\mathbf{K}$. Recall that $\mathbf{K}(\zeta)$ is well-defined pointwise for any distribution $\zeta \in \mathcal{C}^\beta(\mathbb{R}\times\mathbb{R}^d)$ with $\beta > -2$, by Lemma 12.

Definition – *Assume $\beta_0 > -2$. A $\mathsf{\Pi}$-map on T is said to be* **K-admissible** *if it satisfies*

$$\mathsf{\Pi}(\mathcal{I}\tau) = \mathbf{K}(\mathsf{\Pi}\tau), \tag{3.9}$$

and

$$\mathsf{\Pi}(X^n \star \tau)(x) = x^n(\mathsf{\Pi}\tau)(x), \tag{3.10}$$

for any $\tau \in \mathcal{B}$ and $n \in \mathbb{N}\times\mathbb{N}^d$. A model $(\mathsf{\Pi},\mathsf{g})$ on $\mathscr{T}$ is said to be $\mathbf{K}$-admissible if its $\mathsf{\Pi}$-map is $\mathbf{K}$-admissible.

The notion of admissible $\mathsf{\Pi}$-map gives flesh to the idea that the operator $\mathcal{I}$ is the regularity structure counterpart of the convolution operator $\mathbf{K}$. The importance of the notion of $\mathbf{K}$-admissible model comes from Theorem 17 in the next section. It shows that when working with $\mathbf{K}$-admissible models M, one can upgrade the intertwining relation (3.9) into an intertwining relation between $\mathbf{K}$ and an operator $\mathcal{K}^{\mathsf{M}}$ on modelled distributions, via the reconstruction map R^{M} associated with M.

While for a general model as in Definition 1 defining a g-map satisfying the constraint (2.20) is decorrelated from the task of defining a $\mathsf{\Pi}$-map satisfying the constraint (2.21) it will turn out that imposing the intertwining relation (3.9) on $\mathsf{\Pi}$ will constrain strongly g. Unlike general models, admissible models on a regularity structure satisfying assumptions **(A-B1)** will turn out to be partly defined by their $\mathsf{\Pi}$ map; this will be proved in Proposition 15. The g-map of an admissible model on a regularity structure satisfying further a mild assumption **(B2)** will turn out to be *entirely* defined by its $\mathsf{\Pi}$ map.

We worked so far with models that are not constrained by anything else than their defining properties (2.20) and (2.21) and it is not clear that one can further impose additional conditions like (3.9). We will construct in Section 3.5 a whole class of admissible models with values in the set of smooth functions. This is all we need for the study of singular stochastic PDEs, as the nonsmooth admissible models involved in this setting are limits of smooth admissible models, and limits of admissible models are admissible. As for now, we keep going and see what can be done with admissible models.

Recall from Lemma 12 the definition of $\partial^n\mathbf{K}(\mathsf{\Pi}_x^{\mathsf{g}}\sigma)(x)$, for any $x \in \mathbb{R}\times\mathbb{R}^d$ and $n \in \mathbb{N}\times\mathbb{N}^d$ such that $|n|_{\mathfrak{s}} < |\sigma| + 2$, and define the model-dependent polynomial-valued function on T

$$\mathscr{J}^{\mathsf{M}}(x)\tau := \sum_{|n|_{\mathfrak{s}}<|\tau|+2} \frac{X^n}{n!}\,\partial^n\mathbf{K}(\mathsf{\Pi}_x^{\mathsf{g}}\tau)(x) \in T_X,$$

for any $\tau \in \mathcal{B}$ and $x \in \mathbb{R}\times\mathbb{R}^d$.

Proposition 13. *For a* $\mathbf{K}$*-admissible model* M *on* $\mathscr{T}$ *one has, for any* $x \in \mathbb{R} \times \mathbb{R}^d, \tau \in T$,

$$\mathsf{\Pi}_x^{\mathsf{g}}\big(\mathcal{I}\tau + \mathcal{J}^{\mathsf{M}}(x)\tau\big) = \mathbf{K}(\mathsf{\Pi}_x^{\mathsf{g}}\tau)$$

Proof – Using identity (3.7) and the admissibility of the model, one has indeed

$$\begin{aligned}\mathsf{\Pi}_x^{\mathsf{g}}(\mathcal{I}\tau) = \big(\mathsf{\Pi}\otimes \mathsf{g}_x^{-1}\big)(\Delta\mathcal{I}\tau) &= \sum_{\sigma\leqslant\tau}\big(\mathsf{\Pi}\otimes \mathsf{g}_x^{-1}\big)(\mathcal{I}\sigma\otimes\tau/\sigma) + \sum_{|\ell|_{\mathfrak{s}}<|\tau|+2}\frac{(\cdot)^\ell}{\ell!}\,\mathsf{g}_x^{-1}(\mathcal{I}_\ell^+\tau)\\ &=: \mathbf{K}(\mathsf{\Pi}\sigma)\mathsf{g}_x^{-1}(\tau/\sigma) + P_x(\tau,\cdot) = \mathbf{K}\big(\mathsf{\Pi}_x^{\mathsf{g}}\tau\big) + P_x(\tau,\cdot),\end{aligned}$$

so $\mathsf{\Pi}_x^{\mathsf{g}}(\mathcal{I}\tau)$ and $\mathbf{K}\big(\mathsf{\Pi}_x^{\mathsf{g}}\tau\big)$ differ by a polynomial $P_x(\tau,\cdot)$ of degree at most $|\tau|+2$. We identify this polynomial with $-\mathcal{J}^{\mathsf{M}}(x)\tau$ noting that $\mathsf{\Pi}_x^{\mathsf{g}}(\mathcal{I}\tau)$ has null derivatives at x up to the order $[|\tau|+2]$, from condition (2.21). ▷

Corollary 14. *For a* $\mathbf{K}$*-admissible model* M *on* $\mathscr{T}$ *one has, for any* $x, y \in \mathbb{R}\times\mathbb{R}^d$,

$$\widehat{\mathsf{g}_{yx}}\big(\mathcal{I}+\mathcal{J}^{\mathsf{M}}(x)\big) = \big(\mathcal{I}+\mathcal{J}^{\mathsf{M}}(y)\big)\widehat{\mathsf{g}_{yx}}. \tag{3.11}$$

Proof – Given $\tau\in T$ and $x,y\in\mathbb{R}\times\mathbb{R}^d$, one has both $\big(\widehat{\mathsf{g}_{yx}}(\mathcal{I}+\mathcal{J}^{\mathsf{M}}(x)) - (\mathcal{I}+\mathcal{J}^{\mathsf{M}}(y))\widehat{\mathsf{g}_{yx}}\big)\tau \in T_X$, and

$$\begin{aligned}\mathsf{\Pi}_y^{\mathsf{g}}\Big(\widehat{\mathsf{g}_{yx}}(\mathcal{I}+\mathcal{J}^{\mathsf{M}}(x))\tau - (\mathcal{I}+\mathcal{J}^{\mathsf{M}}(y))\widehat{\mathsf{g}_{yx}}\tau\Big) &= \mathsf{\Pi}_x^{\mathsf{g}}\big((\mathcal{I}\tau+\mathcal{J}^{\mathsf{M}}(x)\tau\big) - \mathsf{\Pi}_y^{\mathsf{g}}\big((\mathcal{I}+\mathcal{J}^{\mathsf{M}}(y))\widehat{\mathsf{g}_{yx}}\tau\big)\\ &= \mathbf{K}\big(\mathsf{\Pi}_x^{\mathsf{g}}\tau\big) - \mathbf{K}\big(\mathsf{\Pi}_y^{\mathsf{g}}\widehat{\mathsf{g}_{yx}}\tau\big) = 0,\end{aligned}$$

from Proposition 13. This implies that

$$\widehat{\mathsf{g}_{yx}}\big(\mathcal{I}+\mathcal{J}^{\mathsf{M}}(x)\big) - \big(\mathcal{I}+\mathcal{J}^{\mathsf{M}}(y)\big)\widehat{\mathsf{g}_{yx}} = 0.$$

▷

Proposition 15. *Let* $\mathscr{T}$ *be a regularity structure satisfying assumption* **(A)** *and assumption* **(B1)**. *The* g*-map of a* $\mathbf{K}$*-admissible model* $(\mathsf{\Pi},\mathsf{g})$ *on* $\mathscr{T}$ *satisfies*

$$\mathsf{g}_x(\mathcal{I}_0^+\tau) = \mathbf{K}(\mathsf{\Pi}\tau)(x),$$

for any $\tau\in\mathcal{B}$ *and all* $x\in\mathbb{R}\times\mathbb{R}^d$.

Proof – We show that there is at most one choice of $\mathsf{g}(\mathcal{I}_k^+\tau)$ such that $(\mathsf{\Pi},\mathsf{g})$ is an admissible model. Applying $\mathsf{\Pi}\otimes\mathsf{g}_x^{-1}$ to the identity (3.7) with $n=0$, one gets from the $\mathbf{K}$-admissibility of $\mathsf{\Pi}$

$$\begin{aligned}\mathsf{\Pi}_x^{\mathsf{g}}(\mathcal{I}\tau) &= \mathbf{K}\big(\mathsf{\Pi}_x^{\mathsf{g}}\tau\big) + \sum_{|\ell|_{\mathfrak{s}}<|\tau|+2}\frac{(\cdot)^\ell}{\ell!}\,\mathsf{g}_x^{-1}(\mathcal{I}_\ell^+\tau)\\ &= \mathbf{K}\big(\mathsf{\Pi}_x^{\mathsf{g}}\tau\big) + \sum_{|\ell|_{\mathfrak{s}}<|\tau|+2}\frac{(\cdot-x)^\ell}{\ell!}\,\mathsf{f}_x(\mathcal{I}_\ell^+\tau),\end{aligned} \tag{3.12}$$

where f and g are related by the formulas

$$\mathsf{f}_x(\mathcal{I}_\ell^+\tau) := \sum_{|m|_{\mathfrak{s}}<|\tau|+2-|\ell|_{\mathfrak{s}}}\frac{x^m}{m!}\,\mathsf{g}_x^{-1}(\mathcal{I}_{\ell+m}^+\tau),\quad \mathsf{g}_x^{-1}(\mathcal{I}_n^+\tau) = \sum_{|m|_{\mathfrak{s}}<|\tau|+2-|n|_{\mathfrak{s}}}\frac{(-x)^m}{m!}\mathsf{f}_x(\mathcal{I}_{n+m}^+\tau).$$

As in the proof of Proposition 13, since the derivatives of $\mathsf{\Pi}_x^{\mathsf{g}}(\mathcal{I}\tau)$ up to order $|\tau|+2$ vanish at x, we have

$$\mathsf{f}_x(\mathcal{I}_n^+\tau) = -\partial^n\mathbf{K}(\mathsf{\Pi}_x^{\mathsf{g}}\tau)(x),$$

hence

$$\mathsf{g}_x^{-1}(\mathcal{I}_n^+\tau) = -\sum_{|m|_{\mathfrak{s}}<|\tau|+2-|n|_{\mathfrak{s}}}\frac{(-x)^m}{m!}\partial^{n+m}\mathbf{K}\big(\mathsf{\Pi}_x^{\mathsf{g}}\tau\big)(x). \tag{3.13}$$

This implies another inductive formula

$$\mathsf{g}_x(\mathcal{I}_k^+\tau) = \sum_{\sigma\leqslant\tau;|k|_{\mathfrak{s}}<|\sigma|+2}\mathsf{g}_x(\tau/\sigma)\,\partial^k\mathbf{K}(\mathsf{\Pi}_x^{\mathsf{g}}\sigma)(x), \tag{3.14}$$

which is proved by applying $\mathsf{g}_x^{-1}\otimes\mathsf{g}_x$ to the identity (3.6) and using (3.13). Since $\beta_0>-2$, if $k=0$ the condition $|k|_{\mathfrak{s}}<|\sigma|+2$ can be removed. Hence we have $\mathsf{g}_x(\mathcal{I}_0^+\tau) = \mathbf{K}(\mathsf{\Pi}\tau)(x)$, by the comodule identity (2.9). ▷

Set

$$\mathscr{J}^{\mathsf{M}+}(x)\tau := \sum_{|n|_{\mathfrak{s}}<|\tau|+2} \frac{X_+^n}{n!}\, \partial^n \mathbf{K}(\Pi_x^{\mathsf{g}}\tau)(x) \in T_X^+ \subset T^+,$$

for any $\tau \in \mathcal{B}$ and $x \in \mathbb{R}\times\mathbb{R}^d$. The following statement is proved exactly as Proposition 13 and Corollary 14; it will be used in the proof of Theorem 18.

Proposition 16. *Given a regularity structure satisfying assumptions* **(A-B1)** *and a* $\mathbf{K}$*-admissible model* (Π, g) *on it then one has, for any* $x, y \in \mathbb{R}\times\mathbb{R}^d$,

$$\mathsf{g}_{yx}\Big(\mathcal{I}_0^+\tau + \mathscr{J}^{\mathsf{M}+}(x)\tau\Big) = \mathbf{K}(\Pi_x^{\mathsf{g}}\tau)(y),$$

and

$$\widehat{\mathsf{g}_{yx}}^{\,+}\Big(\mathcal{I}_0^+ + \mathscr{J}^{\mathsf{M}+}(x)\Big) = \Big(\mathcal{I}^+ + \mathscr{J}^{\mathsf{M}+}(y)\Big)\widehat{\mathsf{g}_{yx}}. \tag{3.15}$$

3.4 Lifting $\mathbf{K}$ as a continuous map from $\mathcal{D}^\gamma(T,\mathsf{g})$ into $\mathcal{D}^{\gamma+2}(T,\mathsf{g})$

For a given $\mathbf{K}$-admissible model $\mathsf{M} = (\Pi, \mathsf{g})$ we define in this section a continuous map $\mathcal{K}^{\mathsf{M}}$ from $\mathcal{D}^\gamma(T,\mathsf{g})$ into $\mathcal{D}^{\gamma+2}(T,\mathsf{g})$, for any positive non-integer regularity exponent γ, intertwined to $\mathbf{K}$ via the reconstruction operator

$$\mathbf{K}\circ\mathsf{R}^{\mathsf{M}} = \mathsf{R}^{\mathsf{M}}\circ\mathcal{K}^{\mathsf{M}}. \tag{3.16}$$

To get a grasp on what $\mathcal{K}^{\mathsf{M}}$ could be one keeps from the reconstruction theorem, Theorem 4, the image that for $\boldsymbol{g}\in\mathcal{D}^{\gamma+2}(T,\mathsf{g})$ and $x\in\mathbb{R}\times\mathbb{R}^d$, the distribution $\mathsf{R}^{\mathsf{M}}\boldsymbol{g} - \Pi_x^{\mathsf{g}}\boldsymbol{g}(x)$ behaves near x like the function $d(\cdot,x)^{\gamma+2}$. For $\boldsymbol{f}\in\mathcal{D}^\gamma(T,\mathsf{g})$, since we have

$$\mathbf{K}\big(\mathsf{R}^{\mathsf{M}}\boldsymbol{f}\big) - \Pi_x^{\mathsf{g}}\Big(\big(\mathcal{I}+\mathscr{J}^{\mathsf{M}}(x)\big)\boldsymbol{f}(x)\Big) = \mathbf{K}\Big(\mathsf{R}^{\mathsf{M}}\boldsymbol{f} - \Pi_x^{\mathsf{g}}\boldsymbol{f}(x)\Big),$$

from Proposition 13, it then looks natural to add to $\big(\mathcal{I}+\mathscr{J}^{\mathsf{M}}(x)\big)\boldsymbol{f}(x)$ the polynomial expansion

$$\big(\mathcal{N}^{\mathsf{M}}\boldsymbol{f}\big)(x) := \sum_{|\ell|_{\mathfrak{s}}<\gamma+2} \frac{X^\ell}{\ell!}\,(\partial^\ell\mathbf{K})\Big(\mathsf{R}^{\mathsf{M}}\boldsymbol{f} - \Pi_x^{\mathsf{g}}\boldsymbol{f}(x)\Big)(x)$$

of $\mathbf{K}\big(\mathsf{R}^{\mathsf{M}}\boldsymbol{f} - \Pi_x^{\mathsf{g}}\boldsymbol{f}(x)\big)$ at point x, at order $\gamma+2$, and expect that

$$\mathbf{K}\big(\mathsf{R}^{\mathsf{M}}\boldsymbol{f}\big) - \Pi_x^{\mathsf{M}}\Big(\big(\mathcal{I}+\mathscr{J}^{\mathsf{M}}(x)\big)\boldsymbol{f}(x) + \big(\mathcal{N}^{\mathsf{M}}\boldsymbol{f}\big)(x)\Big)$$

behaves like $d(\cdot,x)^{\gamma+2}$ near x. (The remark after Lemma 12 justifies the good definition of the quantities $(\partial^\ell\mathbf{K})\big(\mathsf{R}^{\mathsf{M}}\boldsymbol{f} - \Pi_x^{\mathsf{g}}\boldsymbol{f}(x)\big)$ in $\big(\mathcal{N}^{\mathsf{M}}\boldsymbol{f}\big)(x)$, for $|\ell|_{\mathfrak{s}} < \gamma+2$.) This does not guarantee that the T-valued map

$$\big(\mathcal{K}^{\mathsf{M}}\boldsymbol{f}\big)(x) := \big(\mathcal{I}+\mathscr{J}^{\mathsf{M}}(x)\big)\boldsymbol{f}(x) + \big(\mathcal{N}^{\mathsf{M}}\boldsymbol{f}\big)(x), \qquad x\in\mathbb{R}\times\mathbb{R}^d, \tag{3.17}$$

is a modelled distribution, but this turns out to be the case! Note that unlike $\mathcal{I}$ or $\mathscr{J}^{\mathsf{M}}(x)$, the T_X-valued function $\mathcal{N}^{\mathsf{M}}\boldsymbol{f}$ is a non-local function of $\boldsymbol{f}$ – i.e. not a function of $\boldsymbol{f}(x)$ only. Note also that one has *formally*

$$\big(\mathcal{K}^{\mathsf{M}}\boldsymbol{f}\big)(x) = \mathcal{I}\boldsymbol{f}(x) + \sum_{|\ell|_{\mathfrak{s}}<\gamma+2} \frac{X^\ell}{\ell!}\,\partial^\ell\mathbf{K}\big(\mathsf{R}^{\mathsf{M}}\boldsymbol{f}\big)(x).$$

This identity gives the intuitive meaning of the polynomial part of $\big(\mathcal{K}^{\mathsf{M}}\boldsymbol{f}\big)(x)$. Decomposition (3.17) is needed to make sense of $\big(\mathcal{K}^{\mathsf{M}}\boldsymbol{f}\big)(x)$ in a rigorous way.

Theorem 17. *Let the regularity structure* $\mathscr{T}$ *satisfy assumptions* **(A-B1)** *and* (Π, g) *be a* $\mathbf{K}$*-admissible model on it. For any positive non-integer regularity exponent* γ*, the map* $\mathcal{K}^{\mathsf{M}}$ *sends continuously* $\mathcal{D}^\gamma(T,\mathsf{g})$ *into* $\mathcal{D}^{\gamma+2}(T,\mathsf{g})$*, and satisfies the intertwining identity* (3.16).

Proof – We use the interwining relation (3.11) to write

$$\begin{aligned}(\mathcal{K}^{\mathsf{M}}\boldsymbol{f})(y)-\widehat{\mathsf{g}_{yx}}(\mathcal{K}^{\mathsf{M}}\boldsymbol{f})(x)&=(\mathcal{K}^{\mathsf{M}}\boldsymbol{f})(y)-\widehat{\mathsf{g}_{yx}}\big(\mathcal{I}+\mathcal{J}^{\mathsf{M}}(x)\big)\boldsymbol{f}(x)-\widehat{\mathsf{g}_{yx}}(\mathcal{N}^{\mathsf{M}}\boldsymbol{f})(x)\\&=(\mathcal{K}^{\mathsf{M}}\boldsymbol{f})(y)-\big(\mathcal{I}+\mathcal{J}^{\mathsf{M}}(y)\big)\widehat{\mathsf{g}_{yx}}\boldsymbol{f}(x)-\widehat{\mathsf{g}_{yx}}(\mathcal{N}^{\mathsf{M}}\boldsymbol{f})(x)\\&=\mathcal{I}\Big(\boldsymbol{f}(y)-\widehat{\mathsf{g}_{yx}}\boldsymbol{f}(x)\Big)+\mathcal{J}^{\mathsf{M}}(y)\Big(\boldsymbol{f}(y)-\widehat{\mathsf{g}_{yx}}\boldsymbol{f}(x)\Big)+\Big((\mathcal{N}^{\mathsf{M}}\boldsymbol{f})(y)-\widehat{\mathsf{g}_{yx}}(\mathcal{N}^{\mathsf{M}}\boldsymbol{f})(x)\Big).\end{aligned}$$

For the $\mathcal{I}$ term, one has the elementary estimate

$$\big\|\mathcal{I}\big(\boldsymbol{f}(y)-\widehat{\mathsf{g}_{yx}}\boldsymbol{f}(x)\big)\big\|_\beta\lesssim\big\|\boldsymbol{f}(y)-\widehat{\mathsf{g}_{yx}}\boldsymbol{f}(x)\big\|_{\beta-2}\leqslant\|\boldsymbol{f}\|_{\mathcal{D}^\gamma}\,d(y,x)^{\gamma+2-\beta}$$

for any $\beta\in A$. The $\mathcal{J}^{\mathsf{M}}$ and $\mathcal{N}^{\mathsf{M}}$ terms take values in the polynomial part T_X of T. Decompose $K(x,y)$ into the integral of q_r by (3.3), and let $\mathcal{J}^{\mathsf{M}}=:\int_0^1\mathcal{J}_r^{\mathsf{M}}dr-(e-1)\mathcal{J}_1^{\mathsf{M}}$, and $\mathcal{N}^{\mathsf{M}}=:\int_0^1\mathcal{N}_r^{\mathsf{M}}dr-(e-1)\mathcal{N}_1^{\mathsf{M}}$, stand for the corresponding operators. Since q_1 has a smoothing property, it is sufficient to consider the integral part only. Fix $n\in\mathcal{N}$, and write $(\tau)_{X^n}$ for the component of $\tau\in T$ in the direction of X^n. We have for

$$(\ominus):=\Big(\mathcal{J}_r^{\mathsf{M}}(y)\big(\boldsymbol{f}(y)-\widehat{\mathsf{g}_{yx}}\boldsymbol{f}(x)\big)+\big(\mathcal{N}_r^{\mathsf{M}}\boldsymbol{f}\big)(y)-\widehat{\mathsf{g}_{yx}}(\mathcal{N}_r^{\mathsf{M}}\boldsymbol{f})(x)\Big)_{X^n}$$

the two decompositions

$$\begin{aligned}(\ominus)&=\sum_{\beta\in A,|n|_{\mathfrak{s}}<\beta+2}\frac{1}{n!}\Big\langle\Pi_y^{\mathsf{g}}\big(\boldsymbol{f}(y)-\widehat{\mathsf{g}_{yx}}\boldsymbol{f}(x)\big)_\beta,\partial_y^nq_r(y,\cdot)\Big\rangle\\&\quad+\Big\{\frac{1}{n!}\Big\langle\mathsf{R}^{\mathsf{M}}\boldsymbol{f}-\Pi_y^{\mathsf{g}}\boldsymbol{f}(y),\partial_y^nq_r(y,\cdot)\Big\rangle-\sum_{|k|_{\mathfrak{s}}<\gamma+2-|n|_{\mathfrak{s}}}\frac{(y-x)^k}{k!}\Big\langle\mathsf{R}^{\mathsf{M}}\boldsymbol{f}-\Pi_x^{\mathsf{g}}\boldsymbol{f}(x),\partial_x^{n+k}q_r(x,\cdot)\Big\rangle\Big\}\\&=:(\ast)_r^1+(\ast)_r^2,\end{aligned}$$

and

$$\begin{aligned}(\ominus)&=\sum_{\beta\in A,|n|_{\mathfrak{s}}<\beta+2}\frac{1}{n!}\Big\langle\Pi_y^{\mathsf{g}}\big(\boldsymbol{f}(y)-\widehat{\mathsf{g}_{yx}}\boldsymbol{f}(x)\big)_\beta,\partial_y^nq_r(y,\cdot)\Big\rangle\\&\quad+\frac{1}{n!}\Big\langle\mathsf{R}^{\mathsf{M}}\boldsymbol{f}-\Pi_x^{\mathsf{g}}\boldsymbol{f}(x),(\partial^nq_r)_{y,x}^{\gamma+2-|n|_{\mathfrak{s}}}\Big\rangle+\frac{1}{n!}\Big\langle\Pi_x^{\mathsf{g}}\boldsymbol{f}(x)-\Pi_y^{\mathsf{g}}\boldsymbol{f}(y),\partial_y^nq_r(y,\cdot)\Big\rangle\\&=\frac{1}{n!}\Big\langle\mathsf{R}^{\mathsf{M}}\boldsymbol{f}-\Pi_x^{\mathsf{g}}\boldsymbol{f}(x),(\partial^nq_r)_{y,x}^{\gamma+2-|n|_{\mathfrak{s}}}\Big\rangle-\sum_{\beta\in A,|n|_{\mathfrak{s}}\geqslant\beta+2}\frac{1}{n!}\Big\langle\Pi_y^{\mathsf{g}}\big(\boldsymbol{f}(y)-\widehat{\mathsf{g}_{yx}}\boldsymbol{f}(x)\big)_\beta,\partial_y^nq_r(y,\cdot)\Big\rangle\\&=:(\bigstar)_r^1+(\bigstar)_r^2,\end{aligned}$$

where

$$(\partial^nq_r)_{y,x}^{\gamma+2-|n|_{\mathfrak{s}}}(z):=\partial_y^nq_r(y,z)-\sum_{|k|_{\mathfrak{s}}<\gamma+2-|n|_{\mathfrak{s}}}\frac{(y-x)^k}{k!}\,\partial_x^{n+k}q_r(x,z).$$

Choose $r_0\in(0,1]$ such that $r_0^{\frac14}\simeq d(y,x)\wedge 1$. We use the $(\ast)$-decomposition to estimate the integral over $0<r<r_0$, and the $(\bigstar)$-decomposition to estimate the integral over $r_0\leqslant r\leqslant 1$.

• For $r\in(0,r_0]$, we have from the bound (3.2) the estimate

$$\begin{aligned}\int_0^{r_0}\big|(\ast)_r^1\big|dr&\lesssim\sum_{\beta\in A,|n|_{\mathfrak{s}}<\beta+2}d(y,x)^{\gamma-\beta}\int_0^{r_0}r^{\frac{\beta-|n|_{\mathfrak{s}}-\mathfrak{l}}{4}}dr\\&\lesssim\sum_{\beta\in A,|n|_{\mathfrak{s}}<\beta+2}d(y,x)^{\gamma-\beta}r_0^{\frac{\beta-|n|_{\mathfrak{s}}+2}{4}}\lesssim d(y,x)^{\gamma+2-|n|_{\mathfrak{s}}}.\end{aligned}$$

Since $|n|_{\mathfrak{s}}<\gamma+2$, from the bound (2.32) in the reconstruction theorem, we get

$$\begin{aligned}\int_0^{r_0}\big|(\ast)_r^2\big|dr&\lesssim\int_0^{r_0}r^{\frac{\gamma-|n|_{\mathfrak{s}}-\mathfrak{l}}{4}}dr+\sum_{|k|_{\mathfrak{s}}<\gamma+2-|n|_{\mathfrak{s}}}d(y,x)^{|k|_{\mathfrak{s}}}\int_0^{r_0}r^{\frac{\gamma-|n|_{\mathfrak{s}}-|k|_{\mathfrak{s}}-\mathfrak{l}}{4}}dr\\&\lesssim r_0^{\frac{\gamma-|n|_{\mathfrak{s}}+2}{4}}+\sum_{|k|_{\mathfrak{s}}<\gamma+2-|n|_{\mathfrak{s}}}d(y,x)^{|k|_{\mathfrak{s}}}r_0^{\frac{\gamma-|n|_{\mathfrak{s}}-|k|_{\mathfrak{s}}+2}{4}}\lesssim d(y,x)^{\gamma+2-|n|_{\mathfrak{s}}}.\end{aligned}$$

• To deal with the integral over $r\in(r_0,1]$, we use the $(\bigstar)$-decomposition. Since this integral does not make sense if $r_0\geqslant 1$, we assume $d(y,x)\leqslant 1$. For $(\bigstar)_r^1$, we have the (anisotropic) integral Taylor formula for the remainder

$$(\partial^n q_r)^{\gamma+2-|n|_{\mathfrak{s}}}_{y,x}(z) = \sum_{\gamma+2-|n|_{\mathfrak{s}}<|\ell|_{\mathfrak{s}}} \frac{(y-x)^\ell}{\ell!}\int_0^1 \varphi_\ell(r')\partial^{n+\ell}q_r(x_{r'},z)dr',$$

where ℓ *runs over a finite set*, $x_{r'} := x + r'(y-x)$, and $\varphi_\ell(r')$ are bounded functions of r'. Note that no index n with $\gamma + 2 - |n|_{\mathfrak{s}} = |\ell|_{\mathfrak{s}}$ exists, because $\gamma \notin \mathbb{Z}$. By decomposing

$$\mathsf{R}^{\mathsf{M}}\boldsymbol{f} - \mathsf{\Pi}^{\mathsf{g}}_x\boldsymbol{f}(x) = \mathsf{R}^{\mathsf{M}}\boldsymbol{f} - \mathsf{\Pi}^{\mathsf{g}}_{x_{r'}}\boldsymbol{f}(x_{r'}) + \mathsf{\Pi}^{\mathsf{g}}_{x_{r'}}\big(\boldsymbol{f}(x_{r'}) - \widehat{\mathsf{g}_{x_{r'}x}}\boldsymbol{f}(x)\big),$$

and using the bounds (2.32) and (3.2), we have

$$\begin{aligned}\int_{t_0}^1 \big|(\bigstar)^1_r\big|\,dr &\lesssim \sum_{\gamma+2-|n|_{\mathfrak{s}}<|\ell|_{\mathfrak{s}}} d(y,x)^{|\ell|_{\mathfrak{s}}}\int_{r_0}^1 \left\{ r^{\frac{\gamma-|n|_{\mathfrak{s}}-|\ell|_{\mathfrak{s}}-\mathfrak{l}}{4}} + \sum_{\beta\in A,\,\beta<\gamma} d(y,x)^{\gamma-\beta} r^{\frac{\beta-|n|_{\mathfrak{s}}-|\ell|_{\mathfrak{s}}-\mathfrak{l}}{4}}\right\} dr \\ &\lesssim \sum_{\gamma+2-|n|_{\mathfrak{s}}<|\ell|_{\mathfrak{s}}} d(y,x)^{|\ell|_{\mathfrak{s}}}\left\{ r_0^{\frac{\gamma-|n|_{\mathfrak{s}}-|\ell|_{\mathfrak{s}}+2}{4}} + \sum_{\beta\in A,\,\beta<\gamma} d(y,x)^{\gamma-\beta} r_0^{\frac{\beta-|n|_{\mathfrak{s}}-|\ell|_{\mathfrak{s}}+2}{4}}\right\} \\ &\lesssim d(y,x)^{\gamma+2-|n|_{\mathfrak{s}}}.\end{aligned}$$

We obtain the same bound for the $(\bigstar)^2_r$-term by a similar argument.

Uniqueness of the reconstruction operator when a is positive gives identity (3.16). $\triangleright$

Note that the intertwining relation (3.16) between $\mathbf{K}$ and $\mathcal{K}^{\mathsf{M}}$ provides indeed an 'upgraded' version of the defining identity (3.9) for a $\mathbf{K}$-admissible model in so far as the former reduces to the latter when applied to the modelled distribution $\boldsymbol{f}(x) = \boldsymbol{h}^\tau(x) = \sum_{\sigma<\tau}\mathsf{g}_x(\tau/\sigma)\sigma$. Indeed, on the one hand we have $\mathsf{R}^{\mathsf{M}}\boldsymbol{h}^\tau = \mathsf{\Pi}\tau$. On the other hand $\mathcal{K}^{\mathsf{M}}\boldsymbol{h}^\tau$ has positive regularity and takes its values in a function-like sector where the model takes values in the space of continuous functions, so Corollary 6 applies and identifies the reconstruction of $\mathcal{K}^{\mathsf{M}}\boldsymbol{h}^\tau$ as $\mathsf{\Pi}^{\mathsf{g}}_x\big(\mathcal{K}^{\mathsf{M}}\boldsymbol{h}^\tau(x)\big)(x)$, equal to $\mathsf{\Pi}(\mathcal{I}\tau)$, as all the x-indexed polynomial terms are null when evaluated at x.

Note also the following fact that will be used in the proof of Theorem 19 describing a whole class of smooth admissible models. For a regularity structure $\mathscr{T} = (T^+, T)$ and a model $(\mathsf{\Pi}, \mathsf{g})$ on it, the data

$$\mathscr{T}^+ := \big((T^+, \Delta^+), (T^+, \Delta^+)\big)$$

define a regularity structure and setting

$$\big(\mathsf{\Pi}^{(\mathsf{g})}\sigma\big)(x) := \mathsf{g}_x(\sigma), \qquad x \in \mathbb{R}\times\mathbb{R}^d,\ \sigma \in T^+,$$

one defines a model $\big(\mathsf{\Pi}^{(\mathsf{g})}, \mathsf{g}\big)$ on $\mathscr{T}^+$. Denote by $\mathcal{D}^\gamma(T^+, \mathsf{g})$ the space of modelled distributions on the regularity structure $\mathscr{T}^+$. For $\boldsymbol{f} \in \mathcal{D}^\gamma(T, \mathsf{g})$, set

$$\Big(\mathcal{N}^{\mathsf{M}+}\boldsymbol{f}\Big)(x) := \sum_{|\ell|_{\mathfrak{s}}<\gamma+2} \frac{X_+^\ell}{\ell!}\,(\partial^\ell\mathbf{K})\Big(\mathsf{R}^{\mathsf{M}}\boldsymbol{f} - \mathsf{\Pi}^{\mathsf{g}}_x\boldsymbol{f}(x)\Big)(x) \in T_X^+ \subset T^+, \tag{3.18}$$

and

$$(\mathcal{K}^{\mathsf{M}+}\boldsymbol{f})(x) := \Big(\mathcal{I}^+ + \mathcal{J}^{\mathsf{M}+}(x)\Big)\boldsymbol{f}(x) + \big(\mathcal{N}^{\mathsf{M}+}\boldsymbol{f}\big)(x) \in T^+.$$

Using identity (3.15) in Proposition 16, one can repeat verbatim the proof of Theorem 17 and obtain the following continuity result for the operator $\mathcal{K}^{\mathsf{M}+}$.

Theorem 18. *Assume we are given a regularity structure and a model $(\mathsf{\Pi}, \mathsf{g})$ on it, with a $\mathbf{K}$-admissible map $\mathsf{\Pi}$. Then for any non-integer positive exponent γ, the map $\mathcal{K}^{\mathsf{M}+}$ sends continuously $\mathcal{D}^\gamma(T, \mathsf{g})$ into $\mathcal{D}^{\gamma+2}(T^+, \mathsf{g})$, and*

$$\mathbf{K}\circ\mathsf{R}^{\mathsf{M}} = \mathsf{R}^{\mathsf{g}}\circ\mathcal{K}^{\mathsf{M}+}, \tag{3.19}$$

where R^{g} is the reconstruction operator on $\mathscr{T}^+$ associated with the model $\big(\mathsf{\Pi}^{(\mathsf{g})}, \mathsf{g}\big)$ on T^+.

Pay attention to the fact that $\mathcal{K}^{\mathsf{M}+}$ takes an element of $\mathcal{D}^\gamma(T, \mathsf{g})$ and gives back an element of $\mathcal{D}^{\gamma+2}(T^+, \mathsf{g})$.

3.5 Building admissible models

We left aside in Section 3.3 the non-elementary question of existence of non-trivial admisible models to concentrate on their properties. We construct in this section a large class of admissible models for which all $(\Pi\tau)_{\tau\in T}$ and $\big(\mathsf{g}(\sigma)\big)_{\sigma\in T^+}$ are smooth functions. In applications to singular stochastic PDEs, such models can be built from regularized realizations of the noise(s) in the equation.

Recall assumption **(B1)** describes the action of the 'recentering operator' Δ on elements of T of the form $\mathcal{I}_k\tau$. We single out for our needs an assumption on Δ that provides a crucial induction structure; it is satisfied by the regularity structures used for the study of singular stochastic PDEs – see Section 9.

Assumption (B2) – *For any $\tau,\sigma\in\mathcal{B}$, the element $\tau/\sigma\in T^+$ is generated by the symbols*

$$\{X_+^{e_i}\}_{0\leqslant i\leqslant d}\cup\big\{\mathcal{I}_n^+\eta\big\}_{\eta\in\mathcal{B},\,|\eta|<|\tau|,\,n\in\mathbb{N}\times\mathbb{N}^d}.$$

Assumptions **(B1)** and **(B2)** are jointly called assumption **(B)**. Under assumptions **(A-B)**, formula (3.14) in the proof of Proposition 15 shows that the g-map of an admissible model (Π,g) is uniquely determined by its Π-map.

