

Analysis of the Anderson operator

I. BAILLEUL¹ and V. N. DANG and A. MOUZARD²

Abstract. We consider the continuous Anderson operator $H = \Delta + \xi$ on a two dimensional closed Riemannian manifold \mathcal{S} . We provide a short self-contained functional analysis construction of the operator as an unbounded operator on $L^2(\mathcal{S})$ and give almost sure spectral gap estimates under mild geometric assumptions on the Riemannian manifold. We prove a sharp Gaussian small time asymptotic for the heat kernel of H that leads amongst others to norm estimates for eigenfunctions and quasimodes. We introduce a new random field, called the *Anderson Gaussian free field*, and prove that the law of its random partition function characterizes the law of the spectrum of H . We also give a simple and short construction of a polymer measure on path space and another diffusion called *Anderson diffusion*. We relate the Wick square of the Anderson Gaussian free field to the renormalized occupation measure of a Poisson process of loops of diffusion paths. We further prove some large deviation results for the Anderson diffusion and its bridges.

Contents

1. Introduction	1
2. Tools for the analysis	5
3. A construction of the Anderson operator	7
4. Heat operator of the Anderson operator	12
5. Anderson Gaussian free field	29
6. Polymer measure and Anderson diffusion	35
A. Meromorphic Fredholm theory with a parameter	43
B. Geometric Littlewood-Paley decomposition	44

1 – Introduction

Let \mathcal{S} be a two dimensional compact boundaryless Riemannian manifold with metric g and associated volume measure μ . White noise on \mathcal{S} is a $\mathcal{D}'(\mathcal{S})$ -valued random variable ξ with Gaussian law with null mean and covariance

$$\mathbb{E}[\xi(\varphi_1)\xi(\varphi_2)] = \int_{\mathcal{S}} \varphi_1\varphi_2 d\mu,$$

for φ_1, φ_2 smooth functions on \mathcal{S} . Almost surely it takes values in the Besov space $\mathcal{B}_{\infty\infty}^{\alpha-2}(\mathcal{S})$, for any $\alpha < 1$, a distribution space, and its law depends only on the metric g on \mathcal{S} . Let $h \in C^\infty(\mathcal{S})$ be a smooth function. Denote by $M_{h\xi}$ the multiplication operator by $h\xi$, and by Δ the Laplace-Beltrami operator associated with the Riemannian metric on \mathcal{S} . The Anderson Hamiltonian is the random operator

$$H := -\Delta + M_{h\xi}, \tag{1.1}$$

perturbation of the Laplace-Beltrami operator by a distribution-valued potential. The smooth function h plays the role of a modulator for the noise, a position dependent coupling constant. The operator H arises naturally as the scaling limit of a number of microscopic discrete operators of interest in statistical physics. The study of the Anderson operator/Hamiltonian presents an additional difficulty compared to its discrete counterparts. Unlike what happens for the Laplace-Beltrami operator Δ or its perturbations by smooth potentials, the low regularity of ξ prevents a straightforward definition of H as a continuous operator from the Sobolev space $H^2(\mathcal{S})$ into $L^2(\mathcal{S})$ since

$$M_{h\xi}(f) = fh\xi$$

is not an element of $L^2(\mathcal{S})$ for a generic $f \in H^2(\mathcal{S})$. One had to wait for the recent development of the theory of paracontrolled calculus and regularity structures before appropriate functional settings were introduced for the study of the Anderson Hamiltonian – corresponding to $h = 1$. Let \mathbb{T}^2 stand for the two dimensional flat torus. Allez and Chouk [2] first used paracontrolled

¹I. B. thanks the CNRS & PIMS for their hospitality, part of this work was written at UBC during a sabbatical there. I. B. also thanks the ANR through its support via the ANR-16-CE40-0020-01 grant.

²A.M. is supported by the Simons Collaboration on Wave Turbulence.

calculus to define a random domain for H and proved that one can define H as an unbounded self-adjoint operator on $L^2(\mathbb{T}^2)$, with discrete spectrum $\lambda_n(\widehat{\xi})$ diverging to $+\infty$ and eigenvalues $\lambda_n(\widehat{\xi})$ that are continuous functions of a measurable functional $\widehat{\xi}$ of ξ taking values in a Banach space. The basic mechanics at work in [2] was improved in Gubinelli, Ugurcan & Zachhuber's work [32] in which a similar result on the three dimensional torus was proved, amongst others. Labbé was also able in [39] to use the tools of regularity structures to get similar results. We refer to these works for detailed accounts of related matters and extensive references to the literature. All these works are set in the torus. The very recent work of Mouzard [41] used the tools of the high order paracontrolled calculus developed by Bailleul & Bernicot in [5, 6, 7] to study the Anderson Hamiltonian on a two dimensional manifold, simplifying a number of technical points compared to [2, 32] and proving that the random spectrum of H satisfies the same Weyl asymptotic law as the spectrum of the Laplace-Beltrami operator.

◦ *Anderson operator.* We give in this work a self-contained construction of the Anderson operator that is different from the previous constructions. It relies on the direct construction of the resolvent operator via a fixed point equation where the analytic Fredholm theory can be used efficiently. We note in particular that the only point from paracontrolled calculus that we use is the fundamental continuity estimate on the corrector first proved by Gubinelli, Imkeller & Perkowski in the flat torus [30], later extended to a manifold (and possibly parabolic) setting by Bailleul & Bernicot in [5]. Recall that h is the coupling function that appears in front of the noise in the definition (1.1) of the Anderson operator. Given a positive regularization parameter r let

$$\xi_r = e^{r\Delta}(\xi)$$

stand for the heat regularized white noise. The family of operators $-\Delta + M_{h\xi_r} - \frac{|\log r|}{4\pi} h^2$ converges in probability as r goes to 0 to a limit random unbounded self-adjoint operator H which is a quadratic functional of the coupling function h and has a discrete spectrum $\sigma(H)$ converging to $+\infty$. This random operator is called the *Anderson operator*. We give in Section 3 a short and self-contained construction of that operator that only requires paracontrolled ansatz at order 1, unlike the previous works [2, 32, 41]. Our construction is essentially functional analytic.

We give in Theorem 19 a detailed description of the solution to the parabolic Anderson equation with singular initial conditions, with the heat kernel $p_t(x, y)$ of H as a particular example. Our main point here is that a fine description of $p_t(x, y)$ actually contains a lot of information on the operator H itself. As a direct illustration we recover in Proposition 31 Mouzard's Weyl law for the spectrum of H from a Tauberian point of view. Information on different norms of the eigenfunctions or quasi-modes of H can also be recovered from a good control of the heat semigroup. Denote by $(u_n)_{n \geq 0}$ the sequence of L^2 normalized eigenfunctions of H with corresponding eigenvalues $\lambda_n(\widehat{\xi})$.

Theorem 1 – For any $a \in (0, 1)$ one has

$$\|u_n\|_{C^a} \lesssim_a |\lambda_n(\widehat{\xi})|^{\frac{1+a}{2}}.$$

We are able to obtain in Proposition 26 lower and upper Gaussian bounds for $p_t(x, y)$, which imply an interesting parabolic Harnack estimate for $(\partial_t + H)$ -harmonic functions. Somewhat independently of the good control on the heat kernel from Theorem 19 we are also able to quantify the spectral gap of H in terms of some isoperimetric constant of the Riemannian manifold (\mathcal{S}, g) generalizing Cheeger's Poincaré inequalities to our setting and also under the assumption that the Riemannian volume form μ satisfies a log-Sobolev inequality – the definitions of the different quantities below will be recalled in Section 4.4. The eigenfunction u_0 – the ground state, is associated with the smallest eigenvalue $\lambda_0(\widehat{\xi})$ of H .

Theorem 2 – One has the following two almost sure estimates on the spectral gap of H .

- Denote by $C(\mathcal{S}, g) > 0$ the Cheeger constant of the Riemannian manifold (\mathcal{S}, g) . Then one has the spectral gap estimate

$$\lambda_1(\widehat{\xi}) - \lambda_0(\widehat{\xi}) \geq \left(\frac{\min u_0}{\max u_0} \right)^4 \frac{C(\mathcal{S}, g)^2}{4} > 0.$$

- Assume that the Riemannian volume measure μ satisfies a log-Sobolev inequality with constant C_{LS} . Then one has the spectral gap estimate

$$\lambda_1(\widehat{\xi}) - \lambda_0(\widehat{\xi}) \geq \left(\frac{\min u_0}{\max u_0} \right)^2 \frac{(\max u_0^4 + \max u_0^{-4})^{-1}}{2C_{LS}} > 0.$$

◦ *Anderson Gaussian free field.* We introduce and study the Anderson Gaussian free field in Section 5. This doubly random field ϕ on \mathcal{S} is defined from the L^2 spectral decomposition of the random operator H in the same way as the Gaussian free field is defined from the L^2 spectral decomposition of the operator Δ . It thus has two layers of randomness. Like the usual Gaussian free field in two dimensions it is almost surely of regularity 0^- . One can define the Wick square $:\phi^2:$ of ϕ as a doubly random variable; its distribution $\mathcal{L}(:\phi^2:)$ depends on H so it is random. The following result is proved in a more precise form in Theorem 38 and Corollary 39.

Theorem 3 – *The law of the random spectrum of H is characterized by $\mathcal{L}(:\phi^2:)$.*

◦ *A polymer measure.* We construct a polymer measure which provides a mathematical model for the random motion of a particle subject to a thermal motion in an extremely disordered potential modeled by white noise. From Feynman-Kac representation formula it is the probability measure Q formally defined at a generic point $w \in C([0, 1], \mathcal{S})$ by its density

$$\exp \left(\int_0^1 \xi(w_t) dt \right)$$

with respect to the Wiener measure $P_{\mathcal{W}}$ on path space over \mathcal{S} , up to a multiplicative normalization constant. The pointwise evaluation of the distribution ξ at the point w_t is however meaningless, which motivates a definition of the polymer measure Q as a limit as $r > 0$ goes to 0 of the measures $Q^{(r)}$ obtained from a regularized noise ξ_r , setting

$$\frac{dQ^{(r)}}{dP_{\mathcal{W}}}(w) \sim \exp \left(\int_0^1 \left(\xi_r + \frac{|\log r|}{4\pi} \right) (w_t) dt \right). \quad (1.2)$$

The contribution of the constant term $\frac{|\log r|}{4\pi}$ to the density is cancelled by the corresponding term in the implicit normalizing constant. One can then equivalently write

$$\frac{dQ^{(r)}}{dP_{\mathcal{W}}}(w) \sim \exp \left(\int_0^1 \xi_r(w_t) dt \right).$$

Note that the measures $Q^{(r)}$ and the limit measure Q are random, as the white noise environment is random. (Both $Q^{(r)}$ and Q depend implicitly on the starting point of the path w , that may be fixed or random.) The measure Q was first constructed in the flat setting of the two dimensional torus by Cannizzaro & Chouk in [15] by using the then newly developed tools of paracontrolled calculus. Their method of proof is not easily adapted to a manifold setting. We give here the first construction of this measure on a closed Riemannian manifold. Our construction is different from that of Cannizzaro & Chouk and we construct the random measure Q as the law of a time inhomogeneous Markov process built from e^{-tH} , properly normalized to get a probability measure. We note that Alberts, Khanin and Quastel [1] have constructed some polymer measure in the case of a spacetime white noise in one spatial dimension, their polymer measure differs from ours since the environment changes randomly with time. The sharp small time asymptotic that we obtain on the kernel of that operator, or the Gaussian bound proved for that kernel, allow for a straightforward use of Kolmogorov's criterion to construct the polymer measure on a space of Hölder paths. It is singular with respect to the Wiener measure on $C([0, 1], \mathcal{S})$ although it has support in all the spaces $C^\gamma([0, 1], \mathcal{S})$, for $\gamma < 1/2$ like the law of Brownian motion.

◦ *Anderson diffusion.* Another probability measure on path space is naturally associated with the Anderson operator. It is the (time homogeneous) Markov process on \mathcal{S} with transition probability

$$\frac{e^{t\lambda_0(\widehat{\xi})} p_t(x, y) u_0(y)}{u_0(x)}.$$

It is formally the diffusion with generator $\Delta - 2(\nabla \ln u_0)\nabla$. The formal character of this operator comes from the fact that the ground state u_0 is only almost surely of Hölder regularity strictly

smaller than 1, so the drift $-2\nabla(\ln u_0)$ is a distribution. In particular, this diffusion falls in the range of SDE with distributional drift in the Young regime since $\nabla(\ln u_0)$ is of Hölder regularity 0^- . Following a long tradition going back to the work of Symanzik on constructive quantum field theory in the 60s, we relate in Section 6.3 the distribution of the square of the Anderson Gaussian free field and the distribution of the renormalized occupation measure $\mathcal{O}_{1/2}$ of a certain Poisson point process of Anderson diffusion loops in \mathcal{S} . The notations will be defined in Section 6.3.

Theorem 4 – *The renormalized occupation measure $\mathcal{O}_{1/2}$ has the same distribution as the Wick square $:\phi^2:$ of the Anderson Gaussian free field.*

Finally we prove large deviation results for the free end point path and bridges for the Anderson diffusion, for a small travelling time. Given a point $x \in \mathcal{S}$ write Q_x for the polymer measure started from x . Given $0 < r \leq 1$ and $0 < \gamma < 1/2$, denote by $Q_x^{(r)}$ the law under Q_x on $C^\gamma([0, 1], \mathcal{S})$ of the process $(w_{sr})_{0 \leq s \leq 1}$. Given another point $y \in \mathcal{S}$ denote by $Q_{x,y}^{(r)}$ the law of the Anderson diffusion path conditioned on starting from x and ending up in y at time r , after linear reparametrization of the time interval $[0, r]$ by the fixed interval $[0, 1]$. Set

$$\mathcal{I}(w) := \int_0^1 |\dot{w}_s|_g^2 ds \quad (1.3)$$

for $w \in H^1([0, 1], \mathcal{S})$, and $\mathcal{I}(w) = \infty$, otherwise. One proves the following large deviation result for the polymer measure and its bridges, where $d(x, y)$ stands for the Riemannian distance between x and y . Recall that $Q_x^{(r)}$ and $Q_{x,y}^{(r)}$ are families of random measures.

Theorem 5 – *Fix two points $x \neq y$ in \mathcal{S} and $0 < \gamma < 1/2$. The following happens almost surely.*

- *The family $(Q_x^{(r)})_{0 < r \leq 1}$ satisfies in $C^\gamma([0, 1], \mathcal{S})$ a large deviation principle with good rate function $\mathcal{I}(\cdot)$.*
- *The family $(Q_{x,y}^{(r)})_{0 < r \leq 1}$ satisfies in $C^\gamma([0, 1], \mathcal{S})$ a large deviation principle with good rate function $\mathcal{I}(\cdot) - d(x, y)^2$.*

So the Anderson diffusion on free and fixed endpoints paths satisfies the same large deviation principle as Wiener measure and the rate function does not see the effect of the white noise potential.

Organization of this work – We have gathered in Section 2 a number of elementary facts that we use in the remainder of the work. Section 3 provides a short self-contained functional analytic construction of the Anderson operator H . Section 4 provides a fine description of the heat kernel of H and applications to the spectral gap and eigenfunction estimates of H amongst others. Section 5 introduces the Anderson Gaussian free field and studies some of its properties. We relate in particular the distribution of the Wick square Anderson Gaussian free field to the distribution of the spectrum of H . Section 6 deals with the polymer measure, its construction and properties, its link with the Anderson Gaussian free field and the large deviation results for this measure and its bridges. The introduction of each section gives more details on its content. Appendix A contains a proof of a parametric version of meromorphic Fredholm theory. Appendix B gives a number of elements on the geometric Littlewood-Paley decomposition that we use.

Notations. We collect here a number of notations that are used throughout the text.

- *We denote by μ the Riemannian volume measure.*
- *We use the notation $C^\gamma(\mathcal{S})$ for the Hölder spaces, and $H^\gamma(\mathcal{S})$ for the Sobolev spaces, for any $\gamma \in \mathbb{R}$, both defined as Besov spaces over \mathcal{S} .*
- *The notation $\mathcal{B}(E, F)$ stands for the space of continuous linear maps from a Banach space E into a Banach space F , with operator norm $\|\cdot\|_{\mathcal{B}(E, F)}$.*
- *For a constant $z \in \mathbb{C}$, we will stick to the usual convention that z stands for the multiplication operator M_z in an identity involving operators.*
- *The notation $O_E(1)$ stands for a bounded E -valued function.*
- *The Laplace-Beltrami operator Δ is defined as a non-positive operator on $L^2(\mathcal{S})$.*

Acknowledgements. *We would like to thank the anonymous reviewer for her/his patience and for pointing out a mistake in the previous proof of our main theorem as well as Hugo Eulry, Tristan Robert and Immanuel Zachhuber for useful discussions on several points of the present work. N.V.D. would like to thank the Institut Universitaire de France for support.*

2 – Tools for the analysis

We will use in the sequel a number of elementary facts on paraproducts and meromorphic Fredholm theory. We recall here what we need from them and refer the reader to [3, 30, 6, 41] for basics and non-basics on paraproduct and resonant operators. We will only use what is recalled here. The reader can skip safely this section and refer to it only when needed.

• **Paraproduct and resonant operators and corrector** – Recall from Littlewood-Paley theory that one can decompose an arbitrary distribution f on the d -dimensional torus as a sum of smooth functions

$$f = \sum_{n \geq -1} P_n f$$

approximately localized in frequency space in annuli of size 2^n . This allows to decompose formally the product of two distributions into

$$fg = \sum_{i < j-1} (P_i f)(P_j g) + \sum_{j < i-1} (P_i f)(P_j g) + \sum_{|i-j| \leq 1} (P_i f)(P_j g),$$

with the first two quantities always converging in the space of distributions on the torus. Based on that model, and set in our 2-dimensional manifold setting, one can decompose the product of any two smooth functions f, g on \mathcal{S} as

$$fg = P_f g + P_g f + \Pi(f, g), \quad (2.1)$$

using paraproduct and resonant operators P and Π with the following continuity properties.