Theorem 19. *Let $\mathscr{T}$ be a regularity structure satisfying assumption* **(A-B)**. *One can associate to any family $\big([\tau];\tau\in\mathcal{B},|\tau|<0\big)$ of smooth functions on $\mathbb{R}\times\mathbb{R}^d$ a unique $\mathbf{K}$-admissible model (Π,g) on $\mathscr{T}$ such that $\Pi\tau=[\tau]$, for all $\tau\in\mathcal{B}$ with $|\tau|<0$.*

Proof – We set the scene for an inductive proof of the statement, taking profit of the induction structure given by assumption **(B2)**. For $\alpha\in A$, define $T^+_{(<\alpha)}$ as the subalgebra of T^+ generated by

$$\{X_+^{e_i}\}_{0\leqslant i\leqslant d}\cup\big\{\mathcal{I}_k^+\tau\big\}_{\tau\in\mathcal{B},\,k\in\mathbb{N}\times\mathbb{N}^d,\,|\tau|<\alpha}.$$

The Δ^+ map sends the space $T^+_{(<\alpha)}$ into $T^+_{(<\alpha)}\otimes T^+_{(<\alpha)}$, and

$$\mathscr{T}_{<\alpha}:=\big(T^+_{(<\alpha)},T_{<\alpha}\big),$$

equipped with the restrictions of the Δ^+ and Δ maps, is a regularity structure for any $\alpha\in A$, from assumption **(B2)**. We define inductively on $\alpha\in A$ maps

$$\Pi_{<\alpha}:T_{<\alpha}\mapsto\mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}^d),$$

and

$$\mathsf{g}^\alpha:T^+_{(<\alpha)}\mapsto\mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}^d),$$

such that

$$\Pi_{<\beta}|_{T_{<\alpha}}=\Pi_{<\alpha},\quad \mathsf{g}^\beta|_{T^+_{(<\alpha)}}=\mathsf{g}^\alpha$$

for any $\alpha<\beta$. We define $\mathsf{g}_x^{\beta_0}(X_+^{e_i})=x_i$, initializing the induction. Write $\mathsf{M}_{<\alpha}$ for the model $(\mathsf{g}^\alpha,\Pi_{<\alpha})$ on $\mathscr{T}_{<\alpha}$, and assume it is $\mathbf{K}$-admissible. If α is positive, denote by $\mathsf{R}^{\mathsf{M}_{<\alpha}}$ the reconstruction operator on $\mathcal{D}^\alpha\big(T_{<\alpha},\mathsf{g}^\alpha\big)$ associated with the model $\mathsf{M}_{<\alpha}$.
Set

$$\beta:=\min\big\{\alpha'>\alpha\,;\,\alpha'\in A\big\}.$$

We now define an extension $(\mathsf{g}^\beta,\Pi_{<\beta})$ of $(\mathsf{g}^\alpha,\Pi_{<\alpha})$ on $\mathscr{T}_{<\beta}$; the new elements of $\mathscr{T}_{<\beta}$ are the elements of T_α and $\mathcal{I}_n^+(T_\alpha)$. Note for that purpose that given a basis vector $\tau\in\mathcal{B}_\alpha$ the function

$$\boldsymbol{h}^\tau:=\sum_{\sigma<\tau}\mathsf{g}^\alpha(\tau/\sigma)\,\sigma$$

is an element of $\mathcal{D}^\alpha(\mathscr{T}_{<\alpha},\mathsf{g}^\alpha)$. Given that g^α and $\Pi_{<\alpha}$ take values in smooth functions, any smooth function is a reconstruction of $\boldsymbol{h}^\tau$ for the model $\mathsf{M}_{<\alpha}$, if $|\tau|=\alpha<0$. (Recall the reconstruction operator is defined uniquely only when acting on modelled distributions of positive regularity. We are here working with a modelled distribution of negative regularity when $\alpha<0$.) Define $\Pi_{<\beta}\tau$ as equal to $[\tau]$, if $|\tau|=\alpha<0$. If $|\tau|>0$, define $\Pi_{<\beta}\tau$ as equal to $\mathsf{R}^{\mathsf{M}_{<\alpha}}(\boldsymbol{h}^\tau)$; this is a smooth function in both cases. (Recall that X^0 is the only element of T of null homogeneity.) The map $\Pi_{<\beta}$ coincides with $\Pi_{<\alpha}$ on $T_{<\alpha}$. The size requirement

$$\big|\big\langle(\Pi_{<\beta})_x^{\mathsf{g}^\beta}\tau,p_t(x,\cdot)\big\rangle\big|=\big|\big\langle(\Pi_{<\beta})_x^{\mathsf{g}^\alpha}\tau,p_t(x,\cdot)\big\rangle\big|\lesssim t^{\alpha/4}$$

on $(\Pi_{<\beta})_x^{\mathsf{g}^\beta}\tau$ is then a reformulation of the fact that $\Pi_{<\beta}\tau$ is in both cases a reconstruction of $\boldsymbol{h}^\tau$ for $\mathsf{M}_{<\alpha}$ – regardless of the fact that we have not yet extended g^α into g^β.
Define then an extension g^β of g^α to $T^+_{(<\beta)}$ by requiring that it is multiplicative, and by setting

$$\mathsf{g}_x^\beta(\mathcal{I}_k^+\tau):=\sum_{\sigma\leqslant\tau;|k|_{\mathfrak{s}}<|\sigma|+2}\mathsf{g}_x^\alpha(\tau/\sigma)\,\partial^k\mathbf{K}\Big(\big(\Pi_{<\beta}\big)_x^{\mathsf{g}^\alpha}\sigma\Big)(x),$$

for all $\tau \in \mathcal{B}_\alpha$, in view of (3.14). Closing the induction step amounts to proving that

$$\big|\mathsf{g}^\beta_{yx}(\mathcal{I}^+_k\tau)\big| \lesssim d(y,x)^{|\tau|+2-|k|_\mathfrak{s}}, \tag{3.20}$$

for every $k \in \mathcal{N}$ with $|k|_\mathfrak{s} < |\tau|+2$. Look for that purpose at $\mathcal{K}^{(\mathsf{M}_{<\alpha})+}\boldsymbol{h}^\tau$; this is an element of $\mathcal{D}^{\alpha+2}\big(T^+_{(<\alpha)},\mathsf{g}^\alpha\big)$, from Theorem 18. Recall the definition (3.18) for $\mathcal{N}^{(\mathsf{M}_{<\alpha})+}$ and that

$$\mathsf{R}^{\mathsf{M}_{<\alpha}}(\boldsymbol{h}^\tau) - \Pi^{\mathsf{g}}_x\big(\boldsymbol{h}^\tau(x)\big) = (\Pi_{<\beta})\tau - \Pi^{\mathsf{g}}_x(\boldsymbol{h}^\tau(x)) = (\Pi_{<\beta})^{\mathsf{g}^\alpha}_x\,\tau.$$

Hence

$$\big(\mathcal{N}^{(\mathsf{M}_{<\alpha})+}\boldsymbol{h}^\tau\big)(x) = \sum_{|k|_\mathfrak{s}<|\tau|+2}\frac{X^k_+}{k!}\,\partial^k\mathbf{K}\big((\Pi_{<\beta})^{\mathsf{g}^\alpha}_x\,\tau\big)(x),$$

so one has

$$\big(\mathcal{K}^{(\mathsf{M}_{<\alpha})+}\boldsymbol{h}^\tau\big)(x) = \mathcal{I}^+_0\big(\boldsymbol{h}^\tau(x)\big) + \sum_{|k|_\mathfrak{s}<|\tau|+2}\mathsf{g}^\beta_x(\mathcal{I}^+_k\tau)\,\frac{X^k_+}{k!}.$$

The X^k_+-component of

$$\Big(\mathcal{K}^{(\mathsf{M}_{<\alpha})+}\boldsymbol{h}^\tau\Big)(y) - \widehat{\mathsf{g}^\alpha_{yx}}^{\,+}\Big(\mathcal{K}^{(\mathsf{M}_{<\alpha})+}\boldsymbol{h}^\tau\Big)(x)$$

is then equal to

$$\mathsf{g}^\alpha_y(\mathcal{I}^+_k\tau) - \sum_{\eta<\tau}\mathsf{g}^\alpha_x(\tau/\eta)\mathsf{g}^\alpha_{yx}(\mathcal{I}^+_k\eta) - \sum_m \mathsf{g}^\alpha_x(\mathcal{I}^+_{k+m}\tau)\frac{(y-x)^m}{m!} = \mathsf{g}^\beta_{yx}(\mathcal{I}^+_k\tau),$$

and of size $d(y,x)^{|\tau|+2-|k|_\mathfrak{s}}$, since $\mathcal{K}^{(\mathsf{M}_{<\alpha})+}\boldsymbol{h}_\tau \in \mathcal{D}^{\alpha+2}\big(T^+_{(<\alpha)},\mathsf{g}^\alpha\big)$. This shows the bound (3.20).
It remains to show that Π is $\mathbf{K}$-admissible. Given that we assume $\beta_0 = \min A > -2$, the elements of T of the form $\mathcal{I}\tau$ have positive homogeneity. So the definition of Π on $\mathcal{I}\tau$ comes under the form of the reconstruction of a modelled distribution $\boldsymbol{h}^{\mathcal{I}\tau}$. Since $\boldsymbol{h}^{\mathcal{I}\tau}$ is function-like, by Corollary 7, it follows from Theorem 18 that

$$\Pi(\mathcal{I}\tau)(x) = \mathsf{R}^{\mathsf{M}}(\boldsymbol{h}^{\mathcal{I}\tau})(x) = \mathsf{g}_x(\mathcal{I}^+_0\tau) = \mathbf{K}(\Pi\tau)(x).$$

▷

4 – Solving singular PDEs within regularity structures

In this section, we formulate singular stochastic PDEs in the sense of modelled distributions. We trade in this section the generality of the above results for the simplicity of an example that contains the main difficulties of the general case. The reader can consult [44] or [13] for a description of the general case. We consider the generalized (KPZ) equation

$$\begin{aligned}\big(\partial_{x_0} - \Delta_{x'} + 1\big)u &= f(u)\zeta + \sum_{i,j=1}^d g_2^{ij}(u)(\partial_{x_i}u)(\partial_{x_j}u) + \sum_{i=1}^d g_1^i(u)(\partial_{x_i}u) + g_0(u)\\ &=: f(u)\zeta + g_2(u)(\partial_{x'}u)^2 + g_1(u)\partial_{x'}u + g_0(u)\\ &=: f(u)\zeta + g(u,\partial_{x'}u)\end{aligned} \tag{4.1}$$

with a noise $\zeta \in \mathcal{C}^{\beta_0}$. (Remember that the minimum homogeneity in a regularity structure associated with a singular stochastic PDE coincides with the minimum of the regularities of the noises in the equation.) This type of equation appears in a number of problems. If $d = 1$ and ζ is a space-time white noise, then (4.1) contains the KPZ equation, which appears in the large scale picture of one-dimensional random interface evolutions. Here u is a scalar valued, but a vector valued case appears in the description of the random motion of a rubber on a manifold [46], a random perturbation of the harmonic flow map on loops. If $d = 2, 3$ and ζ is a space white noise, then (4.1) contains the generalized PAM

$$(\partial_t - \Delta_x)u = f(u)\zeta.$$

Given any positive real number γ, denote by $(\mathcal{K}'_\gamma)^{\mathsf{M}}$ the lift of the $\mathbf{K}'$ operator in the polynomial part of the regularity structure

$$\big((\mathcal{K}'_\gamma)^{\mathsf{M}}\boldsymbol{f}\big) := \sum_{|\ell|_\mathfrak{s}<\gamma}\partial^\ell\mathbf{K}'\big(\mathsf{R}^{\mathsf{M}}\boldsymbol{f}\big)\,\frac{X^\ell}{\ell!} \in T_X,$$

for $\boldsymbol{f} \in \mathcal{D}^\gamma(T, \mathsf{g})$. Under such settings, the generalized (KPZ) equation (4.1) is lifted to the equation of S-valued modelled distributions $\boldsymbol{v} \in \mathcal{D}^\gamma(T, \mathsf{g})$

$$\boldsymbol{v} = \big(\mathcal{K}^{\mathsf{M}} + (\mathcal{K}'_\gamma)^{\mathsf{M}}\big)\Big(f^\star(\boldsymbol{v})\Xi + g^\star\big(\boldsymbol{v}, \partial\boldsymbol{v}\big)\Big) + \boldsymbol{p} \tag{4.2}$$

with some T_X-valued modelled distribution $\boldsymbol{p}$. This section is dedicated to showing that equation (4.2) has a unique solution on a small time interval $(0, t_0)$; this is the content of Theorem 22.

The restriction to each band $[0, t_0] \times [-R, R]$ of spacetime white noise has a norm growing indefinitely as R goes to infinity for each fixed $t_0 > 0$. To avoid working with unbounded spacial domains and functional spaces involving spacial weights we will assume that all the objects are $\mathbb{Z}^d$-periodic in space. The function $\boldsymbol{p}$ in (4.2) plays the role of the regularity structure lift of the propagator of the initial condition u_0. The use of time weights to take care of the free propagation $(e^{t(\Delta-1)}u_0)_{t>0}$ of the initial condition in a regularity structures setting is made necessary by the classical sharp estimate

$$\big\|\partial_x^k e^{t(\Delta-1)}u_0\big\|_\infty \lesssim t^{-\frac{\alpha+|k|_{\mathfrak{s}}}{2}}\,\|u_0\|_{\mathcal{C}^\alpha(\mathbb{R}^d)} \tag{4.3}$$

Theorem 22 is proved under spacial periodic boundary condition and in the space of modelled distributions involving temporal weights exploding in $t = 0^+$. We introduce the former in Section 4.1 and the latter in Section 4.2. We examine in Section 4.3 the notion of non-anticipative operator, involved in the analysis of equation (4.2).

The letter $\mathcal{E}$ will stand for the remainder of this section for

$$\mathbb{R} \times \mathbb{T}^d.$$

4.1 Spatially periodic models

We work on the models and modelled distributions spacially $\mathbb{Z}^d$-periodic. All the results and estimates proved above hold true in the periodic case. For any $x = (x_0, x') \in \mathbb{R} \times \mathbb{R}^d$ and $m \in \mathbb{Z}^d$, denote by $x + m := (x_0, x' + m)$.

Definition – *A* **model** $\mathsf{M} = (\mathsf{g}, \Pi)$ *is said to be* $\mathbb{Z}^d$-**periodic** *if for any $m \in \mathbb{Z}^d$,*

$$\mathsf{g}^m_{yx} := \mathsf{g}_{y+m, x+m} = \mathsf{g}_{yx}, \quad \big\langle \Pi^{\mathsf{g}^m}_{x+m}\,,\, \varphi(\cdot + m)\big\rangle = \big\langle \Pi^{\mathsf{g}}_x\,,\, \varphi(\cdot)\big\rangle,$$

for all $x, y \in \mathbb{R} \times \mathbb{R}^d$ and all $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$.

The canonical model (Π, g) on the polynomial regularity structure (T_X^+, T_X) is $\mathbb{Z}^d$-periodic in the above sense. Note that $\mathsf{g}_x(X_+^n) = x^n$ and $(\Pi X^n)(x) = x^n$ are *not* $\mathbb{Z}^d$-periodic functions. This is the reason why we do not impose periodic conditions on g and Π. It is elementary to see that if M is a $\mathbb{Z}^d$-periodic model on $\mathscr{T}$ and $\boldsymbol{f} \in \mathcal{D}^\gamma(T, \mathsf{g})$ is $\mathbb{Z}^d$-periodic, with γ positive, then $\mathbf{R}^{\mathsf{M}}\boldsymbol{f}$ is also $\mathbb{Z}^d$-periodic, in the sense that

$$\big\langle \mathbf{R}^{\mathsf{M}}\boldsymbol{f}, \varphi(\cdot + m)\big\rangle = \big\langle \mathbf{R}^{\mathsf{M}}\boldsymbol{f}, \varphi(\cdot)\big\rangle,$$

for all $m \in \mathbb{Z}^d$ and $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$ – see Proposition 3.38 in [44]. *All objects in remainder of this section are implicitly assumed to be $\mathbb{Z}^d$-periodic.*

4.2 Modelled distributions with singularity at $x_0 = 0$

We use a time weight to treat the boundary condition at $x_0 = 0$.

Definition – *Fix two exponents $\eta \leqslant \gamma \in \mathbb{R}$. One defines the space $\mathcal{D}^{\gamma,\eta}(T, \mathsf{g})$ of* **modelled distributions with singularity of weight** η **at** $x_0 = 0$*, as the space of functions $\boldsymbol{f}$ from $\mathbb{R} \times \mathbb{R}^d \backslash \{x_0 = 0\}$ into $T_{<\gamma}$ such that*

$$\begin{aligned}
[\![\boldsymbol{f}]\!]_{\mathcal{D}^{\gamma,\eta}} &:= \max_{\beta<\gamma}\sup_{a>0}\Big\{a^{\left(\frac{\beta-\eta}{2}\vee 0\right)}\sup_{|x_0|\geqslant a}\big\|\boldsymbol{f}(x)\big\|_\beta\Big\} < \infty, \\
\|\boldsymbol{f}\|_{\mathcal{D}^{\gamma,\eta}} &:= \max_{\beta<\gamma}\sup_{a>0}\left\{a^{\frac{\gamma-\eta}{2}}\sup_{|x_0|,|y_0|\geqslant a}\frac{\big\|\boldsymbol{f}(y) - \widehat{\mathsf{g}_{yx}}\boldsymbol{f}(x)\big\|_\beta}{d(y,x)^{\gamma-\beta}}\right\} < \infty.
\end{aligned} \tag{4.4}$$

Set $|\!|\!|\boldsymbol{f}|\!|\!|_{\mathcal{D}^{\gamma,\eta}} := [\![\boldsymbol{f}]\!]_{\mathcal{D}^{\gamma,\eta}} + \|\boldsymbol{f}\|_{\mathcal{D}^{\gamma,\eta}}$.

One also talks of singular modelled distributions. An example of singular modelled distributions is obtained as follows. Given $\eta \in \mathbb{R}$ and $v \in \mathcal{C}^\eta(\mathbb{T}^d)$, the T_X-valued function

$$(\boldsymbol{P}_\gamma v)(x) := \mathbf{1}_{x_0>0} \sum_{|k|_{\mathfrak{s}}<\gamma} \partial^k \big(e^{x_0(\Delta_{x'}-1)} v\big)(x) \, \frac{X^k}{k!} \tag{4.5}$$

belongs to $\mathcal{D}^{\gamma,\eta}(T,\mathrm{g})$ for any $\gamma \geqslant \eta$. This is a consequence of the Schauder estimates satisfied by the heat semigroup recalled in (4.3) – see e.g. [44, Lemma 7.5]. The reconstruction theorem, Theorem 4, is extended to singular modelled distributions as follows. See Appendix C.1 for a detailed proof.

Theorem 20. *Let* $\mathsf{M} = (\Pi, \mathrm{g})$ *be a model over* $\mathscr{T}$ *such that* $-2 < \beta_0 < 0$. *Assume that* $-2 < \eta \leqslant \gamma$, *with* $\gamma \neq 0$. *Then there exists a continuous linear operator*

$$\mathsf{R}^{\mathsf{M}} : \mathcal{D}^{\gamma,\eta}(T,\mathrm{g}) \to \mathcal{C}^{\eta\wedge\beta_0}(\mathbb{R}\times\mathbb{R}^d)$$

such that, for any $\boldsymbol{f} \in \mathcal{D}^{\gamma,\eta}(T,\mathrm{g})$, *the bound*

$$\Big|\Big\langle \mathsf{R}^{\mathsf{M}}\boldsymbol{f} - \Pi_x^{\mathrm{g}}\boldsymbol{f}(x), p_t(x,\cdot)\Big\rangle\Big| \lesssim \|\Pi^{\mathrm{g}}\| \, |\!|\!|\boldsymbol{f}|\!|\!|_{\mathcal{D}^{\gamma,\eta}} \, (|x_0| \vee t^{1/4})^{(\eta\wedge\beta_0)-\gamma} t^{\gamma/4}, \tag{4.6}$$

holds uniformly in $\boldsymbol{f} \in \mathcal{D}^{\gamma,\eta}(T,\mathrm{g}), x \in \mathbb{R}\times\mathbb{R}^d$ *and* $0 < t \leqslant 1$. *Such an operator is unique if the exponent* γ *is positive.*

The operators discussed in previous sections can be extended to the spaces $\mathcal{D}^{\gamma,\eta}(T,\mathrm{g})$, as follows. All of the following maps are locally Lipschitz continuous. For the detailed proofs, see [44, Propositions 6.12, 6.13, 6.15, and 6.16]. Denote by $\mathcal{D}^{\gamma,\eta}_\alpha(T,\mathrm{g})$ the space of modelled distributions $\boldsymbol{f} \in \mathcal{D}^{\gamma,\eta}(T,\mathrm{g})$ of the form

$$\boldsymbol{f} = \sum_{\alpha\leqslant|\tau|<\gamma} f_\tau \tau. \tag{4.7}$$

- (Proposition 8') If a regular product $\star : V \times W \to T$ is given, then
$$\mathcal{D}^{\gamma_1,\eta_1}_{\alpha_1}(V,\mathrm{g}) \times \mathcal{D}^{\gamma_2,\eta_2}_{\alpha_2}(W,\mathrm{g}) \ni (\boldsymbol{f}_1,\boldsymbol{f}_2) \mapsto \mathcal{Q}_{<\gamma}(\boldsymbol{f}_1 \star \boldsymbol{f}_2) \in \mathcal{D}^{\gamma,\eta}_{\alpha_1+\alpha_2}(T,\mathrm{g}),$$
where $\gamma = (\gamma_1+\alpha_2)\wedge(\gamma_2+\alpha_1)$ and $\eta = (\eta_1+\alpha_2)\wedge(\eta_2+\alpha_1)\wedge(\eta_1+\eta_2)$.
- (Proposition 9') Let $0 \leqslant \eta$. If an associative regular product $\star : V\times V \to V$ and a smooth function F is given, then
$$\mathcal{D}^{\gamma,\eta}(T,\mathrm{g}) \ni \boldsymbol{f} \mapsto F^\star(\boldsymbol{f}) \in \mathcal{D}^{\gamma,\eta}(T,\mathrm{g}).$$
- (Proposition 10') Let $\gamma > 1$. If a derivative $D : T \to T$ is given, then
$$\mathcal{D}^{\gamma,\eta}(T,\mathrm{g}) \ni \boldsymbol{f} \mapsto D\boldsymbol{f} \in \mathcal{D}^{\gamma-1,\eta-1}(T,\mathrm{g}).$$
- (Theorem 17') Let $-2 < \eta\wedge\beta_0$. If Π is $\mathbf{K}$-admissible,
$$\mathcal{D}^{\gamma,\eta}(T,\mathrm{g}) \ni \boldsymbol{f} \mapsto \mathcal{K}^{\mathsf{M}}\boldsymbol{f} \in \mathcal{D}^{\gamma+2,\eta\wedge\beta_0+2}(T,\mathrm{g}).$$

4.3 Non-anticipative operators

We consider the modelled distributions defined on the domain

$$\mathcal{E}_{(0,t)} := (0,t)\times\mathbb{T}^d,$$

for given a positive time t. Denote by $\mathcal{D}^{\gamma,\eta}_{(0,t)}(T,\mathrm{g})$ the set of functions $\boldsymbol{f} : \mathcal{E}_{(0,t)} \to T_{<\gamma}$ such that the bounds (4.4) hold with the domain of x, y restricted to $\mathcal{E}_{(0,t)}$. Denote by

$$|\!|\!|\boldsymbol{f}|\!|\!|_{\mathcal{D}^{\gamma,\eta}_{(0,t)}} := [\![\boldsymbol{f}]\!]_{\mathcal{D}^{\gamma,\eta}_{(0,t)}} + \|\boldsymbol{f}\|_{\mathcal{D}^{\gamma,\eta}_{(0,t)}}$$

the associated norms. From the definition of quantities (4.4), the estimate

$$|\!|\!|\boldsymbol{f}|\!|\!|_{\mathcal{D}^{\gamma,\eta-\kappa}_{(0,t)}} \lesssim \max_{\beta<\eta-\kappa} \sup_{x\in\mathcal{E}_{(0,t)}} \big\|\boldsymbol{f}(x)\big\|_\beta + t^{\kappa/2} \, |\!|\!|\boldsymbol{f}|\!|\!|_{\mathcal{D}^{\gamma,\eta}_{(0,t)}} \tag{4.8}$$

follows for any $\kappa > 0$ small enough. The small factor $t^{\kappa/2}$ is used in the fixed point problem in the next section.

A function f on $(\mathbb{R}\times\mathbb{R}^d)^2$ is said to be *non-anticipative* if $f\big((x_0,x'),(y_0,y')\big) = 0$, whenever $x_0 < y_0$. The kernel $K_{\mathbf{L}}$ of the resolution operator $\mathbf{L}^{-1}$ is of the form

$$K_{\mathbf{L}}(x,y) = \mathbf{1}_{x_0>y_0}\, p_{x_0-y_0}(x'-y'),$$

where p_t is the kernel of $e^{t(\Delta-1)}$, thus $K_{\mathbf{L}}$ is non-anticipative. That property makes the space of modelled distributions on $\mathbb{R}\times\mathbb{R}^d_{(0,t)}$ stable under the integration map. For any $\boldsymbol{f}\in\mathcal{D}^{\gamma,\eta}_{(0,t)}(T,\mathsf{g})$, we call $\widetilde{\boldsymbol{f}}\in\mathcal{D}^{\gamma,\eta}(T,\mathsf{g})$ a *positive time extension* of $\boldsymbol{f}$ if

$$\widetilde{\boldsymbol{f}} = \boldsymbol{f} \text{ on } \mathcal{E}_{(0,t)}, \quad \text{and} \quad \widetilde{\boldsymbol{f}} = 0 \text{ on } (-\infty,0)\times\mathbb{T}^d.$$

Such an extension $\widetilde{\boldsymbol{f}}$ always exists, and in fact, one can construct a continuous linear extension map on $\mathcal{D}^{\gamma,\eta}_{(0,t)}(T,\mathsf{g})$, taking values in $\mathcal{D}^{\gamma,\eta}(T,\mathsf{g})$, as proved in Martin's work [60, Theorem 5.3.16]. See also Theorem 48 in Appendix C.1 for the sketch of a proof.

Proposition 21. *Pick $\rho>0$ and $-2<\eta\leqslant\beta_0<0$. Then one has, for any $\boldsymbol{w}\in\mathcal{D}^{\rho,\eta}_t(T,\mathsf{g})$,*

$$\Big\| \big(\mathcal{K}^{\mathsf{M}} + (\mathcal{K}'_{\rho+2})^{\mathsf{M}}\big)(\widetilde{\boldsymbol{w}}) \Big\|_{\mathcal{D}^{\rho+2,\eta+2}_{(0,t)}} \lesssim \|\boldsymbol{w}\|_{\mathcal{D}^{\rho,\eta}_{(0,t)}}.$$

In the left hand side, the restriction of $\big(\mathcal{K}^{\mathsf{M}} + (\mathcal{K}'_{\rho+2})^{\mathsf{M}}\big)(\widetilde{\boldsymbol{w}})$ to $\mathcal{E}_{(0,t)}$ does not depend on the choice of positive time extensions $\widetilde{\boldsymbol{w}}$. Hence we are allowed to write

$$\big(\mathcal{K}^{\mathsf{M}} + (\mathcal{K}'_{\rho+2})^{\mathsf{M}}\big)(\widetilde{\boldsymbol{w}})\big|_{\mathcal{E}_{(0,t)}} = \big(\mathcal{K}^{\mathsf{M}} + (\mathcal{K}'_{\rho+2})^{\mathsf{M}}\big)(\boldsymbol{w}).$$

If further η is sufficiently near to -2 so that $A\cap(0,\eta+2]=\varnothing$, then one has

$$\Big\| \big(\mathcal{K}^{\mathsf{M}} + (\mathcal{K}'_{\gamma+2})^{\mathsf{M}}\big)(\boldsymbol{w}) \Big\|_{\mathcal{D}^{\rho+2,\eta+2-\kappa}_{(0,t)}} \lesssim t^{\kappa/2}\, \|\boldsymbol{w}\|_{\mathcal{D}^{\rho,\eta}_{(0,t)}} \tag{4.9}$$

for any $\kappa\in(0,\eta+2)$. Finally, if the reconstruction of $\boldsymbol{w}$ happens to be continuous on $\mathcal{E}_{(0,t)}$, then one has

$$\mathsf{R}^{\mathsf{M}}\Big(\mathcal{K}^{\mathsf{M}} + (\mathcal{K}'_{\rho+2})^{\mathsf{M}}\Big)(\boldsymbol{w}) = \int_{\mathcal{E}_{(0,t)}} K_{\mathbf{L}}(x,y)\, \mathsf{R}^{\mathsf{M}}\boldsymbol{w}(y)\, dy \tag{4.10}$$

Proof – By Theorem 20, one has $\mathsf{R}^{\mathsf{M}}\widetilde{\boldsymbol{w}}\in\mathcal{C}^\eta(\mathcal{E})$. Since $\mathbf{K}'$ maps $\mathcal{C}^\eta(\mathcal{E})$ into $\mathcal{C}^{\rho+2}(\mathcal{E})$, one has $(\mathcal{K}'_{\rho+2})^{\mathsf{M}}\widetilde{\boldsymbol{w}}\in\mathcal{D}^{\rho+2,\rho+2}(T,\mathsf{g})\subset\mathcal{D}^{\rho+2,\eta+2}(T,\mathsf{g})$. Thus one has

$$\Big\| \big(\mathcal{K}^{\mathsf{M}} + (\mathcal{K}'_{\rho+2})^{\mathsf{M}}\big)(\widetilde{\boldsymbol{w}}) \Big\|_{\mathcal{D}^{\rho+2,\eta+2}} \lesssim \|\widetilde{\boldsymbol{w}}\|_{\mathcal{D}^{\rho,\eta}}.$$

Choosing a continuous extension $\widetilde{\boldsymbol{w}}$ of $\boldsymbol{w}$ as in Theorem 48 in Appendix C.1, the right hand side is bounded above by $\|\boldsymbol{w}\|_{\mathcal{D}^{\rho,\eta}_{(0,t)}}$. It remains to show that the restriction of $\big(\mathcal{K}^{\mathsf{M}} + (\mathcal{K}'_{\rho+2})^{\mathsf{M}}\big)(\widetilde{\boldsymbol{w}})$ to $\mathcal{E}_{(0,t)}$ does not depend on the choice of extensions $\widetilde{\boldsymbol{w}}$. By definition, one has

$$\begin{aligned}\Big(\mathcal{K}^{\mathsf{M}}+(\mathcal{K}'_{\rho+2})^{\mathsf{M}}\Big)(\widetilde{\boldsymbol{w}})(x) &= \mathcal{I}\big(\widetilde{\boldsymbol{w}}(x)\big) + \mathcal{J}^{\mathsf{M}}(x)\widetilde{\boldsymbol{w}}(x)\\ &\quad + \sum_{|\ell|_{\mathfrak{s}}<\rho+2} \frac{X^k}{k!}\mathbf{K}'\big(\Pi^{\mathsf{g}}_x\widetilde{\boldsymbol{w}}(x)\big)(x) + \sum_{|\ell|_{\mathfrak{s}}<\rho+2} \frac{X^\ell}{\ell!}\,\partial^\ell K_{\mathbf{L}}\Big(\mathsf{R}^{\mathsf{M}}\widetilde{\boldsymbol{w}} - \Pi^{\mathsf{g}}_x\widetilde{\boldsymbol{w}}(x)\Big)(x).\end{aligned}$$

Since $K_{\mathbf{L}}$ is non-anticipative, the quantity $\partial^\ell K_{\mathbf{L}}(\cdots)(x)$ above happens to depend on the restriction of $\widetilde{\boldsymbol{w}}$ to $(-\infty,t]\times\mathbb{T}^d$, from Corollary 5. Since $\widetilde{\boldsymbol{w}}=0$ on $(-\infty,0]\times\mathbb{R}^d$, the above quantity does not depend on the choice of $\widetilde{\boldsymbol{w}}$.

Next we prove (4.9). In view of (4.8), it is sufficient to consider the $\mathbf{1}$-component of $\boldsymbol{f} = \big(\mathcal{K}^{\mathsf{M}} + (\mathcal{K}'_{\rho+2})^{\mathsf{M}}\big)(\widetilde{\boldsymbol{w}})$. In the above computation, we see that

$$\boldsymbol{f}_{\mathbf{1}}(x) = K_{\mathbf{L}}\big(\mathsf{R}^{\mathsf{M}}\widetilde{\boldsymbol{w}}\big)(x).$$

Since $\mathsf{R}^{\mathsf{M}}\widetilde{\boldsymbol{w}}\in\mathcal{C}^\eta(\mathcal{E})$, we have $\boldsymbol{f}_{\mathbf{1}}\in\mathcal{C}^{\eta+2}(\mathcal{E})$ by Schauder estimate, so it is Hölder continuous. Since $\boldsymbol{f}_{\mathbf{1}}=0$ on $(-\infty,0)\times\mathbb{R}^d$ from the non-anticipativity of $K_{\mathbf{L}}$, it also vanishes at $x_0=0$

and we have

$$\sup_{x\in\mathcal{E}_{(0,t)}} \|\boldsymbol{f}(x)\|_0 \lesssim \sup_{x\in\mathcal{E}_{(0,t)}} \|\boldsymbol{f}_{\mathbf{1}}(x) - \boldsymbol{f}_{\mathbf{1}}(0,x')\|_0 \lesssim t^{(\eta+2)/2}.$$

▷

4.4 Fixed point solution

Definition – *A* **regularity structure** $\mathscr{T}$ *is said to be* **associated with equation** (4.1) *if it satisfies assumptions* **(A-B)** *and contains subcomodules* $S, \partial S, F, N$ *of* T *satisfying assumption* **(A1)** *and the following constraints.*

- *The symbol* Ξ *and the set* ∂S *are contained in* N.
- *The sector* S *is function-like and regular products*
$$S\times\cdots\times S \to F, \quad \partial S\times \partial S\to N, \quad F\times N\to T$$
are given and satisfy assumption **(A3)**. *We denote them all by the same symbol* $\star$.
- *Abstract integration operators*
$$\mathcal{I}: T\to S, \quad \mathcal{I}_{e_i}: T\to \partial S, \qquad (1\leqslant i\leqslant d)$$
are given and satisfy Assumption **(B)**.
- *Derivative operators*
$$\partial_i : S\to\partial S, \qquad (1\leqslant i\leqslant d)$$
are given and satisfy
$$\mathsf{\Pi}\circ\partial_i = \partial_{x_i}\circ\mathsf{\Pi}, \quad \textit{and} \quad \partial_i X^k = k_i X^{k-e_i}\mathbf{1}_{k\geqslant e_i}, \quad \textit{and} \quad \partial_i\mathcal{I}\tau = \mathcal{I}_{e_i}\tau.$$

The element Ξ represents the noise ζ. The spaces S and ∂S are used to represent the solution u and its derivative $\partial_{x'}u$, respectively. (The letter S is chosen for 'solution'.) The space F are used to represent $f(u)$ and $g_n(u)$, with $n=0,1,2$. (The letter F is chosen for 'function'.) The space N is used to represent the 'singular' elements ζ, $\partial_{x'}u$, and $(\partial_{x'}u)^2$. (The letter N is chosen for 'noise'.) The only role played by the intermediate spaces $\partial S, F, N$ is to clarify on which spaces the product $\star$ is defined; they play no other role. We will see in Section 9 how to construct explicitly a regularity structure associated with the generalized (KPZ) equation. The $\star$ product is used to define nonlinear images of singular modelled distributions as in Section 2.4. In this setting, the regularity structure lift of the generalized (KPZ) equation is formulated under the form

$$\begin{aligned}\boldsymbol{v} &= \big(\mathcal{K}^{\mathsf{M}} + (\mathcal{K}'_\gamma)^{\mathsf{M}}\big)\Big(f^\star(\boldsymbol{v})\Xi + g^\star\big(\boldsymbol{v},\partial\boldsymbol{v}\big)\Big) + \boldsymbol{p} \\ &=: \Phi(\mathsf{M},\boldsymbol{v}).\end{aligned} \tag{4.11}$$

Pick a $\mathbf{K}$-admissible model $\mathsf{M} = (\mathsf{\Pi},\mathsf{g})$ on $\mathscr{T}$ and $\boldsymbol{p}\in\mathcal{D}^{\gamma,\eta}(T_X,\mathsf{g})$. Assume that $\Phi(\mathsf{M},\cdot)$ sends $\mathcal{D}^{\gamma,\eta}(S,\mathsf{g})$ into itself, which turns out to be the case as proved below under the conditions of Theorem 22.

Definition – *A* **solution to equation** (4.2) **on the time interval** $(0,t_0)$ *is a fixed point of the map* $\Phi(\mathsf{M},\cdot): \mathcal{D}^{\gamma,\eta}_{(0,t_0)}(S,\mathsf{g}) \to \mathcal{D}^{\gamma,\eta}_{(0,t_0)}(S,\mathsf{g})$.

Theorem 22. *Let* $\mathscr{T}$ *be a regularity structure associated with the generalized (KPZ) equation satisfying conditions* **(A-B)**, *with* $\beta_0\in(-2,-1)$. *Pick* $\eta\in(0,\beta_0+2]$ *and* $\gamma>-\beta_0$. *Then for any* $\boldsymbol{p}\in\mathcal{D}^{\gamma,\eta}(T_X,\mathsf{g})$, *there exists a positive time* t_0 *such that equation* (4.2) *has a unique solution* $\boldsymbol{u}$ *on the time interval* $(0,t_0)$. *The time* t_0 *can be chosen to be a lower semicontinuous function of* M *and* $\boldsymbol{p}$.

Proof – Recall that $\mathcal{D}^{\gamma,\eta}_{\alpha}(T,\mathsf{g})$ denotes the set of modelled distributions of the form (4.7). Starting from $\boldsymbol{v}\in\mathcal{D}^{\gamma,\eta}(S,\mathsf{g})$, we show that

$$f^\star(\boldsymbol{v})\Xi + g^\star(\boldsymbol{v},\partial\boldsymbol{v}) \in \mathcal{D}^{\gamma+\beta_0,2\eta-2}(T,\mathsf{g}).$$

From the singular version of Proposition 9 given at the end of Section 4.2, one has $f^\star(\boldsymbol{v}), g_i^\star(\boldsymbol{v})\in\mathcal{D}^{\gamma,\eta}_0(F,\mathsf{g})$ for $i=0,1,2$. Since $\Xi\in\mathcal{D}^{\infty,\infty}_{\beta_0}(N,\mathsf{g})$, one has $f^\star(\boldsymbol{v})\Xi\in\mathcal{D}^{\gamma+\beta_0,\eta+\beta_0}(T,\mathsf{g})\subset\mathcal{D}^{\gamma+\beta_0,2\eta-2}(T,\mathsf{g})$

from the singular version of Proposition 8 given at the end of Section 4.2. Noting that the smallest homogeneity in the subcomodule ∂S is $\beta_0+1<0$ (which is the homogeneity of $\partial\mathcal{I}\Xi$), one has $\partial\boldsymbol{v}\in\mathcal{D}^{\gamma-1,\eta-1}_{\beta_0+1}(\partial S,\mathsf{g})$ and $(\partial\boldsymbol{v})^{\star 2}\in\mathcal{D}^{\gamma+\beta_0,2\eta-2}_{2\beta_0+2}(N,\mathsf{g})$. Thus $g^\star(\boldsymbol{v},\partial\boldsymbol{v})\in\mathcal{D}^{\gamma+\beta_0,2\eta-2}(T,\mathsf{g})$. From Proposition 21, one has

$$\begin{aligned}\|\Phi^{\mathsf{M}}(v)\|_{\mathcal{D}^{\gamma,\eta}_{(0,t)}} &\lesssim t^{\eta/2}\|f^\star(\boldsymbol{v})\Xi+g^\star(\boldsymbol{v},\partial\boldsymbol{v})\|_{\mathcal{D}^{\gamma+\beta_0,2\eta-2}_{(0,t)}}+\|\boldsymbol{p}\|_{\mathcal{D}^{\gamma,\eta}_{(0,t)}}\\ &\lesssim t^{\eta/2}F\Big(\|\boldsymbol{v}\|_{\mathcal{D}^{\gamma,\eta}_{(0,t)}}\Big)+\|\boldsymbol{p}\|_{\mathcal{D}^{\gamma,\eta}_{(0,t)}}\end{aligned}$$

for some locally bounded function F. Then one can associate with each positive radius λ a time horizon $t(\lambda)$ such that $\Phi(\mathsf{M},\cdot)$ sends the ball of $\mathcal{D}^{\gamma,\eta}_{(0,t(\lambda))}(S,\mathsf{g})$ of radius λ into itself. From the local Lipschitz continuity result and inequality (4.8), the map $\Phi(\mathsf{M},\cdot)$ is also a contraction on the ball of $\mathcal{D}^{\gamma,\eta}_{(0,t(\lambda))}(T,\mathsf{g})$ of radius λ. As such, it has a unique fixed point on the ball of radius λ. An elementary argument gives the uniqueness of a global fixed point, as in the proof of Theorem 4.7 in [43]. $\rhd$

Thinking of $\boldsymbol{p}$ as the regularity structure lift of the free propagation of an initial condition on $\mathbb{T}^d$, assuming in $\boldsymbol{p}\in\mathcal{D}^{\gamma,\eta}(T_X,\mathsf{g})$ allows us to work with an initial condition of Hölder regularity η – recall the constraint $\eta\in(0,\beta_0+2]$. Note that the map is uniformly contracting on a small enough time interval for $g\in C^4$ ranging in a bounded set. Denote by $t_0(\boldsymbol{p},\mathsf{M})$ the time horizon from Theorem 22. In order to compare fixed points of $\Phi(\mathsf{M},\cdot)$ associated with different admissible models on $\mathscr{T}$ – hence different maps on different spaces, we introduce the following distance. For the usual modelled distributions on $\mathbb{R}\times\mathbb{R}^d$, given two models $\mathsf{M}=(\mathsf{\Pi},\mathsf{g})$ and $\mathsf{M}'=(\mathsf{\Pi}',\mathsf{g}')$ on $\mathscr{T}$, a regularity exponent $\gamma\in\mathbb{R}$, and $\boldsymbol{f}\in\mathcal{D}^\gamma(T,\mathsf{g})$ and $\boldsymbol{f}'\in\mathcal{D}^\gamma(T,\mathsf{g}')$, set

$$d\big(\boldsymbol{f},\boldsymbol{f}'\big):=\sup_{x,y\in\mathbb{R}\times\mathbb{R}^d}\max_{\beta<\gamma}\left\{\big\|\boldsymbol{f}(x)-\boldsymbol{f}'(x)\big\|_\beta+\frac{\Big\|\Big\{\boldsymbol{f}(y)-\widehat{\mathsf{g}_{yx}}\boldsymbol{f}(x)\Big\}-\Big\{\boldsymbol{f}'(y)-\widehat{\mathsf{g}'_{yx}}\boldsymbol{f}'(x)\Big\}\Big\|_\beta}{d(y,x)^{\gamma-\beta}}\right\}.$$

The associated distance between an element of $\mathcal{D}^{\gamma,\eta}_{(0,t)}(T,\mathsf{g})$ and an element of $\mathcal{D}^{\gamma,\eta}_{(0,t)}(T,\mathsf{g}')$ is defined similarly. One can then prove the following statement in terms of this metric by making explicit in the reconstruction theorem and the lifting theorem that the operators $\mathbf{R}^{\mathsf{M}}$ and $\mathcal{K}^{\mathsf{M}}+(\mathcal{K}'_\gamma)^{\mathsf{M}}$ depend in a locally Lipschitz way on M with respect to the pseudo-distance d_γ on the space of models over $\mathscr{T}$ introduced in (2.24). We do not give the details here and refer the reader to the corresponding results in [44], Theorem 3.10 and Theorem 5.12 therein.

Proposition 23. *Given any time $t_0'<t_0(\boldsymbol{p},\mathsf{M})$, the restriction to $[0,t_0']\times\mathbb{T}^d$ of $\boldsymbol{u}$ defines locally a continuous function of the $\mathbf{K}$-admissible model M.*

This result holds more generally for all the equations that can be treated using regularity structures. Emphasize that this continuity result is fundamental. In a random setting where the noise is random and the models of interest are constructed as measurable functionals of the noise the continuity allows to transport automatically support theorems or large deviation results about random models into corresponding results about the solutions of the regularity structure lifts of the equations under study. See Hairer & Schönbauer's work [50] on support theorems, Hairer & Weber's work [51] on large deviation results, or Hairer & Mattingly's work [49] on the strong Feller property for solutions of singular stochastic PDEs, for a sample.

The last statement of this section makes the link between solving equation (4.1) with a smooth noise ζ and the corresponding problem in the regularity structure equipped with the canonical model M^ζ associated with the smooth noise. The latter is constructed in Section 6.1 and the only thing we presently need to know about it is that its reconstruction map $\mathbf{R}^{\mathsf{M}^\zeta}$ is multiplicative and sends the noise symbol Ξ on the smooth function ζ. For positive exponents $\gamma\in(-\beta_0,2)$ and $\eta\in(0,\beta_0+2]$, pick $v\in\mathcal{C}^\eta(\mathbb{T}^d)$ and denote by $\boldsymbol{P}_\gamma v$ the lift in the polynomial structure of the heat propagator acting on v, defined by (4.5).

Proposition 24. *Let $\boldsymbol{u}\in\mathcal{D}^{\gamma,\eta}_{(0,t_0)}(T,\mathsf{g})$ stand for the solution in a sufficiently small time interval $[0,t_0]$ of the fixed point problem*

$$\boldsymbol{u}=\Big(\mathcal{K}^{\mathsf{M}^\zeta}+(\mathcal{K}'_\gamma)^{\mathsf{M}^\zeta}\Big)\Big(f^\star(\boldsymbol{u})\Xi+g^\star\big(\boldsymbol{u},\partial\boldsymbol{u}\big)\Big)+\boldsymbol{P}_\gamma v. \tag{4.12}$$

Then $\mathbf{R}^{\mathsf{M}^\zeta}\boldsymbol{u}$ *coincides with the solution to the well-posed equation* (4.1) *with initial condition* v.

Proof – As in (4.10), the function $u := \mathbf{R}^{\mathsf{M}^\zeta}\boldsymbol{u}$ satisfies the equation

$$u(z) = \int_{(0,t)\times\mathbb{T}^d} K_{\mathbf{L}}(z,w)\mathbf{R}^{\mathsf{M}^\zeta}\Big(f(\boldsymbol{u})\Xi + g\big(\boldsymbol{u},\partial\boldsymbol{u}\big)\Big)(w)\,dw + \big(Pv\big)(z),$$

with Pv the free propagation of the initial condition. We take advantage of the fact that M^ζ is a smooth model to write

$$\mathbf{R}^{\mathsf{M}^\zeta}(\boldsymbol{w})(z) = \Pi^\zeta_z\big(\boldsymbol{w}(z)\big)(z)$$

for any modelled distribution $\boldsymbol{w}\in\mathcal{D}^\alpha(T,\mathsf{g}^\zeta)$ with $\alpha>0$ – see identity (2.33). We use the multiplicative character of the map $\mathbf{R}^{\mathsf{M}^\zeta}$ to write

$$\mathbf{R}^{\mathsf{M}^\zeta}\Big(f(\boldsymbol{u})\Xi + g\big(\boldsymbol{u},\partial\boldsymbol{u}\big)\Big) = \Big(\mathbf{R}^{\mathsf{M}^\zeta}f(\boldsymbol{u})\Big)\zeta + \Big(\mathbf{R}^{\mathsf{M}^\zeta}g_2(\boldsymbol{u})\Big)\Big(\mathbf{R}^{\mathsf{M}^\zeta}\partial\boldsymbol{u}\Big)^2 + \Big(\mathbf{R}^{\mathsf{M}^\zeta}g_1(\boldsymbol{u})\Big)\Big(\mathbf{R}^{\mathsf{M}^\zeta}\partial\boldsymbol{u}\Big) + \mathbf{R}^{\mathsf{M}^\zeta}\big(g_0(\boldsymbol{u})\big).$$

The conclusion then follows as a consequence of Corollary 7 and Proposition 10, which give

$$\mathbf{R}^{\mathsf{M}^\zeta}\big(f(\boldsymbol{u})\big) = f\big(\mathbf{R}^{\mathsf{M}^\zeta}\boldsymbol{u}\big), \quad \mathbf{R}^{\mathsf{M}^\zeta}(\partial\boldsymbol{u}) = \partial_x\mathbf{R}^{\mathsf{M}^\zeta}\boldsymbol{u}.$$

▷

Arrived at that stage we have a model-dependent notion of solution u to the generalized (KPZ) equation, under the form

$$u = \mathbf{R}^{\mathsf{M}}(\boldsymbol{u}), \quad \boldsymbol{u} = \Phi(\mathsf{M},\boldsymbol{u}),$$

indexed by the set of $\mathbf{K}$-admissible models on the regularity structure $\mathscr{T}$ associated with the equation. Theorem 19 gives us a whole family of smooth $\mathbf{K}$-admissible models which we can use. However the $\mathbf{K}$-admissible models of interest are not smooth as we wish they satisfy the identity $\Pi\Xi = \zeta$ for a non-smooth noise ζ. The combinatorial structure of the elements of $\mathscr{T}$ detailled in Section 9 allow to associate to any regularized version ζ_ε of ζ a model Π^ε such that $\Pi^\varepsilon\Xi = \zeta_\varepsilon$, in a canonical way. However, these models diverge as $\varepsilon>0$ goes to 0. The tools needed to construct ε-dependent smooth models that have limits as ε goes to 0 are developed in the next section at the same level of generality as Section 2 and Section 3. The so called renormalization operation involved in the construction of these converging $\mathbf{K}$-admissible models will be given a dynamical meaning in Section 6.

5 – Renormalization structures

We introduce in this section the fundamental notion of renormalization structure, and a notion of compatibility between regularity and renormalization structures. We emphasized in the previous paragraph that it is generically not possible to define canonical $\mathbf{K}$-admissible models Π^ζ as limits of canonical $\mathbf{K}$-admissible models Π^ε associated with regularized noises ζ_ε, if the noise(s) is (are) not sufficiently regular. On a technical level, the non-convergence of the models Π^ε is related to the fact that the canonical model is defined by convolution of kernels that explode on the diagonal. Limit models need to be constructed by probabilistic means as limits in probability of models built from regularized noises, *using a moving window*, as in Meta-Theorem 1 in Section 1. The implementation of this moving window picture involves renormalization structures. Note that we do not need to know the details of the renormalization operation; the only properties that we need are encoded in the definition of a renormalization structure and the compatibility condition with a regularity structure given below. An example of renormalization structure will be given in Section 9, where the renormalization operation will be intimately related to the Taylor expansion procedure.

Renormalization structures are defined in Section 5.1. If we call the concrete regularity structures from Section 2.2 right regularity structures, then renormalization structures

$$\mathscr{U} = \big((U^-,\delta^-),(U,\delta)\big)$$

look like left regularity structures, with the difference that elements of the space U^- have non-positive homogeneities. A fundamental notion of compatibility between renormalization and regularity structures is introduced in Section 5.2; it accounts for the fact that the renormalization operation induces a renormalization operation on T^+ and 'commutes' with the recentering operators Δ and Δ^+. This property allows to associate with each model M on $\mathscr{T}$ and each character k on U^- a new model ${}^k\mathsf{M}$ on $\mathscr{T}$. This is the main result of Section 5.2, Theorem 26. A large class of characters k produces $\mathbf{K}$-admissible models ${}^k\mathsf{M}$ if M is $\mathbf{K}$-admissible.

5.1 Definition

A renormalization structure is made up of two ingredients. First, it is a vector space U with a basis whose elements are built by induction from elementary elements and multilinear operators giving new elements. The use of the symbol τ for a generic basis vector emphasizes this recursive, tree-like, definition. Each basis vector τ is a placeholder for a function $[\tau]$ from $(0,1]$ into a Banach space, typically $\mathbb{R},\mathbb{C}$, a Hölder space or an algebra, whose structure as an element of the target space is encoded in the structure of τ. In the cases of interest, the functions $[\tau]$ have no limit in 0^+ and the basic problem is to remove in a 'consistent' way the diverging pieces of these $[\tau]$ so as to end up with a collection of functions parametrized by $\varepsilon > 0$ having a limit where ε goes to 0. The functions $[\tau]$ are then said to have been renormalized. What 'consistent' means is part of what follows.

Roughly speaking, the basic operation for renormalizing a placeholder τ consists in removing from τ its different diverging pieces, in all possible sensible ways. This is the second ingredient of a renormalization structure. Tuples of pieces of elements of U are not necessarily elements of U; we store them in a side space U^-. Endowing U^- with an algebra structure allows to store the removed pieces of τ as an element of U^- under the form of a product. We require nonetheless that any τ amputated from diverging pieces is an element of U; this is a restriction on which pieces of any $\tau \in U$ can be removed. We thus have a splitting map

$$\delta : U \to U^- \otimes U,$$

with $\delta\tau$ the sum of all the elements from $U^- \otimes U$ corresponding to removing from τ all possible diverging allowed pieces, possibly several at a time. The removed pieces may themselves have diverging subpieces, and it makes sense to assume we have another splitting map

$$\delta^- : U^- \to U^- \otimes U^-,$$

that extracts them on the left hand side of the tensor product $U^- \otimes U^-$. That the remaining piece is still in U^- rather than in another space is a consistency requirement.

Definition – *A* **renormalization structure** *is a pair of graded vector spaces*

$$U =: \bigoplus_{\beta\in B} U_\beta, \qquad U^- =: \bigoplus_{\alpha\in B^-} U^-_\alpha$$

such that the following holds.

- *The vector spaces U^-_α and U_β are finite dimensional.*
- *The space U^- is a connected graded bialgebra with unit $\mathbf{1}_-$, counit $\mathbf{1}'_-$, coproduct*

  $$\delta^- : U^- \to U^- \otimes U^-,$$

  and grading $B^- \subset (-\infty, 0]$.
- *The index set B for U is a locally finite subset of $\mathbb{R}$ bounded below. The space U is a left comodule over U^-, that is U is equipped with a splitting map $\delta : U \to U^- \otimes U$, which satisfies*

  $$(\mathrm{Id}\otimes\delta)\delta = (\delta^-\otimes\mathrm{Id})\delta, \quad \textit{and} \quad (\mathbf{1}'_-\otimes\mathrm{Id})\delta = \mathrm{Id}. \tag{5.1}$$

  Moreover, for any $\beta \in B$, one has

  $$\delta U_\beta \subset \bigoplus_{\alpha\leqslant 0} U^-_\alpha \otimes U_{\beta-\alpha}. \tag{5.2}$$

We denote by

$$\mathscr{U} := \Big((U,\delta),(U^-,\delta^-)\Big)$$

a renormalization structure.

Similarly to the regularity structure, let $\mathcal{U}_\alpha^-$ and $\mathcal{U}_\beta$ be bases of U_α^- and U_β, respectively, and set

$$\mathcal{U}^- := \bigcup_{\alpha\in B^-} \mathcal{U}_\alpha^-, \quad \mathcal{U} := \bigcup_{\beta\in B} \mathcal{U}_\beta.$$

Note that, unlike in the definition of a concrete regularity structure satisfying Assumption **(A1)**, we do not require that $\mathcal{U}_0$ is one dimensional in the definition of a renormalization structure. Since all $\alpha \in B^-$ are non-positive, one has $\beta - \alpha \geqslant \beta$ in (5.2). Proposition 45 in Appendix B can be applied to the negative grading B^- of U^-, and says that U^- is a Hopf algebra; we denote by S_- its antipode. Write

$$\delta\tau =: \sum_{\varphi \trianglelefteq \tau} \varphi \otimes \tau/^-\varphi; \tag{5.3}$$

we call δ a *renormalization splitting*. Similarly to what we saw in Section 2.2 for the Hopf algebra (T^+,Δ^+), the δ^- splitting of the Hopf algebra (U^-,δ^-) induces a convolution group law on the set G^- of characters on U^-

$$(k_1 * k_2)\tau := (k_1 \otimes k_2)\delta^-\tau, \quad (\tau \in U^-).$$

The inverse of a character k for the convolution product is explicit and given by $k \circ S_-$. Given a character k on U^-, we define a function $\widetilde{k} : U \to U$, setting

$$\widetilde{k} := (k \otimes \mathrm{Id})\delta.$$

This is a representation of the group G^- in $GL(U)$, as a direct consequence of the comodule property in (5.1).

5.2 Compatible renormalization and regularity structures

We introduce a 'compatibility' property between regularity and renormalization structures. We use the notations from Appendix B. In particular, given an algebra A and two spaces E, F, we define a linear map $\mathcal{M}^{(13)}$ from the algebraic tensor product $A\otimes E\otimes A\otimes F$ to the algebraic tensor product $A \otimes E \otimes F$ setting

$$\mathcal{M}^{(13)}\Big(a_1 \otimes e \otimes a_2 \otimes f\Big) := (a_1a_2) \otimes e \otimes f.$$

Recall we write $\mathscr{T} = \big((T^+,\Delta^+),(T,\Delta)\big)$ for a regularity structure and S_+ for the antipode map on T^+.

Definition 25. *A regularity structure $\mathscr{T}$ is said to be* **compatible** *with a renormalization structure $\mathscr{U}$ if the following three compatibility conditions hold true.*

- **(a)** *The spaces T and U coincide as linear spaces and the bases $\mathcal{B}$ and $\mathcal{U}$ coincide. (Each element $\tau \in \mathcal{B}$ is in particular homogeneous in both T and U, but it may belong to $\mathcal{B}_{\beta_1}$ and $\mathcal{U}_{\beta_2}$ with $\beta_1 \neq \beta_2$.) Moreover,*

$$\delta T_\beta \subset U^- \otimes T_\beta, \text{ for all } \beta \in A. \tag{5.4}$$

- **(b)** *There exists an algebra morphism*

$$\delta^+ : T^+ \to U^- \otimes T^+$$

such that

$$\big(\mathrm{Id}\otimes\delta^+\big)\delta^+ = \big(\delta^-\otimes\mathrm{Id}\big)\delta^+, \quad \textit{and} \quad \big(\mathbf{1}'_-\otimes\mathrm{Id}\big)\delta^+ = \mathrm{Id} \tag{5.5}$$

and

$$\delta^+T_\alpha^+ \subset U^- \otimes T_\alpha^+, \quad \textit{for all } \alpha \in A^+. \tag{5.6}$$

- **(c)** *The compatibility conditions*

$$\big(\mathrm{Id}\otimes\Delta^{(+)}\big)\delta^{(+)} = \mathcal{M}^{(13)}\big(\delta^{(+)}\otimes\delta^+\big)\Delta^{(+)} \tag{5.7}$$

and

$$\big(\mathrm{Id}\otimes \mathbf{1}'_+\big)\delta^+ = \mathbf{1}'_+(\cdot)\mathbf{1}_- \tag{5.8}$$

hold.

This definition captures the fact that the renormalization procedure encoded in $\mathscr{U}$ induces a renormalization operation on T^+ and commutes with the recentering operators Δ and Δ^+. We will see at the end of this section that the six conditions from Definition 25 hold iff condition (5.4) and condition (5.7), in its form without the (+) labels, hold, under a reasonable assumption on δ^+ that holds true for the regularity and renormalization structures associated with (systems of) singular stochastic PDEs.

Compare conditions (5.4) and (5.2). Emphasize here as in item **(a)** that the notion of homogeneity is relative to the the grading used to define it. An element of $T = U$ may thus have different homogeneities, depending on whether it is considered as an element of T or U. By condition **(a)**, the space T is a left U^--comodule. The map δ^+ in **(b)** accounts for the effect in T^+ of the renormalization process. By (5.5), the space T^+ is also a left U^--comodule. Hence for given a character k on U^-, we can define linear maps $\widetilde{k} : T \to T$ and $\widetilde{k}^+ : T^+ \to T^+$, by

$$\widetilde{k} = \big(k\otimes \mathrm{Id}\big)\delta, \quad \text{and} \quad \widetilde{k}^+ = \big(k\otimes \mathrm{Id}\big)\delta^+.$$

Identities (5.4) and (5.6) ensure that homogeneities of elements of T and T^+ are stable under these actions. Condition **(c)**, read with the + labels, somehow says that the renormalization operation encoded in $\widetilde{k}$ commutes with the Taylor expansion operation on the coefficients of any modelled distribution, encoded in Δ^+. Condition **(c)**, read without the + labels, says something similar for modelled distributions. Note that the Hopf algebra T^+ is a left U^--comodule bialgebra. By Proposition 46, we have the following compatibility condition on the antipode

$$\delta^+ \circ S_+ = \big(\mathrm{Id}\otimes S_+\big)\circ \delta^+. \tag{5.9}$$

Recall that given a model $\mathsf{M} = (\mathsf{g},\Pi)$ on $\mathscr{T}$, the anchored interpretation operator Π^{g}_x associated with M is given for any $x\in\mathbb{R}^d$, by

$$\Pi^{\mathsf{g}}_x = (\Pi\otimes \mathsf{g}_x^{-1})\Delta.$$

The next statement and its proof are part of Theorem 6.15 in Bruned, Hairer and Zambotti's work [16] on the algebraic renormalization of regularity structures. It tells us that the $\widetilde{k}$ and $\widetilde{k}^+$ maps have jointly a natural and simple action on the space of models on $\mathscr{T}$.

Theorem 26. *Let a renormalization structure $\mathscr{U} = (U,U^-)$ be compatible with a regularity structure $\mathscr{T} = (T^+,T)$. Given any character k on U^-, and any model $\mathsf{M} = (\mathsf{g},\Pi)$ on $\mathscr{T}$, define ${}^k\mathsf{M} = ({}^k\Pi, {}^k\mathsf{g})$, on $\mathscr{T}$ setting*

$${}^k\mathsf{M} := \Big(\mathsf{g}\circ\widetilde{k}^+, \Pi\circ\widetilde{k}\Big). \tag{5.10}$$

One has

$$\big(\mathsf{g}_y\circ\widetilde{k}^+\big) * \big(\mathsf{g}_x\circ\widetilde{k}^+\big)^{-1} = \mathsf{g}_{yx}\circ\widetilde{k}^+, \tag{5.11}$$

and

$$\Big(\big(\Pi\circ\widetilde{k}\big)\otimes\big(\mathsf{g}_x\circ\widetilde{k}^+\big)^{-1}\Big)\Delta = \Pi^{\mathsf{g}}_x\circ\widetilde{k}, \tag{5.12}$$

for any $x,y\in\mathbb{R}^d$. Moreover, the size conditions (2.20) *and* (2.21) *hold for ${}^k\mathsf{M} = ({}^k\mathsf{g}, {}^k\Pi)$, so ${}^k\mathsf{M}$ is a model.*

Proof – One has

$$\begin{aligned}\big(\mathsf{g}_y\circ\widetilde{k}^+\big) * \big(\mathsf{g}_x\circ\widetilde{k}^+\big)^{-1} &= \big((k\otimes\mathsf{g}_y)\delta^+\big)\otimes\Big((k\otimes\mathsf{g}_x)\delta^+\circ S_+\Big)\Delta^+ \\ &\overset{(5.9)}{=} \big((k\otimes\mathsf{g}_y)\delta^+\big)\otimes\big((k\otimes\mathsf{g}_x^{-1})\delta^+\big)\Delta^+ \\ &= \big(\mathsf{g}_y\otimes\mathsf{g}_x^{-1}\big)\circ\big(\widetilde{k}^+\otimes\widetilde{k}^+\big)\Delta^+ \\ &\overset{(5.7)}{=} \big(\mathsf{g}_y\otimes\mathsf{g}_x^{-1}\big)\circ(k\otimes\Delta^+)\delta^+ \\ &= \mathsf{g}_{yx}\circ\widetilde{k}^+.\end{aligned}$$

and

$$\Big((\Pi \circ \widetilde{k}) \otimes \big(\mathsf{g}_x \circ \widetilde{k}^+\big)^{-1} \Big)\Delta \overset{(5.9)}{=} \Big((k \otimes \Pi)\delta \otimes (k \otimes \mathsf{g}_x^{-1})\delta^+ \Big)\Delta$$
$$= \big(\Pi \otimes \mathsf{g}_x^{-1}\big) \circ \big(\widetilde{k} \otimes \widetilde{k}^+\big)\Delta$$
$$\overset{(5.7)}{=} \big(\Pi \otimes \mathsf{g}_x^{-1}\big) \circ (k \otimes \Delta)\delta$$
$$= \Pi_x^{\mathsf{g}} \circ \widetilde{k}.$$

The size conditions (2.20) and (2.21) on ${}^k\mathsf{M}$ follow now from formulas (5.11) and (5.12), and from the fact that the maps $\widetilde{k}$ and $\widetilde{k}^+$ preserve the spaces T_β and T_α^+, respectively, as a consequence of the stability conditions (5.4) and (5.6). ▷

Together with Corollary 6 this statement implies in particular that if the model M takes values in the space of *continuous functions* then the reconstruction operator ${}^k\mathbf{R}$ associated with the renormalized model is related to the reconstruction operator $\mathbf{R}$ associated with the unrenormalized model by the relation

$${}^k\mathbf{R} = \mathbf{R} \circ \widetilde{k}.$$

This point will be used crucially in Section 6.3, where we will give a dynamical picture of the renormalization of models.

We consider in the remainder of this section the case of interest for the study of (systems of) singular stochastic PDE(s) where the regularity structure $\mathscr{T}$ is built from integration operators and satisfies Assumption **(B)**. Unfortunately, even if a model M is $\mathbf{K}$-admissible, ${}^k\mathsf{M}$ is not always $\mathbf{K}$-admissible for any $k \in G^-$. We put forward an assumption under which one builds $\mathbf{K}$-admissible models using elements k of a non-trivial subgroup G^-_{ad} of G^-. Assume $\mathcal{B} = \mathcal{U}$ and let $\mathcal{F}$ stand for the family of operators

$$\mathcal{F} := \{\mathcal{I}_p\}_{|p|_{\mathfrak{s}} \leqslant 1} \cup \{X^n \star\}_{n \in \mathbb{N}^{d+1}\setminus\{0\}},$$

acting on the basis $\mathcal{B} = \mathcal{U}$.

Assumption (C1) – *The regularity structure $\mathscr{T}$ is built from integration operators and satisfies assumption* **(B1)** *and the renormalization structure $\mathscr{U}$ is compatible with $\mathscr{T}$. Moreover, the following holds.*

- *The algebra U^- is generated by the basis elements $\mathcal{U}_{<0} := \bigcup_{\alpha<0} \mathcal{U}_\alpha$ and the unit $\mathbf{1}_-$.*
- *Let J^- be the ideal of U^- generated by the set $\big(\mathcal{F}(\mathcal{U})\big) \cap \mathcal{U}_{<0}$. The linear map $\delta : U \to U^- \otimes U$ satisfies, for any operator $F \in \mathcal{F}$ and $\tau \in \mathcal{U}$,*
$$\delta(F\tau) - (\mathrm{Id} \otimes F)\delta\tau \in J^- \otimes U. \tag{5.13}$$
- *We define a projection operator $P_- : U \to U^-$ setting $P_-\tau := \tau \mathbf{1}_{\tau \in \mathcal{U}_{<0}}$, for any $\tau \in \mathcal{U}$. The linear map $\delta^- : U^- \to U^- \otimes U^-$ is defined by $\delta^- = (\mathrm{Id} \otimes P_-)\delta$ on $\mathcal{U}_{<0}$ and its multiplicative extension.*

Define the subset G^-_{ad} of G^- by

$$G^-_{\mathrm{ad}} := \Big\{ k \in G^- \,;\, k\big(F(\tau)\big) = 0 \text{ for any } F(\tau) \in \big(\mathcal{F}(\mathcal{U})\big) \cap \mathcal{U}_{<0} \Big\}.$$

Proposition 27. *The set G^-_{ad} is a subgroup of G^-, and for any $k \in G^-_{\mathrm{ad}}$ and any $\mathbf{K}$-admissible model M, one has ${}^k\mathsf{M}$ is also $\mathbf{K}$-admissible. The group G^-_{ad} is called the* **renormalization group**.

The definition of the group G^-_{ad} gives the meaning to assumption (5.13). Up to irrelevant terms for $k \in G^-_{\mathrm{ad}}$, the renormalization operations in U^- or U of an 'integral' is the integral of its renormalized integrand, and multiplication by a polynomial has no effect on the renormalization process.

Proof – Note that $k(J^-) = 0$ for any $k \in G^-_{\mathrm{ad}}$. Let τ be an element of $\mathcal{U}_\alpha$ such that $F\tau \in \mathcal{U}_{<0}$ for some $F \in \mathcal{F}$.

(a) Given $k, h \in G^-_{\mathrm{ad}}$ and $\tau \in \mathcal{U}_-$, since identity (5.13) ensures that $(k * h)(F\tau) = (k \otimes h)\delta^-(F\tau) = 0$, for all $F \in \mathcal{F}$, we have $k * h \in G^-_{\mathrm{ad}}$. Next we show that $k^{-1} = k \circ S_- \in G^-_{\mathrm{ad}}$. Denote by $\mathcal{M}^-$

the multiplication operator in U^-. Since $\delta\tau \in \mathbf{1}_- \otimes \tau + \sum_{\alpha<\beta} U^-_{\alpha-\beta} \otimes U_\beta$, by applying the operator $\mathcal{M}^-(\mathrm{Id} \otimes S_- P_-)$ to (5.13), we have

$$S_-(F\tau) \in \sum_{\alpha<\beta} \mathcal{M}^-\Big(U^-_{\alpha-\beta} \otimes S_-\big(P_- F U_\beta)\big)\Big) + J^-,$$

which implies $k^{-1}(FU_\alpha) = k(S_- F U_\alpha) = 0$, by an induction on α.

(b) Let $F = \mathcal{I}_p$. By (5.13),

$$^k\Pi(\mathcal{I}_p\tau) = (k \otimes \Pi)\delta\mathcal{I}_p\tau = (k \otimes \Pi\mathcal{I}_p)\delta\tau = \partial^p \mathbf{K}(k \otimes \Pi)\delta\tau = \partial^p\mathbf{K}(^k\Pi\tau).$$

Since we have a similar identity for $F = X^n\star$, we obtain that $^k\mathsf{M}$ is admissible. ▷

We end this section by showing that the definition of compatible renormalization and regularity structures takes then a simple form under the following additional mild assumption. It essentially says that multiplication by a polynomial is not the source of renormalization problems and it is only the integrand of an integral that needs to be renormalized.

Assumption (C2) – *The algebra morphism $\delta^+ : T^+ \to U^- \otimes T^+$, is determined by the identities*

$$\delta^+ X^\ell_+ = \mathbf{1}_- \otimes X^\ell_+, \quad \delta^+(\mathcal{I}^+_p\tau) = (\mathrm{Id} \otimes \mathcal{I}^+_p)\delta\tau. \tag{5.14}$$

Proposition 28. *Under assumption* **(C2)**, *assume $\mathscr{T}$ satisfies identity* (5.4) *and the version of identity* (5.7) *without the + labels. Then the other conditions in Definition 25 follow automatically.*

Proof – The comodule property (5.5) follows from (5.1) and the definition (5.14). Indeed,

$$\begin{aligned}\big(\mathrm{Id} \otimes \delta^+\big)\delta^+\mathcal{I}^+_n\tau &= \big(\mathrm{Id} \otimes \delta^+\mathcal{I}^+_n\big)\delta\tau = \big(\mathrm{Id} \otimes \mathrm{Id} \otimes \mathcal{I}^+_n\big)(\mathrm{Id} \otimes \delta)\delta\tau \\ &= \big(\mathrm{Id} \otimes \mathrm{Id} \otimes \mathcal{I}^+_n\big)(\delta^- \otimes \mathrm{Id})\delta\tau = \big(\delta^- \otimes \mathrm{Id}\big)\big(\mathrm{Id} \otimes \mathcal{I}^+_n\big)\delta\tau \\ &= \big(\delta^- \otimes \mathrm{Id}\big)\delta^+(\mathcal{I}^+_n\tau).\end{aligned}$$

The counit part of (5.5) and (5.8) are left to readers. The condition (5.6) follows from (5.4) and the definition (5.14). The (+)-labelled version of (5.7) is checked for $\mathcal{I}^+_n\tau \in \mathcal{B}^+$ as follows.

$$\begin{aligned}\mathcal{M}^{(13)}\big(\delta^+ \otimes \delta^+\big)\Delta^+(\mathcal{I}^+_n\tau) &= \mathcal{M}^{(13)}\left((\delta^+\mathcal{I}^+_n \otimes \delta^+)\Delta\tau + \sum_{\ell \in \mathbb{N}\times\mathbb{N}^d} \delta^+\frac{X^\ell_+}{\ell} \otimes \delta^+(\mathcal{I}^+_{n+\ell}\tau)\right) \\ &= \mathcal{M}^{(13)}\left(((\mathrm{Id} \otimes \mathcal{I}^+_n)\delta \otimes \delta^+)\Delta\tau + \sum_{\ell \in \mathbb{N}\times\mathbb{N}^d} \mathbf{1}_- \otimes \frac{X^\ell_+}{\ell} \otimes (\mathrm{Id} \otimes \mathcal{I}^+_{n+\ell})\delta\tau\right) \\ &= \big(\mathrm{Id} \otimes \mathcal{I}^+_n \otimes \mathrm{Id}\big)\mathcal{M}^{(13)}(\delta \otimes \delta^+)\Delta\tau + \sum_{\ell \in \mathbb{N}\times\mathbb{N}^d, \varphi \trianglelefteq \tau} \varphi \otimes \frac{X^\ell_+}{\ell!} \otimes \mathcal{I}^+_{n+\ell}(\tau/^-\varphi) \\ &= \big(\mathrm{Id} \otimes \mathcal{I}^+_n \otimes \mathrm{Id}\big)(\mathrm{Id} \otimes \Delta)\delta\tau + \sum_{\ell \in \mathbb{N}\times\mathbb{N}^d, \varphi \trianglelefteq \tau} \varphi \otimes \frac{X^\ell_+}{\ell!} \otimes \mathcal{I}^+_{n+\ell}(\tau/^-\varphi),\end{aligned}$$

and

$$\begin{aligned}\big(\mathrm{Id} \otimes \Delta^+\big)\delta^+(\mathcal{I}^+_n\tau) &= (\mathrm{Id} \otimes \Delta^+\mathcal{I}^+_n)\delta\tau \\ &= \sum_{\varphi \trianglelefteq \tau} \varphi \otimes \left((\mathcal{I}^+_n \otimes \mathrm{Id})\Delta(\tau/^-\varphi) + \sum_{\ell \in \mathbb{N}\times\mathbb{N}^d} \frac{X^\ell_+}{\ell!} \otimes \mathcal{I}^+_{n+\ell}(\tau/^-\varphi)\right),\end{aligned}$$

hence we have

$$\mathcal{M}^{(13)}\big(\delta^+ \otimes \delta^+\big)\Delta^+(\mathcal{I}^+_n\tau) = \big(\mathrm{Id} \otimes \Delta^+\big)\,\delta^+(\mathcal{I}^+_n\tau).$$

▷

6 – Multi-pre-Lie structure and renormalized equations

We now turn to the application of the results of the preceding sections to the study of singular stochastic PDEs. We will concentrate in this section on the study of the generalized (KPZ) equation

$$\begin{aligned}\big(\partial_{x_0} - \Delta_{x'} + 1\big)u &= f(u)\zeta + g_2(u)(\partial_{x'}u)^2 + g_1(u)(\partial_{x'}u) + g_0(u) \\ &= f(u)\zeta + g(u, \partial_{x'}u),\end{aligned} \tag{6.1}$$

with a given initial condition. It already involves the main difficulties of the most general situation, with the advantage of leaving aside a number of purely technical and notational matters compared to the most general situation. We saw in Section 4 that a modelled distribution

$$\boldsymbol{u} = \sum_{\tau\in\mathcal{B}} u_\tau \tau \in \mathcal{D}^{\gamma,\eta}_{(0,t_0)}(T, \mathsf{g}),$$

with $\gamma \in (-\beta_0, 2)$ and $\eta \in (0, \beta_0 + 2]$, is a solution to the lift in the regularity structure $\mathscr{T}$ of equation (6.1) if, and only if, it satisfies the fixed point problem

$$\begin{aligned} \boldsymbol{u} &\simeq \mathcal{I}\Big(f(\boldsymbol{u})\Xi + g_2(\boldsymbol{u})(\partial\boldsymbol{u})^2 + g_1(\boldsymbol{u})\partial\boldsymbol{u} + g_0(\boldsymbol{u})\Big) \\ &\simeq \frac{f^{(k)}(u)}{k!}\, u_{\tau_1}\cdots u_{\tau_k}\, \mathcal{I}\big(\tau_1\cdots\tau_k\Xi\big) + \frac{g_2^{(k)}(u)}{k!}\, u_{\tau_1}\cdots u_{\tau_k} u_{\sigma_1} u_{\sigma_2} \mathcal{I}\big(\tau_1\cdots\tau_k\partial\sigma_1\partial\sigma_2\big) \\ &\quad + \frac{g_1^{(k)}(u)}{k!}\, u_{\tau_1}\cdots u_{\tau_k} u_{\sigma_1}\, \mathcal{I}\big(\tau_1\cdots\tau_k\partial\sigma_1\big) + \frac{g_0^{(k)}(u)}{k!}\, u_{\tau_1}\cdots u_{\tau_k}\, \mathcal{I}\big(\tau_1\cdots\tau_k\big), \end{aligned} \tag{6.2}$$

up to model-dependent non-trivial polynomial components, with τ_i, σ_j monomials or trees, and implicit sums. We see on this identity that T needs at least to be stable by the operations

$$\big(\tau_1,\dots,\tau_k,\sigma_1,\sigma_2\big) \mapsto \mathcal{I}\big(\tau_1\cdots\tau_k\big),\ \mathcal{I}\big(\tau_1\cdots\tau_k\Xi\big),\ \mathcal{I}\big(\tau_1\cdots\tau_k\partial\sigma_1\big),\ \mathcal{I}\big(\tau_1\cdots\tau_k\partial\sigma_1\partial\sigma_2\big);$$

this naturally endows the elements of T with a tree/inductive structure. This fact is common to all the equations that can be treated by the methods of regularity structures. This leads us to set the framework of *rooted decorated trees* as a convenient encoding of the elements of T in Section 6.1. The importance of this algebraic setting comes from the fact that the vector space V spanned by the set of all rooted trees with vertex and edge decorations in given sets happens to be a universal object in a class of algebraic structures called multi-pre-Lie algebras. Morphisms of such multi-pre-Lie algebras defined on V are thus determined by their restrictions to a set of generators. We show in Section 6.2 that the modelled distribution solution of the regularity structure lift of equation (6.1) involves precisely such a morphism, with values in the space of vector fields; see Proposition 33.

The regularity structure associated with equation (6.1) is built from V, with T and T^+ subsets of V. The canonical $\mathbf{K}$-admissible model $\mathsf{M}^\varepsilon = \big(\Pi^\varepsilon, \mathsf{g}^\varepsilon\big)$ associated with a regularized noise ζ_ε is defined from a naive interpretation of each decorated tree τ in $\Pi\tau$. The model takes values in the space of smooth functions and one shows in Proposition 24 of Section 6.2 that u is a solution to equation (6.1) with ζ_ε in the role of ζ, over a time interval $(0, t_0)$, if, and only if, u is the M^ε-reconstruction of $\boldsymbol{u} \in \mathcal{D}^{\gamma,\eta}_{(0,t_0)}(T, \mathsf{g}^\varepsilon)$, solution of the regularity structure lift of (6.1) associated with the model M^ε. (Note, en passant, that all the model-dependent terms in the lifted equation are in the polynomial part of the equation; we do not see them on equation (6.2).)

Building $\mathscr{T}$ within V, any renormalization structure $\mathscr{U}$ compatible with $\mathscr{T}$ and satisfying assumption **(C)** will also be built within V, with U and U^- subsets of V. We saw in Section 5 how to construct a family of $\mathbf{K}$-admissible models associated with elements k of the renormalization group G^-_{ad}, from a single $\mathbf{K}$-admissible model, $\mathsf{M}^\varepsilon = (\Pi^\varepsilon, \mathsf{g}^\varepsilon)$ for instance. It is not clear however that the ${}^k\mathsf{M}^\varepsilon$-reconstruction of the solution $\boldsymbol{u} \in \mathcal{D}^{\gamma,\eta}_{(0,t_0)}(T, {}^k\mathsf{g}^\varepsilon)$ to the ${}^k\mathsf{M}^\varepsilon$-dependent regularity structure lift of equation (6.1) is the solution of a PDE driven by ζ_ε. Theorem 40 shows that this is indeed the case. This is the main result of this section, first proved in the seminal work [13] of Bruned, Chandra, Chevyrev and Hairer, as everything else in this section. The proof builds on the fact that the dual renormalization maps $\widetilde{k}^*$ that one can associate to any $k \in G^-_{\mathrm{ad}}$ happens to be multi-pre-Lie morphisms under a compatibility condition on the multi-pre-Lie structure and the renormalization operator δ, found here under the form of Assumption **(D2)**.

The assumptions **(D1-D3)** to be found in this section are all met in the case of a general subcritical system of singular stochastic PDEs, and we verify them by hand in Section 9 where we construct the regularity and renormalization structures associated with the generalized (KPZ) equation. We emphasize them here as ‘assumptions’ to stress the mechanics at work in the most general case.

6.1 Free E-multi-pre-Lie algebra generated by N

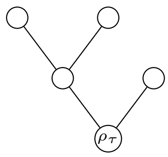
We introduce in the first paragraph the space of edge and node decorated trees. Decoration spaces are associated to any given a system of singular stochastic PDEs and the associated space of decorated trees provides the background scene from which one defines the regularity and renormalization structures associated with the system. The multi-pre-Lie structure of the space of decorated trees is introduced in another paragraph and its dual operator described explicitly.

§ Decorated trees

Definition – *Let $\mathfrak{T}_{\mathrm{n}}$ (called a* node type set*) and $\mathfrak{T}_{\mathrm{e}}$ (called an* edge type set*) be abstract sets.*

- *A* **rooted tree** *τ is a finite connected tree without loops, with a node set N_τ and an edge set E_τ, and with a distinguished node ρ_τ, called the* **root***. The root defines a natural order on each edge, from the root to the leaves. In particular, each edge $e \in E_\tau$ is written as the form $e = (u, v)$, where $u, v \in N_\tau$ are endpoints of e and v is a child of u.*

  We identify two trees τ and σ if they are graph isomorphic, so we always write a graph by putting ancestors lower and descendants upper. The root is put at the bottom. Here is an example.

- *A* **typed rooted tree** *is a rooted tree with type maps $\mathfrak{t}_{\mathrm{n}} : N_\tau \to \mathfrak{T}_{\mathrm{n}}$ and $\mathfrak{t}_{\mathrm{e}} : E_\tau \to \mathfrak{T}_{\mathrm{e}}$. Moreover, a* **rooted decorated tree** *is a typed rooted tree τ with two maps*

  $$\mathfrak{n} : N_\tau \to \mathbb{N}^{d+1}, \quad \mathfrak{e} : E_\tau \to \mathbb{N}^{d+1}.$$

  We denote a generic typed rooted tree by Greek letters like τ, and a generic rooted decorated trees with two decorations $\mathfrak{n}, \mathfrak{e}$ by $\tau_{\mathfrak{e}}^{\mathfrak{n}}$ or a bold letter $\boldsymbol{\tau}$.

We will consider later rooted trees τ equipped with three decorations $\mathfrak{n}, \mathfrak{o}, \mathfrak{e}$ – see Section 9.1 for the precise definitions. In this section, we hide the $\mathfrak{o}$-decoration in the node type map, so we consider the type sets

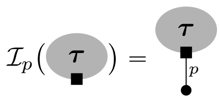
$$\mathfrak{T}_{\mathrm{n}} = \{\bullet, \circ\} \cup \{{\color{red}\bullet}^{\cdot,\alpha}\}_{\alpha\in\mathbb{R}}.$$

The node type $\bullet$ represents the monomial $\mathbf{1} = X^0$, and $\circ$ represents the noise Ξ. The third node type ${\color{red}\bullet}^{\cdot,\alpha}$ is a node with the $\mathfrak{o}$-decoration α. The set $\mathfrak{T}_{\mathrm{e}}$ labels the set of differential operators involved in the system of equations under study. There is a single operator $\partial_{x_0} - \Delta_{x'}$ in the example of the generalized (KPZ) equation (6.1), so the set $\mathfrak{T}_{\mathrm{e}}$ consists of only one element, associated with the integration operator $\mathcal{I}$ in that case. Would we consider a system of singular stochastic PDEs involving different operators, different operators $\mathcal{I}$'s would be associated with each of them and the set $\mathfrak{T}_{\mathrm{e}}$ would collect them all.

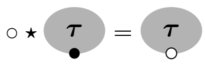
An element $X^n \in T$ is denoted by $\bullet^n$, that is a graph with only node with the type $\bullet$ and the $\mathfrak{n}$-decoration $n \in \mathbb{N}^{d+1}$. An edge with $\mathfrak{e}$-decoration $p \in \mathbb{N}^{d+1}$ represents the operator $\mathcal{I}_p$, with the notations of Section 3.2, for one of the operators $\mathcal{I}$ involved in the equation.

All operations appearing in the equation (6.2) are graphically defined as follows. In the following pictures, types and decorations are omitted unless necessary, and the root is denoted by a square.

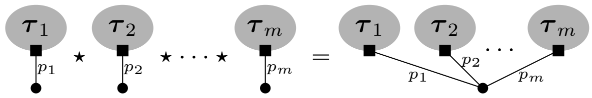
- The integration $\boldsymbol{\tau} \mapsto \mathcal{I}_p(\boldsymbol{\tau})$ is given by the map connecting the root of $\boldsymbol{\tau}$ with a new node, which becomes a root of the tree $\mathcal{I}_p(\boldsymbol{\tau})$, and giving the $\mathfrak{e}$-decoration $p \in \mathbb{N}^{d+1}$ to the connecting edge.

- The product $\mathfrak{t}^n \star \boldsymbol{\tau}$ for $\boldsymbol{\tau}$ with $\mathfrak{t}_{\mathrm{n}}(\rho_\tau) = \bullet$ is given changing the node type of ρ_τ to $\mathfrak{t}^n$. For example,

- The product of trees $\mathcal{I}_{p_j}(\boldsymbol{\tau}_j)$ $(j = 1, \ldots, m)$ is given by the *tree product*, that is joining their roots.



Thus we see that the rooted trees obtained by the above operations are sufficient to describe the fixed point problem (6.2). The symbol $\bullet^{\cdot,\alpha}$ does not come from the fixed point problem (6.2), but its use is made clear in Section 6.3. As we concentrate in this section on the generalized (KPZ) equation the edge type set $\mathfrak{T}_{\mathrm{e}}$ will consist of a single element, suggestively denoted by $\mathcal{I}$. There is no difficulty in working with a finite edge type set.