Proposition 6 – *The following two continuity results hold true.*

– *For any $\alpha_1, \alpha_2 \in \mathbb{R}$ the paraproduct operator*

$$P : (f, g) \mapsto P_f g,$$

maps continuously $C^{\alpha_1}(\mathcal{S}) \times C^{\alpha_2}(\mathcal{S})$ into $C^{\alpha_1 \wedge 0 + \alpha_2}(\mathcal{S})$. For $\alpha_1 \neq 0$, it also maps continuously the space $C^{\alpha_1}(\mathcal{S}) \times H^{\alpha_2}(\mathcal{S})$ and $H^{\alpha_1}(\mathcal{S}) \times C^{\alpha_2}(\mathcal{S})$ into $H^{\alpha_1 + \alpha_2}(\mathcal{S})$.

– *The resonant operator*

$$\Pi : (f, g) \mapsto \Pi(f, g),$$

is symmetric and well-defined as a continuous operator from $C^{\alpha_1}(\mathcal{S}) \times C^{\alpha_2}(\mathcal{S})$ into $C^{\alpha_1 + \alpha_2}(\mathcal{S})$, and from $C^{\alpha_1}(\mathcal{S}) \times H^{\alpha_2}(\mathcal{S})$ into $H^{\alpha_1 + \alpha_2}(\mathcal{S})$, iff $\alpha_1 + \alpha_2 > 0$.

Identity (2.1) thus makes sense for all $f \in C^{\alpha_1}(\mathcal{S}), g \in C^{\alpha_2}(\mathcal{S})$, or $f \in C^{\alpha_1}(\mathcal{S}), g \in H^{\alpha_2}(\mathcal{S})$, provided $\alpha_1 + \alpha_2 > 0$. The reader will find more details on these paraproduct and resonant operators in Appendix B. The next fundamental result is the backbone of Gubinelli, Imkeller & Perkowski' seminal work [30] on singular stochastic PDEs. Its extension to a manifold setting was worked out in Bailleul & Bernicot's work [5] in a general parabolic setting – see Mouzard's work [41] for the mixed elliptic Sobolev/Hölder result.

Proposition 7 – *The trilinear operator*

$$C(a, b, c) := \Pi(P_a b, c) - a\Pi(b, c)$$

is continuous from $C^{\alpha_1}(\mathcal{S}) \times C^{\alpha_2}(\mathcal{S}) \times C^{\alpha_3}(\mathcal{S})$ into $C^{\alpha_1 + \alpha_2 + \alpha_3}(\mathcal{S})$, and from $H^{\alpha_1}(\mathcal{S}) \times C^{\alpha_2}(\mathcal{S}) \times C^{\alpha_3}(\mathcal{S})$ into $H^{\alpha_1 + \alpha_2 + \alpha_3}(\mathcal{S})$, if $\alpha_2 + \alpha_3 < 0$, $\alpha_1 + \alpha_2 + \alpha_3 \in (0, 1)$ and $\alpha_1 \in (0, 1)$.

• **Operators built from $(-\Delta + z_0)^{-1}$** – It is well-known that space white noise on \mathcal{S} takes almost surely its values in the Hölder space $C^{\alpha' - 2}(\mathcal{S})$, for any $\alpha' < 1$. The reader can then think of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which this random variable is defined as $\Omega = C^{\alpha' - 2}(\mathcal{S})$, for

an ad hoc regularity exponent. Fix

$$0 < 2 - 2\alpha' < \alpha < \alpha' < 1,$$

and let ξ stand for a white noise on \mathcal{S} . Fix also a smooth real valued ‘coupling’ function h on \mathcal{S} . Denote by $\sigma(\Delta)$ the spectrum of the Laplace-Beltrami operator Δ . Given $z_0 \notin \sigma(\Delta)$ we will use occasionally the paraproduct-like operator \bar{P} defined by the intertwining relation

$$(-\Delta + z_0)\bar{P}_f g := P_f((-\Delta + z_0)g).$$

It was proved in Bailleul and Bernicot’s work in the parabolic setting [6] and later generalized in the elliptic setting in [41] that this operator has the same regularity properties as the operator P . This operator \bar{P} depends on z_0 , which will be fixed throughout, so we do not record it in the notation. We note that it enjoys continuity estimates whose constants do not depend on $z_0 \geq 1$. It was proved in [6] that the (modified) corrector

$$\bar{C}(a, b, c) := \Pi(\bar{P}_a b, c) - a\Pi(b, c) \quad (2.2)$$

enjoys the same continuity property as C stated in Proposition 7, with z_0 -uniform constants for $z_0 \geq 1$ since $-\Delta$ is a non-negative operator. Set

$$M^-(f) := P_f(h\xi), \quad M^+(f) := P_{h\xi}f + \Pi(f, h\xi). \quad (2.3)$$

While the operator M^- is well-defined and sends continuously $H^\gamma(\mathcal{S})$ into $H^{\gamma+\alpha'-2}(\mathcal{S})$ for any $\gamma \in \mathbb{R}^*$ the operator M^+ is only defined on the spaces $C^\gamma(\mathcal{S})$ and $H^\gamma(\mathcal{S})$ for $\gamma > 2 - \alpha'$, due to the resonant operator in the definition of M^+ . Set

$$\Gamma^{-1}(f) := f + (\Delta + z_0)^{-1}M^-(f) = f + \bar{P}_f(X_{h,z_0}),$$

where

$$X_{h,z_0} := (\Delta + z_0)^{-1}(h\xi).$$

The operator Γ^{-1} is well-defined on all of $\mathcal{D}'(\mathcal{S})$. Pick

$$\frac{2}{3} < s < \alpha < \frac{\alpha+1}{2} < \alpha' < 1. \quad (2.4)$$

We single out here an elementary fact whose proof is left to the reader.

Lemma 8 – *For every regularity exponent $\gamma \in \mathbb{R}$ and every positive η there exists a constant $m_\eta(\xi)$ such that for every real parameter $z_0 \geq m_\eta(\xi)$ one has*

$$\|(-\Delta + z_0)^{-1}\|_{\mathcal{B}(H^\gamma(\mathcal{S}), H^{\gamma+2-\eta}(\mathcal{S}))} < \eta,$$

and the continuous map

$$(-\Delta + z_0)^{-1}M^- : H^\gamma(\mathcal{S}) \rightarrow H^{\gamma \wedge 0 + \alpha}(\mathcal{S})$$

has a norm smaller than 1.

We use the fact that $\xi \in C^{\alpha'-2}(\mathcal{S})$ and $\alpha < \alpha'$, in the proof of the second item of the lemma. It follows that, for every $0 < \beta \leq \alpha$, the map Γ^{-1} from $H^\beta(\mathcal{S})$ into itself is invertible for z_0 positive and large enough. Taking z_0 even larger if needed, the map Γ^{-1} is also invertible as a map from $C^\beta(\mathcal{S})$ into itself, for all $0 < \beta \leq \alpha$. In both cases the norm of Γ^{-1} is bounded above by 2 uniformly in $z_0 \geq z_0(\xi)$. For the readers familiar with the other constructions of the Anderson Hamiltonian, the inverse Γ of Γ^{-1} is nothing but the Γ -map introduced by Gubinelli, Ugurcan and Zachhuber in [32] and used in [41]. The operators Γ^{-1} and Γ allow us here to unveil the analytic structure of the resolvent.

• **Meromorphic Fredholm theory with a parameter** – The analytic Fredholm theory provides conditions under which one can invert a family of Fredholm operators acting on some Hilbert space. Let U be a connected open subset of the complex plane \mathbb{C} . Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Recall that a family $(A(z))_{z \in U}$ of linear maps from \mathcal{H} into itself is said to be holomorphic iff the map A is \mathbb{C} -differentiable in U . This is equivalent to requiring that the \mathbb{C} -valued function $z \mapsto \langle y, A(z)x \rangle$ is holomorphic for any x, y in \mathcal{H} . The family $(A(z))_{z \in U}$ is said to be *finitely meromorphic* if for any $z \in U$, there exists a finite collection of operators $(A_j)_{1 \leq j \leq n_0}$ of *finite rank* and a holomorphic family $A_0(\cdot)$, defined near z , such that one has

$$A(z') = A_0(z') + (z' - z)^{-1}A_1 + \cdots + (z' - z)^{-n_0}A_{n_0},$$

for z' near z . This implies in particular that the poles are isolated. We shall need a version with parameters of the meromorphic Fredholm Theorem where $A(z, \mathbf{a})$ depends continuously on a parameter \mathbf{a} , element of a metric space.

Theorem 9 – *Let $U \subset \mathbb{C}$ be a connected open subset of the complex plane. Let (\mathcal{A}, d) be a metric space and $(K(z, \mathbf{a}))_{z \in U, \mathbf{a} \in \mathcal{A}}$ be a finitely meromorphic family of compact operators depending continuously on $\mathbf{a} \in \mathcal{A}$. If for every $\mathbf{a}_0 \in \mathcal{A}$ the operator $(\text{Id} - K(z_0, \mathbf{a}))^{-1}$ exists at some point $z_0 \in U$, for all \mathbf{a} in a neighborhood of \mathbf{a}_0 , then the family*

$$(z' \in U) \mapsto (\text{Id} - K(z', \mathbf{a}))^{-1}$$

is a well-defined meromorphic family of operators with poles of finite rank which depends continuously on $\mathbf{a} \in \mathcal{A}$.

A proof of this statement is given in Appendix A. Before moving to the construction of the Anderson operator recall here that a sequence $(h_n)_{n \geq 0}$ of Banach space-valued meromorphic functions, defined on a common open subset of \mathbb{C} , converge to a limit meromorphic function h if h_n converges uniformly to h on every compact set that does not contain any pole of h .

3 – A construction of the Anderson operator

Let ξ stand for a space white noise on the Riemannian manifold \mathcal{S} and let h stand for a smooth real valued function on \mathcal{S} . We denote by Δ the Laplace-Beltrami operator associated with the Riemannian metric on \mathcal{S} . Recall one can construct ξ as a random series $\sum_{n \geq 0} \gamma_n f_n$, where the f_n are the eigenfunctions of the Laplace-Beltrami operator and the γ_n are a family of independent centered Gaussian random variables with unit variance. We *define* in this section the unbounded operator $H = -\Delta + M_{h\xi}$ on $L^2(\mathcal{S})$ by its resolvent map $R(z)$, a meromorphic function of z . We identify the map R as the unique solution of a fixed point equation. The naive formulation of the fixed point equation involves however a multiplication problem that is the signature of the singular character of the operator H . A renormalization process is needed to tackle this problem. We smoothen the noise ξ with the heat kernel $e^{r\Delta}$ and add some r -dependent diverging terms in the operator to make the resolvent associated with this modified operator converge as r tends to 0. The resolvent R is then defined from a renormalized version of a naive fixed point equation using the meromorphic Fredholm theory.

A reader already familiar with one of the previous constructions of the Anderson operator [2, 39, 32, 41] may skip this section and keep in mind that we construct the resolvent of this operator as a meromorphic function defined on all of \mathbb{C} .

To disentangle the multiplication problem involved in the definition of the operator H and its resolvent it turns out to be useful to split the multiplication operator $M_{h\xi}$ into

$$M_{h\xi} f = fh\xi = P_f(h\xi) + \left(P_{h\xi} f + \Pi(f, h\xi) \right) = M^- f + M^+ f,$$

using the operators M^- and M^+ from (2.3). This allows to separate well-defined terms of low regularity from ill-defined terms of a priori better regularity. This approach allows to get around the tricky use of strongly paracontrolled distributions from [2, 32], and to avoid the use of the subtle quasi-duality between paraproduct and resonant operators from [32, 41]. However, we mention that one cannot find precise information on the domain of H with this method nor a precise description of the Sobolev spaces attached to the operator H .

Pick z_0 positive and big enough. We will tune it later to make some ξ -dependent quantities small using Lemma 8.

3.1 Definition and approximation of the resolvent

We first formulate in Section 3.1.1 a fixed point equation for the resolvent that involves an ill-defined term, as expected from the singular nature of the Anderson operator. This analytically ill-defined term only involves the noise and it can be given sense by a renormalization procedure of Wick type described in Proposition 10. This is where the fact that the noise is random is put to work

as the renormalized term is constructed by probabilistic means as a random variable. Rewriting in Section 3.1.2 the fixed point equation with the ill-defined term replaced by its well-defined renormalized counterpart provides an equation that can be solved uniquely in an appropriate space of meromorphic operator-valued functions. The renormalization procedure is interpreted in Section 3.1.3 as giving an r -indexed family of resolvent operators associated with a diverging r -indexed family of operators. We will work throughout with some regularity exponents

$$\frac{2}{3} < s < \alpha < \frac{\alpha + 1}{2} < \alpha' < 1. \quad (3.1)$$

3.1.1 – *The naive fixed point equation for the resolvent.* One has at a formal level:

$$\begin{aligned} R(z) &= (-\Delta + M_\xi - z)^{-1} \\ &= \left(\text{Id} + (-\Delta + z_0)^{-1} M^- + (-\Delta + z_0)^{-1} (M^+ - z + z_0) \right)^{-1} (-\Delta + z_0)^{-1} \\ &= \left(\text{Id} + \Gamma(-\Delta + z_0)^{-1} (M^+ - z + z_0) \right)^{-1} \Gamma(-\Delta + z_0)^{-1} \\ &= \left(\text{Id} - ((-\Delta + z_0)\Gamma^{-1} + M^+ - z + z_0)^{-1} (M^+ - z + z_0) \right) \Gamma(-\Delta + z_0)^{-1} \\ &= \Gamma(-\Delta + z_0)^{-1} - ((-\Delta + z_0)\Gamma^{-1} + M^+ - z + z_0)^{-1} (M^+ - z + z_0) \Gamma(-\Delta + z_0)^{-1} \\ &= \Gamma(-\Delta + z_0)^{-1} - R(z) (M^+ - z + z_0) \Gamma(-\Delta + z_0)^{-1} \end{aligned}$$

using the identity $(1 + y^{-1}x)^{-1} = 1 - (y + x)^{-1}x$, for operators x, y and $(-\Delta + z_0)\Gamma^{-1} = -\Delta + z_0 + M^-$. This is the raw version of the fixed point equation that should define $R(z)$. As Γ takes its values at best in $C^{\alpha'}(\mathcal{S})$ we spot a problem in the term $M^+\Gamma$ since the resonant term in

$$M^+(\Gamma u) = P_{h\xi}(\Gamma u) + \Pi(\Gamma u, h\xi),$$

is not well-defined because $\alpha' + (\alpha' - 2) < 0$. The identity

$$\Gamma = \text{Id} - (-\Delta + z_0)^{-1} M^- \Gamma$$

multiplied by M^+ allows to rewrite the naive fixed point equation for $R(z)$ as

$$R(z) = \Gamma(-\Delta + z_0)^{-1} - R(z) \left(M^+ - M^+(-\Delta + z_0)^{-1} M^- \Gamma - (z - z_0)\Gamma \right) (-\Delta + z_0)^{-1}, \quad (3.2)$$

and to isolate precisely the problem in the expression $M^+(-\Delta + z_0)^{-1} M^-$, that is in the resonant term

$$\Pi(\bar{P}_u X_{h,z_0}, h\xi), \quad u \in H^s(\mathcal{S}) \quad (3.3)$$

that comes from the M^+ operator. Using the corrector operator \bar{C} from (2.2) one has

$$M^+(-\Delta + z_0)^{-1} M^-(u) = P_{h\xi}(\bar{P}_u X_{h,z_0}) + u \Pi(X_{h,z_0}, h\xi) + \bar{C}(u, X_{h,z_0}, h\xi),$$

with a well-defined \bar{C} term since $\alpha' + \alpha' + (\alpha' - 2) > 0$. We isolate in the u -independent and noise-dependent term $\Pi(X_{h,z_0}, h\xi)$ the only ill-defined term in the previous sum – the sum of the regularity exponents of X_{h,z_0} and $h\xi$ add up to a negative constant.

An elementary renormalization process allows however to give a proper meaning to such a term. Set

$$\xi_r := e^{r\Delta} \xi$$

for the heat regularized space white noise and

$$X_{h,r,z_0} := (-\Delta + z_0)^{-1} (h\xi_r).$$

The next statement identifies the singular part of the diverging resonant term

$$\Pi(X_{h,r,z_0}, h\xi_r);$$

its proof follows the usual pattern for similar Wick renormalization proofs. It is given in Appendix B. As explained in Lemma 8, the parameter z_0 will be chosen positive enough depending on the size of the noise ξ in our analysis. In fact, it will depend on the size of the enhanced noise

$$\hat{\xi} := \left(\xi, \mathcal{R}\{\Pi(X_{h,z_0}, h\xi)\} \right) \in C^{\alpha'-2}(\mathcal{S}) \times C^{2\alpha'-2}(\mathcal{S}).$$

For this purpose it is important to keep track of the dependance of the renormalized product of the parameter z_0 as stated below.

Proposition 10 – We have in $\mathcal{C}^{2\alpha'-2}(\mathcal{S})$ the identity

$$\mathbb{E}[\Pi(X_{h,r,z_0}, h\xi_r)] = \frac{|\log r| h^2}{4\pi} + O(1) \quad (3.4)$$

and the random variables

$$\Pi(X_{h,r,z_0}, h\xi_r) - \frac{|\log r| h^2}{4\pi}$$

converge in probability in the space $\mathcal{C}^{2\alpha'-2}(\mathcal{S})$, as $\varepsilon > 0$ goes to 0, to a limit random variable denoted by

$$\mathcal{R}\{\Pi(X_{h,z_0}, h\xi)\}.$$

Moreover $\mathcal{R}\{\Pi(X_{h,z_0}, h\xi)\}$ goes to 0 in probability in the space $\mathcal{C}^{2\alpha'-2}(\mathcal{S})$ as $z_0 > 0$ diverges to $+\infty$.

The letter ‘ \mathcal{R} ’ is chosen for ‘*renormalized*’. Identity (3.4) improves upon the corresponding statement in [41] by showing that the singular part of the resonance is a constant when $h \equiv 1$, rather than a function. A similar fact was proved in the closely related work [19] of Dahlqvist, Diehl & Driver on the parabolic Anderson model equation in a closed two dimensional Riemannian manifold. (They developed for their purpose a first order version of regularity structures in that setting rather than using paracontrolled calculus.)

3.1.2 – The renormalized fixed point equation for the resolvent. We define the renormalized version of the operator $\mathbf{M}^+(-\Delta + z_0)^{-1}\mathbf{M}^-$ setting for all $u \in H^s(\mathcal{S})$

$$\mathcal{R}\{\mathbf{M}^+(-\Delta + z_0)^{-1}\mathbf{M}^-\}(u) := \mathbf{P}_{h\xi}((-\Delta + z_0)^{-1}\mathbf{M}^-u) + u\mathcal{R}\{\Pi(X_{h,z_0}, h\xi)\} + \bar{\mathbf{C}}(u, X_{h,z_0}, h\xi).$$

The assumptions (3.1) on the regularity exponents guarantee that the linear operator $\mathcal{R}\{\mathbf{M}^+(-\Delta + z_0)^{-1}\mathbf{M}^-\}$ is linear continuous from $L^2(\mathcal{S})$ into $H^{2\alpha-2}(\mathcal{S})$. The renormalized counterpart of the fixed point equation (3.2) for $R(z)$ reads

$$R(z) = \Gamma(-\Delta + z_0)^{-1} - R(z) \left(\mathbf{M}^+ - \mathcal{R}\{\mathbf{M}^+(-\Delta + z_0)^{-1}\mathbf{M}^-\}\Gamma - (z - z_0)\Gamma \right) (-\Delta + z_0)^{-1},$$

that is

$$R(z) \left\{ \text{Id} + \left(\mathbf{M}^+ - \mathcal{R}\{\mathbf{M}^+(-\Delta + z_0)^{-1}\mathbf{M}^-\}\Gamma - (z - z_0)\Gamma \right) (-\Delta + z_0)^{-1} \right\} = \Gamma(-\Delta + z_0)^{-1} \quad (3.5)$$

Choosing $z_0 > 0$ random and big enough ensures with Lemma 8 and Proposition 10 the bound

$$\left\| \left(\mathbf{M}^+ - \mathcal{R}\{\mathbf{M}^+(-\Delta + z_0)^{-1}\mathbf{M}^-\}\Gamma \right) (-\Delta + z_0)^{-1} \right\|_{\mathcal{B}(E, H^{2\alpha-2}(\mathcal{S}))} < 1, \quad (3.6)$$

with $E = L^2(\mathcal{S})$ or $H^{2\alpha-2}(\mathcal{S})$. One notes further that the operator

$$(\dots)(-\Delta + z_0)^{-1} \in \mathcal{B}(H^{2\alpha-2}(\mathcal{S}), H^{2\alpha-2}(\mathcal{S}))$$

in the preceding inequality is *compact* as it actually maps $H^{2\alpha-2}(\mathcal{S})$ into $H^{2\alpha'-2}(\mathcal{S})$ and $\alpha < \alpha'$. Equation (3.5) then defines a map

$$R(z_0) = \Gamma(-\Delta + z_0)^{-1} \left\{ \text{Id} + \left(\mathbf{M}^+ - \mathcal{R}\{\mathbf{M}^+(-\Delta + z_0)^{-1}\mathbf{M}^-\}\Gamma \right) (-\Delta + z_0)^{-1} \right\}^{-1}. \quad (3.7)$$

We invite the reader to check that the assumptions of Theorem 9 on meromorphic Fredholm theory with a parameter are met, with $\widehat{\xi} \in C^{\alpha'-2}(\mathcal{S}) \times C^{2\alpha'-2}(\mathcal{S})$ in the role of the parameter. The meromorphic Fredholm theory applied to the meromorphic $\widehat{\xi}$ -indexed family of compact operators

$$\text{Id} + \left(\mathbf{M}^+ - \mathcal{R}\{\mathbf{M}^+(-\Delta + z_0)^{-1}\mathbf{M}^-\}\Gamma - (z - z_0)\Gamma \right) (-\Delta + z_0)^{-1} \in \mathcal{B}(H^{2\alpha-2}(\mathcal{S}), H^{2\alpha-2}(\mathcal{S}))$$

allows to define

$$R(z) = \Gamma(-\Delta + z_0)^{-1} \left\{ \text{Id} + \left(\mathbf{M}^+ - \mathcal{R}\{\mathbf{M}^+(-\Delta + z_0)^{-1}\mathbf{M}^-\}\Gamma - (z - z_0)\Gamma \right) (-\Delta + z_0)^{-1} \right\}^{-1}$$

as a meromorphic function of $z \in \mathbb{C}$ with values in $\mathcal{B}(H^{2\alpha-2}(\mathcal{S}), \Gamma(H^{2\alpha}(\mathcal{S})))$ that depends continuously on $\widehat{\xi}$. Since $H^{2\alpha}(\mathcal{S})$ is continuously embedded into $C^{2\alpha-1}(\mathcal{S})$, the restriction of R to $L^2(\mathcal{S})$ defines a meromorphic function with values in $\mathcal{B}(L^2(\mathcal{S}), C^{2\alpha-1}(\mathcal{S}))$.

3.1.3 – The regularized renormalized fixed point equation. The convergence result of Proposition 10 and the fixed point equation giving the meromorphic function $R(\cdot)$ can be put together to provide approximations of $R(z)$ by the resolvent of some bounded operators. It is convenient for that purpose to use Skorohod representation theorem for weak convergence (hence convergence in probability) and assume that the convergence in Proposition 10 is almost sure. This can be done by a change of probability space Ω on which white noise is defined – see e.g. Theorem 4.30 in Kallenberg’s book [36] for Skorohod theorem. Denote by Ω_1 the measurable subset of Ω of probability 1 where the almost sure convergence holds. Since we are only interested in almost sure statements, what happens on the null set $\Omega \setminus \Omega_1$ is irrelevant.

Given a positive regularization parameter r set

$$\Gamma_r^{-1}(f) := f + (-\Delta + z_0)^{-1} P_f(h\xi_r)$$

and

$$M_r^-(f) := P_f(h\xi_r), \quad M_r^+(f) := P_{h\xi_r} f + \Pi(f, h\xi_r).$$

We denote by Γ_r the inverse of Γ_r^{-1} . One proves the following statement in Appendix B.

Lemma 11 – *For $r > 0$ the operator M_r^- is a smoothing operator and the operator M_r^+ is a pseudodifferential operator of order 0.*

The operator Γ_r is also a pseudo-differential operator of order 0. Denote here by

$$c_{h,r} := \frac{|\log r| h^2}{4\pi}$$

the diverging part of $\Pi(X_{h,r,z_0}, h\xi_r)$ – this is a function on \mathcal{S} independent of z_0 whose associated multiplication operator is denoted by $M_{c_{h,r}}$. Set

$$\mathcal{R}_r \{M^+(-\Delta + z_0)^{-1} M^-\}(u) := M_r^+(-\Delta + z_0)^{-1} M_r^- u - c_{h,r} u.$$

The convergence result from Proposition 10 implies that the map $\mathcal{R}_r \{M^+(-\Delta + z_0)^{-1} M^-\}$ is converging to the map $\mathcal{R} \{M^+(-\Delta + z_0)^{-1} M^-\}$ in $\mathcal{B}(L^2(\mathcal{S}), H^{2\alpha-2}(\mathcal{S}))$, for all chance elements $\omega \in \Omega_1$. It follows that for all $\omega \in \Omega_1$, the (ω -dependent) operators

$$R_r(z) := \Gamma_r(-\Delta + z_0)^{-1} \left\{ \text{Id} + \left(M^+ - \mathcal{R}_r \{M^+(-\Delta + z_0)^{-1} M^-\} \Gamma_r - (z - z_0) \Gamma_r \right) (-\Delta + z_0)^{-1} \right\}^{-1}$$

converge as r goes to 0 to the (ω -dependent) operator $R(z)$ in $\mathcal{B}(L^2(\mathcal{S}), C^{2\alpha-1}(\mathcal{S}))$, as a meromorphic function of z by the analytic Fredholm theory. Rewinding the algebraic process that led to this expression of $R_r(z)$ requires the use of the following elementary statement whose proof is given in Appendix B.

Lemma 12 – *Pick $a \in \mathbb{R}$. Let P be an invertible elliptic pseudo-differential operator of order a and $Q(z)$ a pseudo-differential operator of positive order b , depending holomorphically on $z \in \mathbb{C}$. If there exists z_0 such that $(\text{Id} + P^{-1}Q(z_0))$ and $(\text{Id} + Q(z_0)P^{-1})$ are invertible from $H^a(\mathcal{S})$ into itself then we have*

$$(P + Q(z))^{-1} = (\text{Id} + P^{-1}Q(z))^{-1} P^{-1} = P^{-1}(\text{Id} + Q(z)P^{-1})^{-1} \quad (3.8)$$

for all $z \in \mathbb{C}$, where both sides of each equality are Fredholm operators from $H^a(\mathcal{S})$ into itself depending meromorphically on $z \in \mathbb{C}$.

Lemma 11 and Lemma 12 justify that we write

$$\begin{aligned} & \left(-\Delta + z_0 + M_r^- \right)^{-1} \circ \\ & \left\{ \text{Id} + \left(M_r^+ - \left(M_r^+(-\Delta + z_0)^{-1} M_r^- - M_{c_{h,r}} \right) \Gamma_r - (z + z_0) \Gamma_r \right) (-\Delta + z_0)^{-1} \right\}^{-1} \end{aligned}$$

$$\begin{aligned}
&= \left(-\Delta + z_0 + M_r^- \right)^{-1} \circ \\
&\quad \left\{ \text{Id} + M_r^+ (-\Delta + z_0)^{-1} - \left(M_r^+ (-\Delta + z_0)^{-1} M_r^- - M_{c_{h,r}} + (z + z_0) \right) (-\Delta + z_0 + M_r^-)^{-1} \right\}^{-1} \\
&= \left\{ -\Delta + z_0 + M_r^- + M_r^+ (-\Delta + z_0)^{-1} \left(-\Delta + z_0 + M_r^- \right) - \right. \\
&\quad \left. \left(M_r^+ (-\Delta + z_0)^{-1} M_r^- - M_{c_{h,r}} + (z + z_0) \right) \right\}^{-1} \\
&= \left(-\Delta - z + M_r^- + M_r^+ + M_{c_{h,r}} \right)^{-1} = \left(-\Delta - z + M_{h\xi_r} + M_{c_{h,r}} \right)^{-1},
\end{aligned}$$

by the usual composition in the pseudo-differential calculus. So $R_r(z)$ is the resolvent of the operator $-\Delta + M_{h\xi_r + c_{h,r}}$, perturbation of minus the Laplace-Beltrami operator Δ by the r -diverging smooth potential $h\xi_r + c_{h,r}$.

Proposition 13 – *The meromorphic maps $R_r(\cdot)$, with values in $\mathcal{B}(L^2(\mathcal{S}), C^{2\alpha-1}(\mathcal{S}))$, converge to the meromorphic map $R(\cdot)$ as $r > 0$ goes to 0, and $R(\cdot)$ has real poles in a half-plane $\{\text{Re}(z) > m\}$, for m negative large enough and random.*

Proof – The R_r have real poles as the potentials ξ_r and $c_{h,r}$ are real valued. The poles of R are limits of the poles of R_r . We see from (3.6) and (3.7) that R has no poles in the half-plane $\{\text{Re}(z) \leq m\}$, for m negative large enough and random. \triangleright

We used Skorohod representation theorem to represent a convergence in probability as an almost sure convergence on a different probability space. The reader should keep in mind that the resolvent of the regularized and renormalized operator $-\Delta + h\xi_r - \frac{\ln r}{4\pi} h^2$ is only converging in probability to a limit resolvent.

3.2 Construction of the operator H

We can construct an operator associated with the map R .

Theorem 14 – *The map R is the resolvent of a closed unbounded self-adjoint operator H on $L^2(\mathcal{S})$ with real discrete spectrum bounded below.*

Proof – Pick a real number z_1 which is not a pole of the limit family $R(\cdot)$. For $r_0 > 0$ small enough, z_1 is not a pole of the resolvent $R_r(\cdot)$ for all $r \in [0, r_0]$, so $R(z_1)$ is the limit in operator norms of the family $R_r(z_1)$ of self-adjoint operators acting on $L^2(\mathcal{S})$, as r goes to 0. This implies that $R(z_1)$ itself is compact self-adjoint as an operator on $L^2(\mathcal{S})$. Denote by

$$\sigma(R(z_1)) = \{(\lambda_n - z_1)^{-1}\}_{n \geq 0} \subset \mathbb{R}$$

its spectrum, with $\lambda_n \leq \lambda_{n+1}$ for all n , and by $(u_n)_{n \geq 0}$ its eigenvalues – they form an orthonormal system of L^2 . Also the meromorphic family of operators $R(z)$ satisfies the resolvent identity

$$R(z) = R(z_1) (\text{Id} + (z - z_1) R(z_1))^{-1}, \quad (3.9)$$

for any z that is not a pole of $R(\cdot)$, where the term $(\text{Id} + (z - z_1) R(z_1))^{-1}$ exists by meromorphic Fredholm theory in $\mathcal{B}(L^2(\mathcal{S}), L^2(\mathcal{S}))$ relying on the compactness of $R(z_1) \in \mathcal{B}(L^2(\mathcal{S}), H^{2\alpha}(\mathcal{S}))$. (This identity is obtained by passing to the limit in the corresponding identity satisfied by R_r using the convergence of R_r to R .) The resolvent identity (3.9) implies that the range of $R(z_1)$ does not depend on z_1 . Define the z -independent vector space

$$\mathfrak{D}(H) := R(z)(L^2(\mathcal{S})).$$

By the resolvent equation (3.9), the meromorphic family of operators $R(\cdot)$ has poles contained in $(\lambda_n)_{n \geq 0}$ and satisfies for all $n \geq 0$ the eigenvalue equation

$$R(z)u_n = (z - \lambda_n)^{-1}u_n.$$

This implies that we can define an unbounded operator $H - z$ on $L^2(\mathcal{S})$, with domain $\mathfrak{D}(H)$, in such a way that $(H - z)R(z)$ is the identity map on $L^2(\mathcal{S})$.

The spectrum of H is bounded below since its resolvent $R(\cdot)$ has no poles in the half-plane $\{\operatorname{Re}(z) \leq m\}$, for m negative enough. Last the operator $H : \mathfrak{D}(H) \subset L^2(\mathcal{S}) \mapsto L^2(\mathcal{S})$ is self-adjoint, hence closed since $\mathfrak{D}(H) = R(z)(L^2(\mathcal{S}))$, and $(H - z)R(z) = \operatorname{Id} : L^2(\mathcal{S}) \mapsto L^2(\mathcal{S})$ and $R(z_1)$ is bounded self-adjoint. \triangleright

Remarks – 1. *Since*

$$(-\Delta + M_{h\xi_r} - M_{c_{h,r}})R_r$$

is the identity map on $L^2(\mathcal{S})$, and R is the limit of the R_r , one can think of H as the limit of the operators $-\Delta + M_{h\xi_r} + M_{c_{h,r}}$ in the resolvent sense.

2. *One has*

$$\mathfrak{D}(H) = \operatorname{Im}(R(z_1)) \subset C^{2\alpha-1}(\mathcal{S}),$$

with elements $f \in \mathfrak{D}(H)$ such that $f + \overline{P}_f X_{h,z_0} \in H^{2\alpha}(\mathcal{S})$, while all element of this form are not necessarily in the domain. This property of elements in the domain of H was the starting point of the constructions of the Anderson operator in [2, 32, 41]. A regularity structures picture is given in [39]. (Note that we learn from the explicit description of $\mathfrak{D}(H)$ in [41] that the domain of H is not an algebra.) The operator H and its domain are the objects of primary interest in these works and one has first to ‘guess’ the domain and check its density in an appropriate space before proving a number of functional inequalities satisfied by H . A fixed point argument is used in [2, 39] to construct the inverse of $H + c$, for c negative and $|c|$ big enough, while the Babuska-Lax-Milgram theorem is used as a substitute in [32, 41].

It follows from the spectral theorem for unbounded self-adjoint operators with compact resolvent that one has the following spectral representation of the heat kernel of H

$$e^{-tH} = \sum_{n \geq 0} e^{-t\lambda_n} u_n \otimes u_n.$$

We emphasize the dependence of the eigenvalues λ_n of H on $\widehat{\xi}$ by writing $\lambda_n(\widehat{\xi})$. We will see in Proposition 17 below that the eigenvalues and their associated eigen-projectors are continuous functions of the enhanced noise $\widehat{\xi}$.

4 – Heat operator for the Anderson operator

The main result of this section, Theorem 19, provides a sharp small time asymptotic Gaussian estimate for the Schwartz kernel $p_t(x, y)$ of e^{-tH} . The existence, regularity and strict positivity of p_t are proved in Section 4.1, with a number of consequences. The sharp asymptotic of p_t obtained in Section 4.2 gives a direct access in Section 4.4 to a proof of Weyl’s law for the distribution of the random eigenvalues of H and a number of estimates on its eigenfunctions. We also prove in that section some Gaussian upper and lower bounds on $p_t(x, y)$ and give almost sure lower bounds on the spectral gap of H under different kinds of geometric assumptions on (\mathcal{S}, g) .

4.1 Heat kernel and properties of H

It is elementary to get qualitative informations on the Schwartz kernel of the heat operator of H . Thinking of α as 1^- the regularity exponent $(2\alpha - 1)$ that appears in the next statement is also of the form 1^- .