Definition – *Let $\mathcal{V}$ be the set of* all *rooted decorated trees with type sets $\mathfrak{T}_{\mathrm{n}} = \{\bullet, \circ\} \cup \{\bullet^{\cdot,\alpha}\}_{\alpha\in\mathbb{R}}$ and $\mathfrak{T}_{\mathrm{e}} = \{\mathcal{I}\}$, and let V be the vector space spanned by $\mathcal{V}$. Moreover, denote by $\big(\boldsymbol{\tau}^* : V \to \mathbb{R}\big)_{\boldsymbol{\tau}\in\mathcal{V}}$ the dual basis of $\mathcal{V}$ and let V^* be the vector space spanned by $\{\boldsymbol{\tau}^*\}_{\boldsymbol{\tau}\in\mathcal{V}}$.*

Throughout this section, we view each element of $\mathcal{V}$ as the rooted tree τ with the composite decorations $(\mathfrak{t}_{\mathrm{n}}, \mathfrak{n}) : N_\tau \to \mathsf{N}$ and $(\mathfrak{t}_{\mathrm{e}}, \mathfrak{e}) : E_\tau \to \mathsf{E}$, where

$$\mathsf{E} := \mathfrak{T}_{\mathrm{e}} \times \mathbb{N}^{d+1} \simeq \mathbb{N}^{d+1}, \quad \mathsf{N} := \mathfrak{T}_{\mathrm{n}} \times \mathbb{N}^{d+1}.$$

The set N is considered as a subset of $\mathcal{V}$ consisting of simple trees

$$\mathsf{N} = \{\mathfrak{t}^n\}_{\mathfrak{t}\in\mathfrak{T}_{\mathrm{n}}, n\in\mathbb{N}^{d+1}} = \big\{\circ^\ell, \bullet^m, \bullet^{n,\alpha}\big\}_{\ell,m,n\in\mathbb{N}^{d+1},\alpha\in\mathbb{R}}.$$

Write $\mathsf{N}^0 := \{\mathfrak{t}^0\}_{\mathfrak{t}\in\mathfrak{T}_{\mathrm{n}}} \simeq \mathfrak{T}_{\mathrm{n}}$. We introduce a few notations. Note that each $\boldsymbol{\tau} \in \mathcal{V}$ has a decomposition of the form

$$\boldsymbol{\tau} = \mathfrak{t}^n \star \mathop{\bigstar}_{i=1}^{a} \mathcal{I}_{p_i}(\boldsymbol{\tau}_i) = \mathfrak{t}^n \star \mathcal{I}_{p_1}(\boldsymbol{\tau}_1) \star \cdots \star \mathcal{I}_{p_a}(\boldsymbol{\tau}_a) \tag{6.3}$$

with $\mathfrak{t}^n \in \mathsf{N}$, $p_1, \ldots, p_a \in \mathbb{N}^{d+1}$, and $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_a \in \mathcal{V}$. Taking care of the number of automorphisms of $\boldsymbol{\tau}$ that live it fixed, for $\boldsymbol{\tau}$ of the form

$$\boldsymbol{\tau} = \mathfrak{t}^n \star \mathop{\bigstar}_{j=1}^{b} \big(\mathcal{I}_{q_j}(\boldsymbol{\sigma}_j)\big)^{\star m_j}, \tag{6.4}$$

with $(q_i, \boldsymbol{\sigma}_i) \neq (q_j, \boldsymbol{\sigma}_j)$ for any $i \neq j$, define inductively

$$S(\boldsymbol{\tau}) := n! \prod_{j=1}^{b} S(\boldsymbol{\sigma}_j)^{m_j}\, m_j!.$$

Then we define the paring $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ between V and V^* by

$$\langle\!\langle \boldsymbol{\tau}, \boldsymbol{\sigma}^* \rangle\!\rangle := S(\boldsymbol{\sigma})\, \boldsymbol{\sigma}^*(\boldsymbol{\tau}) \tag{6.5}$$

for $\boldsymbol{\tau}, \boldsymbol{\sigma} \in \mathcal{V}$. We see V^* as a part of the algebraic dual of V. (As V is infinite-dimensional V^* is not equal to the full algebraic dual of V.) The 'copy' space V^* will play an important role in the second half part of this section.

§ **Canonical model**

Given a *smooth* noise $\zeta \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^d)$, we define the canonical operator $\mathsf{\Pi}^\zeta$ on the whole of V requiring that it is multiplicative and setting, for all $z = (t, x) \in \mathbb{R} \times \mathbb{R}^d$,

$$\mathsf{\Pi}^\zeta(\circ^n)(z) = z^n \zeta(z), \quad \mathsf{\Pi}^\zeta(\bullet^n)(z) = \mathsf{\Pi}^\zeta(\bullet^{n,\alpha})(z) = z^n,$$

and

$$(\mathsf{\Pi}^\zeta)(\mathcal{I}_p \boldsymbol{\tau}) = \partial^p \mathbf{K}(\mathsf{\Pi}^\zeta \boldsymbol{\tau}),$$

for all $n, \alpha, \boldsymbol{\tau}, p$. The regularity structures we will work with have spaces T and T^+ that are subsets of V. Since all functions $\mathsf{\Pi}^\zeta \boldsymbol{\tau}$ are smooth, the restriction of $\mathsf{\Pi}^\zeta$ to $T_{<0}$ defines the **canonical model**

$$\mathsf{M}^\zeta = \big(\mathsf{\Pi}^\zeta, \mathsf{g}^\zeta\big)$$

on the regularity structure $\mathscr{T}$, from Theorem 19. Things are explicit here as the multiplicativity and the $\mathbf{K}$-admissibility properties fix the definition of Π^{ζ} on all decorated trees in V. Emphasize the fact that since the map Π^{ζ} is multiplicative its associated reconstruction map is also multiplicative.

§ Multi-pre-Lie algebras

We first recall the definition of a multi-pre-Lie algebra, refering the reader to Foissy's article [31] for basics on multi-pre-Lie algebras. All we need to know on the subject is the following definition and the result of Proposition 30 below.

Definition – *Let* E *be a set. A vector space* W*, equiped with a family* $(\rhd_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}}$ *of bilinear maps from* $W\times W$ *into* W*, is called an* E-**multi-pre-Lie algebra** *if one has*

$$(a \rhd_{\mathsf{e}} b) \rhd_{\mathsf{e}'} c - a \rhd_{\mathsf{e}} (b \rhd_{\mathsf{e}'} c) = (b \rhd_{\mathsf{e}'} a) \rhd_{\mathsf{e}} c - b \rhd_{\mathsf{e}'} (a \rhd_{\mathsf{e}} c),$$

for all $a,b,c\in W$*, and* $\mathsf{e},\mathsf{e}'\in\mathsf{E}$.

The two arguments of a pre-Lie product $a \rhd_{\mathsf{e}} b$ do not play a symmetric role, and we think here of a as acting on b via the operator $\rhd_{\mathsf{e}}$; we read $a \rhd_{\mathsf{e}} b$ from left to right. Here is an example of E-multi-pre-Lie algebra. Take E finite, identified with $\{1,\dots,|\mathsf{E}|\}$, and consider the space of smooth functions on $\mathbb{R}^{|\mathsf{E}|}$. Then the family of differentiation operators

$$G \rhd_{\mathsf{e}} H := G\partial_{x_{\mathsf{e}}} H$$

defines an E-multi-pre-Lie algebra. If E consists of a single element $\rhd$, this operator is called a *pre-Lie product*, and a vector space equipped with a pre-Lie product is called a *pre-Lie algebra.* Any pre-Lie algebra is Lie-admissible, in the sense that the map $(a,b)\mapsto a\rhd b - b\rhd a$ defines a Lie bracket. The relevance of the multi-pre-Lie structure in the study of singular stochastic PDEs comes from Proposition 33 in the next section, as it identifies the components u_τ of solutions $\boldsymbol{u} = \sum u_\tau \tau$ regularity structures lifts of a singular stochastic PDEs as E-multi-pre-Lie algebra morphisms. The next statement is fundamental, and can be proved as Corollary 9 in Foissy's work [31] – it was first proved in Proposition 4.21 of Bruned, Chandra, Chevyrev and Hairer's work [13]. A proof can be found in Appendix C.2.

Proposition 29. *The space* V^* *with the operators* $\{\overset{\mathsf{e}}{\curvearrowright}\}_{\mathsf{e}\in\mathsf{E}}$ *is the free* E*-multi-pre-Lie algebra generated by* N.

Any morphism from V^* into an E-multi-pre-Lie algebra is thus determined by its restriction to the generators N of V^*. This is the universal property of the free E-multi-pre-Lie algebra with generators N. In particular, if two E-multi-pre-Lie morphisms from V^* into another E-multi-pre-Lie algebra coincide on the generators of V^* then they are equal.

We define now the multi-pre-Lie structure in the space V^*. The reason for working on V^* rather than on V will appear clearly in Section 6.2 and Section 6.3. The spaces V and V^* being infinite dimensional, the symbol $\otimes$ denotes below the algebraic tensor product of these spaces with themselves, with no reference to any completion.

Definition – *Given* $\mathsf{e}\in\mathsf{E}$*, a node* v *of a decorated tree* $\boldsymbol{\sigma}\in\mathcal{V}$*, and* $\boldsymbol{\tau}\in\mathcal{V}$*, denote by*

$$\boldsymbol{\tau} \overset{\mathsf{e}}{\rightarrow}_{(v)} \boldsymbol{\sigma},$$

the element of $\mathcal{V}$ *obtained by grafting* $\boldsymbol{\tau}$ *on the node* v *of* $\boldsymbol{\sigma}$*, along an edge of* $\mathfrak{e}$*-decoration* e*. Define also*

$$\boldsymbol{\tau} \overset{\mathsf{e}}{\curvearrowright}_{(v)} \sigma^{\mathfrak{n}}_{\mathfrak{e}} := \sum_{m\in\mathbb{N}^{d+1};\, m\leqslant \mathfrak{n}(v)\wedge\mathsf{e}} \binom{\mathfrak{n}(v)}{m} \boldsymbol{\tau} \xrightarrow{\mathsf{e}-m}_{(v)} \sigma^{\mathfrak{n}-m\mathbf{1}_v}_{\mathfrak{e}} \in V,$$

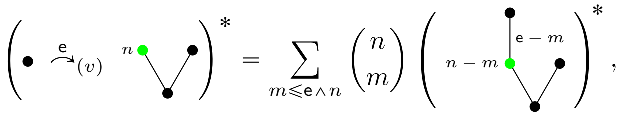
$$\boldsymbol{\tau} \overset{\mathsf{e}}{\curvearrowright} \sigma^{\mathfrak{n}}_{\mathfrak{e}} := \sum_{v\in N_\sigma} \boldsymbol{\tau} \overset{\mathsf{e}}{\curvearrowright}_{(v)} \sigma^{\mathfrak{n}}_{\mathfrak{e}} \in V,$$

where $\mathbf{1}_v$ *is the indicator function of* v*. Finally, define a linear map* $\overset{\mathsf{e}}{\curvearrowright} : V^*\otimes V^* \to V^*$ *by*

$$\boldsymbol{\tau}^* \overset{\mathsf{e}}{\curvearrowright} \boldsymbol{\sigma}^* := (\boldsymbol{\tau} \overset{\mathsf{e}}{\curvearrowright} \boldsymbol{\sigma})^*, \qquad \boldsymbol{\tau},\boldsymbol{\sigma}\in\mathcal{V},$$

where the map $(\cdot)^* : V \to V^*$ *is the linear extension of the map* $\mathcal{V}\ni\boldsymbol{\tau}\mapsto\boldsymbol{\tau}^*\in V$.

Here is an example



where v is colored in green.

The space $T = U$ of the regularity and renormalization structures associated with the generalized (KPZ) equation is a subspace of V with each space $T_\beta, T_\alpha^+, U_{\beta'}, U_{\alpha'}^-$ spanned by finitely many rooted decorated trees. Denote by $\pi_U : V \to U$ the canonical projection. The next assumption essentially means that a piece of a basis element $\tau \in \mathcal{B}$ is also a basis element and that the project map π_U behaves consistently with respect to all the grafting products $\overset{\mathsf{e}}{\curvearrowright}$.

Assumption (D1) – *The homogeneous basis $\mathcal{B}$ of T and U is a subset of $\mathcal{V}$ with the following properties.*

- *If $\boldsymbol{\tau} = \tau_{\mathfrak{e}}^{\mathfrak{n}} \in \mathcal{B}$, then $\tau_{\mathfrak{e}}^{\mathfrak{m}} \in \mathcal{B}$ for any $\mathfrak{m} : N_\tau \to \mathbb{N}^{d+1}$.*
- *If $\boldsymbol{\tau} = \mathfrak{t}^n \star \bigstar_{i=1}^a \mathcal{I}_{p_i}(\boldsymbol{\tau}_i) \in \mathcal{B}$, then $\mathfrak{t}, \boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_a \in \mathcal{B}$.*
- *For any $\boldsymbol{\tau}, \boldsymbol{\sigma} \in \mathcal{V}$ and $\mathsf{e} \in \mathsf{E}$,*

$$\pi_U\big(\boldsymbol{\tau} \overset{\mathsf{e}}{\curvearrowright} (\pi_U \boldsymbol{\sigma})\big) = \pi_U\big((\pi_U \boldsymbol{\tau}) \overset{\mathsf{e}}{\curvearrowright} \boldsymbol{\sigma}\big) = \pi_U\big(\boldsymbol{\tau} \overset{\mathsf{e}}{\curvearrowright} \boldsymbol{\sigma}\big)$$

Set

$$U^* := \mathrm{span}\big\{\boldsymbol{\tau}^*\,;\, \boldsymbol{\tau} \in \mathcal{B}\big\}$$

and denote by $\pi_{U^*} : V^* \to U^*$ be the canonical projection. Then we define the map

$$\overset{\mathsf{e}}{\curvearrowright}_\flat : U^* \otimes U^* \to U^*$$

setting

$$\boldsymbol{\tau}^* \overset{\mathsf{e}}{\curvearrowright}_\flat \boldsymbol{\sigma}^* := \pi_{U^*}\big(\boldsymbol{\tau}^* \overset{\mathsf{e}}{\curvearrowright} \boldsymbol{\sigma}^*\big).$$

The proof of the following statement is proved in Appendix C.2.

Proposition 30. *Under Assumption* **(D1)***, the space U^* with the operators $\{\overset{\mathsf{e}}{\curvearrowright}_\flat\}_{\mathsf{e}\in\mathsf{E}}$ is the E-multi-pre-Lie algebra generated by $\mathsf{N} \cap \mathcal{B}$.*

Finally we define an operator playing the role of 'antiderivative'.

Definition – *For each $i \in \{0, 1, \dots, d\}$, define the linear map $\uparrow_i: V^* \to V^*$ by*

$$\uparrow_i (\tau_{\mathfrak{e}}^{\mathfrak{n}})^* = \sum_{v\in N_\tau} (\tau_{\mathfrak{e}}^{\mathfrak{n}+e_i\mathbf{1}_v})^*.$$

The map $\uparrow_i$ sends U^* into itself under Assumption **(D1)**. We denote by

$$\downarrow_i: V \to V$$

the dual map of $\uparrow_i: V^* \to V^*$ under the pairing (6.5), that is,

$$\langle\!\langle \downarrow_i \boldsymbol{\tau}, \boldsymbol{\sigma}^* \rangle\!\rangle = \langle\!\langle \boldsymbol{\tau}, \uparrow_i \boldsymbol{\sigma}^* \rangle\!\rangle.$$

Moreover, we extend the pairing (6.5) into a pairing between $V \otimes V$ and $V^* \otimes V^*$ setting

$$\langle\!\langle \boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_2, \boldsymbol{\sigma}_1^* \otimes \boldsymbol{\sigma}_2^* \rangle\!\rangle := \langle\!\langle \boldsymbol{\tau}_1, \boldsymbol{\sigma}_1^* \rangle\!\rangle \langle\!\langle \boldsymbol{\tau}_2, \boldsymbol{\sigma}_2^* \rangle\!\rangle.$$

Under such pairings, denote by

$$\upharpoonleft\!\downharpoonright_{\mathsf{e}}: V \to V \otimes V,$$

the dual map of $\overset{\mathsf{e}}{\curvearrowright}: V^* \otimes V^* \to V^*$, that is,

$$\langle\!\langle \upharpoonleft\!\downharpoonright_{\mathsf{e}} \boldsymbol{\eta}, \boldsymbol{\tau}^* \otimes \boldsymbol{\sigma}^* \rangle\!\rangle := \langle\!\langle \boldsymbol{\eta}, \boldsymbol{\tau}^* \curvearrowright_{\mathsf{e}} \boldsymbol{\sigma}^* \rangle\!\rangle, \tag{6.6}$$

for any $\boldsymbol{\tau}, \boldsymbol{\sigma}, \boldsymbol{\eta} \in \mathcal{V}$ and $\mathsf{e} \in \mathsf{E}$. The following explicit formulas for $\uparrow_i$ and $\upharpoonleft\!\downharpoonright_{\mathsf{e}}$ are helpful to get a graphical image. It is used only in the proof of Theorem 44 giving an explicit construction of the regularity and renormalization structures associated with the generalized (KPZ) equation.

Lemma 31. *For any $i \in \{0,1,\dots,d\}$ and any $\boldsymbol{\tau} = \tau_{\mathfrak{e}}^{\mathfrak{n}} \in \mathcal{V}$, one has*

$$\downarrow_i \left(\tau_{\mathfrak{e}}^{\mathfrak{n}}\right) = \sum_{v \in N_\tau,\, e_i \leqslant \mathfrak{n}(v)} \mathfrak{n}(v)\, \tau_{\mathfrak{e}}^{\mathfrak{n} - e_i \mathbf{1}_v}.$$

Moreover, for any $\boldsymbol{\tau} = \tau_{\mathfrak{e}}^{\mathfrak{n}} \in \mathcal{V}$ and any $\mathsf{e} \in \mathsf{E}$, one has

$$\Uparrow_{\mathsf{e}} \left(\tau_{\mathfrak{e}}^{\mathfrak{n}}\right) = \sum_{e=(v,w) \in E_\tau;\, \mathfrak{e}(e) \leqslant \mathsf{e}} \frac{1}{(\mathsf{e} - \mathfrak{e}(e))!} (C_e\tau)_{\mathfrak{e}}^{\mathfrak{n}} \otimes (P_e\tau)_{\mathfrak{e}}^{\mathfrak{n} + (\mathsf{e} - \mathfrak{e}(e))\mathbf{1}_v},$$

where $C_e\tau$ and $P_e\tau$ are the two connected components of the graph $\tau\backslash\{e\}$, with $P_e\tau$ containing the root of τ. The decoration $\mathbf{1}_v$ is an indicator function on N_τ.

Proof – We show that the equation (6.6) holds for the map $\Uparrow_{\mathsf{e}}$ defined by second formula. The first formula is proved by a similar argument. Note that, for any elements $\boldsymbol{\tau} = \mathsf{t}^n \star \bigstar_{i=1}^a \mathcal{I}_{p_i}(\boldsymbol{\tau}_i) \in \mathcal{V}$ and $\boldsymbol{\sigma} = \mathsf{u}^m \star \bigstar_{j=1}^b \mathcal{I}_{q_j}(\boldsymbol{\sigma}_j) \in \mathcal{V}$, one has

$$\langle\!\langle \boldsymbol{\tau}, \boldsymbol{\sigma}^* \rangle\!\rangle = \mathbf{1}_{\mathsf{t}=\mathsf{u}, n=m, a=b}\, n! \sum_{s \in S_a} \prod_{i=1}^a \mathbf{1}_{p_i = q_{s(i)}} \langle\!\langle \boldsymbol{\tau}_i, \boldsymbol{\sigma}_{s(i)}^* \rangle\!\rangle, \tag{6.7}$$

where S_a is the symmetric group of the set $\{1,2,\dots,a\}$. For $\boldsymbol{\eta}$ of the form $\boldsymbol{\eta} = \mathsf{t}^n \star \bigstar_{i=1}^a \mathcal{I}_{p_i}(\boldsymbol{\eta}_i)$, by definition one has

$$\begin{aligned}\Uparrow_{\mathsf{e}} \boldsymbol{\eta} &= \sum_i \frac{1}{(\mathsf{e} - p_i)!} \boldsymbol{\eta}_i \otimes \mathsf{t}^{n + \mathsf{e} - p_i} \star \bigstar_{j;j\neq i} \mathcal{I}_{p_j}(\boldsymbol{\eta}_j) + \sum_i \sum \boldsymbol{\eta}_i^1 \otimes \mathsf{t}^n \star \mathcal{I}_{p_i}(\boldsymbol{\eta}_i^2) \star \bigstar_{j;j\neq i} \mathcal{I}_{p_j}(\boldsymbol{\eta}_j) \\ &=: \Uparrow_{\mathsf{e}}^1 \boldsymbol{\eta} + \Uparrow_{\mathsf{e}}^2 \boldsymbol{\eta},\end{aligned}$$

where $\Uparrow_{\mathsf{e}} \boldsymbol{\eta}_i = \sum \boldsymbol{\eta}_i^1 \otimes \boldsymbol{\eta}_i^2$ in the first equality. Hence it is sufficient to show that

$$\langle\!\langle \boldsymbol{\eta}, \boldsymbol{\tau}^* \overset{\mathsf{e}}{\curvearrowright}_1 \boldsymbol{\sigma}^* \rangle\!\rangle = \langle\!\langle \Uparrow_{\mathsf{e}}^1 \boldsymbol{\eta}, \boldsymbol{\tau}^* \otimes \boldsymbol{\sigma}^* \rangle\!\rangle, \tag{6.8}$$

$$\langle\!\langle \boldsymbol{\eta}, \boldsymbol{\tau}^* \overset{\mathsf{e}}{\curvearrowright}_2 \boldsymbol{\sigma}^* \rangle\!\rangle = \langle\!\langle \Uparrow_{\mathsf{e}}^2 \boldsymbol{\eta}, \boldsymbol{\tau}^* \otimes \boldsymbol{\sigma}^* \rangle\!\rangle. \tag{6.9}$$

It is not difficult to show (6.8) directly from (6.7). For (6.9), it is sufficient to consider $\boldsymbol{\sigma} = \mathsf{t}^n \star \bigstar_{i=1}^a \mathcal{I}_{q_i}(\boldsymbol{\sigma}_i)$, and for such $\boldsymbol{\sigma}$ one has

$$\langle\!\langle \boldsymbol{\eta}, \boldsymbol{\tau}^* \overset{\mathsf{e}}{\curvearrowright}_2 \boldsymbol{\sigma}^* \rangle\!\rangle = n! \sum_{i=1}^d \sum_{s \in S_a} \mathbf{1}_{q_{s(i)} = p_i} \langle\!\langle \boldsymbol{\eta}_i, \boldsymbol{\tau}^* \overset{\mathsf{e}}{\curvearrowright} \boldsymbol{\sigma}_{s(i)}^* \rangle\!\rangle \prod_{j;j\neq i} \mathbf{1}_{q_{s(j)} = p_j} \langle\!\langle \boldsymbol{\eta}_j, \boldsymbol{\sigma}_{s(j)}^* \rangle\!\rangle$$

and

$$\langle\!\langle \Uparrow_{\mathsf{e}}^2 \boldsymbol{\eta}, \boldsymbol{\tau}^* \otimes \boldsymbol{\sigma}^* \rangle\!\rangle = n! \sum_{i=1}^d \sum_{s \in S_I} \langle\!\langle \boldsymbol{\eta}_i^1, \boldsymbol{\tau}^* \rangle\!\rangle \mathbf{1}_{q_{s(i)} = p_i} \langle\!\langle \boldsymbol{\eta}_i^2, \boldsymbol{\sigma}_{s(i)}^* \rangle\!\rangle \prod_{j;j\neq i} \mathbf{1}_{q_{s(j)} = p_j} \langle\!\langle \boldsymbol{\eta}_j, \boldsymbol{\sigma}_{s(j)}^* \rangle\!\rangle.$$

Since

$$\sum \langle\!\langle \boldsymbol{\eta}_i^1, \boldsymbol{\tau}^* \rangle\!\rangle \langle\!\langle \boldsymbol{\eta}_i^2, \boldsymbol{\sigma}_{s(i)}^* \rangle\!\rangle = \langle\!\langle \Uparrow_{\mathsf{e}} \boldsymbol{\eta}, \boldsymbol{\tau}^* \otimes \boldsymbol{\sigma}_{s(i)}^* \rangle\!\rangle,$$

the identity (6.9) follows if (6.6) holds for $\boldsymbol{\sigma} = \boldsymbol{\sigma}_i$, which leads an induction on the number of edges contained in $\boldsymbol{\sigma}$. The case $\boldsymbol{\sigma} = \mathsf{t}^n \in \mathsf{N}$ is an easy exercise. ▷

6.2 Modelled distributions solutions of singular PDEs

The approximate description (6.2) of the fixed point problem (4.12) leads to an explicit formula for the coefficients of the solution $\boldsymbol{u}$. Noting that $\gamma < 2$, the solution $\boldsymbol{u}$ of (6.2) is of the form

$$\boldsymbol{u} = u_{\mathbf{1}} \mathbf{1} + \sum_{i=1}^d u_{X_i} X_i + \sum_{\boldsymbol{\tau} \in \mathcal{B} \cap T_{<0}} u_{\mathcal{I}(\boldsymbol{\tau})}\, \mathcal{I}(\boldsymbol{\tau}). \tag{6.10}$$

Inserting such an expansion into (6.2), we see that all coefficients $u_{\mathcal{I}(\boldsymbol{\tau})}$ are functions of

$$u_0 := u_{\mathbf{1}} \in \mathbb{R}, \quad \text{and} \quad u_1 := (u_{X_i})_{i=1}^d \in \mathbb{R}^d.$$

Define a derivation D on functions G of (u_0, u_1) setting

$$D_0 G := 0, \quad D_i G := u_{X_i} \partial_{u_0} G \qquad (1 \leqslant i \leqslant d),$$

and $D^n := \prod_{i=0}^d D_i^{n_i}$ for $n = (n_i)_{i=0}^d \in \mathbb{N}^{d+1}$.

Definition – *Define the linear map F from V^* to the space of functions of (u_0, u_1) as follows. For the primitive trees in N^0 set*

$$\begin{aligned} F(\circ^*) &:= f(u_0), \\ F(\bullet^*) &:= g(u_0,u_1) := g_2(u_0)(u_1)^2 + g_1(u_0)u_1 + g_0(u), \\ F\big((\bullet^{0,\alpha})^*\big) &:= 0. \end{aligned} \tag{6.11}$$

For a generic tree

$$\boldsymbol{\tau} = \mathrm{t}^n \star \bigstar_{i=1}^{a} \mathcal{I}_{p_i}(\boldsymbol{\tau}_i),$$

define inductively

$$F(\boldsymbol{\tau}^*)(u_0,u_1) := \left\{ \prod_{i=1}^{a} F(\boldsymbol{\tau}_i^*)(u_0,u_1) \right\} \left\{ D^n \prod_{i=1}^{a} \partial_{u_{p_i}} \right\} F\big((\mathrm{t}^0)^*\big)(u_0,u_1), \tag{6.12}$$

where ∂_{u_p} denotes the derivative with respect to the X^p-component u_{X^p} of $\boldsymbol{u}$ – hence $\partial_{u_p} := 0$ if $|p|_{\mathfrak{s}} \geqslant 2$.

Lemma 32. *If the modelled distribution $\boldsymbol{u}$ of the form* (6.10) *solves the fixed point problem* (4.12)*, then one has*

$$u_{\mathcal{I}(\boldsymbol{\tau})} = \frac{1}{S(\boldsymbol{\tau})} F(\boldsymbol{\tau}^*)(u_0,u_1). \tag{6.13}$$

Proof – We consider here

$$\boldsymbol{\tau} = \circ^n \star \bigstar_{j=1}^{b} \big(\mathcal{I}(\boldsymbol{\sigma}_j)\big)^{\star m_j};$$

the other cases are proved by similar arguments. The element $\boldsymbol{\tau}$ appears in the term $f^\star(\boldsymbol{u}) \star Z$ in the right hand side of (4.12). Inserting the expansion (6.10) into $f^\star(\boldsymbol{u}) \star Z$, its $\boldsymbol{\tau}$-component is calculated as

$$\begin{aligned} \sum_{\substack{\boldsymbol{\tau}_1,\dots,\boldsymbol{\tau}_a, \\ \circ^n \star \bigstar_{j=1}^{b} (\mathcal{I}(\boldsymbol{\sigma}_j))^{\star m_j} = \bigstar_{i=1}^{a} \boldsymbol{\tau}_i}} \frac{f^{(a)}(u_0)}{a!} u_{\boldsymbol{\tau}_1} \dots u_{\boldsymbol{\tau}_a} &= \frac{f^{(|n|+m_1+\dots+m_b)}(u_0)}{n!\, m_1! \cdots m_b!} (u_1)^n u_{\mathcal{I}(\boldsymbol{\sigma}_1)}^{m_1} \cdots u_{\mathcal{I}(\boldsymbol{\sigma}_b)}^{m_b} \\ &= \frac{1}{n!\, m_1! \cdots m_b!} u_{\mathcal{I}(\boldsymbol{\sigma}_1)}^{m_1} \cdots u_{\mathcal{I}(\boldsymbol{\sigma}_b)}^{m_b} D^n \partial_{u_0}^{m_1+\dots+m_b} F(\circ^*). \end{aligned}$$

This should be equal to $u_{\boldsymbol{\tau}}$ from identity (4.12). Assuming $u_{\mathcal{I}(\boldsymbol{\sigma}_j)} = F(\boldsymbol{\sigma}_j^*)/S(\boldsymbol{\sigma}_j)$ inductively, we have

$$u_{\mathcal{I}(\boldsymbol{\tau})} = \frac{1}{S(\boldsymbol{\tau})} F(\boldsymbol{\tau}^*).$$

$\rhd$

Modelled distributions satisfying identity (6.13) are called '*coherent*' in [13].

Recall $\mathsf{E} \simeq \mathbb{N}^{d+1}$, and define the family of differential operators

$$G \rhd_{\mathsf{e}} H := G\, \partial_{u_{\mathsf{e}}} H \qquad (\mathsf{e} \in \mathsf{E}),$$

acting on smooth functions of (u_0, u_1), with $u_1 = (u_{X_i})_{i=1}^d$. The family $\{\rhd_{\mathsf{e}}\}_{\mathsf{e}\in\mathsf{E}}$ defines an E-multi-pre-Lie algebra structure.

Proposition 33. *The map F is an E-multi-pre-Lie algebra morphism: For any $\mathsf{e} \in \mathsf{E}$ and any decorated trees $\boldsymbol{\tau}, \boldsymbol{\sigma}$ in $\mathcal{B}$, one has*

$$F\big(\boldsymbol{\tau}^* \overset{\mathsf{e}}{\curvearrowright} \boldsymbol{\sigma}^*\big) = F(\boldsymbol{\tau}^*) \rhd_{\mathsf{e}} F(\boldsymbol{\sigma}^*). \tag{6.14}$$

Proof – Assume $\boldsymbol{\sigma}$ is of the form

$$\boldsymbol{\sigma} = \mathrm{t}^n \star \bigstar_{i=1}^{a} \mathcal{I}_{p_i}(\boldsymbol{\sigma}_i).$$

Then by definition,

$$\boldsymbol{\tau} \overset{\mathsf{e}}{\curvearrowright} \boldsymbol{\sigma} = \sum_{\ell} \binom{n}{\ell} \mathrm{t}^{n-\ell} \star \mathcal{I}_{\mathsf{e}-\ell}(\boldsymbol{\tau}) \star \bigstar_{i=1}^{a} \mathcal{I}_{p_i}(\boldsymbol{\sigma}_i) + \sum_{i=1}^{a} \mathrm{t}^{n} \star \mathcal{I}_{p_i}\big(\boldsymbol{\tau} \overset{\mathsf{e}}{\curvearrowright} \boldsymbol{\sigma}_i\big) \star \bigstar_{j;j\neq i} \mathcal{I}_{p_j}(\boldsymbol{\sigma}_j)$$
$$=: \boldsymbol{\tau} \overset{\mathsf{e}}{\curvearrowright}_1 \boldsymbol{\sigma} + \boldsymbol{\tau} \overset{\mathsf{e}}{\curvearrowright}_2 \boldsymbol{\sigma}.$$

Hence

$$F\big(\boldsymbol{\tau}^* \overset{\mathsf{e}}{\curvearrowright} \boldsymbol{\sigma}^*\big) = F\big(\boldsymbol{\tau}^* \overset{\mathsf{e}}{\curvearrowright}_1 \boldsymbol{\sigma}^*\big) + F\big(\boldsymbol{\tau}^* \overset{\mathsf{e}}{\curvearrowright}_2 \boldsymbol{\sigma}^*\big),$$

where $\boldsymbol{\tau}^* \overset{\mathsf{e}}{\curvearrowright}_i \boldsymbol{\sigma}^* = \big(\boldsymbol{\tau} \overset{\mathsf{e}}{\curvearrowright}_i \boldsymbol{\sigma}\big)^*$. On the other hand, by Leibniz rule,

$$F(\boldsymbol{\tau}^*) \rhd_{\mathsf{e}} F(\boldsymbol{\sigma}^*) = F(\boldsymbol{\tau}^*) \left\{ \prod_{i=1}^{a} F(\boldsymbol{\sigma}_i^*) \right\} \partial_{u_{\mathsf{e}}} D^n \prod_{i=1}^{a} \partial_{u_{p_i}} F(\mathrm{t}^*)$$
$$+ F(\boldsymbol{\tau}^*) \sum_{i=1}^{a} \partial_{u_{\mathsf{e}}} F(\boldsymbol{\sigma}_i^*) \left\{ \prod_{j;j\neq i} F(\boldsymbol{\sigma}_i^*) \right\} D^n \prod_{i=1}^{a} \partial_{u_{p_i}} F(\mathrm{t}^*).$$

It is elementary to show that $\partial_{u_{\mathsf{e}}} D^n F = \sum_\ell \binom{n}{\ell} D^{n-\ell} \partial_{u_{\mathsf{e}-\ell}} F$. Hence the first term of the right hand side coincides with $F(\boldsymbol{\tau}^* \overset{\mathsf{e}}{\curvearrowright}_1 \boldsymbol{\sigma}^*)$. The second term turns out to coincides with $F\big(\boldsymbol{\tau}^* \overset{\mathsf{e}}{\curvearrowright}_2 \boldsymbol{\sigma}^*\big)$ if (6.15) holds for $\boldsymbol{\tau}^*$ and $\boldsymbol{\sigma}_i^*$, which leads an induction on the number of edges contained in $\boldsymbol{\sigma}^*$. ▷

Assumption **(D1)** is a necessary condition for the basis $\mathcal{B}$. The next assumption means that $\mathcal{B}$ is sufficiently large to describe all terms in the right hand side of (6.2).

Assumption (D2) – *The homogeneous basis $\mathcal{B}$ of T and U contains all* **strongly conform** *trees – see Section 9.1 for the definition.*

Assumptions **(D1)** and **(D2)** is jointly called assumption **(D)**. Assumption **(D2)** ensures that $F(\boldsymbol{\tau}) = 0$ for any $\boldsymbol{\tau} \in \mathcal{V} \backslash \mathcal{B}$. Indeed, if $\boldsymbol{\tau}$ is not strongly conform and does not have any node with $\bullet^{\cdot,\alpha}$ decoration, then $\boldsymbol{\tau}$ have an edge $\mathcal{I}_p$ with $|p|_{\mathfrak{s}} \geqslant 2$ or have a node with at least three leaving edges $\mathcal{I}_p$ with $|p|_{\mathfrak{s}} = 1$. Since $F(\bullet^*)$ is at most quadratic with respect to u_1, we have $F(\boldsymbol{\tau}) = 0$. We define

$$\Upsilon := F|_{U^*}.$$

By assumption 2, we can conclude that Υ is E-multi-pre-Lie algebra morphism on the E-multi-pre-Lie algebra $(U^*, \{\overset{\mathsf{e}}{\curvearrowright}_\flat\}_{\mathsf{e}\in\mathsf{E}})$.

Proposition 34. *The map Υ is an E-multi-pre-Lie algebra morphism: For any $\mathsf{e} \in \mathsf{E}$ and any decorated trees $\boldsymbol{\tau}, \boldsymbol{\sigma}$ in $\mathcal{B}$, one has*

$$\Upsilon\big(\boldsymbol{\tau}^* \overset{\mathsf{e}}{\curvearrowright}_\flat \boldsymbol{\sigma}^*\big) = \Upsilon(\boldsymbol{\tau}^*) \rhd_{\mathsf{e}} \Upsilon(\boldsymbol{\sigma}^*). \tag{6.15}$$

Proof – Since $\Upsilon \circ \pi_{U^*} = F \circ \pi_{U^*} = F$,

$$\Upsilon\big(\boldsymbol{\tau}^* \overset{\mathsf{e}}{\curvearrowright}_\flat \boldsymbol{\sigma}^*\big) = F\big(\boldsymbol{\tau}^* \overset{\mathsf{e}}{\curvearrowright} \boldsymbol{\sigma}^*\big) = F(\boldsymbol{\tau}^*) \rhd_{\mathsf{e}} F(\boldsymbol{\sigma}^*) = \Upsilon(\boldsymbol{\tau}^*) \rhd_{\mathsf{e}} \Upsilon(\boldsymbol{\sigma}^*).$$

▷

The next proposition is proved by an induction similar to the induction used in the proof of Proposition 33, noting that D_i satisfies Leibniz rule.

Proposition 35. *For any $i \in \{0, 1, \ldots, d\}$ and $\boldsymbol{\tau} \in \mathcal{V}$, one has*

$$F(\uparrow_i \boldsymbol{\tau}^*) = D_i F(\boldsymbol{\tau}^*).$$

and for $\boldsymbol{\tau} \in \mathcal{B}$,

$$\Upsilon(\uparrow_i \boldsymbol{\tau}^*) = D_i \Upsilon(\boldsymbol{\tau}^*). \tag{6.16}$$

6.3 Renormalization structure over a multi-pre-Lie algebra

We now come to the main result of [13] giving a dynamical meaning to the renormalization operations on models associated with elements $k \in G^-_{\mathrm{ad}}$ of the renormalization group and more generally to elements $k \in G^-$. We keep working on the example of the generalized (KPZ) equation.

In Theorem 44 in Section 9, we show that one can choose U stable under all the splitting maps $\Lsh_{\mathsf{e}}$, that is

$$\Lsh_{\mathsf{e}}(U) \subset U \otimes U \tag{6.17}$$

for any $\mathsf{e} \in \mathsf{E}$. The restricted map

$$\Lsh_{\mathsf{e}}^{\flat} := (\Lsh_{\mathsf{e}})|_U : U \to U \otimes U$$

is then the dual of the map $\overset{\mathsf{e}}{\curvearrowright}_{\flat}$. The following assumption is thus to be understood as a constraint on which renormalization schemes δ can be used.

Assumption (D3) – *For any $\mathsf{e} \in \mathsf{E}$, the space U is stable under $\Lsh_{\mathsf{e}}$, and one has*

$$\big(\mathrm{Id} \otimes \Lsh_{\mathsf{e}}^{\flat}\big)\delta = \mathcal{M}^{(13)}\big(\delta \otimes \delta\big)\, \Lsh_{\mathsf{e}}^{\flat}\,. \tag{6.18}$$

and

$$\delta \circ \downarrow_i = \big(\mathrm{Id} \otimes \downarrow_i\big)\delta \tag{6.19}$$

Identity (6.18) is the E-multi-pre-Lie version of the compatibility condition (5.7) between the splitting map Δ of a regularity structure and a renormalization splitting δ. Recall that any character k of U^- defines a linear map $\widetilde{k} = (k \otimes \mathrm{Id})\delta : U \to U$. Denote by $\widetilde{k}^* : U^* \to U^*$ the dual map of $\widetilde{k}$ under the pairing (6.5). Anticipating over Section 9, say here that ${\color{red}\bullet}^{0,\alpha}$ is used to denote the result of extracting from a decorated tree τ the entire tree, but keeping track of the homogeneity $\alpha = |\tau|$ of the tree that was removed. Using the duality relation defining $\widetilde{k}^*$ and the definition of $\widetilde{k}$ we see that

$$\widetilde{k}^*(\circ^*) = \circ^*, \quad \widetilde{k}^*(\bullet^*) = \bullet^*,$$

and

$$\widetilde{k}^*\big({\color{red}\bullet}^{0,\alpha}\big) = 0$$

for $\alpha > 0$, and

$$\widetilde{k}^*\big({\color{red}\bullet}^{0,\alpha}\big) = \sum_{\tau \in \mathcal{B}, |\tau| = \alpha} \frac{k(\tau)}{S(\tau)}\, \tau^*,$$

for $\alpha < 0$. The following result is part of Proposition 4.18 in Bruned, Chandra, Chevyrev and Hairer's work [13]. It is the reason why we insisted on making a difference between U and U^*, to emphasize the dual action of $\widetilde{k}$.