Proposition 15 – *The heat semigroup e^{-tH} of the Anderson Hamiltonian H has a positive kernel $p_t(x, y)$ with respect to the Riemannian volume measure on \mathcal{S} . This kernel is a continuous function of $(t, x, y) \in (0, \infty) \times \mathcal{S}^2$ and each $p_t(\cdot, \cdot)$ is $(2\alpha - 1)$ -Hölder, uniformly in $t \in [t_0, t_1]$, for $0 < t_0 \leq t_1 < \infty$.*

Proof – *Existence of the heat kernel.* We follow the classical approach, as exposed for instance in Section 5.2 of Davies’ textbook [21]. Recall that the graph norm of H on its domain $\mathfrak{D}(H)$ is defined by

$$\|u\|_H^2 := \|u\|_{L^2}^2 + \|Hu\|_{L^2}^2,$$

and that it turns $\mathfrak{D}(H)$ into a Hilbert space. We note first that for $f \in L^2(\mathcal{S})$, the element $e^{-tH}f$ belongs to the domain $\mathfrak{D}(H)$ of H , for all $t > 0$, by the spectral theorem, so $x \in \mathcal{S} \mapsto (e^{-tH}f)(x)$

is a $(2\alpha - 1)$ -Hölder function of $x \in \mathcal{S}$ for each $t > 0$. Since $t \mapsto e^{-tH}f$ is an analytic function of t on the half plane $\{\operatorname{Re}(t) > 0\}$ with values in the Hilbert space $(\mathfrak{D}(H), \|\cdot\|_H)$, we have that $(t, x) \mapsto (e^{-tH}f)(x)$, is a continuous function on $[t_0, t_1] \times \mathcal{S}$, for each compact interval $[t_0, t_1] \subset (0, \infty)$, analytic in the first time variable and Hölder in the second space variable. As the linear form $f \mapsto (e^{-tH}f)(x)$, is bounded on $L^2(\mathcal{S})$ for each $t > 0$ and $x \in \mathcal{S}$ there exists $a(t, x) \in L^2(\mathcal{S})$ such that

$$(e^{-tH}f)(x) = \langle f, a(t, x) \rangle_{L^2}.$$

The map

$$\left((t, x) \in (0, 1] \times \mathcal{S} \right) \mapsto a(t, x) \in L^2(\mathcal{S}),$$

being weakly Hölder continuous is norm Hölder continuous with strictly smaller Hölder exponent – a consequence of some general principle used by Davies [20, Section 1.5 p. 26] as follows : If we have a function $f : (t, x) \in [t_1, t_2] \times \mathcal{S} \mapsto f(t, x) \in L^2(\mathcal{S})$ such that for all $\psi \in L^2$, $(t, x) \mapsto \langle f(t, x), \psi \rangle \in \mathcal{C}^\alpha$, $\alpha > 0$, then $f : (t, x) \in [t_1, t_2] \times \mathcal{S} \mapsto f(t, x) \in L^2(\mathcal{S})$ is $(\alpha - \rho)$ -Hölder for all $\rho > 0$. A first application of the uniform boundedness principle to the family $f(t, x)_{t \in [t_1, t_2] \times \mathcal{S}}$ which is weakly bounded in $L^2(\mathcal{S})$ allows to deduce that $(t, x) \in [t_1, t_2] \times \mathcal{S} \mapsto f(t, x) \in L^2(\mathcal{S})$ is strongly bounded. The family

$$\operatorname{dist}((t_1, x_1), (t_2, x_2))^{-\alpha} \left| \langle f(t_1, x_1) - f(t_2, x_2), \psi \rangle_{L^2} \right|$$

is bounded for all $\psi \in L^2(\mathcal{S})$, then it implies by the uniform boundedness principle that

$$\sup_{(t_1, x_1), (t_2, x_2)} \operatorname{dist}((t_1, x_1), (t_2, x_2))^{-\alpha} (f(t_1, x_1) - f(t_2, x_2))$$

is bounded in L^2 . It follows that for all $\rho > 0$ the limit as $\operatorname{dist}((t_1, x_1), (t_2, x_2)) \rightarrow 0^+$

$$\lim \operatorname{dist}((t_1, x_1), (t_2, x_2))^{-\alpha+\rho} (f(t_1, x_1) - f(t_2, x_2)) \rightarrow 0 \in L^2(\mathcal{S}),$$

hence f is $(\alpha - \rho)$ -Hölder continuous as an $L^2(\mathcal{S})$ valued function.

We then have for all test functions $h_1, h_2 \in C^\infty(\mathcal{S})$

$$\begin{aligned} \langle e^{-tH}h_1, h_2 \rangle_{L^2} &= \left\langle e^{-\frac{t}{2}H}h_1, e^{-\frac{t}{2}H}h_2 \right\rangle_{L^2} \\ &= \int p_t(x, y) h_1(x) h_2(y) dx dy \end{aligned}$$

with

$$p_t(x, y) := \langle a(t/2, x), a(t/2, y) \rangle_{L^2}$$

a continuous function of its arguments. One gets the $(2\alpha - 1)$ -Hölder regularity of $p_t(x, y)$ as a function of x , for t, y fixed, noting that since the map $(x \in \mathcal{S}) \mapsto a(t, x) \in L^2(\mathcal{S})$ is weakly $(2\alpha - 1)$ -Hölder continuous it is also norm $(2\alpha - 1 - \rho)$ -Hölder continuous for all $\rho > 0$ – here again a consequence of the uniform boundedness principle. The joint regularity of $p_t(x, y)$ as a function of (x, y) follows, for $0 < t_0 \leq t \leq t_1 < \infty$. The reader will find in Section 4.3 an independent derivation of the above results which does not rely on abstract functional analytic arguments and is stronger in the sense we give quantitative estimates on the kernel $p_t(x, y)$.

Positivity. The fact that $p_t(x, y)$ is positive is established in Section 4.3.9 following ideas in Cannizzaro, Friz & Gassiat in their proof of Theorem 5.1 in [16] and our sharp description of the structure of the Schwartz kernel of e^{-tH} , in particular our proof works for all initial data in $L^2(\mathcal{S})$.

▷

We note here that Dahlqvist, Diehl and Driver only considered in [19] the parabolic Anderson model equation with smooth initial condition, so their results do not provide any insight on the heat kernel of the Anderson operator. A reader who has seen the parabolic paracontrolled structure used to solve the parabolic Anderson model equation may be puzzled by the fact that $e^{-tH}f$ is in the domain of H for any $f \in L^2(\mathcal{S})$ at positive times t , while it is essentially given by a seemingly different structure $(\partial_t - \Delta + z_0)^{-1}(\mathbb{P}_u \xi)$, for some u , up to a remainder term. Commuting the paraproduct and the resolution operator $(\partial_t - \Delta + z_0)^{-1}$ produces a remainder term, so the elliptic paracontrolled structure of $e^{-tH}f$, for $f \in L^2(\mathcal{S})$, pops out from the parabolic structure as a consequence of the identity

$$\begin{aligned}
(\partial_t - \Delta + z_0)^{-1}(\xi)(t) &= \int_0^t e^{-(t-s)(-\Delta+z_0)} \xi \, ds \\
&= \int_0^t e^{-r(-\Delta+z_0)} \xi \, dr = (-\Delta + z_0)^{-1} \xi - \int_t^\infty e^{-r(-\Delta+z_0)} \xi \, dr.
\end{aligned} \tag{4.1}$$

We take profit here from the fact that the noise ξ is time-independent and the integral over (t, ∞) is a smooth remainder term when $t > 0$.

The next statement follows from the positivity of the heat kernel of H and the Krein-Rutman theorem [52, Thm A.1 p. 123].

Corollary 16 – *Almost surely the lowest eigenvalue $\lambda_0(\widehat{\xi})$ of H is simple with a positive eigenvector.*

(Note that this question was also considered in Chouk & van Zuijlen's work [18], however their proof seems incomplete since they used Cannizzaro, Friz & Gassiat' strong maximum principle [16] which requires a continuous initial condition rather than an arbitrary initial condition in $L^2(\mathcal{S})$. Proceeding as in the subsection 4.3.9 fixes that point.) We now state another corollary of Proposition 15 that will be important for us later. It only relies on the convergence in the resolvent sense of the renormalized operators to the Anderson Hamiltonian and was already known from previous construction, see for example [39], if one consider only L^2 convergence of the ground state. For the convergence in Hölder spaces, the result is new however it can also be obtained with the description from [41] as well as our approach.

Proposition 17 – *The eigenvalues $\lambda_0(\widehat{\xi}_r)$ and $\lambda_1(\widehat{\xi}_r)$ of the regularized renormalized operator H_r are converging to $\lambda_0(\widehat{\xi})$ and $\lambda_1(\widehat{\xi})$, respectively, as $r > 0$ goes to 0. The ground state $u_{0,r}$ of H_r is converging in $C^{2\alpha-1}(\mathcal{S})$ to the ground state u_0 of H as r goes to 0.*

Proof – Pick an eigenvalue λ of H and a small disc D around λ whose intersection with $\sigma(H)$ equals $\{\lambda\}$. Since the regularized and renormalized resolvent R_r converges to R as a Fredholm meromorphic map and $R(z)$ is invertible for $z \in \partial D$, we know that for r small enough, the operators $R_r(z)$ are well-defined and invertible for $z \in \partial D$. Moreover it follows from the uniform convergence of $R_r(z)$ to $R(z)$ on ∂D that the family of spectral projectors

$$\Pi_r^D := \frac{i}{2\pi} \int_{\partial D} R_r(z) \, dz$$

is well-defined for $r > 0$ small enough and converges in $\mathcal{B}(L^2(\mathcal{S}), H^{2\alpha-1}(\mathcal{S}))$, so the limit operator reads

$$\Pi^D := \frac{i}{2\pi} \int_{\partial D} R(z) \, dz : L^2(\mathcal{S}) \mapsto H^{2\alpha-1}(\mathcal{S}).$$

We know from Rouché's Theorem [22, Thm C.12] applied to the operator valued meromorphic function $(\text{Id} + (z - z_1)R_r(z_1))^{-1}$, $z_1 \notin \mathbb{R}$, (this meromorphic Fredholm operator has same poles with multiplicity as $R(z)$) that $\sigma(H_r) \cap D$ has fixed multiplicity for r small enough since the poles of R_r and R contained in the disc D have the same multiplicity. Furthermore, as Π_r^D is a self-adjoint spectral projector, one has $\Pi_r^D \circ \Pi_r^D = \Pi_r^D$. It follows that $(\Pi^D)^2 = \Pi^D$ and Π^D is a self-adjoint projector such that one has for any $n \geq 0$

$$\Pi^D u_n = \frac{i}{2\pi} \int_{\partial D} R(z) u_n \, dz = \frac{i}{2\pi} \int_{\partial D} (\lambda_n(\widehat{\xi}) - z)^{-1} u_n \, dz = u_n \mathbf{1}_{\lambda=\lambda_n(\widehat{\xi})}.$$

This implies that Π^D acts as the identity when restricted on the eigenspace of λ and vanishes on all eigenfunctions u_n of eigenvalue $\lambda_n(\widehat{\xi}) \neq \lambda$. By continuity of $\Pi^D \in \mathcal{B}(L^2(\mathcal{S}), L^2(\mathcal{S}))$ this implies that Π^D vanishes on the orthogonal of the eigenspace of λ hence Π^D is the orthogonal projector on the eigenspace of λ .

As a consequence of this discussion $\lambda_0(\widehat{\xi}_r)$ and $\lambda_1(\widehat{\xi}_r)$ are both converging to $\lambda_0(\widehat{\xi})$ and $\lambda_1(\widehat{\xi})$. By construction the lowest eigenvalues $\lambda_0(\widehat{\xi}_r)$ are simple for all $r \geq 0$ however one needs a stronger result than the convergence of $\Pi_{\lambda_0(\widehat{\xi}_r)}$ to $\Pi_{\lambda_0(\widehat{\xi})}$ in $\mathcal{B}(L^2(\mathcal{S}), L^2(\mathcal{S}))$ to get the convergence of the ground state in $C^{2\alpha-1}(\mathcal{S})$. Using the convergence of the kernel of e^{-H_r} to the kernel of e^{-H} in the space $\mathcal{B}(L^2(\mathcal{S}), C^{2\alpha-1}(\mathcal{S}))$ that is a consequence of the continuous dependance on $\widehat{\xi}$ from Theorem 19 below, we see that if one picks a small disc $D_0(\widehat{\xi})$ with center $\lambda_0(\widehat{\xi})$ so that $D_0(\widehat{\xi}) \cap \sigma(H) = \{\lambda_0(\widehat{\xi})\}$, one has the convergence of

$$\Pi_r^0 = e^{\lambda_0(\widehat{\xi}_r)} e^{-H_r} \Pi_r^{D_0(\widehat{\xi})}$$

to

$$\Pi_{\lambda_0(\widehat{\xi})} = e^{\lambda_0(\widehat{\xi})} e^{-H} \Pi_{\lambda_0(\widehat{\xi})}$$

in $\mathcal{B}(L^2(\mathcal{S}), C^{2\alpha-1}(\mathcal{S}))$ using that $e^H \Pi^D = e^\lambda \Pi^D$. This implies the convergence of $u_{0,r}$ to u_0 in $C^{2\alpha-1}(\mathcal{S})$. Indeed, there exists a constant $m_r > 0$ converging to 0 such that for all $v \in L^2(\mathcal{S})$, one has

$$\|\langle u_{0,r}, v \rangle u_{0,r} - \langle u_0, v \rangle u_0\|_{C^{2\alpha-1}} \leq m_r \|v\|_{L^2(\mathcal{S})}$$

using that the first eigenvalues are simple thus the projections are just the scalar product with the ground states. Since $(u_{0,r})_{r>0}$ is bounded in $L^2(\mathcal{S})$, it converges weakly to $u'_0 \in L^2(\mathcal{S})$ up to an extraction. For any $z \in D_0(\widehat{\xi}) \setminus \{\lambda_0(\widehat{\xi})\}$ and $v \in L^2(\mathcal{S})$, we have

$$\begin{aligned} \langle (H+z)^{-1} u'_0, v \rangle &= \langle u'_0, (H+z)^{-1} v \rangle = \lim_{r \rightarrow 0} \langle u_{0,r}, (H+z)^{-1} v \rangle \\ &= \lim_{r \rightarrow 0} (\lambda_0(\widehat{\xi}_r) + z)^{-1} \langle u_{0,r}, v \rangle = (\lambda_0(\widehat{\xi}) + z)^{-1} \langle u'_0, v \rangle \end{aligned}$$

thus $u'_0 = u_0$. Applying the previous bound with $v = u_0$ yields

$$\|\langle u_{0,r}, u_0 \rangle u_{0,r} - u_0\|_{C^{2\alpha-1}} \leq m_r$$

and completes the proof. The proof shows that the spectral projectors are continuous functions of $\widehat{\xi}$. \triangleright

We note that the proof of Theorem 19 below does not use the result of Proposition 17, so the above proof is not circular. The image νe^{-tH} by e^{-tH} of a Borel finite measure ν on \mathcal{S} has density

$$\int_{\mathcal{S}} p_t(x, \cdot) \nu(dx)$$

with respect to the Riemannian volume measure on \mathcal{S} . One says that ν is invariant by the semigroup $(e^{-tH})_{t>0}$ if $\nu e^{-tH} = \nu$, for all $t > 0$.

Corollary 18 – *Each random variable $\lambda_n(\widehat{\xi})$ has a law that is absolutely continuous with respect to Lebesgue measure on \mathbb{R} , with a positive density. So the kernel of H is almost surely trivial and the semigroup $(e^{-tH})_{t>0}$ has no invariant Borel probability measure.*

Proof – Given any random variable X , the random variable $X + N$ is absolutely continuous with respect to the Lebesgue measure if N is. This can be seen as a regularisation of the characteristic function via a convolution. Thus, it suffices for example to see that the law of the random variables $\lambda_n(\widehat{\xi})$ is absolutely continuous with respect to the law of $\lambda_n(\widehat{\xi}) + N$ with N a $\mathcal{N}(0, 1)$ variable independent of ξ . Since the translation of the potential by a constant only induces a translation of the spectrum, we have

$$\lambda_n(\widehat{\xi} + N) = \lambda_n(\widehat{\xi}) + N$$

hence $\lambda_n(\widehat{\xi} + N)$ is absolutely continuous with respect to the Lebesgue measure. The Cameron-Martin theorem gives that ξ is absolutely continuous with respect to $\xi + N$ and since the eigenfunctions are measurable functions of ξ , we have that $\lambda_n(\widehat{\xi})$ is absolutely continuous with respect to $\lambda_n(\widehat{\xi} + N)$. This gives the first part of the statement.

Since the unbounded operator H is symmetric in $L^2(\mathcal{S})$, the heat kernel of H is a symmetric function of its space arguments. So a Borel invariant probability measure has a non-negative density with respect to the Riemannian volume measure given by

$$f = \int_{\mathcal{S}} p_t(x, \cdot) \nu(dx)$$

for any $t > 0$ which is $L^2(\mathcal{S})$ as a continuous function on a compact. Using the basis of eigenfunctions $(u_n)_{n \geq 0}$, we get

$$f = \sum_{n \geq 0} c_n u_n$$

and the invariance of the measure implies $e^{-tH} f = f$ for any $t > 0$ hence f is in the domain of H and

$$e^{-t\lambda_n(\widehat{\xi})} c_n = c_n$$

for all $n \geq 0$. This last relation implies that f belongs to the kernel of H . Conversely, a non-null element of the kernel of H defines an invariant Borel signed measure. The previous absolute

continuity result implies that any eigenvalue of H has null probability of being null and complete the proof. \triangleright

It is not clear however that tuples of k eigenvalues have a law that is absolutely continuous with respect to Lebesgue measure in \mathbb{R}^k .

4.2 An asymptotic for the heat kernel of Anderson operator

The qualitative estimate on the heat kernel p of H provided by Proposition 15 is not sufficient for our needs, which are quantitative. Fix a finite positive time horizon T . Denote by $p_t^\Delta(x, y)$ the Schwartz kernel of the usual heat operator $e^{t\Delta}$.

Theorem 19 – Given $y \in \mathcal{S}$ consider the difference

$$(t \in (0, T]) \mapsto (p_t - p_t^\Delta)(\cdot, y).$$

(1) For all $\delta \in (0, 1)$, there is $\rho > 0, T > 0$ such that

$$\sup_{t \in [0, T]} t^{\frac{1}{2} + \delta} \|(p_t - p_t^\Delta)(\cdot, y)\|_{\mathcal{C}^\rho(\mathcal{S})} < +\infty$$

(2) For all $\kappa > 0$ and $U \in L^2$, for all $a \in (0, 1)$ and $0 < t \leq T$

$$\|(e^{-tH} - e^{t\Delta})U\|_{\mathcal{C}^a} \lesssim t^{-\frac{a-\kappa}{2}} \|U\|_{L^2}.$$

Both $t^{\frac{a+\kappa}{2}} (e^{-tH} - e^{t\Delta}) \in B(L^2(\mathcal{S}), \mathcal{C}^a(\mathcal{S}))$ and $t^{\frac{1}{2} + \delta} (p_t - p_t^\Delta)(\cdot, y) \in \mathcal{C}^\rho(\mathcal{S})$ depend continuously on $\hat{\xi}$

4.3 Proof of Theorem 19.

The key idea of the proof is to use the Γ operator, similar to Γ introduced in the first part, to conjugate e^{-tH} into a better behaved semigroup e^{-tH^\sharp} which is one the main idea of [41] and [42]. It is possible to deduce the second claim of Theorem 19 directly from the detailed description of the domain of the operator H together with the Sobolev spaces of the Anderson operator of [41] and [42], see subsection 4.3.7. In addition to Theorem 19, we obtain general Schauder estimates and the strong maximum principle for e^{-tH} .

In the sequel we denote by

$$H^\sharp := \Gamma^{-1} H \Gamma$$

the conjugate of H by Γ . We will see that H^\sharp is a better behaved perturbation of $-\Delta$ than H and

$$e^{-tH} = \Gamma e^{-tH^\sharp} \Gamma^{-1}.$$

This is the approach followed in [42] to study the Schrödinger group associated with H . While the use of the second order paracontrolled expansion was crucial therein, we do not need it here.

4.3.1 Controlling the difference operator $H^\sharp + \Delta$ – We use the standard notation $\mathcal{B}_{p,q}^s(\mathcal{S})$ for the Besov spaces. The goal of the present paragraph is to study the regularity properties in Besov space of the difference operator

$$(H^\sharp + \Delta) : \mathcal{B}_{p,p}^{1+\delta}(\mathcal{S}) \mapsto \mathcal{B}_{p,p}^{-\kappa}(\mathcal{S})$$

for $\kappa > 0$, $\delta \in (0, 1)$ and $p \in [1, \infty]$. This is an important fact we shall use several times in the sequel; in particular the case $p < 2$ will be needed.

Lemma 20 – The operator $H\Gamma + \Delta$ maps continuously $\mathcal{B}_{p,p}^{1+\delta}(\mathcal{S})$ into $\mathcal{B}_{p,p}^{-\kappa}(\mathcal{S})$ for all $\kappa > 0, \delta \in (0, 1), p \in [1, \infty]$. It follows that $H^\sharp + \Delta : \mathcal{B}_{p,p}^{1+\delta}(\mathcal{S}) \mapsto \mathcal{B}_{p,p}^{-\kappa}(\mathcal{S})$ continuously.

Proof – A direct calculation [41, p. 21] using the definition of Γ and the product decomposition yields the expression

$$H\Gamma v = -\Delta v + P_\xi \Gamma v + \Gamma v \mathcal{R}\Pi(\xi, X) + \Pi(v, \xi) + [P_{\Gamma v}, \Delta]X + C(\Gamma v, X, \xi)$$

where C is the corrector, therefore

$$(H\Gamma + \Delta)v = P_\xi \Gamma v + \Gamma v \mathcal{R}\Pi(\xi, X) + \Pi(v, \xi) + [P_{\Gamma v}, \Delta]X + C(\Gamma v, X, \xi).$$

Note now the following facts.

- Since $v \in \mathcal{B}_{p,p}^{1+\delta}(\mathcal{S})$ one has $\Gamma v \in \mathcal{B}_{p,p}^{1-\frac{\kappa}{2}}(\mathcal{S})$ so $P_\xi \Gamma v \in \mathcal{B}_{p,p}^{1-\frac{\kappa}{2}-1-\frac{\kappa}{2}}(\mathcal{S}) = \mathcal{B}_{p,p}^{-\kappa}(\mathcal{S})$.
- The renormalized resonant term $\mathcal{R}\Pi(\xi, X)$ belongs to $\mathcal{C}^{-\kappa}(\mathcal{S})$ hence the product with $\Gamma v \in \mathcal{B}_{p,p}^{1-\frac{\kappa}{2}}(\mathcal{S})$ belongs to $\mathcal{B}_{p,p}^{-\kappa}(\mathcal{S})$.
- The resonant term $\Pi(v, \xi) \in \mathcal{B}_{p,p}^2(\mathcal{S})$ since $\xi \in \mathcal{B}_{\infty,\infty}^{1-\frac{\delta}{2}}(\mathcal{S})$, $v \in \mathcal{B}_{p,p}^{1+\frac{\delta}{2}}(\mathcal{S})$ by the usual properties of the resonant product of Besov distributions.
- The term $[P_{\Gamma v}, \Delta]X$ belongs to $\mathcal{B}_{p,\infty}^{1-\frac{\kappa}{2}-1}(\mathcal{S}) = \mathcal{B}_{p,\infty}^{-\frac{\kappa}{2}}(\mathcal{S}) \subset \mathcal{B}_{p,p}^{-\kappa}(\mathcal{S})$ since $\Gamma v \in \mathcal{B}_{p,p}^{1-\frac{\kappa}{2}}(\mathcal{S})$ and $X \in \mathcal{C}^{1-\frac{\kappa}{2}}(\mathcal{S})$ and we prove in Appendix B that for $u \in \mathcal{B}_{p,p}^\beta(\mathcal{S})$, $\beta \in (0, 1)$, $X \in \mathcal{C}^\alpha(\mathcal{S})$ the commutator $[P_u, \Delta]X \in \mathcal{B}_{p,\infty}^{\alpha+\beta}(\mathcal{S})$.
- The corrector $C(\Gamma v, X, \xi)$ has regularity $1 - \frac{\kappa}{2} + 1 - \frac{\kappa}{2} - 1 - \frac{\kappa}{2} = 1 - \frac{3\kappa}{2}$. In fact, it is simple to prove that the corrector $C(\Gamma v, X, \xi)$ should belong to $\mathcal{B}_{p,\infty}^{1-\frac{3\kappa}{2}}(\mathcal{S})$.

All these terms inject in $\mathcal{B}_{p,p}^{-\kappa}(\mathcal{S})$ which yields the claim. To deduce the analytical properties of $H^\sharp + \Delta$ we write

$$H^\sharp + \Delta = \Gamma^{-1}H\Gamma + \Delta = \Gamma^{-1}(H\Gamma + \Delta) + (1 - \Gamma^{-1})\Delta.$$

Recall that $1 - \Gamma^{-1} = P_\bullet X$ hence $(1 - \Gamma^{-1})\Delta$ sends continuously $H^{1+\delta}(\mathcal{S})$ into $H^{-2\delta}(\mathcal{S})$. The result then follows from the fact that $(H\Gamma + \Delta)u \in H^{-\kappa}(\mathcal{S})$ and Γ^{-1} is a bounded operator on $H^{-\kappa}(\mathcal{S})$. \triangleright

4.3.2 Besov spaces, continuous functions and Dirac delta distributions – We start with an elementary result.

Lemma 21 – *On a Riemannian surface (\mathcal{S}, g) , the Dirac delta distribution belongs to all the Besov spaces $\mathcal{B}_{p,p}^{\frac{2}{p}-2-\varepsilon}(\mathcal{S})$ for which $p \in [1, +\infty]$ and $\varepsilon > 0$. For $\delta > 0$ the Besov space $\mathcal{B}_{p,p}^{1+\delta}(\mathcal{S})$ injects continuously in $C^0(\mathcal{S})$ as soon as $(1 + \delta)p > 2$.*

Proof – We know that $\delta \in \mathcal{C}^{-2-\varepsilon}(\mathcal{S}) = \mathcal{B}_{\infty,\infty}^{-2-\varepsilon}(\mathcal{S})$. First note that for every $x \in \mathcal{S}$ and every chart $\kappa : \Omega \subset \mathcal{S} \mapsto \kappa(\Omega) \subset \mathbb{R}^2$ where $x \in \Omega$, $\delta_x \circ \kappa^{-1} = C\delta_{\kappa(x)}$ where C is a constant (we use the result that a distribution supported on a point of order 0 is a multiple of the δ function). Therefore the regularity of $\delta_x, x \in \mathcal{S}$ is the same as the regularity of $\delta_{\{0\}}^{\mathbb{R}^2}$. But $\delta_{\{0\}}^{\mathbb{R}^2} \in \mathcal{C}^{-2-\varepsilon}$ since it is a distribution homogeneous of degree -2 under scaling

$$\delta(\lambda \cdot) = \lambda^{-2}\delta(\cdot).$$

The distributions $\delta_x, x \in \mathcal{S}$, also belongs to $\mathcal{B}_{1,1}^{-\varepsilon}(\mathcal{S})$ since $\mathcal{B}_{1,1}^{-\varepsilon}(\mathcal{S})$ is the dual of $\mathcal{B}_{\infty,\infty}^\varepsilon(\mathcal{S})$ which is contained in $C^0(\mathcal{S})$. Hence the first result on the Dirac distribution δ follows by interpolation of Besov spaces. The second claim follows from the Besov space injections in $\mathcal{B}_{\infty,\infty}^\varepsilon(\mathcal{S})$. \triangleright

In the sequel, the whole point is that we can always choose the Besov exponent $p < 2$ but close to 2 so that $\mathcal{B}_{p,p}^{1+\delta}(\mathcal{S})$ injects in continuous functions.

4.3.3 The bootstrap argument – Our goal is to control the C^0 -norm of the difference $e^{t\Delta}(\delta_y) - e^{-tH^\sharp}(\delta_y)$. After Lemma 21 it suffices to estimate

$$\|e^{t\Delta}(\delta_y) - e^{-tH^\sharp}(\delta_y)\|_{\mathcal{B}_{p,p}^{1+\rho}}$$

with the constraint $p > \frac{2}{1+\rho}$.

Assume presently that we have a Schauder estimate for e^{-tH^\sharp} of the form

$$\|e^{-tH^\sharp}(U)\|_{\mathcal{B}_{p,p}^\beta} \lesssim t^{-\frac{\beta}{2}} \|U\|_{\mathcal{B}_{p,p}^\beta} \quad (4.2)$$

for some $\beta < \frac{2}{p} - 2$ and $p \in [1, 2)$. Now we choose $\rho > 0$ such that $p > \frac{2}{1+\rho}$ so that $\mathcal{B}_{p,p}^{1+\rho}(\mathcal{S})$ is contained in $C^0(\mathcal{S})$. Then thanks to the Duhamel formula we have the following control on the

difference kernel

$$\begin{aligned}
\|e^{-tH^\sharp}(\delta_y) - e^{t\Delta}(\delta_y)\|_{\mathcal{B}_{p,p}^{1+\rho}} &\lesssim \int_0^t |t-s|^{-\frac{1+\rho+\kappa}{2}} \|(H^\sharp + \Delta) e^{-sH^\sharp}(\delta_y)\|_{\mathcal{B}_{p,p}^{-\kappa}} ds \\
&\lesssim \int_0^t |t-s|^{-\frac{1+\rho+\kappa}{2}} \|e^{-sH^\sharp}(\delta_y)\|_{\mathcal{B}_{p,p}^\gamma} ds \\
&\lesssim \int_0^t |t-s|^{-\frac{1+\rho+\kappa}{2}} s^{-\frac{\gamma-\beta}{2}} \|\delta_y\|_{\mathcal{B}_{p,p}^\beta} ds \lesssim t^{-\frac{1+\rho+\kappa}{2} - \frac{\gamma-\beta}{2} + 1}
\end{aligned} \tag{4.3}$$

for $\kappa > 0$, the condition for the integral to be defined near $s = 0^+$ is that $\gamma - \beta < 2$. Together with the constraint $\beta < \frac{2}{p} - 2$, this implies that $\beta < \frac{2}{p} - 2$ and $1 < \gamma < \frac{2}{p}$ which gives the main constraint on the Schauder estimates. For the integral at $s = t$ to be well-defined, we need $1 + \rho + \kappa < 2$. Together with $p > \frac{2}{1+\rho}$, this implies that $\frac{2}{p} - 1 < \rho < 1$. Moreover, we would like that the leading term of the asymptotic of $e^{-tH^\sharp}(\delta_y)$ comes from $e^{t\Delta}(\delta_y) = \mathcal{O}(t^{-1})$ so we should require that

$$\underbrace{\frac{\rho + \kappa - 1}{2}}_{< \frac{\kappa}{2}} + \underbrace{\frac{\gamma - \beta}{2}}_{< 1} < 1;$$

this condition is always satisfied when the constant κ is chosen small enough.

4.3.4 Schauder estimates for e^{-tH^\sharp} in Besov spaces – In this part we prove the Schauder estimate (4.2) under the assumption that $\gamma \in (1, 2)$ and $0 < \gamma - \beta < 2$. For our particular application to the bootstrap argument we will be interested in the specific case where $\beta < \frac{2}{p} - 2$, $1 < \gamma < \frac{2}{p}$, $p \in [1, 2)$. In the choice of exponent γ , we **do not require** that γ, p are chosen in such a way that $\mathcal{B}_{p,p}^\gamma(\mathcal{S})$ sits in $C^0(\mathcal{S})$.

From the Schauder estimates in Besov spaces contained in [8, Prop 2.4], we know that

$$\|e^{t\Delta}(\delta_y)\|_{\mathcal{B}_{p,p}^\gamma} \lesssim t^{-\frac{\gamma-\beta}{2}} \|\delta_y\|_{\mathcal{B}_{p,p}^\beta},$$

see also [43, Lem 2.6]. The above estimate could be called a Schauder estimate for the classical heat kernel whereas in the present paragraph we are interested in proving a Schauder estimate for the operator e^{-tH^\sharp} instead of $e^{t\Delta}$. Let $(E, |\cdot|)$ be a Banach space. For $r < 0$ set

$$t^r C((0, T], E) := \left\{ v \in C((0, T], E) ; \sup_{0 < s \leq T} s^{|r|} |v(s)| < \infty \right\}.$$

We define the following Banach space

$$\mathcal{E}_T = C((0, T], \mathcal{B}_{p,p}^\beta(\mathcal{S})) \cap t^{-\frac{\gamma-\beta}{2}} C((0, T], \mathcal{B}_{p,p}^\gamma(\mathcal{S}))$$

endowed with the norm

$$\|U\|_{\mathcal{E}_T} := \sup_{t \in [0, T]} \|U(t, \cdot)\|_{\mathcal{B}_{p,p}^\beta} + \sup_{t \in (0, T]} t^{\frac{\gamma-\beta}{2}} \|U(t, \cdot)\|_{\mathcal{B}_{p,p}^\gamma}.$$

It is a weighted space in the time variable which encodes blow-up in time of the more regular norm in space. We trade in this definition some space regularity for blow-up in time. For $v \in \mathcal{B}_{p,p}^\beta(\mathcal{S})$ the function $e^{t\Delta}(v)$ is for instance an element of the Banach space \mathcal{E}_T . For $v \in \mathcal{B}_{p,p}^\beta(\mathcal{S})$ define the map

$$F_v : u \in \mathcal{E}_T \mapsto \left(t \in [0, T] \mapsto e^{t\Delta}v + \int_0^t e^{(t-s)\Delta} (H^\sharp + \Delta) u(s) ds \right)$$

We need to prove that F maps \mathcal{E}_T into itself and that F is a contraction. The fixed point of F will be nothing but the element $e^{-tH^\sharp}(\delta_y)$. Indeed by the Duhamel formula we have

$$e^{-tH^\sharp}v = e^{t\Delta}v + \int_0^t e^{(t-s)\Delta} (H^\sharp + \Delta) e^{-sH^\sharp}(v) ds$$

hence setting $u = e^{-tH^\sharp}(v)$ yields a fixed point equation of the form

$$e^{-tH^\sharp}(v) = F_v(e^{-tH^\sharp}(v)).$$

With v fixed in the sequel of this proof we write F for F_v . We will use the following useful bound, for any element $u \in \mathcal{E}_T$

$$\|u(s)\|_{\mathcal{B}_{p,p}^\gamma} \leq s^{-\frac{\gamma-\beta}{2}} \left(s^{\frac{\gamma-\beta}{2}} \|u(s)\|_{\mathcal{B}_{p,p}^\gamma} \right) \leq s^{-\frac{\gamma-\beta}{2}} \|u\|_{\mathcal{E}_T}.$$