Proposition 36. *Under the compatibility assumption* **(D3)**, *given any character k on U^-, the map $\widetilde{k}^*$ is an E-multi-pre-Lie morphism: For any edge type $\mathsf{e} \in \mathsf{E}$, and any $\boldsymbol{\tau}, \boldsymbol{\sigma} \in \mathcal{B}$, one has*

$$\widetilde{k}^*(\boldsymbol{\tau}^*) \overset{\mathsf{e}}{\curvearrowright}_{\flat} \widetilde{k}^*(\boldsymbol{\sigma}^*) = \widetilde{k}^*(\boldsymbol{\tau}^* \overset{\mathsf{e}}{\curvearrowright}_{\flat} \boldsymbol{\sigma}^*),$$

and

$$\widetilde{k}^* \circ \uparrow_i = \uparrow_i \circ\, \widetilde{k}^* \qquad (1 \leqslant i \leqslant d). \tag{6.20}$$

Proof – We prove the dual identities writing

$$\Lsh_{\mathsf{e}} \circ \widetilde{k} = (k \otimes \Lsh_{\mathsf{e}})\delta \overset{(6.18)}{=} \Big(k \otimes \mathrm{Id} \otimes \mathrm{Id}\Big)\mathcal{M}^{(13)}\big(\delta \otimes \delta\big)\, \Lsh_{\mathsf{e}} = \Big(\big(k \otimes \mathrm{Id}\big)\delta \otimes \big(k \otimes \mathrm{Id}\big)\delta\Big)\, \Lsh_{\mathsf{e}} = \big(\widetilde{k} \otimes \widetilde{k}\big)\, \Lsh_{\mathsf{e}},$$

and

$$\downarrow_i \circ\, \widetilde{k} = (k \otimes \downarrow)\delta \overset{(6.19)}{=} (k \otimes \mathrm{Id})\delta \circ \downarrow_i = \widetilde{k} \circ \downarrow_i\,.$$

$\rhd$

Pick a character k on U^-. For primitive trees $\mathsf{t} \in \mathsf{N}^0$ define

$$F^{(k)}(\mathsf{t}^*) := F\Big(\widetilde{k}^*(\mathsf{t}^*)\Big), \tag{6.21}$$

so we have

$$F^{(k)}(\circ^*) = f(u_0), \quad F^{(k)}(\bullet^*) = g(u_0, u_1), \quad F^{(k)}\big(({\color{red}\bullet}^{0,\alpha})^*\big) = \mathbf{1}_{\alpha<0} \sum_{\boldsymbol{\tau} \in \mathcal{B} \cap U_\alpha} \frac{k(\boldsymbol{\tau})}{S(\boldsymbol{\tau})}\, F(\boldsymbol{\tau}^*).$$

For a tree

$$\boldsymbol{\tau} = \mathsf{t}^n \star \bigstar_{i=1}^{a} \mathcal{I}_{p_i}(\boldsymbol{\tau}_i),$$

define inductively the functions of (u_0, u_1)

$$F^{(k)}(\boldsymbol{\tau}^*)(u_0,u_1) := \left\{\prod_{i=1}^{a} F^{(k)}(\boldsymbol{\tau}_i^*)(u_0,u_1)\right\}\left\{D^n \prod_{i=1}^{a} \partial_{u_{p_i}}\right\} F^{(k)}(\mathfrak{t}^*)(u_0,u_1), \tag{6.22}$$

similarly to (6.12). The map $F^{(k)}$ is an E-multi-pre-Lie morphism with respect to $\{\overset{\mathsf{e}}{\curvearrowright}\}_{\mathsf{e}\in\mathsf{E}}$ by the same proof as the proof of Proposition 33. Recall from Section 2.4 the definition of the nonlinear image of a T-valued function. We remark in the following proposition that the family $\{F^{(k)}(\boldsymbol{\tau}^*)\}_{\boldsymbol{\tau}\in\mathcal{B}}$ is essentially contained in the original family $\{F(\boldsymbol{\tau}^*)\}_{\boldsymbol{\tau}\in\mathcal{B}}$. The notations in the next statement are all defined in Section 9.2; the reader can skip it now and come back to it later.

Proposition 37. *Let k be a character of U^-. For any $\boldsymbol{\tau}\in\mathcal{B}$, the function $F^{(k)}(\boldsymbol{\tau}^*)$ is represented as a linear combination of the functions $F(\boldsymbol{\sigma}^*)$, where $\boldsymbol{\sigma} = \sigma_{\mathfrak{e}}^{\mathfrak{n}}$ runs over all elements of $\mathcal{B}$ without $\bullet^{\cdot,\alpha}$ decorations and such that*

$$\boldsymbol{\tau} = (\sigma/^{\mathrm{red}}\varphi)_{\mathfrak{e}+\mathfrak{e}'_{\partial\varphi}}^{[\mathfrak{n}-\mathfrak{n}_\varphi]_\varphi,\mathfrak{o}(\mathfrak{n}_\varphi+\pi\mathfrak{e}'_{\partial\varphi},\mathfrak{e})}$$

for some subforest φ of σ, $\mathfrak{n}_\varphi : N_\varphi \to \mathbb{N}^{d+1}$ with $\mathfrak{n}_\varphi \leqslant \mathfrak{n}$, and $\mathfrak{e}'_{\partial\varphi} : \partial\varphi \to \mathbb{N}^{d+1}$.

Proof – It is sufficient to consider $\boldsymbol{\tau}$ of the form

$$\boldsymbol{\tau} = \bullet^{n,\alpha} \star \mathop{\bigstar}_{i=1}^{a} \mathcal{I}_{p_i}(\boldsymbol{\tau}_i).$$

Then by definition, $F^{(k)}(\boldsymbol{\tau}^*)$ is a linear combination of the functions of the form

$$\left\{\prod_{i=1}^{a} F(\boldsymbol{\sigma}_i^*)\right\} D^n \prod_{i=1}^{a} \partial_{u_{p_i}} F(\boldsymbol{\eta}^*),$$

where $\boldsymbol{\eta}\in\mathcal{B}\cap U_\alpha$. By an elementary formula $D^n\partial_{u_p} = \sum_{\ell\leqslant p\wedge n}(-1)^\ell\binom{n}{\ell}\partial_{u_{p-\ell}}D^{n-\ell}$ and by Proposition 35, the above function is a linear combination of the functions

$$\left\{\prod_{i=1}^{a} F(\boldsymbol{\sigma}_i^*)\right\} \prod_{i=1}^{a} \partial_{u_{p_i-\ell_i}} F(\uparrow^{n-\sum_{i=1}^a \ell_i} \boldsymbol{\eta}^*).$$

By a similar proof to that of Proposition 33, we can show that the above function is equal to

$$F\Big(\sum_{v_1,\dots,v_a\in N_\eta} (\boldsymbol{\tau}_1 \overset{p_1-\ell_1}{\curvearrowright}_{(v_1)} (\boldsymbol{\tau}_2 \overset{p_2-\ell_2}{\curvearrowright}_{(v_2)} \cdots (\boldsymbol{\tau}_a \overset{p_a-\ell_a}{\curvearrowright}_{(v_a)} \boldsymbol{\eta})\cdots))^*\Big)$$

The trees inside F produce $\boldsymbol{\tau}$ when we contract η as in the explicit formula of D^- in Section 9.2. $\rhd$

As a result of Proposition 37, we have that $F(\boldsymbol{\tau}) = 0$ for any $\boldsymbol{\tau}\in\mathcal{V}\backslash\mathcal{B}$, since $\boldsymbol{\tau}$ is an element of $\mathcal{B}$ if and only if $\boldsymbol{\tau}$ is a contraction of a strongly conform trees without $\bullet^{\cdot,\alpha}$ decoration. This ensures that the map

$$\Upsilon^{(k)} := F^{(k)}|_{U^*},$$

is an E-multi-pre-Lie morphism on U^* with respect to $\{\overset{\mathsf{e}}{\curvearrowright}_\flat\}_{\mathsf{e}\in\mathsf{E}}$. Moreover, denoting by S the subspace of T spanned by $\{\bullet^n\}_{n\in\mathbb{N}^{d+1}} \cup \mathcal{I}(\mathcal{B})$, we have that for any $\mathsf{t}\in\mathsf{N}^0$ and any function $\boldsymbol{u} : \mathbb{R}^{d+1}\to S$, the function

$$\big(F^{(k)}(\mathsf{t}^*)\big)^\star(\boldsymbol{u},\partial\boldsymbol{u}) \star \mathsf{t} : \mathbb{R}^{d+1} \to V$$

is actually T-valued.

Corollary 38. *One has $\Upsilon\circ\widetilde{k}^* = \Upsilon^{(k)}$, for all $k\in G^-$.*

Proof – It follows from Propositions 33 and 36 that the map $\Upsilon\circ\widetilde{k}^*$ is an ($\overset{\mathsf{e}}{\curvearrowright}_\flat$ vs $\rhd_{\mathsf{e}}$) E-multi-pre-Lie morphism. Because of Proposition 37, the map $\Upsilon^{(k)}$ is also an ($\overset{\mathsf{e}}{\curvearrowright}_\flat$ vs $\rhd_{\mathsf{e}}$) E-multi-pre-Lie morphism. Hence it is sufficient to show that they are equal on the generators of U^*, that is,

$$F\big(\widetilde{k}^*(\mathsf{t}^n)^*\big) = F^{(k)}\big((\mathsf{t}^n)^*\big)$$

for any $(\mathsf{t}, n) \in \mathfrak{T}_{\mathrm{n}} \otimes \mathbb{N}^{d+1}$. The case $n = 0$ is given by definition (6.21). For $n = (n_i)_{i=0}^d \in \mathbb{N}^{d+1}$, by writing $\uparrow^n := \prod_{i=0}^d \uparrow_i^{n_i}$, we have

$$F\big(\widetilde{k}^*(\mathsf{t}^n)^*\big) = F\big(\widetilde{k}^*(\uparrow^n \mathsf{t}^*)\big) \overset{(6.20)}{=} F\big(\uparrow^n (\widetilde{k}^*\mathsf{t}^*)\big) \overset{(6.16)}{=} D^n F\big(\widetilde{k}^*\mathsf{t}^*\big) \overset{(6.21)}{=} D^n F^{(k)}(\mathsf{t}^*)$$
$$= F^{(k)}\big((\mathsf{t}^n)^*\big),$$

using (6.16) and (6.20) in the last equality. ▷

Given a modelled distribution $\boldsymbol{u} \in \mathcal{D}^{\gamma,\eta}(T, \mathsf{g})$ with $\gamma > 0$, set

$$\mathcal{F}(\boldsymbol{u}) := \mathcal{Q}_{\leqslant 0}\left(\sum_{\mathsf{t} \in \mathsf{N}^0} F(\mathsf{t})(\boldsymbol{u}, \partial\boldsymbol{u})\, \mathsf{t}\right) = f^\star(\boldsymbol{u}) \star \Xi + g^\star(\boldsymbol{u}, \partial\boldsymbol{u})$$

and

$$\mathcal{F}^{(k)}(\boldsymbol{u}) := \mathcal{Q}_{\leqslant 0}\left(\sum_{\mathsf{t} \in \mathsf{N}^0} \big(F^{(k)}(\mathsf{t}^*)\big)^\star(\boldsymbol{u}, \partial\boldsymbol{u}) \star \mathsf{t}\right). \tag{6.23}$$

Note the appearance in (6.23) of a number of noise symbols $\bullet^{0,\alpha}$, with $\alpha < 0$, that have no counterpart in $\mathcal{F}(\boldsymbol{u})$.

Lemma 39. *If $\boldsymbol{u}$ is a solution of equation (4.12) then*

$$\widetilde{k}\big(\mathcal{F}(\boldsymbol{u})\big) = \mathcal{F}^{(k)}\big(\widetilde{k}(\boldsymbol{u})\big). \tag{6.24}$$

Proof – Lemma 32 implies that $\langle\!\langle \mathcal{F}(\boldsymbol{u}), \boldsymbol{\tau}^* \rangle\!\rangle = \Upsilon(\boldsymbol{\tau}^*)$, for any $\boldsymbol{\tau} \in \mathcal{B} \cap T_{\leqslant 0}$. Noting that $\widetilde{k}$ and $\widetilde{k}^*$ preserve the grading of T, as a consequence of the compatibility condition (5.4), we have

$$\langle\!\langle \widetilde{k}(\mathcal{F}(\boldsymbol{u})), \boldsymbol{\tau}^* \rangle\!\rangle = \langle\!\langle \mathcal{F}(\boldsymbol{u}), \widetilde{k}^*(\boldsymbol{\tau}^*) \rangle\!\rangle = \Upsilon\big(\widetilde{k}^*(\boldsymbol{\tau}^*)\big) = \Upsilon^{(k)}(\boldsymbol{\tau}^*)$$

for any $\boldsymbol{\tau} \in \mathcal{B} \cap T_{\leqslant 0}$. Note that assumption **(C)** yields $\widetilde{k}(X^n) = X^n$, and $\widetilde{k}\big(\mathcal{I}(\boldsymbol{\tau})\big) = \mathcal{I}\big(\widetilde{k}(\boldsymbol{\tau})\big)$. Hence

$$\widetilde{k}(\boldsymbol{u}) = \widetilde{k}\left\{u_0\mathbf{1} + \sum_{i=1}^d u_{X_i} X_i + \mathcal{I}(\mathcal{F}(\boldsymbol{u}))\right\} = u_0\mathbf{1} + \sum_{i=1}^d u_{X_i} X_i + \sum_{\boldsymbol{\tau}} \frac{1}{S(\boldsymbol{\tau})} \Upsilon^{(k)}(\boldsymbol{\tau}')\, \mathcal{I}(\boldsymbol{\tau}).$$

Then by a similar computation as in the proof of Lemma 32, we see that, for trees $\boldsymbol{\tau}$ of the form (6.4), the $\boldsymbol{\tau}$-component of $\mathcal{F}^{(k)}\big(\widetilde{k}(\boldsymbol{u})\big)$ is obtained by

$$\Big(\mathcal{F}^{(k)}\big(\widetilde{k}(\boldsymbol{u})\big)\Big)_{\boldsymbol{\tau}} = \frac{1}{n!\, m_1! \cdots m_a!} \prod_{j=1}^a \big(\widetilde{k}(\boldsymbol{u})\big)_{\mathcal{I}_{q_j}(\boldsymbol{\sigma}_j)}^{m_j} \left\{D^n \prod_{j=1}^a \partial_{u_{q_j}}^{m_j}\right\} \Upsilon^{(k)}(\mathsf{t}^*) = \frac{1}{S(\boldsymbol{\tau})} \Upsilon^{(k)}(\boldsymbol{\tau}^*),$$

where the last equality is from the definition (6.22) of $\Upsilon^{(k)}$. This yields

$$\langle\!\langle \mathcal{F}^{(k)}\big(\widetilde{k}(\boldsymbol{u})\big), \boldsymbol{\tau}^* \rangle\!\rangle = \Upsilon^{(k)}(\boldsymbol{\tau}^*),$$

hence

$$\langle\!\langle \widetilde{k}(\mathcal{F}(\boldsymbol{u})), \boldsymbol{\tau}^* \rangle\!\rangle = \langle\!\langle \mathcal{F}^{(k)}\big(\widetilde{k}(\boldsymbol{u})\big), \boldsymbol{\tau}^* \rangle\!\rangle$$

for any $\boldsymbol{\tau} \in \mathcal{B} \cap T_{\leqslant 0}$. ▷

The next statement provides a dynamical picture of the renormalization operation on models. As its proof will make it clear, it is a consequence of identity (6.24) and Theorem 26, giving in particular the reconstruction operator of a renormalized smooth model in terms of the unrenormalized smooth model, together with the multiplicativity property of the canonical model associated with a smooth noise.

Theorem 40. *Let ζ be a smooth noise with canonical model $\mathsf{M}^\zeta = (\Pi^\zeta, \mathsf{g}^\zeta)$. Given a character $k \in G_{\mathrm{ad}}^-$, denote by ${}^k\mathsf{M}^\zeta = \big(\Pi^\zeta \circ \widetilde{k}, \mathsf{g}^\zeta \circ \widetilde{k}^+\big)$ its associated renormalized $\mathbf{K}$-admissible model. Pick $\eta \in (0, \beta_0 + 2]$ and $\gamma > -\beta_0$. Given an initial condition $v \in \mathcal{C}^\eta(\mathbb{T}^d)$, let $\boldsymbol{u}^{(k)} \in \mathcal{D}^\gamma\big(T, \mathsf{g}^\zeta \circ \widetilde{k}^+\big)$ stand for the solution on $(0, t_0)$ to the equation*

$$\boldsymbol{u}^{(k)} = \Big(\mathcal{K}^{{}^k\mathsf{M}^\zeta} + (\mathcal{K}'_\gamma)^{{}^k\mathsf{M}^\zeta}\Big)\Big(f\big(\boldsymbol{u}^{(k)}\big)\Xi + g\big(\boldsymbol{u}^{(k)}, \partial\boldsymbol{u}^{(k)}\big)\Big) + \mathcal{P}_\gamma v.$$

Then

$$u^{(k)} := \mathbf{R}^{{}^k\mathsf{M}^\zeta}\big(\boldsymbol{u}^{(k)}\big)$$

is the solution on $(0,t_0)$ to the well-posed equation

$$\big(\partial_t - \Delta_x + 1\big)u^{(k)} = f\big(u^{(k)}\big)\zeta + g\big(u^{(k)}, \partial_x u^{(k)}\big) + \sum_{\tau\in\mathcal{B}\cap U_\alpha,\,\alpha<0} \frac{k(\boldsymbol{\tau})}{S(\boldsymbol{\tau})}\, F(\boldsymbol{\tau})\big(u^{(k)}, \partial_x u^{(k)}\big)$$

started from v.

Proof – The proof is similar to the proof of Proposition 24. The function $u^{(k)}$ satisfies the equation

$$u^{(k)}(z) = \int_{(0,t)\times\mathbb{T}^d} K_{\mathbf{L}}(z,w)\mathsf{R}^{{}^k\mathsf{M}^\zeta}\Big(f(\boldsymbol{u}^{(k)})\Xi + g\big(\boldsymbol{u}^{(k)}, \partial\boldsymbol{u}^{(k)}\big)\Big)(w)dw + \big(Pv\big)(z),$$

Since ${}^k\mathsf{M}^\zeta$ is a smooth model one has

$$\mathsf{R}^{{}^k\mathsf{M}^\zeta}(\boldsymbol{w})(z) = \Pi_z^\zeta\Big(\widetilde{k}\big(\boldsymbol{w}(z)\big)\Big)(z).$$

for any modelled distribution $\boldsymbol{w}\in\mathcal{D}^\alpha\big(T, \mathsf{g}^\zeta\circ\widetilde{k}^+\big)$ with $\alpha>0$. Applying Lemma 39 to $\boldsymbol{u}^{(k)}$ one has

$$\mathsf{R}^{{}^k\mathsf{M}^\zeta}\Big[f(\boldsymbol{u}^{(k)})\Xi + g\big(\boldsymbol{u}^{(k)}, \partial\boldsymbol{u}^{(k)}\big)\Big](z) = \Pi_z^\zeta\Big[\widetilde{k}\Big(\mathcal{F}(\boldsymbol{u}^{(k)}(z)\Big)\Big](z) = \Pi_z^\zeta\Big[\mathcal{F}^{(k)}\Big(\widetilde{k}\big(\boldsymbol{u}^{(k)}(z)\big)\Big)\Big](z).$$

We see from the definition of the $F^{(k)}$ that the term $\mathcal{F}^{(k)}(\boldsymbol{w})$ is a sum of functions of the form

$$H(\boldsymbol{w})R(\partial\boldsymbol{w}),$$

for smooth functions $H:\mathbb{R}\to\mathbb{R}$ and polynomials R that are at most quadratic. We now use the fact that since the map Π^ζ is *multiplicative* so is its associated reconstruction operator. The latter has value $\Pi_z^\zeta(\cdot)(z)$ at point z, so we have

$$\Pi_z^\zeta\Big[H\big(\boldsymbol{w}(z)\big)R\big(\partial\boldsymbol{w}(z)\big)\Big](z) = \Pi_z^\zeta\big[H(\boldsymbol{w}(z))\big](z)\,\Pi_z^\zeta\big[R(\partial\boldsymbol{w})(z)\big](z),$$

with $\boldsymbol{w}(z) = \widetilde{k}\big(\boldsymbol{u}^{(k)}(z)\big)$. So Corollary 7 tells us that

$$\Pi_z^\zeta\Big[H\Big(\widetilde{k}\big(\boldsymbol{u}^{(k)}(z)\big)\Big)\Big](z) = H\Big(\Big(\widetilde{k}\big(\boldsymbol{u}^{(k)}(z)\big)\Big)_{\mathbf{1}}\Big) = H\big(\boldsymbol{u}_{\mathbf{1}}^{(k)}(z)\big) = H\big(u^{(k)}(z)\big),$$

and

$$\Pi_z^\zeta\Big[\widetilde{k}\big(\partial\boldsymbol{u}^{(k)}\big)\Big](z) = \Big(\mathsf{R}^{{}^k\mathsf{M}^\zeta}\big(\partial\boldsymbol{u}^{(k)}\big)\Big)(z) = \Big(\partial_x\mathsf{R}^{{}^k\mathsf{M}^\zeta}\boldsymbol{u}^{(k)}\Big)(z) = \big(\partial_x u^{(k)}\big)(z),$$

giving in the end

$$\Pi_z^\zeta\Big[\mathcal{F}^{(k)}\Big(\widetilde{k}\big(\boldsymbol{u}^{(k)}(z)\big)\Big)\Big](z) = \mathcal{F}^{(k)}\Big(u^{(k)}, \partial_x u^{(k)}\Big)(z).$$

▷

Remark – *The preceding proof underlines the fundamental role played by the multiplicative property of the centered naive interpretation operators Π_x^ζ. The canonical smooth model Π^ζ is not the only multiplicative model that one can associate with a smooth noise ζ and the class of models associated with 'preparation maps' introduced by Bruned in* [12] *provides a general setting where to obtain the renormalized equation for a class of renormalization procedures including the procedure implemented here* [6].

7 – The BHZ character

Among all the characters k on U^- that can be used to build a renormalization map $\widetilde{k}$, Bruned, Hairer and Zambotti proved in [16] that there is a unique random character that is centered and translation invariant, in a probabilistic sense, when the smooth noise ζ in the preceding section is random, centered and translation invariant. We describe it in this section and name it 'BHZ character', after the initials of Bruned, Hairer and Zambotti.

We assume throughout this section that we work with regularity and renormalization structures satisfying assumptions **(A-C)**. To have a picture in mind, think of the structures associated with the generalized (KPZ) equation (6.1). Elements of $T=U$ are thus given by node and edge decorated trees. Recall from Assumption **(C1)** that U^- is an algebra generated by $\mathcal{U}_{<0}$ and a unit $\mathbf{1}_-$, and if one extends first the splitting map $\delta: U\to U^-\otimes U$ into an algebra morphism $\hat\delta:\mathbb{R}[U]\to U^-\otimes\mathbb{R}[U]$, then the splitting map δ^- satisfies

$$\delta^- = (\mathrm{Id}\otimes P_-)\hat\delta_{|U^-},$$

for an algebra morphism projection map $P_- : \mathbb{R}[U] \to U^-$. (See Section 9.3 for the details; this projection map sends in particular $\bullet$ and all the ${\color{red}\bullet}^{0,\alpha}$ to $\mathbf{1}_-$). Denote by $\mathbf{1}_{\mathsf{U}}$ the unit of $\mathbb{R}[U]$, seen as the empty graph. (We use a distinct notation for $\mathbf{1}_{\mathsf{U}}$ and $\mathbf{1}_-$ to emphasize that they do not live in the same space.) For basis elements $\boldsymbol{\tau} = \tau_{\mathfrak{e}}^{\mathfrak{n}}$ and $\boldsymbol{\sigma} = \sigma_{\mathfrak{f}}^{\mathfrak{m}}$ of $\mathbb{R}[U]$, we write

$$\boldsymbol{\sigma} \prec \boldsymbol{\tau}$$

if σ is a strict *subgraph* of τ, or $\sigma = \tau$ and $\mathfrak{m} \leqslant \mathfrak{n}$ with $\mathfrak{m} \neq \mathfrak{n}$. (Do not get misled by the notation, σ may be a product of disjoint subtrees of τ.) Recall from Section 5.2 the notation $\mathcal{F}$ for the family of integral operators and multiplication by a non-null monomial. The following assumption describes the properties from the splitting map δ that are relevant here. The renormalization structure built in Section 9 for the generalized (KPZ) satisfies it.

Assumption (E) –

(a) *For any $\boldsymbol{\tau} \in \mathcal{U}_{<0}$, one has the splitting formula*

$$\delta\boldsymbol{\tau} \in P_-(\boldsymbol{\tau}) \otimes {\color{red}\bullet}^{0,|\tau|} + U_{\boldsymbol{\tau}}^- \otimes U,$$

where

$$U_{\boldsymbol{\tau}}^- := \mathrm{span}\Big\{P_-(\boldsymbol{\sigma}) \in U^- \,;\, \boldsymbol{\sigma} \prec \boldsymbol{\tau}\Big\}.$$

(b) *For any $\boldsymbol{\tau} \in \mathcal{U}_{<0}$ and $F \in \mathcal{F}$ such that $F(\boldsymbol{\tau}) \in \mathcal{U}_{<0}$, one has*

$$\delta\big(F(\boldsymbol{\tau})\big) \in (\mathrm{Id} \otimes F)\delta\boldsymbol{\tau} + P_-\big(F(\boldsymbol{\tau})\big) \otimes {\color{red}\bullet}^{0,|F(\tau)|} + J_{\boldsymbol{\tau}}^- \otimes U,$$

where $J_{\boldsymbol{\tau}}^-$ is the ideal of U^- generated by $\Big\{P_-\big(F(\boldsymbol{\sigma})\big) \,;\, F \in \mathcal{F},\, \boldsymbol{\sigma} \in \mathcal{U}_{<0},\, \boldsymbol{\sigma} \prec \boldsymbol{\tau}\Big\}$.

Property (b) is a refinement of the property (5.13) in Assumption **(C1)**. Note that ${\color{red}\bullet}^{0,\beta} \in \mathcal{B}_\beta$, for any $\beta \in \mathbb{R}$. Basis elements of U with 0-homogeneity are not necessarily unique, unlike in Assumption **(A1)** on concrete regularity structures. Condition (b) above ensures the existence of the following map, defined by induction on the order relation $\prec$. Each element τ of U^- has by definition a unique representative τ_U in $\mathbb{R}[U]$. Denote by $\mathcal{M}$ the multiplication operator on $\mathbb{R}[U]$ and extend it naturally on $U^- \otimes \mathbb{R}[U]$ setting $\mathcal{M}(\tau \otimes \sigma) = \mathcal{M}(\tau_U \otimes \sigma)$; it takes values in $\mathbb{R}[U]$.

Definition – *Under assumption* **(E)**, *the* **negative twisted antipode** *is an algebra morphism*

$$S'_- : U^- \to \mathbb{R}[U]$$

given recursively by $S'_-\mathbf{1}_- = \mathbf{1}_{\mathsf{U}}$ and, for every basis element $\tau \in \mathcal{U}_{<0}$ by

$$S'_-\big(P_-(\tau)\big) = -\mathcal{M}_-\big(S'_- \otimes \mathrm{Id}\big)\Big(\delta\tau - P_-(\tau) \otimes {\color{red}\bullet}^{0,|\tau|}\Big). \tag{7.1}$$

The $P_-(\tau)$ generating U^- as an algebra for τ ranging in $\mathcal{U}_{<0}$, identity (7.1) characterizes indeed uniquely an algebra morphism. The intuitive meaning of this recursive definition should be clear. One extracts from τ all possible subdiverging quantities φ_1, but also extracts from φ_1 all its subdiverging quantities, and so on. This formula is close to the Dyson-Salam renormalization formula for the antipode in Hopf algebras [29]; like the latter, it can be rewritten as a sum over forests of diverging sub-forests, as in Zimmermann forest formula. This will not be useful here, and the only thing that matters here is property (7.1). The forest representation is however useful for the analysis of the convergence of renormalized models [20].

Do not be mislead by the name of S'_-: This is *not* the antipode of a Hopf algebra structure. Bruned, Hairer and Zambotti named it like that because its defining relation (7.1) looks like the defining relation (B.1) for the antipode in a Hopf algebra.

Recall from Section 6.2 the definition of the naive interpretation operator $\mathsf{\Pi}^\zeta$ corresponding to a smooth noise ζ in $\mathbb{R} \times \mathbb{R}^d$. We consider a random smooth noise ζ, invariant by translation and centered. Define the character h^ζ on $\mathbb{R}[U]$ by setting $h^\zeta(\mathbf{1}_{\mathsf{U}}) := 1$ and

$$h^\zeta(\tau) := \mathbb{E}\big[\mathsf{\Pi}^\zeta\tau\big](0) \tag{7.2}$$

for $\tau \in \mathcal{U}$, and define a character on U^- setting

$$k^\zeta := h^\zeta \circ S'_-.$$

The associated 'BPHZ renormalized' interpretation operator ${}^{k^\zeta}\Pi^\zeta$ is defined on U by

$$ {}^{k^\zeta}\Pi^\zeta\tau = \big(k^\zeta \otimes \Pi^\zeta\big)\delta\tau = \Big(\big(h^\zeta \circ S'_-\big) \otimes \Pi^\zeta\Big)\delta\tau. $$

The acronym BPHZ stands for Bogoliubov, Parasiuk, Hepp and Zimmermann, who made deep contributions to the renormalization problem in quantum field theory. We call 'BHZ character', after Bruned, Hairer and Zambotti, the character k^ζ on U^-. The reason for introducing the negative twisted antipode operator lies entirely in the following simple computations used in the proof of the next statement claiming that the BPHZ renormalization associated with the BHZ character recenters probabilistically the Π map at all points in spacetime. Its proof is taken from the proof of Theorem 6.17 in Bruned, Hairer and Zambotti's work [16] on the algebraic renormalization of regularity structures.

Theorem 41. *We work with compatible regularity and renormalization structures under assumptions* **(A-C)** *and Assumption* **(E)***. The character k^ζ belongs to G^-_{ad}, and one has*

$$ \mathbb{E}\Big[\big({}^{k^\zeta}\Pi^\zeta\boldsymbol{\tau}\big)(x)\Big] = 0 \tag{7.3} $$

for any $\boldsymbol{\tau} \in \mathcal{U}_{<0}$ and $x \in \mathbb{R}\times\mathbb{R}^d$.

Proof – First we show that $k^\zeta \in G^-_{\mathrm{ad}}$. Let $F \in \mathcal{F}$ and $\boldsymbol{\tau} \in \mathcal{U}_{<0}$ be such that $F\boldsymbol{\tau} \in \mathcal{U}_{<0}$. If $F = X^n\star$ then $F\boldsymbol{\tau}$ is of the form $\tau_{\mathfrak{e}}^{\mathfrak{n}}$ and it follows from the definition of the canonical model that $\mathbb{E}[\Pi^\zeta\tau_{\mathfrak{e}}^{\mathfrak{n}}](x)$ is a monomial of x with non-null degree $\sum_{v\in N_\tau}\mathfrak{n}(v)$; so it is null at point 0 and $h^\zeta(F\boldsymbol{\tau}) = 0$. Recall from Section 3.1 that we defined the operator $\mathbf{K}$ so that

$$ \int_{\mathbb{R}\times\mathbb{R}^d} y^p K(x,y)\,dy = 0 $$

for all $p \in \mathbb{N}\times\mathbb{N}^d$ with $|p|_{\mathfrak{s}} \leqslant 1$. This fact was not used so far. We use it here to have

$$ h^\zeta\big(\mathcal{I}_p\boldsymbol{\tau}\big) = 0, $$

for all $p \in \mathbb{N}\times\mathbb{N}^d$ with $|p|_{\mathfrak{s}} \leqslant 1$; so $h^\zeta(F\boldsymbol{\tau}) = 0$ for all $F \in \mathcal{F}$. Since assumption **(E)** guarantees that we have

$$ S'_-(F\boldsymbol{\tau}) \in -\mathcal{M}_-\big(S'_- \otimes F\big)\delta\boldsymbol{\tau} + \mathcal{M}_-\big(S'_- J_{\boldsymbol{\tau}} \otimes U\big), $$

we can conclude that $h^\zeta\big(S'_-(F\boldsymbol{\tau})\big) = 0$, by an induction on the size of the graph τ. Hence $k^\zeta \in G^-_{\mathrm{ad}}$.

– The negative twisted antipode S'_- is defined so as to have identity (7.3) for $x = 0$. Indeed, since $\Pi^\zeta(\bullet^{0,\beta}) \equiv 1$ for all β, one has from the defining relation (7.1) for the twisted antipode, for any $\boldsymbol{\tau} \in \mathcal{U}_{<0}$,

$$ \begin{aligned} \mathbb{E}\Big[\big({}^{k^\zeta}\Pi^\zeta\boldsymbol{\tau}\big)(0)\Big] &= \sum_{\boldsymbol{\varphi}\trianglelefteq\boldsymbol{\tau}} h^\zeta\big(S'_-(\boldsymbol{\varphi})\big)\,\mathbb{E}\big[\big(\Pi^\zeta(\boldsymbol{\tau}/^-\boldsymbol{\varphi})\big)(0)\big] \\ &= \sum_{\boldsymbol{\varphi}\trianglelefteq\boldsymbol{\tau}} h^\zeta\big(S'_-(\boldsymbol{\varphi})\big)\,h^\zeta(\boldsymbol{\tau}/^-\boldsymbol{\varphi}) \\ &= h^\zeta\Big(\mathcal{M}\big(S'_-\otimes \mathrm{Id}\big)\delta\boldsymbol{\tau}\Big) = h^\zeta\big(S'_-\tau\big)\Big(h^\zeta\big(\bullet^{0,|\boldsymbol{\tau}|}\big) - 1\Big) = 0. \end{aligned} $$

Recall the homogeneity and grading notions on U and T are different. It is the homogeneity of $\boldsymbol{\tau}$, seen as an element of T, that appears in $\bullet^{0,|\boldsymbol{\tau}|}$. It is elementary to go from $\mathbb{E}\big[\big({}^{k^\zeta}\Pi^\zeta\tau\big)(0)\big] = 0$, to $\mathbb{E}\big[\big({}^{k^\zeta}\Pi^\zeta\tau\big)(x)\big] = 0$, for all $x \in \mathbb{R}\times\mathbb{R}^d$, using the probabilistic translation invariance property of Π^ζ. $\rhd$

Remark – *There is no other character k on U^- than $h^\zeta\circ S'_-$, such that the renormalized naive interpretation operator ${}^k\Pi^\zeta := (k\otimes\Pi^\zeta)\delta$, has property (7.3) of Theorem 41. Note that the cointeraction identity between δ and δ^- implies that we have*

$$ {}^{k_1\star k_2}\Pi^\zeta = {}^{k_1}\big({}^{k_2}\Pi^\zeta\big), \tag{7.4} $$

for any two characters k_1, k_2 on U^-. The uniqueness claim then amounts to proving that for any non-null character $k \neq 1$, there exists an element $\tau \in U$ such that $\mathbb{E}\big[\big({}^{k\star k^\zeta}\Pi^\zeta\boldsymbol{\tau}\big)(0)\big] \neq 0$. See the second part of the proof of Theorem 6.18 in [16].

Assume now that $\zeta = \xi_\varepsilon$ is the regularized version of a random irregular noise ξ, centered and translation invariant, and write Π^ε for Π^{ξ_ε}. The BHZ character h from (7.2) becomes ε-dependent as well. Set

$$k_\varepsilon := h^{\xi_\varepsilon} \circ S'_- \tag{7.5}$$

Identity (7.4) tells us that if the maps ${}^{k_\varepsilon}\mathsf{\Pi}^\varepsilon$ converge to a limit when ε goes to zero, then for any character k on U^-, the renormalized interpretation map ${}^{k\star k_\varepsilon}\mathsf{\Pi}^\varepsilon$ is also converging. There is thus a whole class of converging renormalization schemes indexed by the group of characters of U^-, if there is a single converging renormalization scheme. If we insist on building **K**-admissible models, this provides a family of convergent models indexed by the renormalization group G^-_{ad}.

Recall the notation $\textsc{Sol}(\xi_\varepsilon; F)$ for the solution to the PDE (1.4) driven by the smooth noise ξ_ε, associated with a given initial condition. The renormalization group aquires a dynamical meaning from Theorem 40 if one notices that

$$\begin{aligned}\Big\{\textsc{Sol}\Big(\xi_\varepsilon; \Upsilon^{(k*k_\varepsilon)}\Big)\Big\}_{k\in G^-} &= \Big\{\textsc{Sol}\Big(\xi_\varepsilon; \big(\Upsilon^{(k_\varepsilon)}\big)^{(k)}\Big)\Big\}_{k\in G^-} \\ &= \Big\{\textsc{Sol}\Big(\xi_\varepsilon; \big(\Upsilon^{(k')}\big)^{(k_\varepsilon)}\Big)\Big\}_{k'\in G^-},\end{aligned}$$

for any fixed positive ε, since one has $\widetilde{k * k_\varepsilon} = \widetilde{k_\varepsilon} \circ \widetilde{k'}$, for $k' = (k_\varepsilon)^{-1} * k * k^\varepsilon$. This remark tells us that the *family* of solutions of the singular stochastic PDE (1.4) is parametrized by the subset $\big(\Upsilon^{(k)}\big)_{k\in G^-}$ of the space $\mathfrak{F}$ of nonlinearities. This remains true at the limit when ε goes to 0. We will see in the Section 8 that this subset is actually a finite dimensional immersed manifold.

Arrived at that stage, the only piece of the story that is missing to complete a proof of the meta-theorems from Section 1 is a proof of the fact that one can indeed construct regularity structures satisfying the different assumptions that we put forward in the course of obtaining the above results, and to prove that the BHZ renormalized smooth **K**-admissible models associated with a regularized noise ξ_ε and the element k_ε from (7.5) converge in probability to a limit model as the regularization parameter tends to 0. We tackle the first point in Section 9. The second point is the object of Chandra & Hairer's work [20]; we do not treat it here.

8 – The manifold of solutions

We take for granted in this section the convergence result of Chandra & Hairer from [20], and work with the limit random admissible model $(\mathsf{\Pi}, \mathsf{g}) = \mathsf{M}$, obtained as a limit in probability of the renormalized naive models ${}^{h_\varepsilon}\mathsf{M}^\varepsilon$ when $\varepsilon > 0$ goes to 0. Recall from equality (6.23) the expression of $\mathcal{F}^{(k)}(\boldsymbol{u})$, for $k \in G^-_{\mathrm{ad}}$. Pick $\eta \in (0, \beta_0 + 2], \gamma > -\beta_0$ and an initial condition $u_0 \in \mathcal{C}^\eta(\mathbb{T}^d)$. Write $\boldsymbol{u}(k) \in \mathcal{D}^\gamma(T, \mathsf{g})$ for the solution to the equation

$$\boldsymbol{u}(k) = \Big(\mathcal{K}^{\mathsf{M}} + (\mathcal{K}'_\gamma)^{\mathsf{M}}\Big)\Big(\mathcal{F}^{(k)}(\boldsymbol{u}(k))\Big) + \mathcal{P}_\gamma u_0 =: \Psi_k\big(\boldsymbol{u}(k)\big),$$

and set

$$u(k) := \mathbf{R}^{\mathsf{M}}\big(\boldsymbol{u}(k)\big).$$

The family of functions $\big(u(k)\big)_{k\in G^-_{\mathrm{ad}}}$ coincides with the limit of the family

$$\Big(\textsc{Sol}\big(\xi_\varepsilon; F^{(k*h_\varepsilon)}\big)\Big)_{k\in G^-_{\mathrm{ad}}}.$$

Note that Ψ_k depends linearly, hence smoothly, on k. We saw in Theorem 22 in Section 4 that given a bounded set of nonlinearities in C^4, there exists a positive time horizon t_0 such that the 'integral' map Ψ_k is a contraction from $\mathcal{D}^{\gamma,\eta}_{(0,t_0)}(T, \mathsf{g})$, uniformly with respect to the nonlinearities in the given bounded set. So the continuous linear map $\big(\mathrm{Id} - \partial_{\boldsymbol{u}}\Psi_k\big)$, from the Banach space $\mathcal{D}^{\gamma,\eta}_{(0,t_0)}(T, \mathsf{g})$ into itself has a continuous inverse, given under the form of the classical Neumann series. The map $\big(\mathrm{Id} - \partial_{\boldsymbol{u}}\Psi_k\big)$ is thus a continuous isomorphism of $\mathcal{D}^{\gamma,\eta}_{(0,t_0)}(T, \mathsf{g})$ by the open mapping theorem. It is then a direct consequence of the implicit function theorem that the unique fixed point $\boldsymbol{u}(k)$ of the equation

$$\boldsymbol{u}(k) = \Psi_k\big(\boldsymbol{u}(k)\big)$$

defines a smooth function of $k \in G^-_{\mathrm{ad}}$.

Proposition 42. *The family* $\{u(k)\}_{k\in G^-_{ad}}$ *forms a finite dimensional immersed submanifold of* $\mathcal{C}^\alpha(\mathbb{R}\times\mathbb{R}^d)$.

Proof – It suffices from the implicit function theorem to see that $D_k u$ has constant rank; this follows from the linearity of the reconstruction map if we can see that $D_k\boldsymbol{u}$ is injective. (The reconstruction map is not injective without further assumptions.) The linear map $D_k\boldsymbol{u}$ sends $T_{\mathrm{Id}}G^-_{\mathrm{ad}}$ into $\mathcal{D}^\gamma(T,\mathsf{g})$. But picking $h\in T_{\mathrm{Id}}G^-_{\mathrm{ad}}$, and setting $\boldsymbol{v}:=(D_k\boldsymbol{u})(h)\in\mathcal{D}^\gamma(T,\mathsf{g})$, the modelled distribution $\boldsymbol{v}$ cannot be null unless $h=0$, since $\boldsymbol{v}$ is the solution to the affine equation

$$\boldsymbol{v}=\Big(\mathcal{K}^{\mathsf{M}}+(\mathcal{K}'_\gamma)^{\mathsf{M}}\Big)\bigg((\mathcal{F}^{(k)})'(\boldsymbol{u})\boldsymbol{v}+\sum_{\tau\in\mathcal{B}\cap U_\alpha,\,\alpha<0}\frac{h(\boldsymbol{\tau})}{S(\boldsymbol{\tau})}F(\boldsymbol{\tau})(\boldsymbol{u},\partial\boldsymbol{u})\bigg).$$

$\rhd$

Remark – The use of the implicit function theorem actually shows that the solution $\boldsymbol{u}$ of the equation

$$\boldsymbol{u}=\Big(\mathcal{K}^{\mathsf{M}}+(\mathcal{K}'_\gamma)^{\mathsf{M}}\Big)\Big(f(\boldsymbol{u})\Xi+g\big(\boldsymbol{u},\partial\boldsymbol{u}\big)\Big)+\mathcal{P}_\gamma u_0, \tag{8.1}$$

is a C^1 function of $f,g\in C^4$, and a smooth function of f,g, if $f,g\in C^\infty$. This gives a direct access to Taylor expansions in small noise, where f is replaced by af, for a small positive parameter a, or if f is the value at $a=0$ of a smooth family $f(a,\cdot)\in C^\infty$, as the solution $\boldsymbol{u}$ happens then to be a smooth function of the expansion parameter a. Elementary classical calculus is used to see that the derivatives of $\boldsymbol{u}$ with respect to the parameter a are solutions of affine equations obtained by formal differentiation of equation (8.1) with respect to the parameter. This kind of questions has a long history, under the name 'stochastic Taylor expansion' in a stochastic calculus setting – after seminal works by Azencott [1] and Ben Arous [11], where it was used together with the stationary phase method on Wiener space to get heat kernel estimates for elliptic and sub-elliptic diffusions. Inahama & Kawabi extended the approach to a rough paths setting in [53], and Friz, Gassiat and Pigato made a first use of this type of ideas in a regularity structures setting in [32]. The present result holds for all subcritical singular stochastic PDEs, with the above straightforward proof.