We will need to control $F(u)$ in the two norms $\mathcal{B}_{p,p}^\gamma$ and $\mathcal{B}_{p,p}^\beta$. We start to control F in the higher regularity norm

$$\begin{aligned} \|F(u)(t)\|_{\mathcal{B}_{p,p}^\gamma} &\leq t^{-\frac{\gamma-\beta}{2}} \|e^{t\Delta}(\delta_y)(\cdot)\|_{\mathcal{E}_T} + C \int_0^t |t-s|^{-\frac{\gamma+\kappa}{2}} \|(H^\sharp + \Delta)u(s)\|_{\mathcal{B}_{p,p}^{-\kappa}} ds \\ &\leq t^{-\frac{\gamma-\beta}{2}} \|e^{t\Delta}(\delta_y)(\cdot)\|_{\mathcal{E}_T} + C \int_0^t |t-s|^{-\frac{\gamma+\kappa}{2}} \|u(s)\|_{\mathcal{B}_{p,p}^\gamma} ds \\ &\leq t^{-\frac{\gamma-\beta}{2}} \|e^{t\Delta}(\delta_y)(\cdot)\|_{\mathcal{E}_T} + C \int_0^t |t-s|^{-\frac{\gamma+\kappa}{2}} s^{-\frac{\gamma-\beta}{2}} \|u\|_{\mathcal{E}_T} ds \\ &\leq t^{-\frac{\gamma-\beta}{2}} \|e^{t\Delta}(\delta_y)(\cdot)\|_{\mathcal{E}_T} + Ct^{-\frac{\gamma+\kappa+\gamma-\beta}{2}+1} \|u\|_{\mathcal{E}_T}, \end{aligned}$$

for some positive constant C . The integral over s is integrable near $s = 0^+$ since $\gamma - \beta < 2$ (this follows from the constraints on α, β) and $\frac{2\gamma+\kappa-\beta}{2} - 1 < \frac{\gamma+\kappa}{2}$, so we can always choose $\kappa > 0$ small enough so that $\frac{2\gamma+\kappa-\beta}{2} - 1 < \frac{\gamma-\beta}{2}$. For such a κ this proves that F maps $t^{-\frac{\gamma-\beta}{2}} C((0, T], \mathcal{B}_{p,p}^\alpha(\mathcal{S}))$ into itself. Then we estimate $F(u)$ in the low regularity norm

$$\begin{aligned} \|F(u)(t)\|_{\mathcal{B}_{p,p}^\beta} &\leq \|e^{t\Delta}(\delta_y)(\cdot)\|_{\mathcal{E}_T} + C \int_0^t |t-s|^{-\frac{\beta-\kappa}{2}} \|(H^\sharp + \Delta)u(s)\|_{\mathcal{B}_{p,p}^{-\kappa}} ds \\ &\leq \|e^{t\Delta}(\delta_y)(\cdot)\|_{\mathcal{E}_T} + C \int_0^t |t-s|^{-\frac{\beta-\kappa}{2}} \|u(s)\|_{\mathcal{B}_{p,p}^\gamma} ds \\ &\leq \|e^{t\Delta}(\delta_y)(\cdot)\|_{\mathcal{E}_T} + C \int_0^t |t-s|^{-\frac{\beta-\kappa}{2}} s^{-\frac{\gamma-\beta}{2}} \|u\|_{\mathcal{E}_T} ds \lesssim t^{-\frac{\beta-\kappa+\gamma-\beta}{2}+1} \|u\|_{\mathcal{E}_T} \end{aligned}$$

the exponent is $\frac{\kappa-\gamma}{2} + 1$ which is positive if κ is small enough. Therefore F maps the space $t^{-\frac{\gamma-\beta}{2}} C((0, T], \mathcal{B}_{p,p}^\alpha(\mathcal{S}))$ into itself and $F : \mathcal{E}_T \mapsto \mathcal{E}_T$ is a well-defined continuous linear map. We now prove that F is a contraction. First we study it in the high regularity norm

$$\|F(u_1)(t) - F(u_2)(t)\|_{\mathcal{B}_{p,p}^\gamma} \leq Ct^{-\frac{2\gamma+\kappa-\beta}{2}+1} \|u_1 - u_2\|_{\mathcal{E}_T}$$

hence we deduce that

$$t^{\frac{\gamma-\beta}{2}} \|F(u_1)(t) - F(u_2)(t)\|_{\mathcal{B}_{p,p}^\gamma} \leq Ct^{1-\frac{\gamma+\kappa}{2}} \|u_1 - u_2\|_{\mathcal{E}_T}.$$

now if we choose T small enough so that $CT^{1-\frac{\gamma+\kappa}{2}} < \frac{1}{3}$, which is always possible since $1 - \frac{\gamma+\kappa}{2} > 0$, we find that

$$t^{\frac{\gamma-\beta}{2}} \|F(u_1)(t) - F(u_2)(t)\|_{\mathcal{B}_{p,p}^\gamma} \leq \frac{1}{3} \|u_1 - u_2\|_{\mathcal{E}_T}$$

for all $t \in [0, T]$. Then for all $t \in [0, T]$

$$\|F(u_1)(t) - F(u_2)(t)\|_{\mathcal{B}_{p,p}^\beta} \leq Ct^{\frac{\kappa-\gamma}{2}+1} \|u_1 - u_2\|_{\mathcal{E}_T},$$

now choosing T small enough so that $CT^{\frac{\kappa-\gamma}{2}+1} < \frac{1}{3}$ we get

$$\|F(u_1)(t) - F(u_2)(t)\|_{\mathcal{B}_{p,p}^\beta} \leq \frac{1}{3} \|u_1 - u_2\|_{\mathcal{E}_T}.$$

Finally we get

$$\|F(u_1) - F(u_2)\|_{\mathcal{E}_T} \leq \frac{2}{3} \|u_1 - u_2\|_{\mathcal{E}_T},$$

hence F is a contraction in the Banach space \mathcal{E}_T and it has a unique fixed point.

Theorem 22 – For $\gamma \in (1, 2)$, $0 < \gamma - \beta < 2$ and $p \in [1, +\infty]$ the semigroup e^{-tH^\sharp} satisfies the estimates

$$\|e^{-tH^\sharp}(v)\|_{\mathcal{B}_{p,p}^\alpha} \lesssim t^{-\frac{\alpha-\beta}{2}} \|v\|_{\mathcal{B}_{p,p}^\beta}.$$

In the particular case where v is a Dirac mass δ_y , applying the previous estimate for $\beta < \frac{2}{p} - 2$, $1 < \gamma < \frac{2}{p}$ and $p \in [1, 2)$ gives

$$\|e^{-tH^\sharp}(\delta_y)\|_{\mathcal{B}_{p,p}^\gamma} \lesssim t^{-\frac{\gamma-\beta}{2}}.$$

If we want integrability in time near $t = 0^+$ we need to choose an exponent γ for which the Besov space $\mathcal{B}_{p,p}^\gamma(\mathcal{S})$ does not inject continuously in the space of continuous functions.

4.3.5 The structure of the kernel of e^{-tH} – Note that for all $\varepsilon > 0$ such that $\varepsilon < 1 + \rho - \frac{2}{p}$,

$p \in (1, 2)$ the Besov space $\mathcal{B}_{p,p}^{1+\rho}(\mathcal{S})$ injects continuously in $\mathcal{C}^\varepsilon(\mathcal{S})$ so

$$\|e^{-tH^\sharp}(\delta_y) - e^{t\Delta}(\delta_y)\|_{\mathcal{C}^\varepsilon} \lesssim \|e^{-tH^\sharp}(\delta_y) - e^{t\Delta}(\delta_y)\|_{\mathcal{B}_{p,p}^{1+\rho}} \lesssim t^{-\frac{1+\rho+\kappa}{2} - \frac{\gamma-\beta}{2} + 1},$$

with the last upper bound from (4.3). If we choose the tuple (p, ρ, γ, β) very close to the tuple $(2, \frac{2}{p} - 1, 1, \frac{2}{p} - 2)$ with κ small enough, the exponent $-\frac{1+\rho+\kappa}{2} - \frac{\gamma-\beta}{2} + 1$ can be made arbitrarily close to $-\frac{1}{2}$ and

$$\|e^{-tH^\sharp}(\delta_y) - e^{t\Delta}(\delta_y)\|_{\mathcal{C}^\varepsilon} \lesssim t^{-\frac{1}{2}-\delta}$$

for all $\delta > 0$ such that $\frac{1}{2} + \delta < 1$.

Now to complete the proof we need to go back to comparing the difference $e^{-tH} - e^{t\Delta}$. One has

$$\begin{aligned} (e^{-tH} - e^{t\Delta})\delta_y &= \Gamma(e^{-tH^\sharp} - e^{t\Delta})\Gamma^{-1}(\delta_y) + (\Gamma e^{t\Delta}\Gamma^{-1} - e^{t\Delta})\delta_y \\ &= \Gamma(e^{-tH^\sharp} - e^{t\Delta})\Gamma^{-1}(\delta_y) - \left(\underbrace{\Gamma e^{t\Delta}(P_{\delta_y}X)}_{\text{underbraced}} + \underbrace{(1-\Gamma)e^{t\Delta}(\delta_y)}_{\text{underbraced}} \right) \end{aligned} \quad (4.4)$$

using the relation $\Gamma^{-1} = \text{id} - P_\bullet X$. Note that in the estimates of $(e^{-tH^\sharp} - e^{t\Delta})\delta_y$ we could replace the delta distribution δ_y by any distribution U_0 in $\mathcal{B}_{p,p}^\beta(\mathcal{S})$ since the choice of the distribution δ_y plays no role in our Schauder estimates in Besov spaces. Since $\delta_y \in \mathcal{B}_{p,p}^\beta(\mathcal{S})$ and $X \in \mathcal{C}^{1-\eta}(\mathcal{S})$ we have $P_{\delta_y}X \in \mathcal{B}_{p,\infty}^{\beta+1-\eta}(\mathcal{S}) \subset \mathcal{B}_{p,p}^\beta(\mathcal{S})$ if η is small enough thus $\Gamma^{-1}(\delta_y) \in \mathcal{B}_{p,p}^\beta(\mathcal{S})$. For $U_0 = \Gamma^{-1}(\delta_y) \in \mathcal{B}_{p,p}^\beta(\mathcal{S})$, using our estimates on $e^{-tH^\sharp} - e^{t\Delta}$ and the continuity of $\Gamma : \mathcal{C}^\varepsilon \mapsto \mathcal{C}^\varepsilon$ one gets

$$\|\Gamma(e^{-tH^\sharp} - e^{t\Delta})U_0\|_{\mathcal{C}^\varepsilon} \lesssim \|(e^{-tH^\sharp} - e^{t\Delta})U_0\|_{\mathcal{C}^\varepsilon} \lesssim t^{-\frac{1}{2}-\delta} \|U_0\|_{\mathcal{B}_{p,p}^\beta}.$$

It remains to treat the two terms underbraced in (4.4). We use for that purpose the Schauder estimates for the classical heat operator acting on Hölder spaces and the fact that $\delta_y \in \mathcal{C}^{-2-\varepsilon}(\mathcal{S})$ for all $\varepsilon > 0$. First we have

$$\|\Gamma e^{t\Delta}(P_{\delta_y}X)\|_{\mathcal{C}^\varepsilon} \lesssim \|e^{t\Delta}(P_{\delta_y}X)\|_{\mathcal{C}^\varepsilon} \lesssim t^{-\frac{1-3\varepsilon}{2}} \|P_{\delta_y}X\|_{\mathcal{C}^{-1-2\varepsilon}}.$$

Then we treat $(1-\Gamma)e^{t\Delta}(\delta_y)$ which yields

$$\|(1-\Gamma)e^{t\Delta}(\delta_y)\|_{\mathcal{C}^\varepsilon} \lesssim \|e^{t\Delta}(\delta_y)\|_{\mathcal{C}^{-1+3\varepsilon}} \lesssim t^{-\frac{1}{2}-2\varepsilon} \|\delta_y\|_{\mathcal{C}^{-2-\varepsilon}}$$

where we used the fact that $1-\Gamma$ has the same regularizing properties as the paraproduct $P_\bullet X$ hence $1-\Gamma : \mathcal{C}^{-1+2\varepsilon}(\mathcal{S}) \mapsto \mathcal{C}^\varepsilon(\mathcal{S})$ continuously. Finally this yields a bound of the form

$$\sup_y \|e^{-tH}(\cdot, y) - e^{t\Delta}(\cdot, y)\|_{\mathcal{C}^\varepsilon(\mathcal{S})} \lesssim t^{-\frac{1+3\rho}{2}}$$

and we have proved the first claim of Theorem 19.

4.3.6 Hölder estimates for initial data in L^2 . For $U \in L^2$ and $\gamma \in (1, 2)$, the growth of the norm

of $\|e^{-tH^\sharp}(U)\|_{H^\gamma}$ when $t > 0$ goes to 0 is given by the previous Schauder-type estimate

$$\|e^{-tH^\sharp}(U)\|_{H^\gamma} \lesssim t^{-\frac{\gamma}{2}} \|U\|_{L^2}.$$

We also have

$$\begin{aligned} \|(e^{-tH^\sharp} - e^{t\Delta})U\|_{H^\gamma} &\lesssim \int_0^t |t-s|^{-\frac{\gamma+\kappa}{2}} \|(H^\sharp + \Delta)e^{-sH^\sharp}(U)\|_{H^{-\kappa}} ds \\ &\lesssim \int_0^t |t-s|^{-\frac{\gamma+\kappa}{2}} s^{-\frac{1+\varepsilon}{2}} \|U\|_{L^2} ds \lesssim t^{-\frac{\gamma}{2} + \frac{1}{2} - \frac{\kappa+\varepsilon}{2}} \|U\|_{L^2} \end{aligned}$$

Using the injection $H^\gamma \hookrightarrow \mathcal{C}^{\gamma-1}$, we get a Hölder bound of the form

$$\|e^{-tH^\sharp}(U)\|_{\mathcal{C}^{\gamma-1}} \lesssim t^{-\frac{\gamma}{2}} \|U\|_{L^2}$$

and

$$\|(e^{-tH^\sharp} - e^{t\Delta})U\|_{\mathcal{C}^{\gamma-1}} \lesssim t^{\frac{1-\gamma}{2} - \frac{\kappa}{2}} \|U\|_{L^2}$$

for all $\kappa < 1 - \gamma$. Now using the fact that both Γ and Γ^{-1} map $\mathcal{C}^{\gamma-1}(\mathcal{S})$ into itself and they are bounded on $L^2(\mathcal{S})$ we get

$$\|e^{-tH}(U)\|_{\mathcal{C}^{\gamma-1}} = \|\Gamma e^{-tH^\sharp} \Gamma^{-1}(U)\|_{\mathcal{C}^{\gamma-1}} \lesssim \|e^{-tH^\sharp}(U)\|_{\mathcal{C}^{\gamma-1}} \lesssim t^{-\frac{\gamma}{2}} \|U\|_{L^2}$$

and

$$\begin{aligned} \|(e^{-tH} - e^{t\Delta})U\|_{\mathcal{C}^{\gamma-1}} &\lesssim \|\Gamma(e^{-tH^\sharp} - e^{t\Delta})\Gamma^{-1}(U)\|_{\mathcal{C}^{\gamma-1}} + \|\Gamma e^{t\Delta}(P_U X)\|_{\mathcal{C}^{\gamma-1}} + \|(1-\Gamma)e^{t\Delta}(U)\|_{\mathcal{C}^{\gamma-1}} \\ &\lesssim t^{\frac{1-\gamma}{2} - \frac{\kappa}{2}} \|U\|_{L^2} \end{aligned}$$

for all $\kappa < 1 - \gamma$ since $P_U X \in H^{1-\kappa}(\mathcal{S})$ for $X \in \mathcal{C}^{1-\kappa}(\mathcal{S}), U \in L^2(\mathcal{S})$ hence

$$\|e^{t\Delta}(P_U X)\|_{H^\gamma} \lesssim t^{-\frac{\gamma-1+\kappa}{2}} \|U\|_{L^2}$$

and

$$\|P_{e^{t\Delta}(U)} X\|_{H^\gamma} \lesssim \|e^{t\Delta}(U)\|_{H^{\gamma-1+\kappa}} \|X\|_{\mathcal{C}^{1-\kappa}} \lesssim t^{-\frac{\gamma-1+\kappa}{2}} \|U\|_{L^2}.$$

These are quantitative bounds on the Hölder norm of solutions to the Anderson heat equation with L^2 initial data. This proves the second claim of Theorem 19 and concludes our discussion.

4.3.7 Alternative proofs using the H -Sobolev spaces and the domain of H –

The aim of this short paragraph is to present an alternative approach to item (2) of Theorem 19 using the Sobolev spaces associated to H . This approach is not self-contained and relies on some of the results of [41].

First, for every $\sigma \in \mathbb{R}$, we define the Sobolev spaces $\mathcal{D}_H^\sigma(\mathcal{S})$ of regularity σ associated to H as the closure of the vector space spanned by the eigenfunctions e_n of H for the norm

$$\|u\|_{\mathcal{D}_H^\sigma} := \left(\sum_n (1 + |\lambda_n|)^\sigma |\langle u, e_n \rangle_{L^2}|^2 \right)^{\frac{1}{2}}.$$

We note something subtle about these spaces: There is a threshold regularity for the elements of $\mathcal{D}_H^\sigma(\mathcal{S})$ since the eigenfunctions of H are not smooth. These Sobolev spaces $\mathcal{D}_H^\sigma(\mathcal{S})$ can be used to describe precisely the domain of H – [41]. They can also be compared with the usual Sobolev spaces for certain ranges of the exponent σ . Indeed, the domain of H is $\Gamma(H^2(\mathcal{S}))$, where $\Gamma : H^\sigma(\mathcal{S}) \mapsto \mathcal{D}^\sigma(\mathcal{S})$ continuously for all $\sigma \in (0, 2)$. By construction both Γ, Γ^{-1} map $H^\sigma(\mathcal{S})$ into itself when $\sigma \in [0, 1)$, therefore we deduce by duality that $H^\sigma(\mathcal{S}) = \mathcal{D}_H^\sigma(\mathcal{S})$ when $\sigma \in (-1, 1)$. These results can be found in [41, section 2.2 p. 1402]. One of the advantages of the Sobolev spaces $\mathcal{D}_H^\sigma(\mathcal{S})$ is that the Schauder-type estimates

$$\|e^{-tH}(u)\|_{\mathcal{D}_H^\alpha} \lesssim t^{-\frac{\alpha-\beta}{2}} \|u\|_{\mathcal{D}_H^\beta}$$

holds for general exponent by definition, for any $\beta < \alpha$.

Lemma 23 – *One has $\delta_y \in \mathcal{D}_H^{-1-\delta}(\mathcal{S})$ for all $\delta > 0$. Therefore we have*

$$\|e^{-tH}(\delta_y)\|_{\mathcal{C}^\delta} \leq \|e^{-tH}(\delta_y)\|_{\mathcal{D}_H^{1+\delta}} \lesssim t^{-1-\delta} \|\delta_y\|_{\mathcal{D}_H^{-1-\delta}}.$$

Proof – Observe that

$$\mathcal{D}^{1+\delta}(\mathcal{S}) = \Gamma(H^{1+\delta}(\mathcal{S})) \subset \Gamma(\mathcal{C}^\delta(\mathcal{S})) = \mathcal{C}^\delta(\mathcal{S})$$

where we used the two-dimensional Sobolev injection $H^{1+\delta}(\mathcal{S}) \subset \mathcal{C}^\delta(\mathcal{S})$ together with the fact that $\Gamma : \mathcal{C}^\delta(\mathcal{S}) \mapsto \mathcal{C}^\delta(\mathcal{S})$ for all $\delta \in (0, 1)$ continuously. Since the Besov space $\mathcal{B}_{1,1}^{-\delta}(\mathcal{S})$ is the dual of the space $\mathcal{C}^\delta(\mathcal{S})$, duality yields the inclusion

$$\mathcal{B}_{1,1}^{-\delta}(\mathcal{S}) \subset \mathcal{D}^{-1-\delta}(\mathcal{S}).$$

Note that the Dirac δ_y distribution sits naturally in the dual of continuous functions hence in the dual of $\mathcal{C}^\delta(\mathcal{S})$

$$\sup_{x \in \mathcal{S}} \sup_{\|\varphi\|_{\mathcal{C}^\delta} = 1} |\langle \delta_y, \varphi \rangle| = 1,$$

we deduce that δ_y is bounded in $\mathcal{B}_{1,1}^{-\delta}(\mathcal{S})$ uniformly in $y \in \mathcal{S}$; one therefore has

$$\sup_{y \in \mathcal{S}} \|\delta_y\|_{\mathcal{D}_H^{-1-\delta}} < +\infty.$$

From the above observations we obtain the bound

$$\|e^{-tH}(\delta_y)\|_{\mathcal{C}^\delta} \leq \|e^{-tH}(\delta_y)\|_{\mathcal{D}_H^{1+\delta}} \lesssim t^{-1-\delta} \|\delta_y\|_{\mathcal{D}_H^{-1-\delta}}.$$

▷

Lemma 23 yields that $e^{-tH}(\delta_y)$ lies in $\mathcal{D}_H^\sigma(\mathcal{S})$ for any $\sigma \geq -1$ with the bound

$$\|e^{-tH}(\delta_y)\|_{\mathcal{D}_H^\sigma} \lesssim t^{-\frac{1+\delta+\sigma}{2}} \|\delta_y\|_{\mathcal{D}_H^{-1-\delta}}.$$

In particular the kernel $e^{-tH}(\cdot, y)$ lies in the domain $\mathfrak{D}(H)$ of the operator H . One deduces item (2) of Theorem 19 from the previous proof writing for $\sigma \in (0, 1)$

$$\|e^{-tH}(U)\|_{\mathcal{C}^\sigma} \lesssim \|e^{-tH}(U)\|_{\mathcal{D}_H^{1+\sigma}} \lesssim t^{-\frac{1+\sigma}{2}} \|U\|_{L^2}.$$

4.3.8 Applications to L^2 -traces – The next statement gives a property of the operator $p - p^\Delta$ that we will use later. Set

$$A := p - p^\Delta.$$

Corollary 24 – *The operator A has a well-defined Schwartz kernel $A((t, x), y)$ such that for all $\delta \in (0, 1)$, there is $\rho > 0, T > 0$ such that*

$$\sup_{y \in \mathcal{S}} \sup_{0 < t \leq T} \sup_{x_1 \neq x_2} t^{-\frac{1}{2}-\delta} \frac{|A((t, x_1), y) - A((t, x_2), y)|}{|x_1 - x_2|^\rho} < \infty, \quad (4.5)$$

For all $t \in (0, T]$ the operator $A(t)$ is trace class in $L^2(\mathcal{S})$ and one has

$$\mathrm{tr}_{L^2}(A(t)) \leq O(t^{-\frac{1}{2}-\delta})$$

for all $\delta \in (0, 1)$.

Proof – The first claim is a consequence of Theorem 19. The key ingredient of our proof is the notion of flat trace Tr^b which is defined for an operator A with continuous kernel as

$$Tr^b(A) := \int_{\mathcal{S}} A(x, x) dx.$$

To prove the second claim, the first step is to show that for all $t > 0$ the operator e^{-tH} is trace class and its L^2 -trace coincides with its flat trace. First note that $e^{-\frac{t}{2}H} = e^{\frac{t}{2}\Delta} + A(t/2)$, where the operators on the right hand side have continuous Schwartz kernel by the properties of A and since $t > 0$ and the heat kernel is smooth at positive times. Since $e^{-\frac{t}{2}H}(x, y) \in C^0(\mathcal{S}^2)$ one has $e^{-\frac{t}{2}H}(x, y) \in L^2(\mathcal{S}^2)$ since \mathcal{S} is compact with finite volume. This implies by [46, Thm VI.23 p. 210] that the operator $e^{-\frac{t}{2}H}$ acting on $L^2(\mathcal{S})$ is Hilbert-Schmidt with

$$Tr_{L^2}\left((e^{-\frac{t}{2}H})^* e^{-\frac{t}{2}H}\right) = \int_{\mathcal{S} \times \mathcal{S}} e^{-\frac{t}{2}H}(x, y) \overline{e^{-\frac{t}{2}H}(y, x)} dx dy.$$

This implies that $e^{-tH} = e^{-\frac{t}{2}H} e^{-\frac{t}{2}H} = (e^{-\frac{t}{2}H})^* e^{-\frac{t}{2}H}$ is **trace class** and that $Tr_{L^2}(e^{-tH})$ is well-defined to be equal to

$$\begin{aligned} \int_{\mathcal{S} \times \mathcal{S}} e^{-\frac{t}{2}H}(x, y) \overline{e^{-\frac{t}{2}H}(y, x)} dx dy &= \int_{\mathcal{S} \times \mathcal{S}} e^{-\frac{t}{2}H}(x, y) e^{-\frac{t}{2}H}(y, x) dx dy \\ &= \int_{\mathcal{S}} e^{-tH}(x, x) dx = \text{Tr}^b(e^{-tH}) \end{aligned}$$

by the Markov property of the kernel $e^{-tH}(x, y)$. Since it is well-known in the case of the classical heat operator that the operator $e^{t\Delta}$ for $t > 0$ is also trace class with $\text{Tr}_{L^2}(e^{t\Delta}) = \text{Tr}^b(e^{t\Delta})$ then the difference $A(t) = e^{-tH} - e^{t\Delta}$ is trace class since trace class operators forms a vector space. Furthermore this implies that $A(t)$ is trace class on $L^2(\mathcal{S})$

$$\text{Tr}_{L^2}(A(t)) = \text{Tr}_{L^2}(e^{-tH}) - \text{Tr}_{L^2}(e^{t\Delta}) = \text{Tr}^b(e^{-tH}) - \text{Tr}^b(e^{t\Delta}) = \text{Tr}^b(A(t))$$

and its L^2 -trace coincides with its flat trace. Now using the first property that $\sup_{y \in \mathcal{S}} \|A(t, \cdot, y)\|_{C^0} \lesssim t^{-\frac{1}{2}-\delta}$, we conclude that $\text{Tr}_{L^2}(A(t)) = \int_{\mathcal{S}} A(t, x, x) dx = \mathcal{O}(t^{-\frac{1}{2}-\delta})$ which is the desired claim. \triangleright

4.3.9 Strong maximum principle for e^{-tH} – The goal of the present short paragraph is to show

the strong maximum principle for the semi-group e^{-tH} as a consequence of our method of proof of Theorem 19. We follow Cannizzaro, Friz & Gassiat's proof [16].

Proposition 25 – *If $U \in L^2$ is non-negative and $t > 0$ one has that $e^{-tH}(U)$ is continuous and has a positive minimum. It follows that the (random) ground state e_0 of H is a positive function.*

Proof – First we observe that for all $0 < t$, $e^{-tH}(U) \in \mathcal{D}_H^\infty$, in particular it is in $\mathcal{C}^{1-\kappa}$ for all $\kappa > 0$. We need to prove that $e^{-tH}(U) \geq 0$. This is done by mollifying H into $H_r = -\Delta + \xi_r - c_r$ and note that

$$e^{-tH_r}(U) = \mathbb{E} \left[e^{-t2c_r - \int_0^t 2\xi_r(B_s) ds} U(B_t) \right] > 0$$

by the Feynman-Kac formula. Letting $r > 0$ converge to 0 we deduce that at the limit $e^{-tH}(U) \geq 0$. From this fact we deduce that the Schwartz kernel $e^{-tH}(x, y) \geq 0$. The time $T_0 > 0$ is given and we would like to prove that $e^{-T_0H}(x, y) > 0$ on \mathcal{S}^2 ; this implies that $e^{-T_0H}(U) > 0$ whenever $U \geq 0, U \neq 0$. Let us denote by D the diameter of the surface \mathcal{S} and we would like to propagate positivity at a speed which is greater than $\frac{D^2}{T_0}$ so that at time T_0 everything is positive. So we choose some ‘velocity’ $v_0 = 2\frac{D^2}{T_0}$. Recall we have the decomposition $e^{-tH} = e^{t\Delta} + A(t, \cdot, \cdot)$. Now note that by the Li-Yau estimates [49, Thm 4.8 p. 172] we have a lower bound of the form

$$\frac{C_1}{t} e^{-\frac{c_2 d^2(x, y)}{t}} \leq e^{t\Delta}(x, y)$$

on the classical heat kernel. Furthermore, we know that $A(t, x, y) \leq C_4 t^{-\frac{1}{2}-\delta}$ for a certain $\delta \in (0, 1)$. Therefore if $d(x, y) \leq \sqrt{v_0 t}$ then

$$e^{t\Delta}(x, y) + A(t, x, y) \geq \frac{C_1}{t} e^{-c_2 v_0} - C_4 t^{-\frac{1}{2}-\delta}$$

so there exists τ which depends on v_0 such that for all $t \leq \tau$ one has $\frac{C_1}{t} e^{-c_2 v_0} - C_4 t^{-\frac{1}{2}-\delta} > 0$ since

$$\lim_{t \rightarrow 0^+} \frac{C_1}{t} e^{-c_2 v_0} - C_4 t^{-\frac{1}{2}-\delta} = +\infty.$$

We deduce that for all $t \leq \tau$, one has $e^{-tH}(x, y) > 0$ when $d(x, y) \leq \sqrt{v_0 t}$. We propagate positivity by composition. Choose $t \leq \tau$ such that $T_0 = nt$ for some integer n . We know that the kernel of $e^{-T_0H} = e^{-ntH}$ is the kernel of $e^{-tH} \circ \dots \circ e^{-tH}$ (n times) is positive when $d(x, y) \leq \sqrt{nv_0 t} = \sqrt{T_0 v_0}$ since composition is thickening the support of the Schwartz kernel. But v is chosen in such a way that $\sqrt{T_0 v_0} = \sqrt{T_0 2\frac{D^2}{T_0}} = \sqrt{2}D$ is greater than the diameter of \mathcal{S} hence the kernel of e^{-T_0H} is positive everywhere. Since the ground state e_0 of e^{-tH} is non-negative and it satisfies $e^{t\lambda_0} e^{-tH}(e_0) = e_0$ for all times $t > 0$ one has $e_0 > 0$. \triangleright

4.4 Moment bounds for the heat kernel and spectral gap

The sharp description of the heat kernel of H provided by Theorem 19 has several useful and non-trivial consequences. We prove two-sided Gaussian estimates for the heat kernel of H . Building on the proof of this fact that we give in Proposition 26 we can provide in Theorem 30 an almost sure spectral estimate for H in terms of u_0 only under a mild geometric assumption on the Riemannian manifold. In Theorem 29, we give an estimate on the spectral gap in terms of isoperimetric constants and the ground state of H which holds for any Riemannian surface (\mathcal{S}, g) .

Proposition 26 – *There exists constants m and c that depend only on the ground state u_0 of H such that one has*

$$\frac{e^{-t\lambda_0(\widehat{\xi})}}{mct} \exp\left(-\frac{cd(y,x)^2}{t}\right) \leq p_t(x,y) \leq \frac{mce^{-t\lambda_0(\widehat{\xi})}}{t} \exp\left(-\frac{d(y,x)^2}{ct}\right) \quad (4.6)$$

for all $0 < t \leq 1$.

For a positive regularization parameter r set

$$H_r f := -\Delta f + h\xi_r f + c_{h,r} f$$

with $c_{h,r} = -\frac{|\log r|}{4\pi} h^2$. We justify Proposition 26 by proving an r -uniform similar estimate for the heat kernel $p_r(t, x, y)$ of H_r . The continuity of $p - p^\Delta$ as a function of $\widehat{\xi}$ in item (i) of Theorem 19 allows us to pass to the limit in the corresponding pointwise inequalities for each fixed positive t . For a fixed positive r we use the idea of conjugating the operator to a simpler operator for which one can use well-known heat kernel bounds with good control on its parameters as functions of r . The reader will find in Section 1.1 of [45] more references on works about diffusions with distributional drifts.

Proof – Pick $1 < \beta < 2$. Fix $r > 0$ and denote by $u_{0,r}$ the L^2 normalized ground state of H_r , with associated eigenvalue $\lambda_{0,r}$; it is a positive function. The conjugated operator

$$M_{u_{0,r}^{-1}}(H_r - \lambda_{0,r})M_{u_{0,r}} = -\Delta - 2\nabla(\log u_{0,r})\nabla \quad (4.7)$$

is known to have a heat kernel with Gaussian lower and upper bounds depending only on the oscillation $\text{osc}(u_{0,r}^2) := \max u_{0,r}^2 - \min u_{0,r}^2$ of $u_{0,r}^2$, as this is a conservative perturbation of the Laplace-Beltrami operator. See for instance Section 4.3 and Section 6.4 of Stroock's book [54]. So there is a continuous positive function $c(\cdot)$ of $\text{osc}(u_{0,r}^2)$ with $c(0) = 1$ such that setting

$$c_r := c(\text{osc}(u_{0,r}^2)), \quad m_r := \frac{\max u_{0,r}}{\min u_{0,r}},$$

one has

$$\frac{e^{-t\lambda_{0,r}}}{m_r c_r t} \exp\left(-\frac{c_r d(y,x)^2}{t}\right) \leq p_r(t, x, y) \leq \frac{m_r c_r e^{-t\lambda_{0,r}}}{t} \exp\left(-\frac{d(y,x)^2}{c_r t}\right) \quad (4.8)$$

for all $0 < t \leq 1$ and $x, y \in \mathcal{S}$. Note that $\min u_{0,r}$ is indeed bounded from below by a positive constant uniform in r using that it converges to u_0 in L^∞ which satisfies this property as proved in Proposition 25. We now see that one can take the constants $\lambda_{0,r}$, m_r and c_r uniform in $r \in (0, 1]$. One gets from Proposition 17 the continuous dependence of $\lambda_{0,r}$ and $u_{0,r} \in C^{2\alpha-1}(\mathcal{S})$ as functions of r . The bounds (4.6) follow from that continuity and the fact that the limit u_0 of $u_{0,r}$ is continuous and positive. \triangleright

We will use the following moment estimate (4.9) below in our study of the polymer measure, in Section 6.

Corollary 27 – *For all positive integer exponents k and $0 < t \leq 1$ one has the moment estimate*

$$\sup_{x \in \mathcal{S}} \left(\int_{\mathcal{S}} p_t(x, y) d(x, y)^k \mu(dy) \right)^{1/k} \lesssim (k!)^{1/k} \sqrt{t}. \quad (4.9)$$

It is well-known from Fabes & Stroock's work [23] that the above two-sided Gaussian bounds are all we need to prove a parabolic Harnack principle which takes here the following form. Denote by $B(x, \rho)$ the closed geodesic ball of \mathcal{S} of centre x and radius ρ .

Corollary 28 – Pick $0 < k_1 < k_2 < 1$ and $k_3 \in (0, 1)$. There exists a constant c depending only on k_1, k_2, k_3 such that for all non-negative $(\partial_t - H)$ harmonic function u on a domain of $(0, 1] \times \mathcal{S}$ of the form $[s - \rho, s] \times B(x, \rho)$, one has

$$u(t, y) \leq cu(s, x),$$

for all $(t, y) \in [s - k_2\rho^2, s - k_1\rho^2] \times B(x, k_3\rho)$.

The conjugation trick used in the proof of Proposition 26 together with a continuity argument turns out to be useful to give lower bounds on the spectral gap of H that seem to be hard to obtain otherwise. We do that under two kinds of assumptions, geometric and functional analytic.

• *Isoperimetric estimate on the spectral gap.* Let ν be a smooth volume measure on \mathcal{S} . Given a subset A of \mathcal{S} and $\kappa > 0$ denote by $A^{(\kappa)} := \{m \in \mathcal{S}; d(m, A) \leq \kappa\}$ its κ -enlargement and set

$$\sigma_\nu(\partial A) := \liminf_{\kappa \searrow 0} \frac{\nu(A^{(\kappa)}) - \nu(A)}{\kappa}.$$

The Cheeger constant of the Riemannian manifold (\mathcal{S}, g) associated with the smooth volume measure ν is defined as

$$C(\nu) := \inf_{A \subset \mathcal{S}} \frac{\sigma_\nu(\partial A)}{\min\{\nu(A), \nu(\mathcal{S} \setminus A)\}}.$$

We do not emphasize the dependence on (\mathcal{S}, g) in the notation as the manifold \mathcal{S} and its Riemannian structure g are fixed in almost all of this work. Recall we denote by μ the Riemannian volume measure on \mathcal{S} .

Theorem 29 – One has almost surely the following estimate on the spectral gap

$$\lambda_1(\widehat{\xi}) - \lambda_0(\widehat{\xi}) \geq \frac{C(u_0^2\mu)^2}{4}.$$

This formula gives back, in particular, the almost sure lower bound $\left(\frac{\min u_0}{\max u_0}\right)^4 \frac{C(\mu)^2}{4}$ for the spectral gap of H , in terms of the Cheeger constant $C(\mu)$ of (\mathcal{S}, g) ; this lower bound is positive. The constant $C(\mu)$ was denoted by $C(\mathcal{S}, g)$ in Theorem 2. It is equal to $2/L$ for a flat torus of size L .

Proof – Proceeding as in the proof of Proposition 26, we see that it suffices to prove that the spectral gap $\lambda_1(\widehat{\xi}_r) - \lambda_0(\widehat{\xi}_r)$ of the conjugated regularized operator $\Delta - 2(\nabla \log u_{0,r})\nabla$ is bounded below by $C(u_{0,r}^2\mu)^2/4$, for the convergence of $u_{0,r}$ to u_0 in $C^{2\alpha-1}(\mathcal{S})$ proved in Proposition 17 implies that $C(u_{0,r}^2\mu)$ is converging to $C(u_0^2\mu)$ as r goes to 0.