9 – Building regularity and renormalization structures

In the end, for the above results to hold, we require from the regularity structure $\mathscr{T}$ and the renormalization structure $\mathscr{U}$ that they satisfy the different assumptions introduced above along the way for different purposes. We summarize them here, with a quick description of what they are useful for.

Assumption	Section	What it is useful for
(A1-2)	2.3	Inclusion of the polynomial structure in our regularity structures.
(A3)	2.4	Product between T_X and T.
(B1)	3.2	Actions of $\Delta^{(+)}$ on $\mathcal{I}_n^{(+)}$.
(B2)	3.5	Induction structure on Δ for building admissible models.
(C)	5.2	Compatibility between the maps $\delta^{(+)}$ and $\mathcal{I}_n^{(+)}$.
(D)	6.3	Compatibility between multi-pre-Lie and renormalization structures.
(E)	7	Structure assumption on U^-, and induction structure on δ.

Following Bruned, Hairer and Zambotti [16], we describe in this section a setting tailor made for the study of the generalized (KPZ) equation where all these conditions hold true. We introduce a homogeneity map on the decorated trees from Section 6.1.

Definition – *Let* $\mathfrak{T}_{\mathrm{n}}$ *and* $\mathfrak{T}_{\mathrm{e}}$ *be abstract finite sets, equipped with homogeneity maps* $|\cdot|:\mathfrak{T}_{\mathrm{n}},\mathfrak{T}_{\mathrm{e}}\to\mathbb{R}$.

- *On the sets* $\mathsf{N}:=\mathfrak{T}_{\mathrm{n}}\times\mathcal{N}$ *and* $\mathsf{E}:=\mathfrak{T}_{\mathrm{e}}\times\mathcal{N}$, *the homogeneity maps are extended by*

$$\begin{cases}|k|_n:=|n|+|k|_{\mathfrak{s}}, & (n,k)\in\mathsf{N},\\ |\ell|_e:=|e|-|\ell|_{\mathfrak{s}}, & (e,\ell)\in\mathsf{E}.\end{cases}$$

- *The* **naive homogeneity** *of a decorated tree $\tau_{\mathfrak{e}}^{\mathfrak{n}}$ is defined by*

$$|\tau_{\mathfrak{e}}^{\mathfrak{n}}|' := \sum_{e\in E_\tau} |\mathfrak{e}(e)|_{\mathfrak{t}_{\mathrm{e}}(e)} + \sum_{n\in N_\tau} |\mathfrak{n}(n)|_{\mathfrak{t}_{\mathrm{n}}(n)}.$$

We start from the sets

$$\mathfrak{T}_{\mathrm{n}} = \{\bullet, \circ\}, \quad \mathfrak{T}_{\mathrm{e}} = \{\mathcal{I}\},$$

for an abstract symbol $\mathcal{I}$ – we use on purpose the same symbol as the abstract integration map from Section 3.2. The node type set $\mathfrak{T}_{\mathrm{n}}$ is enlarged later. The two elements $\bullet$ and $\circ$ of $\mathfrak{T}_{\mathrm{n}}$ represent the monomial $\mathbf{1} = X^0$ and the noise Z, respectively. The set $\mathfrak{T}_e$ consists of only one integration operator $\mathcal{I}$. Each element has homogeneity

$$|\bullet| = 0, \quad |\circ| = \beta_0, \quad |\mathcal{I}| = 2,$$

where $\beta_0 \in (-2, 0)$ is the regularity of the noise ζ in the equation. (Would the equation under study involve several noises with different regularities we would introduce several $\circ$ symbols with the corresponding homogeneities.) Given that the polynomial structure is needed to encode at a regularity structure level the term $\mathcal{P}u_0$ describing the propagation of the initial condition, and the piece of $\mathcal{K}^{\mathsf{M}} + (\mathcal{K}')^{\mathsf{M}}$ taking values in the polynomial regularity structure, the use of trees with a node decoration encoding multiplication by polynomials appears as natural. On the other hand, the use of edge decorations for equations that do not involve derivatives of the solution in their formulation, like the generalized (PAM) equation

$$(\partial_t - \Delta_x)u = f(u)\zeta,$$

may look strange. The necessity to use edge decorations to encode derivatives of quantities of the form $\mathcal{I}(\cdot)$, even in such a case, comes from the renormalization process implemented in this setting, as the latter involves Taylor expansions.

As said in section 6, the final form of a generic element of our regularity structures will be the datum of a decorated tree together with a coloring and an additional decoration $\mathfrak{o} : N_\tau \to \mathbb{Z}[\beta_0]$, which plays an important role in the compatibility condition between regularity and renormalization structures from Definition 25. In a nutshell, this additional decoration will keep track of the naive homogeneity of the 'diverging' trees that will be extracted by the renormalization map δ. This is what will allow to have a δ map satisfying he fundamental conditions

$$\delta T_\beta^{(+)} \subset U^- \otimes T_\beta^{(+)} \qquad (\beta \in A^{(+)})$$

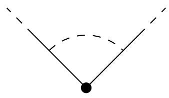
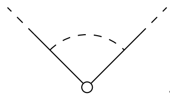
involved in the definition of compatible regularity and renormalization structures. So one should not be surprised that we will use the naive homogeneity to define the gradings in U and U^- and a different notion of homogeneity in T and T^+, taking into account the $\mathfrak{o}$-decorations. The discussion will be general enough for the reader to see what needs to be added to deal with the general case.

9.1 Rules and extended decoration

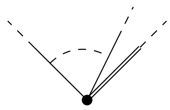
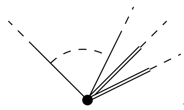
Working with the set of all decorated trees as a candidate for a regularity structure is not reasonable and we first identify a few notions that help clarifying the matter. Recall the abstract self-explaining formulation

$$\boldsymbol{v} = \mathcal{I}\Big(f^\star(\boldsymbol{v})\Xi + g^\star(\boldsymbol{v}, \partial\boldsymbol{v})\Big) + (T_X) \tag{9.1}$$

of the generalized (KPZ) equation. In the present tree setting the $\star$ product is given by the joining operator $\mathscr{J}$ on trees. If one wants to make sense of Picard iteration within the concrete regularity structure, one needs to make sense of a number of recursive relations – recall the subcomodules introduced in Section 4 and see the pictures in Section 6.1. General constraints of this type come under the name of **rule**, that is the definition for each node type n, of constraints on which kind of tuples of edges $\{e^i = (e_-^i, e_+^i)\}_i$ can have $\mathfrak{t}_{\mathrm{node}}(e_-^i) = n$, for all i, in a tree allowed by the rule. The choice of a rule is determined by the equation under consideration. Consider the right hand side of equation (9.1). Making sense of the nonlinear term $f^\star(\boldsymbol{v})\Xi + g_0^\star(\boldsymbol{v})$ requires that one can find $\mathscr{J}\big(\mathcal{I}(\cdot), \dots, \mathcal{I}(\cdot)\big)X^n$ or $\mathscr{J}\big(\mathcal{I}(\cdot), \dots, \mathcal{I}(\cdot)\big)X^n\Xi$, within the trees allowed by the rule, that is the corresponding nodes are of the form

 or  .

Making sense of the other terms $g_2^\star(\boldsymbol{v})(\partial\boldsymbol{v})^2 + g_1^\star(\boldsymbol{v})(\partial\boldsymbol{v})$ requires that one can find $\mathscr{J}\big(\mathcal{I}(\cdot),\dots,\mathcal{I}(\cdot),\mathcal{I}_{e_i}(\cdot)\big)X^n$, or $\mathscr{J}\big(\mathcal{I}(\cdot),\dots,\mathcal{I}(\cdot),\mathcal{I}_{e_i}(\cdot),\mathcal{I}_{e_j}(\cdot)\big)X^n$ for some $i,j = 1,\dots,d$ within the trees allowed by the rule, so each node of the corresponding elements of C has the form

 or  .

The operators $\mathcal{I}_{e_i}$ are represented by the double line in the above picture. Given a rule, a decorated **conform tree** is a tree such that all nodes of the tree, except perhaps the root, satisfy the rule. Denote by

$$C$$

the set of conform trees. If all node of the tree satisfy the rule, the tree is called **strongly conform**. We denote by

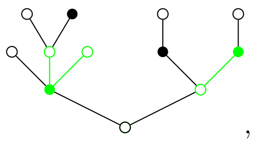
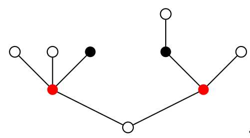
$$SC$$

the set of strongly conform trees. A rule is said to be **normal** if any subtree of a strongly conform tree is also strongly conform.

To construct regularity and renormalization structures, the rooted decorated trees obtained from the above iterations are not sufficient. Another important operation is the **contraction** of rooted trees, involved in the definition of the splitting maps Δ and δ. Given a typed rooted tree τ and a family φ of disjoint typed subtrees of τ, we use the notation

$$\tau/^{\mathrm{red}}\varphi$$

to denote the typed rooted tree obtained by identifying each subtree τ_i with a single node $\bullet$ with red color in the quotient tree. Here is an example, with φ in green,

$\tau =$  , $\tau/^{\mathrm{red}}\varphi =$  .

We allow such an operation for the set SC of strongly conform trees. Precisely, if each connected component of φ belongs to SC, then we assume that $\tau/^{\mathrm{red}}\varphi \in SC$. Hence each element of SC is a rooted decorated tree with a node type set

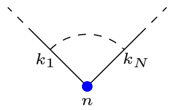
$$\mathfrak{T}_{\mathrm{node}}^{SC} = \{\bullet, {\color{red}\bullet}, \circ\}.$$

The analytic role of ${\color{red}\bullet}$ is the same as that of $\bullet$. In particular, the homogeneity of ${\color{red}\bullet}$ is 0. This is an example of the coloring of the tree. Only decorated trees without red color appear in the analysis of the well-posedness problem (9.1), but colors are used in the definition of the splitting maps in the renormalization structure.

Recall from assumption **(B1)** and Section 3.5 that the algebra T^+ is spanned by elements of the form

$$X^n \prod_{i=1}^N \mathcal{I}_{k_i}^+(\tau_i), \tag{9.2}$$

where $n \in \mathbb{N}\times\mathbb{N}^d$, $k_1,\dots,k_N \in \mathcal{N}$, and $\tau_1,\dots,\tau_N \in SC$. It is convenient to consider an element like (9.2) as a tree by interpreting $\mathcal{I}_k^+$ as the planting operator like $\mathcal{I}_k$ and the product $\prod$ as the tree product $\mathscr{J}$. To distinguish such trees from elements of SC, we give a blue color to their roots, encoding in this way the $+$ sign in $\mathcal{I}^+$.



(9.3)

The set C consists of such trees, where we see that the rule is broken at the root. This is because C is only conform, not strongly conform. Each element of C is thus a rooted decorated tree with a node type set

$$\mathfrak{T}^C_{\text{node}} = \{\bullet, \textcolor{red}{\bullet}, \textcolor{blue}{\bullet}, \circ\}.$$

A node of a conform tree has the type $\textcolor{blue}{\bullet}$ if and only if it is a root. The homogeneity of $\textcolor{blue}{\bullet}$ is 0. The trees with a blue root will only be involved in the description of the space T^+.

A rule is said to be **subcritical** if for any $\gamma \in \mathbb{R}$, only finitely many elements of SC have naive homogeneity less than γ. A **complete** rule will guarantee that a rooted decorated tree obtained from the contraction of a strongly conform tree by extracting ‘diverging’ pieces, and changing the decorations accordingly, will still be strongly conform. Proposition 5.21 in [16] ensures that *any normal subcritical rule can be extended into a normal subcritical complete rule.* We take this result for granted and do not reprove it here. The above rule on the set of decorated trees is normal, subcritical and complete.

To construct compatible regularity and a renormalization structures we introduce an additional decoration. Denote by N^{red}_τ the subset of N_τ consisting of the nodes with type $\textcolor{red}{\bullet}$.

Definition – *A tree with* **extended decoration** *is a rooted decorated tree* $\tau^{\mathfrak{n}}_{\mathfrak{e}}$ *with a map*

$$\mathfrak{o} : N^{\text{red}}_\tau \to \mathbb{Z}[\beta_0].$$

We write $\boldsymbol{\tau} = \tau^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}}$ *for a generic tree with extended decoration. The* **extended homogeneity** *of such a tree is defined by*

$$\big|\tau^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}}\big| := \sum_{e \in E_\tau} \big|\mathfrak{e}(e)\big| + \sum_{n \in N_\tau} \big|\mathfrak{n}(n)\big| + \sum_{n \in N^{\text{red}}_\tau} \mathfrak{o}(n).$$

We extend the naive homogeneity to the set of decorated trees with an extended decoration setting

$$\big|\tau^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}}\big|' := \big|\tau^{\mathfrak{n}}_{\mathfrak{e}}\big|'.$$

Note here that only trees without decoration, that is $\mathfrak{o} = 0$, appear in the analysis of the well-posedness problem (9.1). Indeed, the trees without $\mathfrak{o}$-decoration are stable under the coproducts (Δ^+, Δ) defined below. The $\mathfrak{o}$-decoration is only involved in the analysis of the renormalization procedure and the associated convergence problem. We define the set

$$\mathcal{SC}$$

of strongly conform trees with $\mathfrak{o}$-decoration as the minimal set which contains SC and such that the vector space spanned by $\mathcal{SC}$ is stable under all the coproducts defined below. (One could also consider $\mathcal{SC}$ as a set of rooted decorated trees with node type set

$$\mathfrak{T}^{\mathcal{SC}}_{\text{node}} = \{\bullet, \circ\} \cup \{\textcolor{red}{\bullet}^{\cdot,\alpha}\}_{\alpha \in \mathbb{Z}[\beta_0]}.$$

We used such an identification in Section 6. In the present section we treat $\mathfrak{o}$ as a decoration, rather than as part of a node type.) Similarly, we define

$$\mathcal{C}$$

as the set of decorated trees with extended decorations of the form (9.2), where $\tau_1, \dots, \tau_N \in \mathcal{SC}$. We use the bold symbol $\boldsymbol{\tau}$ to denote a generic element of $\mathcal{SC}$ or $\mathcal{C}$. The above rule on the set of extended decorated trees is normal, subcritical and complete. (The subcriticality of the rule on this set of trees with extended decorations comes from the fact that, for any fixed $\gamma \in \mathbb{R}$, the decoration α of trees with extended homogeneity less than γ, will only range in the set of homogeneities of subtrees of strongly conform trees $\tau^{\mathfrak{n}}_{\mathfrak{e}}$ with homogeneity less than γ.)

9.2 Coproducts

We define coproducts in the spaces of rooted decorated trees. This requires first that we define what we mean by 'subtrees' and 'subforests'. Recall that the type sets

$$\mathfrak{T}^{SC}_{\text{node}} = \{\bullet, {\color{red}\bullet}, \circ\}, \quad \mathfrak{T}^{C}_{\text{node}} = \big\{\bullet, {\color{red}\bullet}, {\color{blue}\bullet}, \circ\big\}, \quad \mathfrak{T}_{\text{edge}} = \{\mathcal{I}\}$$

are fixed. Given a typed rooted tree τ, a nonempty connected subgraph of τ is called a **subtree** if it inherits from τ its type map. Any possibly empty family of disjoint subtrees of τ is called a **subforest**. Given a rooted tree τ and a subforest $\varphi = \big\{\tau_1, \dots, \tau_m\big\}$, we use the notation

$$\tau/\varphi$$

to denote the rooted tree obtained by identifying each subtree τ_i with a single node with node type $\bullet$ in the quotient tree. Precisely, writing $y \sim_\varphi z$ if y and z are in the same connected component of φ, we define τ/φ as the tree consisting of the node set $N_\tau/\sim_\varphi$ and the edge set $E_\tau \backslash E_\varphi$. Moreover, we write

$$\tau/^{\text{red}}\varphi, \quad \text{or} \quad \tau/^{\text{blue}}\varphi$$

if we give a corresponding color to the nodes of φ in the quotient tree.

- For any function $f : N_\tau \to \mathbb{N} \times \mathbb{N}^d$, define the function $[f]_\varphi$ on $N_{\tau/\varphi}$ by
$$[f]_\varphi\big([x]\big) := \sum_{y \sim_\varphi x} f(y),$$
where $[x]$ denotes the equivalence class of $x \in N_\tau$.
- Denote by $\partial\varphi$ the leaves of φ, that is, the set of edges $(x, y) \in E_\tau$ such that $x \in N_\varphi$ and $y \in N_\tau \backslash N_\varphi$. For any function $g : \partial\varphi \to \mathbb{N} \times \mathbb{N}^d$, define the function πg on N_τ by setting
$$(\pi g)(x) := \sum_{e=(x,y)\in\partial\varphi} g(e).$$
- For any decorations $\mathfrak{n}_\varphi$ and $\mathfrak{e}_\varphi$ on φ, define the function $\mathfrak{o}(\varphi, \mathfrak{n}_\varphi, \mathfrak{e}_\varphi) : N_{\tau/\varphi} \to \mathbb{Z}[\beta_0]$ by
$$\mathfrak{o}\big(\varphi, \mathfrak{n}_\varphi, \mathfrak{e}_\varphi\big)\big([\tau_j]\big) = \Big|(\tau_j)^{\mathfrak{n}_\varphi|_{\tau_j}}_{\mathfrak{e}_\varphi|_{\tau_j}}\Big|'$$
for each $1 \leqslant j \leqslant m$, and $\mathfrak{o}(\varphi, \mathfrak{n}_\varphi, \mathfrak{e}_\varphi) = 0$ outside $[\varphi]$.

Define

$$\begin{aligned} T &:= \operatorname{span}(\mathcal{SC}), \\ \mathsf{T}^+ &:= \operatorname{span}(\mathcal{C}), \\ \mathsf{U}^- &:= \mathbb{R}[\mathcal{SC}]. \end{aligned}$$

Note that T^+ is an algebra with the tree product and unit $\mathbf{1}_+ := {\color{blue}\bullet}^0$, and U^- is an algebra with the forest product and unit $\mathbf{1}_- := \varnothing$. The space T^+ will be built from the side space T^+ and the space U^- from the side space U^-. Similarly, the different splitting maps defining a regularity structure and a renormalization structure are built from splitting maps taking values in, or defined on, the spaces $T, \mathsf{T}^+, \mathsf{U}^-$.

Definition – *We introduce three splitting operators.*

1. *The linear map*
$$\mathsf{D} : T \to T \otimes \mathsf{T}^+$$
is defined for $\tau^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}} \in \mathcal{SC}$ *by*
$$\mathsf{D}\tau^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}} := \sum_{\mu} \sum_{\mathfrak{n}_\mu, \mathfrak{e}'_{\partial\mu}} \frac{1}{\mathfrak{e}'_{\partial\mu}!} \binom{\mathfrak{n}}{\mathfrak{n}_\mu} \mu^{\mathfrak{n}_\mu + \pi\mathfrak{e}'_{\partial\mu}, \mathfrak{o}|_\mu}_{\mathfrak{e}} \otimes \big(\tau/^{\text{blue}}\mu\big)^{[\mathfrak{n}-\mathfrak{n}_\mu]_\mu, \mathfrak{o}|_{\tau\backslash\mu}}_{\mathfrak{e}+\mathfrak{e}'_{\partial\mu}}, \tag{9.4}$$
where the first sum is over all subtrees μ of τ which contains the root of τ, and the second sum is over functions $\mathfrak{n} : N_\mu \to \mathbb{N} \times \mathbb{N}^d$, with $\mathfrak{n}_\mu \leqslant \mathfrak{n}$ and functions $\mathfrak{e}'_{\partial\mu} : \partial\mu \to \mathbb{N} \times \mathbb{N}^d$. The algebra morphism
$$\mathsf{D}^+ : \mathsf{T}^+ \to \mathsf{T}^+ \otimes \mathsf{T}^+,$$

is defined by formula (9.4) for $\tau_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}} \in \mathcal{C}$.

2. *The algebra morphism*
$$\mathsf{D}^- : \mathsf{U}^- \to \mathsf{U}^- \otimes \mathsf{U}^-$$
is defined by $\mathsf{D}^-(\mathbf{1}_-) := \mathbf{1}_- \otimes \mathbf{1}_-$, *and for* $\tau_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}} \in \mathcal{SC}$
$$\mathsf{D}^-\big(\tau_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}}\big) := \sum_{\varphi} \sum_{\mathfrak{n}_\varphi, \mathfrak{e}'_{\partial\varphi}} \frac{1}{\mathfrak{e}'_{\partial\varphi}!} \binom{\mathfrak{n}}{\mathfrak{n}_\varphi} \{\mathscr{M}\varphi\}_{\mathfrak{e}}^{\mathfrak{n}_\varphi + \pi\mathfrak{e}'_{\partial\varphi}, \mathfrak{o}|_\varphi} \otimes \big(\tau/^{\mathrm{red}}\varphi\big)_{\mathfrak{e}+\mathfrak{e}'_{\partial\varphi}}^{[\mathfrak{n}-\mathfrak{n}_\varphi]_\varphi, [\mathfrak{o}]_\varphi + \mathfrak{o}(\varphi, \mathfrak{n}_\varphi + \pi\mathfrak{e}'_{\partial\varphi}, \mathfrak{e})},$$
where the first sum is over all subforests φ *of* τ *which contains all red nodes of* τ, *and the sum over* $\mathfrak{n}_\varphi$ *and* $\mathfrak{e}'_{\partial\varphi}$ *is taken as in item* 1 *of the present definition.*

3. *The algebra morphism*
$$\overline{\mathsf{D}}^- : \mathsf{T}^+ \to \mathsf{U}^- \otimes \mathsf{T}^+$$
is defined by the same formula as D^-, *with the first sum restricted to subforests* φ *which are disjoint from the root of* τ.

As suggested by the target spaces of the preceding maps, the splitting map Δ will be constructed from D and the map Δ^+ from D^+, the maps δ and δ^- from D^-, and the map δ^+ from $\overline{\mathsf{D}}^-$. Only trees with blue roots appear in the right hand side of the tensor products defining D. This is consistent with the fact that the trees with blue roots will represent later elements of T^+. The restriction on the choice of φ in the definition of $\overline{\mathsf{D}}^-$ ensures that it takes values in $\mathsf{U}^- \otimes \mathsf{T}^+$ – recall that the root of a conform tree is $\bullet$, and that the multiplicative property
$$\overline{\mathsf{D}}^-(\boldsymbol{\tau}\boldsymbol{\sigma}) = \big(\overline{\mathsf{D}}^-\boldsymbol{\tau}\big)\big(\overline{\mathsf{D}}^-\boldsymbol{\sigma}\big)$$
holds. This reflects the fact that the product of two functions
$$\mathsf{g}(\boldsymbol{\tau})\,\mathsf{g}(\boldsymbol{\sigma}), \quad \boldsymbol{\tau}, \boldsymbol{\sigma} \in \mathcal{C}$$
does not cause any new renormalization.

Remark – *Keep in mind that the elements of* U^- *are meant to be evaluated by characters of* U^-, *and turned to numbers, while elements of* U *are meant to be turned to distributions. This is done jointly in a renormalized naive model* $(k \otimes \Pi^\zeta)\delta$. *Recall that the problem of renormalization comes from the fact that the kernel of the operator* $\mathbf{K}$ *explodes on the diagonal. The building block of the renormalization operations* δ *and* δ^- *is best understood in the light of the following archetype problem. Let* $g : ([0,1]^d)^n \to \mathbb{R}$ *be a function that is smooth outside the deep diagonal* $\mathsf{diag} := \big\{\mathbf{z} = (z_1, \dots, z_n) \in ([0,1]^d)^n ; z_1 = \cdots = z_n\big\}$, *near which it behaves as* $|\mathbf{z} - (z_1, \dots, z_1)|^{-a}$, *for an exponent* $a > d$. *The function* g *is not integrable in any neighbourhood of the deep diagonal, so it only makes sense as a distribution on* $([0,1]^d)^n \backslash \mathsf{diag}$
$$\int_{([0,1]^d)^n} g(\mathbf{z}) f(\mathbf{z}) d\mathbf{z},$$
for f *smooth, with support with empty intersection with the deep diagonal. Can we define a distribution* Λ *on* $([0,1]^d)^n$ *that extends this distribution? This can be done defining* Λ *on* $([0,1]^d)^n$
$$(\Lambda, \psi) = \int_{([0,1]^d)^n} g(\mathbf{z}) \left(\psi(\mathbf{z}) - \psi(\mathbf{z}_1) - \cdots - \frac{(\mathbf{z} - \mathbf{z}_1)^{[a-d]}}{[a-d]!} \psi^{([a-d])}(\mathbf{z}_1) \right) d\mathbf{z},$$
for any smooth function ψ *on* $([0,1]^d)^n$. *This formula defines indeed a distribution, which coincides with the distribution associated with* g *outside the deep diagonal, since the* $\psi^{[\ell]}(\mathbf{z}_1)$ *are null for functions with compact support with null intersection with* diag. *Taylor expansion appears as the building block of this extension procedure. In this parallel,* τ *has two pieces,* g *and* f, *so the role of* φ *in* D^- *would be played by either of them, and the role of the projector* p_- *in* δ^-, *defined below, would select only the diverging term. The term* $\varphi^n \otimes (\tau/\varphi)_n$ *in* $\delta^-\tau$ *would precisely correspond to a term* $g(\mathbf{z})\, \frac{(\mathbf{z}-\mathbf{z}_1)^n}{n!}\, f^{(n)}(\mathbf{z}_1)$ *in the integral defining* Λ. *A formula like the above defining relation for* D^- *appears if one deals with a multiple integral where several subintegrals define functions of their external variables of the same kind as* g, *and one uses a similar kind of extension procedure as above.*

The following lemma is proved in Appendix C.3.

Lemma 43. *One has the coassociativity formulas*

$$\begin{aligned}(\mathsf{D}\otimes \mathrm{Id})\mathsf{D} &= (\mathrm{Id}\otimes \mathsf{D}^+)\mathsf{D},\\ (\mathsf{D}^+\otimes \mathrm{Id})\mathsf{D}^+ &= (\mathrm{Id}\otimes \mathsf{D}^+)\mathsf{D}^+,\\ (\mathsf{D}^-\otimes \mathrm{Id})\mathsf{D}^- &= \big(\mathrm{Id}\otimes \mathsf{D}^-\big)\mathsf{D}^-,\\ (\mathsf{D}^-\otimes \mathrm{Id})\overline{\mathsf{D}}^- &= \big(\mathrm{Id}\otimes \overline{\mathsf{D}}^-\big)\overline{\mathsf{D}}^-.\end{aligned}$$

Moreover, one has the cointeraction formulas

$$\begin{aligned}\mathcal{M}^{(13)}\big(\mathsf{D}^-\otimes\overline{\mathsf{D}}^-\big)\mathsf{D} &= (\mathrm{Id}\otimes\mathsf{D})\mathsf{D}^-,\\ \mathcal{M}^{(13)}\big(\overline{\mathsf{D}}^-\otimes\overline{\mathsf{D}}^-\big)\mathsf{D}^+ &= (\mathrm{Id}\otimes\mathsf{D}^+)\overline{\mathsf{D}}^-.\end{aligned}$$

9.3 Regularity and renormalization structures

We define the Hopf algebra parts of regularity and renormalization structures, from the side spaces T^+ and U^-. We use the shorthand notation $\mathcal{I}^+_n(\tau)$ to denote the tree $\mathcal{I}_n(\tau)$ with a blue root, with $\mathcal{I}_n(\tau)$ standing for $\mathcal{I}(\tau)$ with decoration n on the edge outgoing from the root. We define subsets $\mathcal{C}^+\subset\mathcal{C}$ and $\mathcal{SC}^-\subset\mathcal{SC}$, by

$$\begin{aligned}\mathcal{C}^+ &:= \Big\{X^n\mathscr{J}\big(\mathcal{I}^+_{m_1}(\boldsymbol{\tau}_1),\dots,\mathcal{I}^+_{m_N}(\boldsymbol{\tau}_b)\big)\in\mathcal{C}\,;\,\big|\mathcal{I}^+_{m_j}(\boldsymbol{\tau}_j)\big|>0,\text{ for any } j=1,\dots,b\Big\},\\ \mathcal{SC}^- &:= \Big\{\boldsymbol{\tau}\in\mathcal{SC}\,;\,|\boldsymbol{\tau}|'<0\Big\}\end{aligned}$$

and set

$$\begin{aligned}T = U &= \mathrm{span}(\mathcal{SC}),\\ T^+ &:= \mathrm{span}(\mathcal{C}^+),\\ U^- &:= \mathbb{R}[\mathcal{SC}^-].\end{aligned}$$

Note the use of the two notions of homogeneity in these definitions, the extended homogeneity $|\cdot|$ for T and T^+, and the naive homogeneity $|\cdot|'$ for U and U^-. Denote by

$$p_+ : \mathsf{T}^+\to T^+$$

the canonical projection, and define an algebra morphism

$$p_- : \mathsf{U}^-\to U^-$$

setting

$$p_-(\boldsymbol{\tau}) := \begin{cases}\mathbf{1}_-, & \text{for } \boldsymbol{\tau}=\mathbf{1}_-, \bullet^{0,\alpha},\\ \boldsymbol{\tau}, & \text{for } \boldsymbol{\tau}\in\mathcal{SC}^-,\\ 0, & \text{for } \boldsymbol{\tau}\in\mathcal{SC}\backslash\Big\{\mathcal{SC}^-\cup\{\bullet^{0,\alpha}\}_{\alpha\in\mathbb{Z}[\beta_0]}\Big\}.\end{cases}$$

Definition – *Define the linear map s*

$$\Delta := \big(\mathrm{Id}\otimes p_+\big)\mathsf{D} : T\to T\otimes T^+,$$

and

$$\Delta^+ := \big(p_+\otimes p_+\big)\overline{\mathsf{D}}^+|_{T^+} : T^+\to T^+\otimes T^+.$$

Define the linear maps

$$\delta := \big(p_-\otimes\mathrm{Id}\big)\mathsf{D}^-|_U : U\to U^-\otimes U,$$

and

$$\delta^- := \big(p_-\otimes p_-\big)\mathsf{D}^-|_{U^-} : U^-\to U^-\otimes U^-.$$

Finally set

$$\delta^+ := (p_-\otimes\mathrm{Id})\overline{\mathsf{D}}^-|_{T^+} : T^+\to U^-\otimes T^+.$$

It follows from the multiplicativity of $p_\pm$ that Δ^+ and $\delta^\pm$ *are algebra morphisms.* The assumption $p_-(\bullet^{0,\mathbf{o}})=\mathbf{1}_-$, is needed to ensure the formulas

$$\begin{aligned}
\delta\boldsymbol{\tau} &= \mathbf{1}_- \otimes \boldsymbol{\tau} + \sum_{|\boldsymbol{\varphi}|<0} \boldsymbol{\varphi} \otimes (\boldsymbol{\tau}/\boldsymbol{\varphi}),\\
\delta^-\boldsymbol{\sigma} &= \mathbf{1}_- \otimes \boldsymbol{\sigma} + \boldsymbol{\sigma} \otimes \mathbf{1}_- + \sum_{|\boldsymbol{\sigma}|<|\boldsymbol{\psi}|<0} \boldsymbol{\psi} \otimes (\boldsymbol{\sigma}/\boldsymbol{\psi})
\end{aligned}$$

for $\boldsymbol{\tau} \in \mathcal{SC}$ and $\boldsymbol{\sigma} \in \mathcal{SC}^-$. Excluding planted trees from $\mathcal{SC}^-$ ensures the identity

$$\delta\mathcal{I}_k(\boldsymbol{\tau}) = (\mathrm{Id} \otimes \mathcal{I}_k)\delta\boldsymbol{\tau}.$$

This is a requirement of assumption **(C)**, which ensures the admissibility of the renormalized model.

Theorem 44. *Set*

$$\begin{aligned}
\mathscr{T} &:= \big((T^+, \Delta^+), (T, \Delta)\big),\\
\mathscr{U} &:= \big((U^-, \delta^-), (U, \delta)\big).
\end{aligned}$$

- *(a)* $\mathscr{T}$ *is a regularity structure satisfying assumptions* **(A)** *and* **(B)**, *with the grading* $|\cdot|$.
- *(b)* $\mathscr{U}$ *is a renormalization structure satisfying assumption* **(E)**, *with the grading* $|\cdot|'$.
- *(c)* $\mathscr{T}$ *and* $\mathscr{U}$ *are compatible and satisfy assumption* **(C)**.
- *(d)* *The compatibility assumption* **(D)** *between the splittings* $\upharpoonleft\!\downharpoonright^{\flat}_{\mathsf{e}}$ *and* δ *holds true.*

Proof – Write as shorthand

$$\begin{aligned}
\mathsf{D}\boldsymbol{\tau} \ \text{ or } \ \mathsf{D}^+\boldsymbol{\tau} &= \sum_i \boldsymbol{\sigma}_i \otimes \boldsymbol{\eta}_i,\\
\mathsf{D}^-\boldsymbol{\tau} \ \text{ or } \ \overline{\mathsf{D}}^-\boldsymbol{\tau} &= \sum_j \boldsymbol{\varphi}_j \otimes \boldsymbol{\psi}_j,
\end{aligned}$$

and note that the following stability formulas of the naive and extended homogeneities. One has

$$|\boldsymbol{\tau}| = |\boldsymbol{\sigma}_i| + |\boldsymbol{\eta}_i|, \tag{9.5}$$

$$|\boldsymbol{\tau}|' = |\boldsymbol{\varphi}_j|' + |\boldsymbol{\psi}_j|', \quad |\boldsymbol{\tau}| = |\boldsymbol{\psi}_j| \tag{9.6}$$

for each i and j. Here we define $|\mathbf{1}_-|' := 0$.

(a) By the first identity of (9.5),

$$(p_+ \otimes p_+)\mathsf{D}^+ p_+ = (p_+ \otimes p_+)\mathsf{D}^+$$

holds on T^+. Then one has the comodule property of Δ as follows.

$$\begin{aligned}
(\Delta \otimes \mathrm{Id})\Delta &= \big(\mathrm{Id} \otimes p_+ \otimes \mathrm{Id}\big)(\mathsf{D} \otimes \mathrm{Id})(\mathrm{Id} \otimes p_+)\mathsf{D}\\
&= \big(\mathrm{Id} \otimes p_+ \otimes p_+\big)(\mathsf{D} \otimes \mathrm{Id})\mathsf{D}\\
&= \big(\mathrm{Id} \otimes p_+ \otimes p_+\big)(\mathrm{Id} \otimes \mathsf{D}^+)\mathsf{D}\\
&= \big(\mathrm{Id} \otimes p_+ \otimes p_+\big)(\mathrm{Id} \otimes \mathsf{D}^+)(\mathrm{Id} \otimes p_+)\mathsf{D}\\
&= (\mathrm{Id} \otimes \Delta^+)\Delta.
\end{aligned}$$

The coassociativity of Δ^+ is obtained similarly. One gets for free the existence of an antipode on T^+ from the fact that T^+ is a connected graded bialgebra – see Proposition 46 in Appendix B.

(b) The comodule properties of δ and δ^- are obtained by the similar way to (a), since

$$(p_- \otimes p_-)\mathsf{D}^- p_- = (p_- \otimes p_-)\mathsf{D}^-$$

holds on U^-, by identity (9.6). By definition, $\mathbf{1}_-$ is the only element in U^- of 0 homogeneity, so U^- is a connected graded bialgebra.

(c) We prove the cointeraction property

$$\mathcal{M}^{(13)}(\delta \otimes \delta^+)\Delta = (\mathrm{Id} \otimes \Delta)\delta;$$

the proofs of other properties are left to readers. See also Proposition 28. The second identity of (9.6) yields

$$\delta^+ \circ p_+ = (\mathrm{Id} \otimes p_+)\delta^+$$

on T^+. Thus we have

$$\begin{aligned}
\mathcal{M}^{(13)}(\delta\otimes\delta^+)\Delta &= \mathcal{M}^{(13)}\big(\delta\otimes(\delta^+\circ p_+)\big)\mathsf{D} \\
&= \mathcal{M}^{(13)}\Big(p_-\otimes \mathrm{Id}\otimes p_-\otimes p_+\Big)\big(\mathsf{D}^-\otimes\overline{\mathsf{D}}^-\big)\mathsf{D} \\
&= \big(p_-\otimes\mathrm{Id}\otimes p_+\big)\mathcal{M}^{(13)}\big(\mathsf{D}^-\otimes\overline{\mathsf{D}}^-\big)\mathsf{D} \\
&= \big(p_-\otimes\mathrm{Id}\otimes p_+\big)(\mathrm{Id}\otimes\mathsf{D})\mathsf{D}^- \\
&= \big(\mathrm{Id}\otimes\Delta\big)\delta.
\end{aligned}$$

(d) Recall the explicit formula for the map $\upharpoonleft\!\downharpoonright_{\mathsf{e}}$, from Lemma 31. It is obvious that U is stable under $\upharpoonleft\!\downharpoonright_{\mathsf{e}}$. Define

$$\upharpoonleft\!\downharpoonright(\tau_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}}) := \sum_{\sigma\in A(\tau)}\sum_{\mathfrak{n}_\sigma,\mathfrak{e}'_{\partial\sigma}}\frac{1}{\mathfrak{e}'_{\partial\sigma}!}\binom{\mathfrak{n}}{\mathfrak{n}_\sigma}(\tau/\sigma)^{\mathfrak{n}-\mathfrak{n}_\sigma,\mathfrak{o}|_{\tau\setminus\sigma}}_{\mathfrak{e}+\mathfrak{e}'_{\partial\sigma}}\otimes\sigma_{\mathfrak{e}}^{\mathfrak{n}_\sigma+\mathfrak{e}'_{\partial\sigma},\mathfrak{o}},$$

where $A(\tau) := \{P_e\tau\}_{e\in E_\tau}$. Comparing this with the definition of Δ^+, it is not difficult to show the equality

$$\mathcal{M}^{(13)}\big(\delta\otimes\delta\big)\upharpoonleft\!\downharpoonright = (\mathrm{Id}\otimes\upharpoonleft\!\downharpoonright)\delta$$

proceeding as in the proof of point (c). Note that the contracted tree τ/σ is always planted. Let p_{e} be the canonical projection on the set of planted trees η with

$$\mathfrak{n}(\rho_\eta) = 0, \quad \mathfrak{e}(e_\eta) = \mathsf{e},$$

where e_η is the only one edge leaving the root ρ_η, and let c be the map sending the tree of the form $\mathcal{I}_n(\tau)$ to τ. Then

$$\upharpoonleft\!\downharpoonright_{\mathsf{e}}^{\flat} = (c\circ p_{\mathsf{e}}\otimes\mathrm{Id})\upharpoonleft\!\downharpoonright .$$

on U. Since it is elementary to show

$$\begin{aligned}
(\mathrm{Id}\otimes c)\delta &= \delta\circ c, \\
(\mathrm{Id}\otimes p_{\mathsf{e}})\delta &= \delta\circ p_{\mathsf{e}},
\end{aligned}$$

the compatibility condition follows by writing

$$\begin{aligned}
\mathcal{M}^{(13)}(\delta\otimes\delta)\upharpoonleft\!\downharpoonright_{\mathsf{e}}^{\flat} &= \mathcal{M}^{(13)}\Big(\big(\mathrm{Id}\otimes c\circ p_{\mathsf{e}}\big)\delta\otimes\delta\Big)\upharpoonleft\!\downharpoonright \\
&= \big(\mathrm{Id}\otimes c\circ p_{\mathsf{e}}\otimes\mathrm{Id}\big)\mathcal{M}^{(13)}(\delta\otimes\delta)\upharpoonleft\!\downharpoonright \\
&= \big(\mathrm{Id}\otimes c\circ p_{\mathsf{e}}\otimes\mathrm{Id}\big)\big(\mathrm{Id}\otimes\upharpoonleft\!\downharpoonright\big)\delta = (\mathrm{Id}\otimes\upharpoonleft\!\downharpoonright_{\mathsf{e}}^{\flat})\delta.
\end{aligned}$$

$\triangleright$

9.4 Examples

Some examples are provided in this section.

– The renormalization of the singular PDE. For simplicity, consider the equation

$$(\partial_t-\Delta_x+1)u = f(u)\zeta + g(u)(\partial_x u)^2$$

with the noise $\zeta\in\mathcal{C}^{-1-\kappa}$ for sufficiently small $\kappa>0$. Theorem 40 yields that, for any $k\in G^-_{\mathrm{ad}}$ one has the renormalized equation

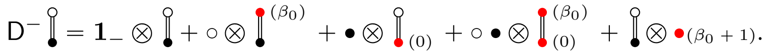
$$\begin{aligned}
(\partial_t-\Delta_x+1)u^{(k)} &= f(u^{(k)})\zeta + g(u^{(k)})(\partial_x u^{(k)})^2 \\
&\quad + k(\circ)f(u^{(k)}) + k(\circ_1)f'(u^{(k)})\partial_x u^{(k)} + 2k\Big(\ \Big)f(u^{(k)})g(u^{(k)})\partial_x u^{(k)} \\
&\quad + k\Big(\ \Big)f(u^{(k)})f'(u^{(k)}) + k\Big(\ \Big)f^2(u^{(k)})g(u^{(k)}).
\end{aligned}$$

More terms are needed when $\zeta\in\mathcal{C}^{-3/2-\kappa}$.

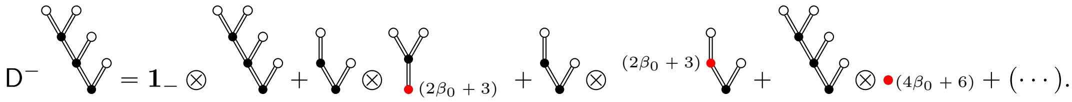
– The table below is the list of strongly conform trees associated with the generalized (KPZ) equation (6.1), without red nodes. Fix $d=1$ for simplicity. Fix also $\beta_0 = -3/2-\kappa$ for sufficiently small $\kappa>0$. The double line $\|$ represents the edge with $\mathfrak{e}$-decoration $(0,1)\in\mathbb{N}\times\mathbb{N}$. The dots $\circ_1$ and $\bullet_1$ represent the nodes with $\mathfrak{n}$-decoration $(0,1)\in\mathbb{N}\times\mathbb{N}$.

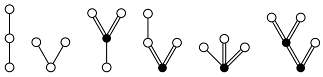
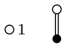
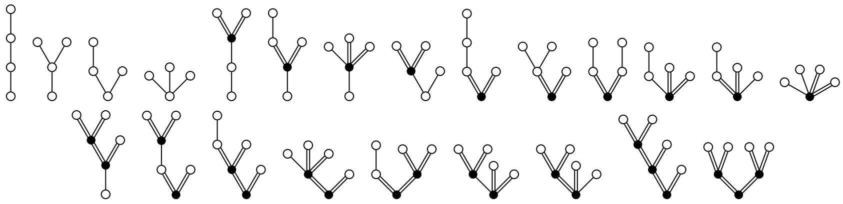
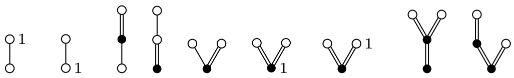
– Here are some examples of the actions of splitting map D^-. The dot $\bullet_{(\alpha)}$ represents the node with $\mathfrak{o}$-decoration $\alpha \in \mathbb{Z}[\beta_0]$.

$$\mathsf{D}^- \circ = \mathbf{1}_- \otimes \circ + \circ \otimes \bullet_{(\beta_0)},$$



For larger trees, it is inconvenient to write down all possible terms. Note that some of them vanishes by the application of p_- or Π^ζ. Omitting them by $(\cdots)$, one has for example



Homogeneity	Rooted decorated trees
$\beta_0 = -3/2 - \kappa$	$\circ$
$2\beta_0 + 2 = -1 - 2\kappa$	
$3\beta_0 + 4 = -1/2 - 3\kappa$	
$\beta_0 + 1 = -1/2 - \kappa$	
$4\beta_0 + 6 = -4\kappa$	
$2\beta_0 + 3 = -2\kappa$	

A – Summary of notations

The following is a summary of notations used in several sections.

Notations	Section	Meaning
$\mathscr{T} = \big((T^+,\Delta^+),(T,\Delta)\big)$	2.2	(Concrete) regularity structure.
$\mathcal{B}^+$, $\mathcal{B}$	2.2	Bases of T^+ and T.
S_+	2.2	Antipode of T^+.
G^+, $\widehat{h}$	2.2	Character group of T^+, and an action of $h \in G^+$ on T.
$\mathbb{R}\times\mathbb{R}^d$, $\mathcal{N}$	2.3	Space or spacetime domain and multi-index set.
$d(\cdot,\cdot)$, $\lvert\cdot\rvert_{\mathfrak{s}}$	2.3	Scaled metric and scaled degree.
$\big(T_{\underline{X}}^+, T_X\big)$, $\big(\mathcal{B}_{\underline{X}}^+,\mathcal{B}_X\big)$	2.3	Polynomial regularity structure and their bases.
$\mathcal{D}_{(0,t)}^{\gamma,\eta}$	4.3	Singular modelled distributions on the time interval $(0,t)$.
$\mathscr{U} = \big((U^-,\delta^-),(U,\delta)\big)$	5.1	Renormalization structure.
G^-, $\widetilde{k}$	5.1	Character group of U^-, and an action of $k \in G^-$ on U.
${}^k\mathsf{M}$	5.2	Renormalized model.
τ, N_τ, E_τ, ρ_τ	6.1	Rooted tree, node set, edge set, and root.
V, V'	6.1	Set of all rooted decorated trees, and its copy set.
$\overset{\mathsf{e}}{\curvearrowright}$, $\overset{\mathsf{e}}{\curvearrowright}_\flat$	6.1	Grafting operator $V^*\otimes V^*\to V^*$, and its projection on U^*.
$\mathsf{M}^\zeta = \big(\Pi^\zeta,\mathsf{g}^\zeta\big)$	6.2	Canonical model associated with a smooth noise ζ.
$\Rsh_{\mathsf{e}}$, $\Rsh_{\mathsf{e}}^\flat$	6.3	Dual map of $\curvearrowright_{\mathsf{e}}$, and its restriction to U.
$S'_- : U^- \to \mathbb{R}[U]$	7	Twisted negative antipode.
$\mathcal{SC}$	9.1	Strongly conform fully decorated trees.
$\mathcal{C}$	9.1	Conform fully decorated trees.

B – Basics from algebra

We recall some basics of bialgebras, Hopf algebras, and comodules without proofs. See [65, 59, 30] for details. Note that, for any two algebras A and B with units $\mathbf{1}_A$ and $\mathbf{1}_B$ respectively, the tensor space $A\otimes B$ is also an algebra with the product

$$(a_1\otimes b_1)\cdot(a_2\otimes b_2) := (a_1a_2)\otimes(b_1,b_2), \quad (a_1,a_2\in A,\ b_1,b_2\in B)$$

and with unit $\mathbf{1}_A\otimes\mathbf{1}_B$.

Definition – *A* **Bialgebra** $(B,\mathcal{M},\mathbf{1},\triangle,\theta)$ *is a 5-tuple of the following components.*

- *An algebra B with product $\mathcal{M}: B\otimes B\to B$, and unit $\mathbf{1}$.*
- *An algebra morphism $\triangle : B\to B\otimes B$ satisfying the* **coassociativity**

$$(\triangle\otimes\mathrm{Id})\triangle = (\mathrm{Id}\otimes\triangle)\triangle.$$

- *An algebra morphism $\theta : B\to\mathbb{R}$, satisfying*

$$(\theta\otimes\mathrm{Id})\triangle = (\mathrm{Id}\otimes\theta)\triangle = \mathrm{Id},$$

where we identify $a\otimes\tau = \tau\otimes a = a\tau$, for any $a\in\mathbb{R}$ and $\tau\in B$.

The map $\triangle$ is called a **coproduct***, and the map θ is called a* **counit***. An algebra morphism $S : B\to B$, such that*

$$\mathcal{M}(\mathrm{Id}\otimes S)\triangle = \mathcal{M}(S\otimes\mathrm{Id})\triangle = \theta(\cdot)\mathbf{1} \tag{B.1}$$

is called an **antipode***. A bialgebra equipped with an antipode S is called a* **Hopf algebra***.*

The counit $\theta(\cdot)$ is traditionally denoted $\varepsilon(\cdot)$. We use a different letter as ε already stands for a regularisation parameter in this work. The following result gives a sufficient condition for a bialgebra to be a Hopf algebra. A bialgebra B is called **graded** if it is a direct sum $\bigoplus_{\lambda\in\Lambda} B_\lambda$ of vector spaces such that

- Λ be a locally finite subset of $[0,\infty)$ such that $0\in\Lambda$ and $\Lambda+\Lambda\subset\Lambda$.
- $\mathbf{1}\in B_0$ and $B_\lambda\cdot B_\mu\subset B_{\lambda+\mu}$, for any $\lambda,\mu\in\Lambda$.
- $\triangle B_\lambda\subset\sum_{\mu,\nu\in\Lambda,\,\mu+\nu=\lambda} B_\mu\otimes B_\nu$.

We call Λ a **grading** in this paper. A graded bialgebra with $B_0=\langle\mathbf{1}\rangle$ is said to be **connected**.

Proposition 45. [65, Exercises pages 228 and 238], [59, Proposition II.1.1 and Corollary II.3.2] *Any connected graded bialgebra is a Hopf algebra. Moreover, one has the following properties.*

- $\theta(\mathbf{1})=1$ *and* $\theta(\tau)=0$ *for any* $\tau\in\bigoplus_{\lambda>0} B_\lambda$.
- $\triangle\mathbf{1}=\mathbf{1}\otimes\mathbf{1}$, *and for any* $\tau\in B_\lambda$ *with* $\lambda>0$,

$$\triangle\tau\in\Bigg\{\tau\otimes\mathbf{1}+\mathbf{1}\otimes\tau+\sum_{\substack{\mu,\nu\in\Lambda\\ \mu+\nu=\lambda,\,0<\mu<\lambda}} B_\mu\otimes B_\nu\Bigg\}.$$

Based on the first assertion, we denote by $\mathbf{1}'$ the counit θ of a connected graded bialgebra. The preceding formula for $\Delta\tau$ gives an inductive formula for the antipode. For $\tau\neq\mathbf{1}$ and $\Delta\tau=\tau\otimes\mathbf{1}+\mathbf{1}\otimes\tau+\sum\tau_1\otimes\tau_2$, one has

$$S(\tau)=-\tau-\sum S(\tau_1)\tau_2.$$

On the dual space B' of the bialgebra B, the **convolution product** is defined by

$$(f*g)\tau:=(f\otimes g)\Delta'\tau,$$

for all $f,g\in B'$, $\tau\in B$, where we identify $a\otimes b=ab$ for any $a,b\in\mathbb{R}$. The coassociativity of Δ' implies the associativity of the convolution

$$(f*g)*h=f*(g*h),$$

for all $f,g\in B'$, and the counit θ is indeed a unit of the convolution product

$$f*\theta=\theta*f=f,$$

for all $f\in B'$. Hence the triplet $(B',*,\theta)$ is a unital ring. Moreover, the subset $G\subset B'$ of algebra morphisms $g:B\to\mathbb{R}$ is stable under the convolution product. The existence of an antipode S implies that G is a group. Indeed, the inverse of $g\in G$ is given by $g^{-1}=g\circ S$. Each element of G is called a **character**, and when B is a Hopf algebra, the set G is called the **character group**.

We recall comodules and comodule bialgebras. Given an algebra A and two spaces E,F, we define on the algebraic tensor product $A\otimes E\otimes A\otimes F$ the $A\otimes E\otimes F$-valued map

$$\mathcal{M}^{(13)}\big(a_1\otimes e\otimes a_2\otimes f\big):=(a_1a_2)\otimes e\otimes f.$$

Definition – *Let* $(B,\mathcal{M},\mathbf{1},\triangle,\theta)$ *be a bialgebra.*

- *A linear space* M *equipped with a linear map* $\delta:M\to B\otimes M$, *with the properties*

$$(\mathrm{Id}_B\otimes\delta)\delta=(\triangle\otimes\mathrm{Id}_M)\delta,\quad \textit{and}\quad (\theta\otimes\mathrm{Id}_M)\delta=\mathrm{Id}_M,$$

  is called a left B-**comodule**. *Similarly, a linear space* N *is called a right* B*-comodule if a linear map* $\rho:N\to N\otimes B$, *exists and satisfies*

$$(\rho\otimes\mathrm{Id}_B)\rho=(\mathrm{Id}_N\otimes\triangle)\rho,\quad \textit{and}\quad (\mathrm{Id}_N\otimes\theta)\rho=\mathrm{Id}_N.$$

- *A bialgebra* M *is called a left* B-**comodule bialgebra** *if* M *is a left* B*-comodule by an algebra morphism* $\delta:M\to B\otimes M$, *such that*

$$\mathcal{M}^{(13)}(\delta\otimes\delta)\triangle_M=(\mathrm{Id}\otimes\triangle_M)\delta,\quad (\mathrm{Id}\otimes\theta_M)\delta=\theta_M(\cdot)\mathbf{1},$$

  where $\triangle_M$ *is a coproduct of* M, *and* θ_M *is a counit of* M.

Proposition 46. [30, Proposition 2] *Let* M *be a* B*-comodule bialgebra. If* M *has an antipode* S_M, *then*

$$\delta\circ S_M=(\mathrm{Id}_B\otimes S_M)\delta.$$

C – Technical proofs

This section is dedicated to proving Theorem 20 and Lemma 43.

C.1 Proof of Theorem 20

In this section p_t denotes the heat kernel of the operator $\mathcal{G} = \partial_{x_0}^2 - \Delta_{x'}^2$ on $\mathbb{R} \times \mathbb{R}^d$. For any $x \in \mathbb{R} \times \mathbb{R}^d$, and $\lambda \in (0,1]$, denote by $\varphi \mapsto \varphi_x^\lambda$ the transformation of functions on $\mathbb{R} \times \mathbb{R}^d$ defined by

$$\varphi_x^\lambda(y) := \lambda^{-d-2}\varphi\Big(\lambda^{-2}(y_0 - x_0), \lambda^{-1}(y' - x')\Big).$$

The following bound appears in the Hairer's original paper [44]. Recall $\beta_0 = \min A$.

Lemma 47. *Let $\mathsf{M} = (\mathsf{g}, \Pi)$ be a model over the regularity structure $\mathscr{T}$ and $\boldsymbol{f} \in \mathcal{D}^\gamma(T, \mathsf{g})$ with $\gamma \in \mathbb{R}$. Assume $\beta_0 > -2$. Then for any Schwartz function $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$, $x \in \mathbb{R} \times \mathbb{R}^d$, and $\lambda \in (0,1]$, one has the bound*

$$\Big|\Big\langle \mathbf{R}^\mathsf{M}\boldsymbol{f} - \Pi_x^\mathsf{g}\boldsymbol{f}(x), \varphi_x^\lambda \Big\rangle\Big| \leqslant C_\varphi \|\Pi^\mathsf{g}\| \, \|\boldsymbol{f}\|_{\mathcal{D}^\gamma} \, \lambda^\gamma,$$

where the constant C_φ depends on the size $\sup_{|k|_\mathfrak{s}, |\ell|_\mathfrak{s} \leqslant N} \|x^k \partial^\ell \varphi\|_{L^\infty(\mathbb{R}\times\mathbb{R}^d)}$ for $N > 0$ large enough.

Proof – Write $\Lambda_x := \mathbf{R}^\mathsf{M}\boldsymbol{f} - \Pi_x^\mathsf{g}\boldsymbol{f}(x)$, to shorten notations. Using $p_0 = \int_0^{\lambda^4} \mathcal{G}p_t dt + p_{\lambda^4}$ and the semigroup property,

$$\begin{aligned}
\langle \Lambda_x, \varphi_x^\lambda \rangle &= \int_{\mathbb{R}\times\mathbb{R}^d} \int_0^{\lambda^4} \big\langle \Lambda_x, \mathcal{G}p_t(\cdot, y)\big\rangle \varphi_x^\lambda(y)\, dtdy + \int_{\mathbb{R}\times\mathbb{R}^d} \big\langle \Lambda_x, p_{\lambda^4}(\cdot, y)\big\rangle \varphi_x^\lambda(y)\, dy \\
&= \int_{\mathbb{R}\times\mathbb{R}^d} \int_0^{\lambda^4} \big\langle \Lambda_x, p_{t/2}(\cdot, z)\big\rangle \big\langle \mathcal{G}p_{t/2}(z, \cdot), \varphi_x^\lambda \big\rangle \, dtdz + \int_{\mathbb{R}\times\mathbb{R}^d} \big\langle \Lambda_x, p_{\lambda^4}(\cdot, y)\big\rangle \varphi_x^\lambda(y)\, dy =: (A) + (B).
\end{aligned}$$

Using the properties of models as in Proposition 2, one has

$$\big|\langle \Lambda_x, p_{t/2}(\cdot, z)\rangle\big| \lesssim t^{\gamma/4} + \sum_{\beta<\gamma} t^{\beta/4} d(z,x)^{\gamma-\beta}.$$

This implies $|(B)| \lesssim \lambda^\gamma$. To consider (A), note that all polynomials with isotropic order less than 2 vanishes by $\mathcal{G}$. Thus one has

$$\begin{aligned}
\Big|\Big\langle \mathcal{G}p_{t/2}(z, \cdot), \varphi_x^\lambda \Big\rangle\Big| &= \Big|\Big\langle \mathcal{G}p_{t/2}(z, \cdot), \varphi_x^\lambda - \varphi_x^\lambda(z) - \nabla(\varphi_x^\lambda)(z)(\cdot - z)\Big\rangle\Big| \\
&\lesssim C_\varphi \int_{\mathbb{R}\times\mathbb{R}^d} \big|\mathcal{G}p_{t/2}(z,y)\big| \, (\lambda^{-1}d(y,z))^{2\varepsilon} \Big\{ \big(|\varphi|^{1-\varepsilon}\big)_x^\lambda(y) + \big(|\varphi|^{1-\varepsilon}\big)_x^\lambda(z) \\
&\quad + \big(\lambda^{-2}|y_0 - z_0|\big)^{1-\varepsilon} \big(|\partial_{x_0}\varphi|^{1-\varepsilon}\big)_x^\lambda(z) + \big(\lambda^{-1}|y' - z'|\big)^{1-\varepsilon} \big(|\nabla_{x'}\varphi|^{1-\varepsilon}\big)_x^\lambda(z)\Big\} \, dy
\end{aligned}$$

for any $\varepsilon \in (0,1)$. Recall $\beta_0 < 2$. Choosing ε such that $-\beta_0 < 2\varepsilon$, one has the estimate

$$\int_{\mathbb{R}\times\mathbb{R}^d} \big|\langle \Lambda_x, p_{t/2}(\cdot, z)\rangle\big| \, \Big|\Big\langle \mathcal{G}p_{t/2}(z, \cdot), \varphi_x^\lambda \Big\rangle\Big| \, dz \lesssim \sum_{a>0} t^{\frac{a-4}{4}} \lambda^{\gamma - a},$$

where a runs over finite number of positive constants. This implies $|(A)| \lesssim \lambda^\gamma$. ▷

We recall now from J. Martin's work [60, Theorem 5.3.16] the existence of a 'Whitney extension' map on locally defined modelled distributions. Assume that the regularity structure $\mathscr{T}$ satisfies assumption **(A1)**. Let $t > 0$ and $\mathbb{R} \times \mathbb{R}_t^d := (-\infty, t] \times \mathbb{R}^d$. Denote by $\mathcal{D}_t^\gamma(T, \mathsf{g})$ the set of functions $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}_t^d \to T_{<\gamma}$ which satisfies the bounds of $[\![\boldsymbol{f}]\!]_{\mathcal{D}_t^\gamma}$ and $\|\boldsymbol{f}\|_{\mathcal{D}_t^\gamma}$ as in Definition 3 with $\mathbb{R} \times \mathbb{R}^d$ replaced by $\mathbb{R} \times \mathbb{R}_t^d$.

Theorem 48. *Let $\mathsf{M} = (\mathsf{g}, \Pi)$ be a model over $\mathscr{T}$ with a regular product $\star$ satisfying assumption* **(A3)**. *Then there exists a continuous liner operator $E : \mathcal{D}_t^\gamma(T, \mathsf{g}) \to \mathcal{D}^\gamma(T, \mathsf{g})$ such that $(E\boldsymbol{f})|_{\mathbb{R}\times\mathbb{R}_t^d} = \boldsymbol{f}$, and the bound*

$$|\!|\!| E\boldsymbol{f} |\!|\!|_{\mathcal{D}^\gamma} \leqslant C |\!|\!| \boldsymbol{f} |\!|\!|_{\mathcal{D}_t^\gamma}$$

holds for a positive constant C independent of $t > 0$ and $\boldsymbol{f} \in \mathcal{D}_t^\gamma(T, \mathsf{g})$. A similar result holds for the modelled distributions defined on $\mathcal{F}_t = [t, \infty) \times \mathbb{R}^d$.

Proof – We consider a different construction from Martin's work [60], but the proof uses the same mechanics. We provide only the sketch of the proof here.
Without loss of generality, we consider $t = 0$. Let $h(x_0, x') = h_{x_0}(x')$ be the kernel of the operator $e^{x_0 \Delta_{x'}}$ with $x_0 > 0$. We define the function

$$\boldsymbol{h}(x) = \sum_{|k|_\mathfrak{s} < \gamma} \frac{\partial^k h(x)}{k!} X^k$$

on $(0, \infty) \times \mathbb{R}^d$, and set $(E\boldsymbol{f})(x) := \boldsymbol{f}(x)$ if $(-\infty, 0] \times \mathbb{R}^d$ and

$$(E\boldsymbol{f})(x) := \int_{\mathbb{R}^d} \boldsymbol{h}\big(x - (0, y')\big) \star \widehat{\mathsf{g}_{x(0,y')}} f(0, y')\, dy'$$

if $x \in (0, \infty) \times \mathbb{R}^d$. The bounds of $\big\|(E\boldsymbol{f})(x)\big\|_\beta$ for each $\beta \in A$ follows from the estimate of $\partial^k h$ like (2.16). For the bounds of $\big\|(E\boldsymbol{f})(y) - \widehat{\mathsf{g}_{yx}}(E\boldsymbol{f})(x)\big\|_\beta$, it is sufficient to consider the case $y_0 \geqslant x_0 \geqslant 0$. If $y_0 > x_0 = 0$, then by the property $\int_{\mathbb{R}^d} \boldsymbol{h}(y - (0, z'))dz' = \mathbf{1}$ and assumption **(A3)**,

$$(E\boldsymbol{f})(y) - \widehat{\mathsf{g}_{yx}} \boldsymbol{f}(x) = \int_{\mathbb{R}^d} \boldsymbol{h}\big(y - (0, z')\big) \star \widehat{\mathsf{g}_{y(0,z')}} \Big(\boldsymbol{f}(0, z') - \widehat{\mathsf{g}_{(0,z')x}} \boldsymbol{f}(x)\Big)\, dz'.$$

From the estimate of $\partial^k h$, one has the required bounds. If $y_0 = x_0 > 0$, by property (2.34) and using the (anisotropic) integral Taylor remainder formula

$$\begin{aligned}
(E\boldsymbol{f})(y) - \widehat{\mathsf{g}_{yx}}(E\boldsymbol{f})(x) &= \int_{\mathbb{R}^d} \Big(\boldsymbol{h}\big(y - (0, z')\big) - \widehat{\mathsf{g}_{yx}} \boldsymbol{h}\big(x - (0, z')\big)\Big) \star \widehat{\mathsf{g}_{y(0,z')}} \boldsymbol{f}(0, z')\, dz' \\
&= \sum_{|k|_\mathfrak{s} < \gamma,\, |\ell|_\mathfrak{s} > \gamma - |k|_\mathfrak{s}} \frac{(y - x)^\ell}{\ell!} \int_0^1 \varphi_\ell(r) \partial^{k+\ell} h\big(x_r - (0, z')\big)\, X^k \star \widehat{\mathsf{g}_{y(0,z')}} \boldsymbol{f}(0, z')\, dz',
\end{aligned}$$

where ℓ runs over a finite set, $x_r = x + r(y - x)$, and $\varphi_\ell(r)$ are bounded functions of r. Since $\int_{\mathbb{R}^d} \partial^{k+\ell} h(x_r - (0, z'))dz' = 0$, we can replace $\widehat{\mathsf{g}_{y(0,z')}} \boldsymbol{f}(0, z')$ by

$$\widehat{\mathsf{g}_{y(0,z')}}\, \boldsymbol{f}(0, z') - \widehat{\mathsf{g}_{yx'_r}} \boldsymbol{f}(x'_r) = \widehat{\mathsf{g}_{yx'_r}} \Big(\widehat{\mathsf{g}_{x'_r(0,z')}} \boldsymbol{f}(0, z') - \boldsymbol{f}(x'_r)\Big).$$

Then one can obtain the required estimate of $(E\boldsymbol{f})(y) - \widehat{\mathsf{g}_{yx}}(E\boldsymbol{f})(x)$. Even though $\widehat{\mathsf{g}_{yx'_r}}$ produces factors $|x_0|^a$ with $a > 0$ which is not compatible with $|y' - x'|$, it is cancelled by a factor $|x_0|^{-a}$ coming from the kernel $\partial^{k+\ell} h$. If $y_0 > x_0 > 0$, by the semigroup property

$$\int_{\mathbb{R}^d} \boldsymbol{h}\big(y - (x_0, w')\big) \star \boldsymbol{h}\big((x_0, w'), (0, z')\big)\, dw' = \boldsymbol{h}\big(y - (0, z')\big),$$

the argument leads to the case $y_0 > x_0 = 0$, since the required estimate is already obtained in $\mathbb{R} \times \mathbb{R}^d_{x_0}$.
$\triangleright$

We turn to the proof of the reconstruction theorem for singular modelled distributions.

Proof of Theorem 20 – The proof is just an analogue of the proof of Proposition 6.9 in Hairer' seminal work [44], so we omit a number of details. The only difference is that $p_t(x, \cdot)$ is not compactly supported. By linearity, assume that $\boldsymbol{f} = 0$ on $\mathbb{R} \times \mathbb{R}^d_0 = (-\infty, 0] \times \mathbb{R}^d$. Applying Theorem 48 to the restriction of $\boldsymbol{f}$ on $\mathcal{F}_a = [a, \infty) \times \mathbb{R}^d$ with $a > 0$, and writing $\widetilde{\mathsf{R}}^\mathsf{M}_a$ for $\mathsf{R}^\mathsf{M} \circ E_a$, for the extension map E_a from $\mathcal{F}_a$ to $\mathbb{R} \times \mathbb{R}^d$, one has the distribution $\widetilde{\mathsf{R}}^\mathsf{M}_a \boldsymbol{f}$ on $\mathbb{R} \times \mathbb{R}^d$ such that the bounds

$$\Big|\Big\langle \widetilde{\mathsf{R}}^\mathsf{M}_a \boldsymbol{f} - \mathsf{\Pi}^\mathsf{g}_x \boldsymbol{f}(x), \varphi^\lambda_x \Big\rangle\Big| \lesssim C_\varphi a^{(\eta - \gamma)/2} \|\mathsf{\Pi}^\mathsf{g}\|\, \|\boldsymbol{f}\|_{\mathcal{D}^\gamma}\, \lambda^\gamma$$

hold for any Schwartz function φ, $x \in \mathbb{R} \times \mathbb{R}^d$, $0 < \lambda \leqslant 1$ such that φ^λ_x is supported on $\{y \in \mathbb{R} \times \mathbb{R}^d; y_0 > a\}$. In particular, the restriction of $\widetilde{\mathsf{R}}^\mathsf{M}_a \boldsymbol{f}$ on $\mathcal{F}_a$ are compatible over all $a > 0$ because of the local property of the reconstruction operator, so the quantity $\big\langle \widetilde{\mathsf{R}}^\mathsf{M} \boldsymbol{f}, \varphi \big\rangle$ is defined for any φ supported on $(0, \infty) \times \mathbb{R}^d$. Since $\boldsymbol{f}$ vanishes on $\mathbb{R} \times \mathbb{R}^d_0$, one defines $\big\langle \widetilde{\mathsf{R}}^\mathsf{M} \boldsymbol{f}, \varphi \big\rangle = 0$ if φ is supported on $(-\infty, 0) \times \mathbb{R}^d$. To consider the paring with $p_t(x, \cdot)$, fix the family $\{\phi_{n,k}\}_{n \in \mathbb{N}, k \in \mathbb{Z}^d}$ of functions of the forms

$$\phi_{n,k} = 2^{-n(d+2)} \phi^{2^{-n}}_{x_{n,k}}, \quad x_{n,k} = \big(2^{-2n}, 2^{-n} k\big) \in \mathbb{R} \times \mathbb{R}^d$$

where ϕ is a smooth function supported on $\big\{x \in \mathbb{R} \times \mathbb{R}^d; d(0, x) < 1\big\}$, and such that $\sum_{n,k} \phi_{n,k}(x) = 1$ if $0 < x_0 < 1/2$. Now fix an integer n_0 such that $2^{-n_0} \simeq t^{1/4} \vee |x_0|^{1/2}$, and set $\widetilde{\phi}_{n_0} = 1 - \sum_{n \geqslant n_0, k \in \mathbb{Z}^d} \phi_{n,k}$. Then one can define

$$\Big\langle \widetilde{\mathsf{R}}^\mathsf{M} \boldsymbol{f} - \mathsf{\Pi}^\mathsf{g}_x \boldsymbol{f}(x), p_t(x, \cdot) \Big\rangle = \sum_{n \geqslant n_0, k \in \mathbb{Z}^d} \Big\langle \widetilde{\mathsf{R}}^\mathsf{M} \boldsymbol{f} - \mathsf{\Pi}^\mathsf{g}_x \boldsymbol{f}(x), p_t(x, \cdot)\, \phi_{n,k} \Big\rangle + \Big\langle \widetilde{\mathsf{R}}^\mathsf{M} \boldsymbol{f} - \mathsf{\Pi}^\mathsf{g}_x \boldsymbol{f}(x), p_t(x, \cdot)\, \widetilde{\phi}_{n_0} \Big\rangle$$

if the right hand side converges. For the second term, since $\widetilde{\phi}_{n_0}$ is supported on $\{y \in \mathbb{R}\times\mathbb{R}^d; y_0 \gtrsim 2^{-2n_0}\}$ and $p_t(x,\cdot)\,\widetilde{\phi}_{n_0}(\cdot) = f_x^{t^{1/4}}(\cdot)$ for some Schwartz function f which is uniform over t, x, n_0, one has

$$\left|\langle \widetilde{\mathsf{R}}^{\mathsf{M}}\boldsymbol{f} - \mathsf{\Pi}_x^{\mathsf{g}}\boldsymbol{f}(x), p_t(x,\cdot)\widetilde{\phi}_{n_0}\rangle\right| \lesssim \left(|x_0| \vee t^{1/2}\right)^{(\eta-\gamma)/2} t^{\gamma/4}.$$

For the first term, one decomposes

$$\begin{aligned}\widetilde{\mathsf{R}}^{\mathsf{M}}\boldsymbol{f} - \mathsf{\Pi}_x^{\mathsf{g}}\boldsymbol{f}(x) &= \left(\widetilde{\mathsf{R}}^{\mathsf{M}}\boldsymbol{f} - \mathsf{\Pi}_{x_{n,k}}^{\mathsf{g}}\boldsymbol{f}(x_{n,k})\right) + \mathsf{\Pi}_{x_{n,k}}^{\mathsf{g}}\boldsymbol{f}(x_{n,k}) - \mathsf{\Pi}_{x_{n,k}}^{\mathsf{g}}\left(\widehat{\mathsf{g}_{x_{n,k}x}}\boldsymbol{f}(x)\right)\\ &=: (a) + (b) + (c).\end{aligned}$$

Since roughly $p_t(x,\cdot)\phi_{n,k} \simeq 2^{-n(d+2)}p_t(x,x_{n,k})\phi_{x_{n,k}}^{2^{-n}}$, one has

$$\left|\langle (a), p_t(x,\cdot)\phi_{n,k}\rangle\right| \lesssim 2^{-n(d+2)}\,p_t(x,x_{n,k})(2^{-n})^{\eta-\gamma}(2^{-n})^{\gamma} = 2^{-n(\eta+2)}2^{-nd}\,p_t(x,x_{n,k}).$$

The sum $2^{-nd}\sum_{k\in\mathbb{Z}^d} p_t(x,x_{n,k})$ is roughly equals to $h_t(x_0)$, where h_t is a one-dimensional heat kernel. Hence

$$\begin{aligned}\sum_{n\geqslant n_0}\sum_{k\in\mathbb{Z}^d} |\langle (a), p_t(x,\cdot)\phi_{n,k}\rangle| &\lesssim 2^{-n_0(\eta+2)}h_t(x_0)\\ &\lesssim \left(|x_0|^{\frac12} \vee t^{\frac14}\right)^{\eta+2}t^{-\frac12}\left(|t^{-\frac12}x_0| \vee 1\right)^{-\frac{\gamma+2}{2}} = t^{\frac{\gamma}{4}}\left(|x_0| \vee t^{\frac12}\right)^{\frac{\eta-\gamma}{2}}.\end{aligned}$$

Using the bound $\|f(x)\|_\beta \lesssim |x_0|^{(\eta-\beta)/2\wedge 0}$, one gets the same bounds as above for (b) and (c), with η replaced by β_0. $\triangleright$

C.2 Proof of Proposition 29 and Proposition 30

We provide a sketch of the proof Proposition 29 and Proposition 30, following [30]. Recall that, the basis $\mathcal{V}$ of V is the set of all rooted trees with node types $\mathfrak{T}_{\mathrm{n}}$ and edge types $\mathfrak{T}_{\mathrm{e}}$, and with decorations $\mathfrak{n} : N_\tau \to \mathbb{N}^{d+1}$ and $\mathfrak{e} : E_\tau \to \mathbb{N}^{d+1}$. Write $\mathsf{N} = \mathfrak{T}_{\mathrm{n}} \times \mathbb{N}^{d+1}$ and $\mathsf{E} = \mathfrak{T}_{\mathrm{e}} \times \mathbb{N}^{d+1}$. Moreover, the basis $\mathcal{B}$ of U is a subset satisfying assumption **(D)**. For simplicity, we prove the following proposition for V and U, instead of V^* and U^*.

Proposition – *(a) The space V (or U) with the operators $\{\overset{\mathsf{e}}{\curvearrowright}\}_{\mathsf{e}\in\mathsf{E}}$ (or $\{\overset{\mathsf{e}}{\curvearrowright}_\flat\}_{\mathsf{e}\in\mathsf{E}}$) is an E-multi-pre-Lie algebra.*

(b) Let $(W, \{\rhd_{\mathsf{e}}\}_{\mathsf{e}\in\mathsf{E}})$ be an E-multi-pre-Lie algebra, and let $\{\varphi_{\mathsf{t}^n}\}_{(\mathsf{t},n)\in\mathsf{N}}$ (or $\{\varphi_{\mathsf{t}^n}\}_{(\mathsf{t},n)\in\mathsf{N}\cap\mathcal{B}}$) $\subset W$. Then there exists a unique E-multi-pre-Lie morphism $\varphi : V$ (or U) $\to W$ such that $\varphi(\mathsf{t}^n) = \varphi_{\mathsf{t}^n}$ for any $\mathsf{t}^n \in \mathsf{N}$ (or $\mathsf{N}\cap\mathcal{B}$).

The E-multi-pre-Lie property of $(V, \{\overset{\mathsf{e}}{\curvearrowright}\}_{\mathsf{e}\in\mathsf{E}})$

$$(\boldsymbol{\tau} \overset{\mathsf{e}}{\curvearrowright} \boldsymbol{\sigma}) \overset{\mathsf{e}'}{\curvearrowright} \boldsymbol{\eta} - \boldsymbol{\tau} \overset{\mathsf{e}}{\curvearrowright} (\boldsymbol{\sigma} \overset{\mathsf{e}'}{\curvearrowright} \boldsymbol{\eta}) = (\boldsymbol{\sigma} \overset{\mathsf{e}'}{\curvearrowright} \boldsymbol{\tau}) \overset{\mathsf{e}}{\curvearrowright} \boldsymbol{\eta} - \boldsymbol{\sigma} \overset{\mathsf{e}'}{\curvearrowright} (\boldsymbol{\tau} \overset{\mathsf{e}}{\curvearrowright} \boldsymbol{\eta})$$

is proved in a similar way to Proposition 4 of [30], so we omit the proof. The same property for $(U, \{\overset{\mathsf{e}}{\curvearrowright}_\flat\}_{\mathsf{e}\in\mathsf{E}})$ follows from it. Indeed, by assumption **(D1)** for the canonical projection $\pi_U : V \to U$, we have

$$(\boldsymbol{\tau} \overset{\mathsf{e}}{\curvearrowright}_\flat \boldsymbol{\sigma}) \overset{\mathsf{e}'}{\curvearrowright}_\flat \boldsymbol{\eta} - \boldsymbol{\tau} \overset{\mathsf{e}}{\curvearrowright}_\flat (\boldsymbol{\sigma} \overset{\mathsf{e}'}{\curvearrowright}_\flat \boldsymbol{\eta}) = \pi_U\left((\boldsymbol{\tau} \overset{\mathsf{e}}{\curvearrowright} \boldsymbol{\sigma}) \overset{\mathsf{e}'}{\curvearrowright} \boldsymbol{\eta} - \boldsymbol{\tau} \overset{\mathsf{e}}{\curvearrowright} (\boldsymbol{\sigma} \overset{\mathsf{e}'}{\curvearrowright} \boldsymbol{\eta})\right).$$

In order to prove *(b)*, we introduce the Guin-Oudom extension of the multi-pre-Lie structure. The following is the content of Section 2.2 of [30].

Definition – *Let $(W, \{\rhd_{\mathsf{e}}\}_{\mathsf{e}\in\mathsf{E}})$ be an E-multi-pre-Lie algebra. Let $(W, \mathsf{e}) = \{(a, \mathsf{e})\}_{a\in W}$ be a copy of the linear space W, and denote*

$$W^{\oplus\mathsf{E}} := \bigoplus_{\mathsf{e}\in\mathsf{E}}(W, \mathsf{e}).$$

Moreover, let $S(W^{\oplus\mathsf{E}})$ be the symmetric algebra of $W^{\oplus\mathsf{E}}$, with unit $\mathbf{1}$. Then one can define the following linear maps.

- *Define the linear map* $\rhd_{\mathsf{e}} : W \otimes S(W^{\oplus \mathsf{E}}) \to S(W^{\oplus \mathsf{E}})$ *inductively as follows.*
$$c \rhd_{\mathsf{e}} \mathbf{1} = 0,$$
$$c \rhd_{\mathsf{e}} \prod_{i=1}^{N} (b_i, \mathsf{e}_i) = \sum_{i=1}^{N} (a \rhd_{\mathsf{e}} b_i, \mathsf{e}_i) \prod_{j \neq i} (b_j, \mathsf{e}_j),$$
where $c, b_1, \dots, b_N \in W$ *and* $\mathsf{e}_1, \dots, \mathsf{e}_N \in \mathsf{E}$.
- *Define the linear map* $\rhd : S(W^{\oplus \mathsf{E}}) \otimes W \to W$ *inductively as follows.*
$$\mathbf{1} \rhd b = b,$$
$$(c, \mathsf{e}) \rhd b = c \rhd_{\mathsf{e}} b,$$
$$\prod_{i=1}^{N} (c_i, \mathsf{e}_i) \rhd b = c_1 \rhd_{\mathsf{e}_1} \left(\prod_{i=2}^{N} (c_i, \mathsf{e}_i) \rhd b \right) - \left(c_1 \rhd_{\mathsf{e}_1} \prod_{i=2}^{N} (c_i, \mathsf{e}_i) \right) \rhd b,$$
where $c, c_1, \dots, c_N, b \in W$ *and* $\mathsf{e}, \mathsf{e}_1, \dots, \mathsf{e}_N \in \mathsf{E}$. *(The last quantity is invariant under the permutations of* $(c_1, \mathsf{e}_1), \dots, (c_N, \mathsf{e}_N)$ *because of the multi-pre-Lie property of* W*, so the extension* $\rhd$ *is well-defined.)*

The above extensions keep the multi-pre-Lie morphism property. Indeed, if $\varphi : V \to W$ is an E-multi-pre-Lie morphism, then defining the extension $\varphi : S(V^{\oplus \mathsf{E}}) \to S(W^{\oplus \mathsf{E}})$ by $\varphi((c, \mathsf{e})) := (\varphi(c), \mathsf{E})$ for any $(c, \mathsf{e}) \in (V, \mathsf{e})$, and denoting by $\curvearrowright$ the extension $\curvearrowright : S(V^{\oplus \mathsf{E}}) \otimes V \to V$, one has
$$\varphi(X \curvearrowright b) = \varphi(X) \rhd \varphi(b)$$
for any $X \in S(V^{\oplus \mathsf{E}})$ and $b \in V$.

Since $\mathfrak{T}_{\mathrm{e}} = \{\mathcal{I}\}$, we identify E with $\mathbb{N}^{d+1}$. The following formula can be proved by a similar argument to Lemma 2.6 of [31] by noting that π_U satisfies assumption **(D1)**.

Lemma 49. *For any* $(\mathfrak{t}, n) \in \mathsf{N} \cap \mathcal{B}$, $\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_I \in \mathcal{B}$, *and* $p_1, \dots, p_N \in \mathbb{N}^{d+1}$, *one has*
$$\pi_U \left(\mathfrak{t}^n \star \mathop{\bigstar}_{i=1}^{a} \mathcal{I}_{p_i}(\boldsymbol{\tau}_i) \right) = \sum_{\substack{q_1 \leqslant p_1, \dots, q_N \leqslant p_N \\ q_1 + \dots + q_N \leqslant n}} (-1)^{|q_1| + \dots + |q_N|} \binom{n}{q_1, \dots, q_N} \prod_{i=1}^{a} (\boldsymbol{\tau}_i, p_i - q_i) \curvearrowright_{\flat} \mathfrak{t}^{n - q_1 - \dots - q_N},$$
where $\curvearrowright_{\flat}$ *denotes the extension* $\curvearrowright_{\flat} : S(U^{\oplus \mathsf{E}}) \otimes U \to U$, *and* $\binom{n}{q_1, \dots, q_N}$ *is the multinomial coefficient*
$$\binom{n}{q_1, \dots, q_N} = \frac{n!}{q_1! \cdots q_N! (n - q_1 - \dots - q_N)!}.$$

Then we can prove the uniqueness part of *(b)* immediately. Indeed, if $\varphi : U \to W$ is an E-multi-pre-Lie morphism, then it satisfies
$$\varphi \left(\mathfrak{t}^n \star \mathop{\bigstar}_{i=1}^{a} \mathcal{I}_{p_i}(\boldsymbol{\tau}_i) \right) = \sum_{\substack{q_1 \leqslant p_1, \dots, q_N \leqslant p_N \\ q_1 + \dots + q_N \leqslant n}} (-1)^{|q_1| + \dots + |q_N|} \binom{n}{q_1, \dots, q_N} \prod_{i=1}^{a} (\varphi(\boldsymbol{\tau}_i), p_i - q_i) \rhd \varphi(\mathfrak{t}^{n - q_1 - \dots - q_N})$$
for any $\mathfrak{t}^n \star \bigstar_{i=1}^{a} \mathcal{I}_{p_i}(\boldsymbol{\tau}_i) \in \mathcal{B}$. The right hand side provides the recursive definition of the map φ, so we can conclude that φ is determined by the values $\varphi(\mathfrak{t}^n)$ for any $\mathfrak{t}^n \in \mathsf{N}$. On the other hand, given $\{\varphi(\mathfrak{t}^n)\}_{(\mathfrak{t}, n) \in \mathsf{N}}$, we can prove that the map φ defined by the above formula satisfies indeed the multi-pre-Lie property. We do not provide the details here because the existence part of *(b)* is not used in this paper.

C.3 Proof of Lemma 43

C.3.1 Reduced coproducts

First we consider trees with $\mathfrak{n}$ and $\mathfrak{e}$-decorations, without $\mathfrak{o}$-decoration. Recall that SC is a set of strongly conform trees and C is a set of conform trees. Set

$$\begin{aligned}{}^\circ T &:= \mathrm{span}(SC),\\ {}^\circ\mathsf{T}^+ &:= \mathrm{span}(C),\\ {}^\circ\mathsf{U}^- &:= \mathbb{R}[SC].\end{aligned}$$

Definition – *We define the following splitting maps.*

1. *The linear map* ${}^\circ\mathsf{D} : {}^\circ T \to {}^\circ T \otimes {}^\circ\mathsf{T}^+$, *is defined for* $\tau^{\mathfrak{n}}_{\mathfrak{e}} \in SC$ *by*
$$ {}^\circ\mathsf{D}(\tau^{\mathfrak{n}}_{\mathfrak{e}}) := \sum_{\mu\in ST(\tau)} \sum_{\mathfrak{n}_\mu,\mathfrak{e}'_{\partial\mu}} \frac{1}{\mathfrak{e}'_{\partial\mu}!}\binom{\mathfrak{n}}{\mathfrak{n}_\mu} \mu^{\mathfrak{n}_\mu+\pi\mathfrak{e}'_{\partial\mu}}_{\mathfrak{e}} \otimes \big(\tau/^{\mathrm{blue}}\mu\big)^{[\mathfrak{n}-\mathfrak{n}_\mu]_\mu}_{\mathfrak{e}+\mathfrak{e}'_{\partial\mu}}, $$
where $ST(\tau)$ *is the set of all* subtrees μ *of* τ *which contain the root of* τ*, and the second sum is over functions* $\mathfrak{n} : N_\mu \to \mathcal{N}$ *with* $\mathfrak{n}_\mu \leqslant \mathfrak{n}$ *and functions* $\mathfrak{e}'_{\partial\mu} : \partial\mu \to \mathcal{N}$*. The algebra morphism*
$$ {}^\circ\mathsf{D}^+ : {}^\circ\mathsf{T}^+ \to {}^\circ\mathsf{T}^+ \otimes {}^\circ\mathsf{T}^+ $$
is defined by the same formula with $\tau^{\mathfrak{n}}_{\mathfrak{e}} \in C$.
2. *The algebra morphism*
$$ {}^\circ\mathsf{D}^- : {}^\circ\mathsf{U}^- \to {}^\circ\mathsf{U}^- \otimes {}^\circ\mathsf{U}^- $$
is defined by ${}^\circ\mathsf{D}^-\mathbf{1}_- = \mathbf{1}_- \otimes \mathbf{1}_-$*, and for* $\tau^{\mathfrak{n}}_{\mathfrak{e}} \in SC$
$$ {}^\circ\mathsf{D}^-(\tau^{\mathfrak{n}}_{\mathfrak{e}}) := \sum_{\varphi\in SF(\tau)} \sum_{\mathfrak{n}_\varphi,\mathfrak{e}'_{\partial\varphi}} \frac{1}{\mathfrak{e}'_{\partial\varphi}!}\binom{\mathfrak{n}}{\mathfrak{n}_\varphi} \{\mathscr{M}\varphi\}^{\mathfrak{n}_\varphi+\pi\mathfrak{e}'_{\partial\varphi}}_{\mathfrak{e}} \otimes \big(\tau/^{\mathrm{red}}\varphi\big)^{[\mathfrak{n}-\mathfrak{n}_\varphi]_\varphi}_{\mathfrak{e}+\mathfrak{e}'_{\partial\varphi}}, $$
where $SF(\tau)$ *is the set of all subforests* φ *of* τ *which contain all red nodes of* τ*, and the sum over* $\mathfrak{n}_\varphi$ *and* $\mathfrak{e}'_{\partial\varphi}$ *is taken as in item* 1.
3. *The algebra morphism*
$$ {}^\circ\overline{\mathsf{D}}^- : {}^\circ\mathsf{T}^+ \to {}^\circ\mathsf{U}^- \otimes {}^\circ\mathsf{T}^+ $$
is defined by the same formula as ${}^\circ\mathsf{D}^-$*, but the first sum is restricted to the set* $\overline{SF}(\tau)$ *of all subforests* $\varphi \in SF(\tau)$ *which is disjoint with the root of* τ.

Our aim is to show the coassociativities of ${}^\circ\mathsf{D}^\pm$ and ${}^\circ\mathsf{D}$ and ${}^\circ\overline{\mathsf{D}}^-$. To avoid a confusing calculation, we separate the coproducts into graph part and decoration part. Define simpler coproducts acting on undecorated trees by

$$ {}^*\mathsf{D}^+\tau := \sum_{\sigma\in ST(\tau)} \sigma\otimes\big(\tau/^{\mathrm{blue}}\sigma\big), \quad {}^*\mathsf{D}^-\tau := \sum_{\varphi\in SF(\tau)} \varphi\otimes\big(\tau/^{\mathrm{red}}\varphi\big), \quad {}^*\overline{\mathsf{D}}^-\tau := \sum_{\varphi\in \overline{SF}(\tau)} \varphi\otimes\big(\tau/^{\mathrm{red}}\varphi\big). $$

Given an undecorated tree τ, denote by $\mathbb{X}_{(n,k)}$ the map giving to the node $n \in N_\tau$ the $\mathfrak{n}$-decoration $k \in \mathcal{N}$, and denote by $\mathbb{I}_{(e,\ell)}$ the map giving to the edge $e \in E_\tau$ the $\mathfrak{e}$-decoration $\ell \in \mathcal{N}$. Then any decorated tree $\tau^{\mathfrak{n}}_{\mathfrak{e}}$ is of the form

$$ \mathbb{F}\tau = \mathbb{F}_1\cdots\mathbb{F}_N\tau, \tag{C.1} $$

where τ is an undecorated tree, and $\mathbb{F}_1,\dots,\mathbb{F}_N$ are family of $\mathbb{X}$-type or $\mathbb{I}$-type operators, applying to pairwise disjoint nodes or edges. Moreover, we define the coproducts of such operators by

$$\begin{aligned} \mathbb{D}\mathbb{X}_{(n,k)} &= \sum_{k'\leqslant k}\binom{k}{k'}\mathbb{X}_{(n,k')}\otimes\mathbb{X}_{(n,k-k')},\\ \mathbb{D}\mathbb{I}_{(e,\ell)} &= \mathbb{I}_{(e,\ell)}\otimes\mathrm{Id} + \sum_{\ell'}\frac{1}{\ell'!}\mathbb{X}_{(e_-,\ell')}\otimes\mathbb{I}_{(e,\ell+\ell')}, \end{aligned}$$

where e_- denotes the node from where the edge e leaves. For the products of pairwise disjoint such operators, define $\mathbb{D}\mathbb{F} := (\mathbb{D}\mathbb{F}_1)\dots(\mathbb{D}\mathbb{F}_N)$.

At this stage, we see that the coproducts ${}^\circ\mathsf{D}^{(\cdot,+,-)}$ apply to the decorated tree (C.1) by the forms

$$ {}^\circ\mathsf{D}^{(\cdot,+)}(\mathbb{F}\tau) = (\mathbb{D}\mathbb{F})({}^*\mathsf{D}^+\tau), \quad {}^\circ\mathsf{D}^-(\mathbb{F}\tau) = (\mathbb{D}\mathbb{F})({}^*\mathsf{D}^-\tau). \tag{C.2} $$

In the right hand side of (C.2), be careful that $\mathbb{DF}$ acts on subtrees and contracted trees. For an $\mathbb{X}$-type operator, if $n\notin N_\sigma$ then set $\mathbb{X}_{(n,k)}\sigma=\mathbf{1}_{k=0}\sigma$. On a contracted tree τ/σ, the $\mathbb{X}$-type operator acts of the form $\mathbb{X}_{([n],k)}$, where $[n]$ denotes the equivalence class in the contraction $\tau\to\tau/\sigma$. Hence

$$(\mathbb{D}\mathbb{X}_{(n,k)})(\sigma\otimes(\tau/\sigma))=\begin{cases}\sum_{k'\leqslant k}\binom{k}{k'}\mathbb{X}_{(n,k')}\sigma\otimes\mathbb{X}_{([n],k-k')}(\tau/\sigma), & n\in N_\sigma,\\ \sigma\otimes\mathbb{X}_{([n],k)}(\tau/\sigma), & n\notin N_\sigma.\end{cases}$$

For an $\mathbb{I}$-type operator, if $e\notin E_\sigma$ or $e\notin E_{\tau/\sigma}$ then set $\mathbb{I}_{(e,\ell)}\sigma=0$ and $\mathbb{I}_{(e,\ell)}(\tau/\sigma)=0$. Combing with the definition of $\mathbb{X}_{(n,k)}$, we have

$$(\mathbb{D}\mathbb{I}_{(e,\ell)})(\sigma\otimes(\tau/\sigma))=\begin{cases}\mathbb{I}_{(e,\ell)}\sigma\otimes(\tau/\sigma), & \text{for } e\in E_\sigma,\\ \sum_{\ell'}\frac{1}{\ell'!}\mathbb{X}_{(e_-,\ell')}\sigma\otimes\mathbb{I}_{(e,\ell+\ell')}(\tau/\sigma), & \text{for } e\in\partial\sigma,\\ \sigma\otimes\mathbb{I}_{(e,\ell)}(\tau/\sigma), & \text{for } e\in E_\tau\backslash(E_\sigma\cup\partial\sigma).\end{cases}$$

These conventions show that the identities (C.2) hold.

C.3.2 Coassociativity

Lemma 50. *One has the coassociativity formulas*

$$\begin{aligned}\big({}^\circ\mathsf{D}\otimes\mathrm{Id}\big){}^\circ\mathsf{D}&=\big(\mathrm{Id}\otimes{}^\circ\mathsf{D}^+\big){}^\circ\mathsf{D},\\ \big({}^\circ\mathsf{D}^+\otimes\mathrm{Id}\big){}^\circ\mathsf{D}^+&=\big(\mathrm{Id}\otimes{}^\circ\mathsf{D}^+\big){}^\circ\mathsf{D}^+,\\ \big({}^\circ\mathsf{D}^-\otimes\mathrm{Id}\big){}^\circ\mathsf{D}^-&=\big(\mathrm{Id}\otimes{}^\circ\mathsf{D}^-\big){}^\circ\mathsf{D}^-,\\ \big({}^\circ\mathsf{D}^-\otimes\mathrm{Id}\big){}^\circ\overline{\mathsf{D}}^-&=\big(\mathrm{Id}\otimes{}^\circ\overline{\mathsf{D}}^-\big){}^\circ\overline{\mathsf{D}}^-.\end{aligned}$$

Proof – We prove the identity

$$({}^\circ\mathsf{D}\otimes\mathrm{Id}){}^\circ\mathsf{D}=(\mathrm{Id}\otimes{}^\circ\mathsf{D}^+){}^\circ\mathsf{D};\tag{C.3}$$

the other identities are proved similarly. By the commutation relation (C.2), we have

$$\big(\mathrm{Id}\otimes{}^\circ\mathsf{D}^+\big){}^\circ\mathsf{D}\,\mathbb{F}\tau=\big(\mathrm{Id}\otimes{}^\circ\mathsf{D}^+\big)(\mathbb{DF})({}^*\mathsf{D}^+\tau)=\big((\mathrm{Id}\otimes\mathbb{D})\mathbb{DF}\big)\big(\mathrm{Id}\otimes{}^*\mathsf{D}^+\big){}^*\mathsf{D}^+\tau.$$

Hence it is sufficient for proving (C.3) to show the two identities

$$\big({}^*\mathsf{D}^+\otimes\mathrm{Id}\big){}^*\mathsf{D}^+=\big(\mathrm{Id}\otimes{}^*\mathsf{D}^+\big){}^*\mathsf{D}^+,\tag{C.4}$$

$$(\mathrm{Id}\otimes\mathbb{D})\mathbb{DF}=(\mathbb{D}\otimes\mathrm{Id})\mathbb{DF}.\tag{C.5}$$

It is not difficult to show (C.4) by the definition of $ST(\tau)$, by noting that

$$ST(\tau/\sigma)=\big\{\eta/\sigma\,;\,\sigma\subset\eta\subset\tau\big\}$$

and $(\tau/\sigma)/(\eta/\sigma)=\tau/\eta$. To show (C.5), because of the multiplicative property of $\mathbb{D}$, it is sufficient to consider $\mathbb{F}=\mathbb{X}_{(n,k)}$ and $\mathbb{I}_{(e,\ell)}$. They are easy exercises, so details are left to readers. ▷

Now we consider the extended decoration.

Lemma 51. *One has the coassociativity formulas*

$$\begin{aligned}(\mathsf{D}\otimes\mathrm{Id})\mathsf{D}&=\big(\mathrm{Id}\otimes\mathsf{D}^+\big)\mathsf{D},\\ \big(\mathsf{D}^+\otimes\mathrm{Id}\big)\mathsf{D}^+&=\big(\mathrm{Id}\otimes\mathsf{D}^+\big)\mathsf{D}^+,\\ (\mathsf{D}^-\otimes\mathrm{Id})\mathsf{D}^-&=\big(\mathrm{Id}\otimes\mathsf{D}^-\big)\mathsf{D}^-,\\ \big(\mathsf{D}^-\otimes\mathrm{Id}\big)\overline{\mathsf{D}}^-&=\big(\mathrm{Id}\otimes\overline{\mathsf{D}}^-\big)\overline{\mathsf{D}}^-.\end{aligned}$$

Proof – We consider the first and third identities; the other identities are proved similarly. In this proof, denote by $\bar\tau=\tau_{\mathfrak{e}}^{\mathfrak{n}}$ a generic decorated tree, and write $\bar\tau^{\mathfrak{o}}$ for $\tau_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}}$. As in Section 2.2 and Section 5.1, we use a shorthand notation

$${}^\circ\mathsf{D}\bar\tau=\sum_{\bar\sigma\leqslant\bar\tau}\bar\sigma\otimes(\bar\tau/\bar\sigma),\quad {}^\circ\mathsf{D}^-\bar\tau=\sum_{\bar\varphi\trianglelefteq\bar\tau}\bar\varphi\otimes(\bar\tau/\bar\varphi).$$

Then we can write

$$\mathsf{D}\bar\tau^{\mathfrak{o}}=\sum_{\bar\sigma}\bar\sigma^{\mathfrak{o}}\otimes(\bar\tau/\bar\sigma)^{\mathfrak{o}|_{\tau\backslash\sigma}},\quad \mathsf{D}^-\bar\tau^{\mathfrak{o}}=\sum_{\bar\varphi}\bar\varphi^{\mathfrak{o}}\otimes(\bar\tau/\bar\varphi)^{\mathfrak{o}+\mathfrak{o}(\bar\varphi)}.$$

Recall that $\mathfrak{o}(\bar\varphi) : N_{\tau/\varphi} \to \mathbb{Z}[\beta_0]$, is a function giving the value $|\bar\tau_j|$, where $\bar\tau_j$ is a connected component if φ, to the node $[\tau_j] \in N_{\tau/\varphi}$. We obtain the coassociativity of D from the coassociativity of ${}^\circ\mathsf{D}$, noting that

$$(\bar\tau/\bar\sigma)/(\bar\eta/\bar\sigma) = \bar\tau/\bar\eta, \quad \mathfrak{o}|_{(\tau\backslash\sigma)\backslash(\eta\backslash\sigma)} = \mathfrak{o}|_{\tau\backslash\eta}$$

for any $\bar\eta \leqslant \bar\sigma \leqslant \bar\tau$. To prove the coassociativity of D^-, noting that

$$(\mathsf{D}^- \otimes \mathrm{Id})\mathsf{D}^-\bar\tau^{\mathfrak{o}} = \sum_{\bar\psi \trianglelefteq \bar\varphi \trianglelefteq \bar\tau} \bar\psi^{\mathfrak{o}} \otimes (\bar\varphi/\bar\psi)^{\mathfrak{o}+\mathfrak{o}(\bar\psi)} \otimes (\bar\tau/\bar\varphi)^{\mathfrak{o}+\mathfrak{o}(\bar\varphi)}$$

and

$$\begin{aligned}(\mathrm{Id} \otimes \mathsf{D}^-)\mathsf{D}^-\bar\tau^{\mathfrak{o}} &= \sum_{\bar\psi \trianglelefteq \bar\tau} \bar\psi^{\mathfrak{o}} \otimes \mathsf{D}^-(\bar\tau/\bar\psi)^{\mathfrak{o}+\mathfrak{o}(\bar\psi)} \\ &= \sum_{\bar\psi \trianglelefteq \bar\varphi \trianglelefteq \bar\tau} \bar\psi^{\mathfrak{o}} \otimes (\bar\varphi/\bar\psi)^{\mathfrak{o}+\mathfrak{o}(\bar\psi)} \otimes (\bar\tau/\bar\varphi)^{\mathfrak{o}+\mathfrak{o}(\bar\psi)+\mathfrak{o}(\bar\varphi/\bar\psi)},\end{aligned}$$

it is sufficient to show that $\mathfrak{o}(\bar\varphi) = \mathfrak{o}(\bar\psi) + \mathfrak{o}(\bar\varphi/\bar\psi)$ as a function on $N_{\tau/\varphi}$. This holds true because $|\bar\varphi|' = |\bar\psi|' + |\bar\varphi/\bar\psi|'$. ▷

C.3.3 Co-interaction

Lemma 52. *One has the co-interaction formulas*

$$\begin{aligned}\mathcal{M}^{(13)}\big({}^\circ\mathsf{D}^- \otimes {}^\circ\overline{\mathsf{D}}^-\big){}^\circ\mathsf{D} &= (\mathrm{Id} \otimes {}^\circ\mathsf{D}){}^\circ\mathsf{D}^-, \\ \mathcal{M}^{(13)}\big({}^\circ\overline{\mathsf{D}}^- \otimes {}^\circ\overline{\mathsf{D}}^-\big){}^\circ\mathsf{D}^+ &= \big(\mathrm{Id} \otimes {}^\circ\mathsf{D}^+\big){}^\circ\overline{\mathsf{D}}^-,\end{aligned} \tag{C.6}$$

and

$$\begin{aligned}\mathcal{M}^{(13)}\big(\mathsf{D}^- \otimes \overline{\mathsf{D}}^-\big)\mathsf{D} &= (\mathrm{Id} \otimes \mathsf{D})\mathsf{D}^-, \\ \mathcal{M}^{(13)}\big(\overline{\mathsf{D}}^- \otimes \overline{\mathsf{D}}^-\big)\mathsf{D}^+ &= (\mathrm{Id} \otimes \mathsf{D}^+)\overline{\mathsf{D}}^-.\end{aligned} \tag{C.7}$$

Proof – Consider the first identity of (C.6) and the first identity of (C.7); the two other identities are proved similarly. By the commutation relations (C.2), identity (C.6) rewrites

$$\mathcal{M}^{(13)}\big((\mathbb{D} \otimes \mathbb{D})\mathbb{D}\mathbb{F}\big)\big({}^*\mathsf{D}^- \otimes {}^*\overline{\mathsf{D}}^-\big){}^*\mathsf{D}^+\tau = \big((\mathrm{Id} \otimes \mathbb{D})\mathbb{D}\mathbb{F}\big)(\mathrm{Id} \otimes {}^*\mathsf{D}^+){}^*\mathsf{D}^-. \tag{C.8}$$

By the multiplicativity of $\mathbb{D}$, it is sufficient to show (C.8) for the operators $\mathbb{F} = \mathbb{X}_{(n,k)}$ and $\mathbb{I}_{(e,\ell)}$. By definition,

$$\big({}^*\mathsf{D}^- \otimes {}^*\overline{\mathsf{D}}^-\big){}^*\mathsf{D}^+\tau = \sum_{\sigma \in ST(\tau)} \sum_{\varphi \in SF(\sigma), \psi \in \overline{SF}(\tau/\sigma)} \varphi \otimes (\sigma/^{\mathrm{red}}\varphi) \otimes \psi \otimes (\tau/^{\mathrm{blue}}\sigma)/^{\mathrm{red}}\psi.$$

Note that φ and ψ are disjoint subforests of τ because of the definition of $\overline{SF}$. Thus we have

$$\begin{aligned}&\mathcal{M}^{(13)}\big((\mathbb{D} \otimes \mathbb{D})\mathbb{D}\mathbb{X}_{(n,k)}\big)\big(\varphi \otimes (\sigma/^{\mathrm{red}}\varphi) \otimes \psi \otimes (\tau/^{\mathrm{blue}}\sigma)/^{\mathrm{red}}\psi\big) \\ &= \mathcal{M}^{(13)} \sum_{k=a+b+c+d} \frac{k!}{a!b!c!d!} \mathbb{X}_{(n,a)}\varphi \otimes \mathbb{X}_{(n,b)}(\sigma/^{\mathrm{red}}\varphi) \otimes \mathbb{X}_{(n,c)}\psi \otimes \mathbb{X}_{(n,d)}\big((\tau/^{\mathrm{blue}}\sigma)/^{\mathrm{red}}\psi\big) \\ &= \sum_{k=a+b+d} \frac{k!}{a!b!d!} \mathbb{X}_{(n,a)}(\varphi\psi) \otimes \mathbb{X}_{(n,b)}(\sigma/^{\mathrm{red}}\varphi) \otimes \mathbb{X}_{(n,d)}\big((\tau/^{\mathrm{blue}}\sigma)/^{\mathrm{red}}\psi\big) \\ &= \big((\mathrm{Id} \otimes \mathbb{D})\mathbb{D}\mathbb{X}_{(n,k)}\big)\big(\varphi\psi \otimes (\sigma/^{\mathrm{red}}\varphi) \otimes \big(\tau/^{\mathrm{blue}}\sigma)/^{\mathrm{red}}\psi\big),\end{aligned}$$

since either of a and c has to be 0 in the second line. It is not difficult to a similar equality for $\mathbb{F} = \mathbb{I}_{(e,\ell)}$. Hence we have

$$\mathcal{M}^{(13)}\big((\mathbb{D} \otimes \mathbb{D})\mathbb{D}\mathbb{F}\big)\big({}^*\mathsf{D}^- \otimes {}^*\overline{\mathsf{D}}^-\big){}^*\mathsf{D}^+\tau = \big((\mathrm{Id} \otimes \mathbb{D})\mathbb{D}\mathbb{F}\big)\mathcal{M}^{(13)}\big({}^*\mathsf{D}^- \otimes {}^*\overline{\mathsf{D}}^-\big){}^*\mathsf{D}^+\tau.$$

Since it is not difficult to show the co-interaction formula

$$\mathcal{M}^{(13)}\big({}^*\mathsf{D}^- \otimes {}^*\overline{\mathsf{D}}^-\big){}^*\mathsf{D}^+\tau = \big(\mathrm{Id} \otimes {}^*\mathsf{D}^+\big){}^*\mathsf{D}^-,$$

identity (C.8) follows as a consequence.
Next we consider (C.7). By definition,

$$\mathcal{M}^{(13)}\big(\mathsf{D}^- \otimes \overline{\mathsf{D}}^-\big)\mathsf{D}\bar\tau^{\mathfrak{o}} = \sum_{\bar\sigma \leqslant \bar\tau, \bar\varphi \trianglelefteq \bar\sigma, \bar\psi \trianglelefteq \bar\tau/\bar\sigma} \bar\varphi^{\mathfrak{o}}\bar\psi^{\mathfrak{o}|_{\tau\backslash\sigma}} \otimes (\bar\sigma/\bar\varphi)^{\mathfrak{o}+\mathfrak{o}(\bar\varphi)} \otimes \big((\bar\tau/\bar\sigma)/\bar\psi\big)^{\mathfrak{o}|_{\tau\backslash\sigma}+\mathfrak{o}(\bar\psi)}$$

and

$$(\mathrm{Id} \otimes \mathsf{D})\mathsf{D}^- \bar{\tau}^{\mathfrak{o}} = \sum_{\bar{\zeta} \trianglelefteq \bar{\tau}, \bar{\eta} \leqslant \bar{\tau}/\bar{\zeta}} \bar{\zeta}^{\mathfrak{o}} \otimes \bar{\eta}^{\mathfrak{o}+\mathfrak{o}(\bar{\zeta})} \otimes \big((\bar{\tau}/\bar{\zeta})/\bar{\eta}\big)^{(\mathfrak{o}+\mathfrak{o}(\bar{\zeta}))|_{(\tau/\zeta)\setminus\eta}}.$$

The cointeraction between ${}^\circ\mathsf{D}$ and ${}^\circ\mathsf{D}^-$ implies that the *change of variables*

$$\bar{\zeta} \leftrightarrow \bar{\varphi}\,\bar{\psi}, \quad \bar{\eta} \leftrightarrow \bar{\sigma}/\bar{\varphi}$$

is possible. Since σ and ψ are disjoint,

$$(\bar{\varphi}\bar{\psi})^{\mathfrak{o}} = \bar{\varphi}^{\mathfrak{o}}\,\bar{\psi}^{\mathfrak{o}|_{\tau\setminus\sigma}}, \quad (\bar{\sigma}/\bar{\varphi})^{\mathfrak{o}+\mathfrak{o}(\bar{\varphi}\bar{\psi})} = (\bar{\sigma}/\bar{\varphi})^{\mathfrak{o}+\mathfrak{o}(\bar{\varphi})},$$
$$\big((\bar{\tau}/\bar{\sigma})/\bar{\psi}\big)^{(\mathfrak{o}+\mathfrak{o}(\bar{\varphi}\bar{\psi}))|_{(\tau/(\varphi\psi))\setminus(\sigma/\varphi)}} = \big((\bar{\tau}/\bar{\sigma})/\bar{\psi}\big)^{\mathfrak{o}|_{\tau\setminus\sigma}+\mathfrak{o}(\bar{\psi})}.$$

Thus (C.7) follows. $\triangleright$

D – Comments

Section 1 – Regularity structures theory has its roots in T. Lyons' theory of rough paths and rough differential equations [58]. This theory deals with controlled ordinary differential equations

$$dx_t = V(x_t)dh_t,$$

with controls h of low regularity, say α-Hölder. For $\alpha > 1/2$, Young integration theory allows to make sense of the equation as a fixed point problem for an integral equation. As one expects a solution path to be α-Hölder, the product $V(x_t)dh_t$ makes sense as a distribution on $\mathbb{R}_+$ iff $\alpha + (\alpha - 1) > 0$, that is $\alpha > 1/2$. One of Lyons' deep insights was to realize that what really governs the dynamics is not the $\mathbb{R}^\ell$-valued control h, say, but rather a finite collection of its iterated integrals. The latter are ill-defined when $\alpha \leqslant 1/2$, and a rough path is the a priori datum of quantities playing their role. Natural algebraic and size constraints on these objects are then sufficient to set the entire theory. These constraints are similar to the constraints that define the g-part of a model. Several reformulations of rough paths theory were given after Lyons' seminal work: Davie's numerical scheme approach [26], Gubinelli's controlled paths approach [38, 39], Friz & Victoir's limit ODE picture [34], and Bailleul's approximate flow-to-flow approach [2], amongst others. Gubinelli's versatile notion of controlled paths was a direct source of inspiration for the construction of regularity structures.

Other tools than regularity structures have been developed for the study of singular stochastic PDEs. None of them offers presently a complete alternative to regularity structures.

- Gubinelli, Imkeller and Perkowski laid in [42] the foundations of paracontrolled calculus, that was developed by Bailleul & Bernicot [3, 4, 5]. While the fundamental notions of regularity structures involve pointwise expansions, paracontrolled calculus uses paraproducts as a mean for making sense of what it means to look like a reference quantity. See [40] for lecture notes on the subject and [41] for an overview on the subject, both by Gubinelli & Perkowski.

  In a nutshell, the starting point of the paracontrolled approach to the study of singular stochastic PDEs is the decomposition of a product of two distributions f, g into

  $$fg = \big(\mathsf{P}_f g + \mathsf{P}_g f\big) + \Pi(f, g).$$

  This decomposition is obtained in a Fourier picture of the product by splitting the convolution into what happens far from the diagonal $\big(\mathsf{P}_f g + \mathsf{P}_g f\big)$ from what happens near the diagonal $\Pi(f, g)$. This decomposition isolates in the resonant term $\Pi(f, g)$ what does not make sense in a general product, the paraproduct terms $\mathsf{P}_f g, \mathsf{P}_g f$ being always well-defined. The definition of $\mathsf{P}_f g$ allows to think of it as a modulation of g by f and give meaning to what it means for a distribution/function u to look like another distribution/function g

  $$u = \mathsf{P}_v g + u^\sharp,$$

for a function v and a distribution/function $v^\sharp$ that is more regular than g. The role of modelled distributions is played in a paracontrolled setting by systems $(u_a)_{a\in\mathscr{A}}$ of paracontrolled distributions/functions

$$u_a = \sum_{\tau\in\mathscr{T};|a\tau|\leqslant n\alpha} \mathsf{P}_{u_{a\tau}}[\tau] + u_a^\sharp \tag{D.1}$$

indexed by the set $\mathscr{A}$ of words over an alphabet $\mathscr{T}$, with remainders $u_a^\sharp$ sufficiently regular. The reference distributions/functions $[\tau]$ somehow play the role of $\Pi\tau$ and the $(u_a)_{a\in\mathscr{A}}$ the role of the $(u_\tau)_{\tau\in T,|\tau|<\gamma}$. (Bailleul & Hoshino's work on the link between paracontrolled calculus and regularity structures make that link clear.) Identity (D.1) is is an analogue of the notion of modelled distribution. While the definition of the latter involves pointwise comparisons, here the comparison is somehow done in 'momentum space', although not in a pointwise sense. The core point of the paracontrolled analysis of a (system of) singular stochastic PDE(s) is that we end up dealing with ill-defined terms of the form

$$\Pi'\big(u, \zeta^{(i)}\big) \tag{D.2}$$

for operators Π' that have similar properties as the resonant operator Π, and possibly multi-dimensional functionals $\zeta^{(i)}$ of the noise ζ. It turns out that while an expression like (D.2) does not make sense for a generic u, it makes sense on a restricted class of u of the form (D.1) *provided* one can make sense of the terms $\Pi'\big(\tau, \zeta^{(i)}\big)$. The analysis of a given (system of) singular stochastic PDE(s) gives an inductive definition of the $\zeta^{(i)}$ and the τ's. Compared to the regularity structures setting, the datum of all the $\Pi'\big(\tau, \zeta^{(i)}\big)$ plays the role of the datum of a model. The inductive/tree structure of the elements of a regularity structure takes here the form of the inductive definition of the τ's and $\zeta^{(i)}$'s. A systematic treatment of renormalization operations within paracontrolled calculus has not been invented yet. The links between the regularity structure and paracontrolled settings detailed in Bailleul & Hoshino's works [8, 9] allow however to transport the renormalization machinery of regularity structures into the setting paracontrolled calculus. What is missing presently is an independent, purely paracontrolled, approach of the renormalization problem.

- Otto & Weber [61] developed jointly with Sauer and Smith [62, 63] a variant of regularity structures that is more in the flavour of rough paths theory. Most concepts and objects from regularity structures have counterparts in their setting. It was specifically designed and used for the analysis of a number of quasilinear singular stochastic PDEs. Some of these equations can be approached using the original first order paracontrolled calculus as in Bailleul, Debussche and Hofmanová's work [7] or a variant of it using paracomposition operators, as in Furlan & Gubinelli's work [36]. See also [10, 37] for extensions of paracontrolled calculus and regularity structures designed for the study of a whole class of quasilinear singular stochastic PDEs.
- Kupiainen & Marcozzi managed in [55, 56] to implement a renormalization group approach to the (KPZ) and Φ^4_3 equations. The starting point of their strategy consists in decomposing the resolution operator $\mathbf{K}^{-1}$ involved in the Picard formulation of the equation as a sum of operators $\mathbf{K}_n^{-1}$ turning distributions into smooth functions that vary essentially only up to scale 2^{-n}. The approximate renormalized dynamics will take the form

$$u_N = \bigg(\sum_{n=0}^N \mathbf{K}_n^{-1}\bigg)\Big(F_N(u_N, \partial u_N\,;\,\zeta)\Big)$$

in a simplified problem where the initial condition was taken to be null. The noise is left untouched, with no problem for defining the nonlinearity in the right hand side since u_N is smooth. So one has

$$u_N = u_N^0 + \cdots + u_N^N,$$

where each term u_N^n is morally varying only up to scale 2^{-n}. The point is now to see that one can choose the nonlinearity F_N in such a way that each u_N^n is the solution of an equation of the form

$$u_N^n = \Big(\sum_{m=n}^{N} \mathbf{K}_n^{-1}\Big)\Big(F_N^n(u_N^n, \partial u_N^n\,;\,\zeta)\Big)$$

for a nonlinearity F_N^n, and is converging in a proper space as N goes to infinity. This is done via the use of rescaling operators, taking profit from the exact scaling property of the heat kernel, by turning the problem of convergence of each u_N^n into the problem of the convergence of the family of rescaled versions of the functions F_N^n – taking profit of the fact that the former is a continuous function of the later. The overall convergence of u_N as N goes to ∞ is somehow similar to the well-known fact that a sum of functions $\sum_n u^n$ converges in an α-Hölder space if u^n is localized in Fourier space on a ball of size 2^n and has uniform norm of order $2^{-\alpha n}$.

Section 2 – The functional setting adopted here draws inspiration from [3, 4, 5] and [61]. The main result of this section is the reconstruction theorem.

Several proofs of the reconstruction theorem are available now, in addition to Hairer's original proof. Gubinelli, Imkeller and Perkowski gave in [42] an alternative construction of the reconstruction map using a paraproduct-like operator. Singh & Teichmann showed in [64] how it can be understood as the continuous extension of an elementary reconstruction operator defined on a set of smooth modelled distributions. Otto and Weber have an analogue of the reconstruction map in their rough paths-like setting [61, 62]. Caravenna & Zambotti's recent work [18] provide a robust version of the reconstruction theorem in a setting free of any reference to regularity structures. The notion of coherent germ turns it into a particularly versatile tool. See [48, 52, 57] for versions of the reconstruction theorem in functional settings different from Hölder spaces.

The reconstruction theorem takes its place in the history of a family of statements producing 'transcendantal' objects, i.e. objects constructed by limiting procedures, from families of objects satisfying constraints involving no limiting procedures. The one-step Euler scheme for solving ordinary differential equations characterizes for instance uniquely their flows under sufficient regularity conditions on the vector fields. In its simplest form, for the equation $\dot{y} = -y$, in $\mathbb{R}$, it yields the elementary identity $(1 - t/n)^n \to e^{-t}$, as n goes to ∞. It takes a more elaborate form in Hille's approximation $\big((\mathrm{Id} + \frac{t}{n}\,G)^{-1}\big)^n$ of the semigroup $(e^{-tG})_{t>0}$ generated by an unbounded operator G under well-known conditions. Chernov's theorem [23] on families of strongly continuous perturbations of the identity used for constructing $(e^{-tG})_{t>0}$ has a similar flavour. So is the C^1-approximate flow-to-flow machinery of [2], that provides a far reaching generalization of Lyons' extension theorem in rough paths theory and Gubinelli and Feyel & de la Pradelle' sewing lemma [38, 27, 28]. All these statements characterize uniquely a transcendantal object as the unique object close to a family of objects satisfying a '$O(1)$ condition', involving no limiting procedure. The characterizing identity (2.29) for the reconstruction is of that form when the reconstruction operator is unique. This kind of situation allows to build a calculus for the transcendantal objects from an elementary calculus on their generators.

The space of models over a given regularity structure is nonlinear. Bailleul & Hoshino showed in [8, 9] how to parametrize this space by a linear space using the tools of paracontrolled calculus. The set of $\mathbf{K}$-admissible models on a given regularity structure turns out in particular to be parametrized by the data for each τ of negative homogeneity of a $|\tau|$-Hölder distribution, describing somehow the most regular part of the distribution $\mathsf{\Pi}\tau$. This has a number of consequences, such as an extension theorem similar to Lyons' extension theorem in rough paths theory.

Proposition 9, giving a definition of the image of a modelled distribution by a nonlinear map, has a counterpart in paracontrolled calculus, generalizing Bony's paralinearisation formula to an arbitrary order – see Section 2 of Bailleul & Bernicot's work [5].

Section 3 – The proof of the continuity result for $\mathcal{K}^{\mathsf{M}}$ is an adaptation of the material from Hairer's groundbreaking work [44] to the functional setting adopted here. It is called by Hairer the multilevel Schauder estimates. The construction of $\mathbf{K}$-admissible models from Section 3.5 is adapted from Bailleul & Hoshino's work [8], which gives amongst others a parametrization of the set of all admissible models on any reasonable concrete regularity structure. See [9] for more results on the structure of the space of models and modelled distributions. Note that in the different components u_τ of a modelled distribution also appear in the paracontrolled approach, in which they are involved in the global description of a possible solution, as opposed to their local meaning in the regularity structure setting.

Section 4 – This section essentially follows the line of the corresponding results in [44], Section 7 therein.

Section 5 – The notion of renormalization structures and compatible regularity and renormalization structures introduced in this section is new. It encodes in a simple way the mechanics at work in Bruned, Hairer and Zambotti's work [16].

Section 6 – This section contains the core insights of Bruned, Chandra, Chevyrev and Hairer's work [13], implemented here on the example of the generalized (KPZ) equation. The relevance of the notion of pre-Lie algebra was first noticed in the work [14] of Bruned, Chevyrev and Friz on rough paths. The article [19] would provide a pre-history of the pre-Lie algebra. The comodule-bialgebra structure of the Butcher-Connes-Kreimer Hopf algebra was first investigated in the work [17] of Calaque, Ebrahimi-Fard and Manchon; it played a key motivating role in the work of Bruned, Chevyrev and Friz. The description of the free pre-Lie algebra in this setting is due to Chapoton and Livernet [22]. The notion of multi-pre-Lie algebra was introduced in the work [13] of Bruned, Chandra, Chevyrev and Hairer, where the free multi-pre-Lie was first described.

The $\mathfrak{o}$-decorations introduced here under the form of $\bullet^{n,\alpha}$ is forced by our construction of compatible regularity and renormalization structures for the generalized (KPZ) equation, given in Section 9. It has no dynamical meaning. Bailleul & Bruned showed in [6] how to obtain the renormalized equation without using extended decorations for a large class of renormalization procedures including the BPHZ scheme.

The setting described here is robust enough to deal with equations driven by multiple noises, or systems of equations driven by multiple noises. We take Funaki's example [35] of the random motion of a rubber on a manifold as an archetype – see also [46, 15]. The unknown u is a spacetime function with values in $\mathbb{R}^d$, solution of the system

$$(\partial_t - \partial_x^2)u = \Gamma(u)\big(\partial_x u, \partial_x u\big) + \Sigma(u)\xi,$$

where $\Gamma(z)$ is a symmetric matrix on $\mathbb{R}^d$, and $\Sigma(z)$ a linear map from $\mathbb{R}^k$ to $\mathbb{R}^d$, for any $z \in \mathbb{R}^d$, and $\xi = (\xi^1, \dots, \xi^k)$ is an k-dimensional tuple of identically distributed independent one-dimensional spacetime white noises. We still have only one operator $(\partial_t - \partial_x^2)$ in this example, so the edge type set is here the same as in the study of the generalised (KPZ) equation. The node set is changed from $\{\circ, \bullet\} \times \mathbb{N}^{d+1}$ to $\{\circ^1, \dots, \circ^k, \bullet\} \times \mathbb{N}^{d+1}$ to account for the fact that we have k noises $\xi^1, \dots, \xi^k$ in the system. Things get a bit messier if the system involves different operators, with different regularising properties, and noises with different regularities. The fundamental ideas involved in the analysis remain the same, while the notations needed to take care of this richer setting become heavier. All this is explained in full details in [16].

Section 7 – This section gives what seems to us to be one of the two core results of [16], Theorem 41 here. More general renormalization schemes were introduced by Bruned in [12].

Section 8 – This short section emphasizes a fact that has not received much attention so far.

Section 9 – This section builds on the fundamental work [16], with a number of simplifications. The notion of subcritical equation is subtle to check in the general case of a system of equations, as one needs to keep track of how a given symbol of a regularity structure 'flows' in the different pieces of a system, involving possibly operators with different regularizing properties. The meaning of subcriticality remains, though.

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• I. Bailleul – Univ. Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France.
E-mail: ismael.bailleul@univ-rennes1.fr

• M. Hoshino – Graduate School of Engineering Science, Osaka University, Japan
E-mail: hoshino@sigmath.es.osaka-u.ac.jp