The Cheeger lower bound on $\lambda_1(\widehat{\xi}_r) - \lambda_0(\widehat{\xi}_r)$ is classical in Riemannian geometry and we give a self-contained proof adapted to our context as follows. We use the notation $\nu_{0,r}$ for the volume measure $u_{0,r}^2\mu$. The point is to see that for all smooth functions $f \in C^\infty(\mathcal{S})$, with median value $m_{0,r}(f)$ with respect to $\nu_{0,r}$, one has

$$\int_{\mathcal{S}} \|\nabla f\| d\nu_{0,r} \geq C(\nu_{0,r}) \int_{\mathcal{S}} |f - m_{0,r}(f)| d\nu_{0,r}. \quad (4.10)$$

If one takes (4.10) for granted for a moment one can apply this inequality to the function $f|f|$ where f is rescaled in such a way that it has unit $L^2(\nu_{0,r})$ -norm and $f^{-1}(0)$ and $(f|f|)^{-1}(0)$ have equal $\nu_{0,r}$ -measure $\nu_{0,r}(\mathcal{S})/2$, so $f|f|$ has a null median. This yields

$$\int_{\mathcal{S}} \|\nabla(f|f|)\| d\nu_{0,r} = 2 \int_{\mathcal{S}} \|f\nabla f\| d\nu_{0,r} \geq C(\nu_{0,r}) \int_{\mathcal{S}} |f|^2 d\nu_{0,r} = C(\nu_{0,r}),$$

and we get from Cauchy-Schwartz inequality that

$$C(\nu_{0,r}) \leq 2\|\nabla f\|_{L^2(\nu_{0,r})}.$$

In the general case if $f \in C^\infty(\mathcal{S}, \mathbb{R})$ is such that $\int_{\mathcal{S}} f d\nu_{0,r} = 0$ and $\int_{\mathcal{S}} f^2 d\nu_{0,r} = 1$, one can use the inequality

$$\int_{\mathcal{S}} (f+c)^2 d\nu_{0,r} = \int_{\mathcal{S}} (f^2 + c^2) d\nu_{0,r} \geq \int_{\mathcal{S}} f^2 d\nu_{0,r}$$

to possibly add a constant to f and trade the assumption that $\int_{\mathcal{S}} f d\nu_{0,r} = 0$ for the assumption that $f^{-1}(0)$ cuts \mathcal{S} in two pieces of equal $\nu_{0,r}$ measure. Applying the above arguments to $\frac{f+c}{\|f+c\|_{L^2(\nu_{0,r})}}$ yields

$$C(\nu_{0,r}) \leq 2 \frac{\|\nabla f\|_{L^2(\nu_{0,r})}}{\|f+c\|_{L^2(\nu_{0,r})}} \leq 2 \frac{\|\nabla f\|_{L^2(\nu_{0,r})}}{\|f\|_{L^2(\nu_{0,r})}}.$$

The representation of the spectral gap of $\Delta + 2(\nabla \log u_{0,r})\nabla$ as a Rayleigh quotient

$$\lambda_1(\widehat{\xi}_r) - \lambda_0(\widehat{\xi}_r) = \inf_{\int_{\mathcal{S}} f d\nu_{0,r}=0} \frac{\int_{\mathcal{S}} \|\nabla f\|^2 d\nu_{0,r}}{\int_{\mathcal{S}} |f|^2 d\nu_{0,r}}$$

then makes it clear that

$$\lambda_1(\widehat{\xi}_r) - \lambda_0(\widehat{\xi}_r) \geq \frac{C(\nu_{0,r})^2}{4}.$$

It remains to prove formula (4.10). Recall from the coarea formula that one has

$$\int_{\mathcal{S}} \|\nabla f\| d\nu_{0,r} = \int_{\mathbb{R}} \sigma_{\nu_{0,r}}(\{f = t\}) dt.$$

From the isoperimetric inequality

$$\sigma_{\nu_{0,r}}(\partial A) \geq C(\nu_{0,r}) \min(\nu_{0,r}(A), \nu_{0,r}(\mathcal{S} \setminus A))$$

we deduce that if 0 is a median of f we have the bounds

$$\begin{aligned} \int_{\mathcal{S}} \|\nabla f\| d\nu_{0,r} &= \int_{f \leq 0} |\nabla f| d\nu_{0,r} + \int_{f > 0} |\nabla f| d\nu_{0,r} \\ &= \int_{-\infty}^0 \sigma_{\nu_{0,r}}(\{f = t\}) dt + \int_0^{\infty} \sigma_{\nu_{0,r}}(\{f = t\}) dt \\ &\geq C(\nu_{0,r}) \left(\int_{-\infty}^0 \nu_{0,r}(\{f \leq t\}) dt + \int_0^{\infty} \nu_{0,r}(\{f > t\}) dt \right) \\ &\geq C(\nu_{0,r}) \int_{\mathcal{S}} |f| d\nu_{0,r} \end{aligned}$$

where we used integration by parts for the last step and disintegration of the volume $\nu_{0,r}$ along level sets of f . \triangleright

• *Log-Sobolev estimate on the spectral gap.* Let ν be a non-negative measure on \mathcal{S} . Recall that the ν -entropy of a positive integrable function f such that $\int_{\mathcal{S}} f |\log f| d\nu < \infty$ is the quantity

$$\text{Ent}_{\nu}(f) := \int_{\mathcal{S}} f \log f d\nu - \left(\int_{\mathcal{S}} f d\nu \right) \log \left(\int_{\mathcal{S}} f d\nu \right).$$

Recall also that we say that a measure ν on \mathcal{S} satisfies a log-Sobolev inequality with constant C_{LS} with respect to the Dirichlet form associated with the Riemannian gradient operator ∇ if

$$\text{Ent}_{\nu}(f^2) \leq 2C_{\text{LS}} \int_{\mathcal{S}} |\nabla f|^2 d\nu$$

for all functions f in the domain of the Dirichlet form. Such an inequality is known to imply a Poincaré inequality with constant $1/C_{\text{LS}}$ and a corresponding spectral gap. Bakry, Gentil & Ledoux's monograph [9] presents several geometric conditions ensuring that μ satisfies a log-Sobolev inequality.

Theorem 30 – *Assume that the Riemannian volume form μ satisfies a log-Sobolev inequality with constant C_{LS} . Then the spectral gap of H satisfies almost surely the lower bound*

$$\lambda_1(\widehat{\xi}) - \lambda_0(\widehat{\xi}) \geq \left(\frac{\min u_0}{\max u_0} \right)^2 \frac{(\max u_0^4 + \max u_0^{-4})^{-1}}{2C_{\text{LS}}}.$$

Proof – Fix a regularization parameter $r > 0$. Denote by m_r the spectral gap of H_r in $L^2(\mu)$ and by m'_r the spectral gap of H_r in $L^2(u_{0,r}^{-2}\mu)$. Then m'_r is equal to the spectral gap of the conjugated

operator $\Delta - 2\nabla(\log u_{0,r})\nabla$ and

$$m_r \geq m'_r \left(\frac{\min u_{0,r}}{\max u_{0,r}} \right)^2.$$

As in the proof of Theorem 29 we recognize in the conjugated operator the Dirichlet form of the Riemannian gradient operator with respect to the weighted Riemannian volume form $u_{0,r}^2 \mu$. As Holley & Stroock well-known stability argument for log-Sobolev inequality ensures that the weighted measure $u_{0,r}^2 \mu$ satisfies, under the assumption of the statement, a log-Sobolev inequality with constant $2C_{\text{LS}} (\max u_{0,r}^4 + \max u_{0,r}^{-4})$ we see that

$$m'_r \geq \frac{(\max u_{0,r}^4 + \max u_{0,r}^{-4})^{-1}}{2C_{\text{LS}}}.$$

(See e.g. Proposition 5.1.6 in [9] for a proof of the stability argument.) We thus have the lower bound

$$\lambda_{1,r} - \lambda_{0,r} = m_r \geq \left(\frac{\min u_{0,r}}{\max u_{0,r}} \right)^2 \frac{(\max u_{0,r}^4 + \max u_{0,r}^{-4})^{-1}}{2C_{\text{LS}}}.$$

We conclude by using the continuity of the eigenvalues as functions of $\widehat{\xi}_r$ and the convergence in $L^\infty(\mathcal{S})$ of $u_{0,r}$ to u_0 – Proposition 17. \triangleright

Note that the lower bounds on the spectral gap of H of Theorem 29 and Theorem 30 both involve only the ground state u_0 .

4.5 Bounds for the eigenvalues and eigenfunctions of H

The sharp description of p_t given by Theorem 19 gives direct access to quantitative information on the spectrum of H and its eigenfunctions. Recall we denote by μ the Riemannian volume measure.

◦ Pick any $t > 0$. By the proof of corollary 24, we have

$$\text{tr}_{L^2}(e^{-tH}) = \text{tr}_{L^2}(e^{t\Delta}) + \text{tr}_{L^2}(A(t)),$$

where A satisfies the assumptions of Lemma 24, we have the asymptotic for all $\delta \in (0, 1)$:

$$\text{tr}_{L^2}(e^{-tH}) = \text{tr}_{L^2}(e^{t\Delta}) + \text{tr}_{L^2}(A(t)) = \frac{\mu(\mathcal{S})}{4\pi t} + O(t^{-\frac{1}{2}-\delta}). \quad (4.11)$$

The following statement was first proved by Mouzard in [41] by using a fine description of the domain of H and a minimax representation of the eigenvalues, based on the link between the operators H and Δ . The statement follows here from the small time equivalent (4.11) for the heat kernel by Karamata's Tauberian Theorem [10, Thm 2.42 p. 94].

Proposition 31 – *We have almost surely the equivalent*

$$\#\left\{ \lambda \in \sigma(H); \lambda \leq a \right\}_{a, +\infty} \sim \frac{\mu(\mathcal{S})}{4\pi} a. \quad (4.12)$$

One thus has almost surely the equivalent

$$\lambda_n(\widehat{\xi}) \sim \lambda_n(0) \sim \frac{4\pi}{\mu(\mathcal{S})} n$$

as n goes to ∞ , with $\lambda_n(0)$ the n^{th} eigenvalue of the Laplace-Beltrami operator Δ . Note that one cannot get this estimate from the Gaussian bounds (4.6). We note further that since there is a random variable c_1 such that one has

$$\text{tr}_{L^2}(e^{-tH}) \leq \frac{c_1(\widehat{\xi})}{t} \quad (4.13)$$

for all $0 < t \leq 1$, and the $\lambda_k(\widehat{\xi})$ are non-decreasing, we have for all $k \geq 1$

$$k e^{-\lambda_k(\widehat{\xi})t} \leq \frac{c_1(\widehat{\xi})}{t},$$

so taking $t = 1/|\lambda_k(\widehat{\xi})|$ when this quantity is less than 1 gives the non-asymptotic lower bound

$$|\lambda_k(\widehat{\xi})| \geq \frac{e}{c_1(\widehat{\xi})} k,$$

for all eigenvalues such that $|\lambda_k(\widehat{\xi})| \geq 1$. The function

$$F_1(x) := \mathbb{P}(c_1(\widehat{\xi}) \geq x)$$

has thus the property that

$$\mathbb{P}(1 \leq |\lambda_k(\widehat{\xi})| \leq \lambda) \leq F_1\left(\frac{ek}{\lambda}\right)$$

for all $k \geq 1$ and $\lambda \geq 1$. The analysis of the proof of Theorem 19 shows that one can choose $c_1(\widehat{\xi})$ of the form

$$c_1(\widehat{\xi}) = e^{c\|\widehat{\xi}\|},$$

for a positive constant c , and

$$\widehat{\xi} \in \left(h\xi, \mathcal{R}\{\Pi(X_{h,z_0}, h\xi)\}\right) \in \mathcal{C}^{\alpha'-2}(\mathcal{S}) \times \mathcal{C}^{2\alpha'-2}(\mathcal{S}).$$

As we know that ξ has a Gaussian tail and $\mathcal{R}\{\Pi(X_{h,z_0}, h\xi)\}$ has an exponential tail – see e.g. Proposition 2.2 in [41], there exists a positive constant b such that

$$F_1(x) \lesssim \frac{1}{x^b}.$$

We record these facts as a statement.

Proposition 32 – *One has*

$$\mathbb{P}(1 \leq |\lambda_k(\widehat{\xi})| \leq \lambda) \lesssim \left(\frac{\lambda}{k}\right)^b$$

for all $k \geq 1$ and $\lambda \geq 1$.

This kind of statement is somewhat ‘orthogonal’ to the exponential tail bounds from Allez & Chouk [2], Labbé [39] and Mouzard [41]; they take here the form

$$e^{-b_1(k)\lambda} \lesssim \mathbb{P}(\lambda_k(\widehat{\xi}) < -\lambda) \lesssim e^{-b_2(k)\lambda} \quad (4.14)$$

when λ is large enough, for some positive constants $b_1(k), b_2(k)$ on which we have relatively poor control as functions of k . We also infer from the bound (4.13) that if $n_k(\widehat{\xi})$ stands for the multiplicity of the eigenvalue $\lambda_k(\widehat{\xi})$ then one has

$$n_k(\widehat{\xi}) \leq e c_1(\widehat{\xi}) |\lambda_k(\widehat{\xi})|$$

for all eigenvalues for which $|\lambda_k(\widehat{\xi})| \geq 1$. The following elementary bound

$$n_k(\widehat{\xi}) \leq c_1(\widehat{\xi}) e^{\lambda_k(\widehat{\xi})}$$

can be interesting for negative eigenvalues. Since $n_0(\widehat{\xi}) = 1$ we infer from that bound that

$$\lambda_0(\widehat{\xi}) \geq -\ln c_1(\widehat{\xi}) \gtrsim -\|\widehat{\xi}\|.$$

(This lower bound is consistent with what one can infer from (3.6) and (3.7).) We recover from the integrability properties of $\|\widehat{\xi}\|$ the upper bound of (4.14) for $\lambda_0(\widehat{\xi})$. We conjecture that H has almost surely a simple spectrum.

◦ Recall $(u_n)_{n \geq 0}$ stands for the orthonormal basis of $L^2(\mathcal{S})$ made up of eigenvectors of H , with corresponding eigenvalues in non-decreasing order.

Theorem 33 – *Fix a positive constant c and $1 < a < 2$ and $\kappa > 0$. One has for all $n \geq 0$ such that $|\lambda_n(\widehat{\xi})| \geq c$ the n -uniform estimate*

$$\|u_n\|_{C^{a-1-\kappa}} \lesssim |\lambda_n(\widehat{\xi})|^{\frac{a}{2}}. \quad (4.15)$$

Proof – On the one hand, we have from the Schauder estimate and the Sobolev embedding

$$\|e^{t\Delta}(u_n)\|_{C^{a-1-\kappa}} \lesssim \|e^{t\Delta}(u_n)\|_{B_{22}^a} \lesssim_c t^{-a/2},$$

valid for all $0 < t \leq 1/c$ (any fixed constant actually). On the other hand, we have from item (2) of Theorem 19

$$e^{-t\lambda_n}(u_n) = e^{t\Delta}(u_n) + O(t^{(1-a-\kappa)/2})$$

in the space $C^{a-1-\kappa}(\mathcal{S})$. Choosing $t = 1/\lambda_n(\widehat{\xi})$ gives the conclusion of the statement. \triangleright

Corollary 34 – *The heat semigroup $(e^{-tH})_{t>0}$ is hypercontractive.*

Proof – Using that there is a finite number of eigenvalues in $(-1, 1)$, it suffices to notice that for any $a > 2$ one has from Theorem 33 and Weyl estimate

$$\begin{aligned} \|e^{-t(H-\lambda_0(\widehat{\xi}))}f\|_{L^p} &\lesssim \|f\|_{L^2} \sum_{n \geq 0} e^{-t(\lambda_n(\widehat{\xi})-\lambda_0(\widehat{\xi}))} \|u_n\|_{L^p} \\ &\lesssim \|f\|_{L^2} \sum_{|\lambda(\widehat{\xi})| < 1} e^{-t(\lambda_n(\widehat{\xi})-\lambda_0(\widehat{\xi}))} \|u_n\|_{L^p} + \|f\|_{L^2} \sum_{|\lambda(\widehat{\xi})| \geq 1} e^{-t(\lambda_n(\widehat{\xi})-\lambda_0(\widehat{\xi}))} |\lambda_n(\widehat{\xi})|^{\frac{p'}{2}} \\ &\leq C_1(t) \|f\|_{L^2} \end{aligned}$$

for any positive time t and a finite positive constant $C_1(t)$. This is known to entail that the semigroup satisfies a log-Sobolev inequality with constants $\frac{2pt}{p-2}$ and $\frac{2p \log C_1(t)}{p-2}$ – see e.g. Theorem 5.2.5 in [9], from which the hypercontractivity property follows. \triangleright

The proof of Theorem 33 is tailor-made to get estimates on eigenfunctions. We use item (2) of Theorem 19 to obtain estimates on eigen clusters or quasimodes in $H^{1-\varepsilon}$ rather than in $B_{q,\infty}^{1-\varepsilon}(\mathcal{S})$. Given $\lambda \in \mathbb{R}$ denote by

$$\pi_{\leq \lambda} : L^2(\mathcal{S}) \rightarrow L^2(\mathcal{S})$$

the spectral projector

$$\pi_{\leq \lambda}(f) := \sum_{\lambda_n \leq \lambda} (f, u_n) u_n,$$

with (f, u_n) standing for the L^2 scalar product of f and u_n .

Theorem 35 – *For $0 < \varepsilon < 1/8$, one has for all $\lambda \in \mathbb{R}_+$ and all $f \in L^2(\mathcal{S})$ the upper bound*

$$\|\pi_{\leq \lambda}(f)\|_{H^{1-\varepsilon}} \lesssim \lambda^{\frac{1-\varepsilon}{2}} \|f\|_{L^2}. \quad (4.16)$$

Proof – We use the fact that the Anderson Sobolev space $\mathcal{D}^{1-\varepsilon}$ coincides with the usual Sobolev space $H^{1-\varepsilon}$. One therefore has

$$\|\pi_{\leq \lambda}(f)\|_{H^{1-\varepsilon}} \lesssim \|\pi_{\leq \lambda}(f)\|_{\mathcal{D}^{1-\varepsilon}} = \left(\sum_{\mu \leq \lambda, \mu \in \sigma(H)} \langle \mu \rangle^{1-\varepsilon} |f_\mu|^2 \right)^{\frac{1}{2}} \leq \lambda^{\frac{1-\varepsilon}{2}} \|f\|_{L^2}.$$

\triangleright

In the spirit of recent works [25, 34, 11, 35] studying Schrödinger operators with singular potentials it would be interesting to combine more involved microlocal techniques together with paracontrolled methods to obtain sharp Weyl laws, some forms of local Weyl laws or improved norm estimates for quasimodes of H . Note that the potential involved in these works is much more regular than white noise.

5 – Anderson Gaussian free field

We fix throughout this section a random variable

$$c = c(\omega) > -\lambda_0(\widehat{\xi}),$$

with $\omega \in \Omega$ the probability space on which the space white noise ξ and its enhancement $\widehat{\xi}$ are defined. The operator $H + c$ is thus positive and one defines a distribution-valued Gaussian field with covariance $(H + c)^{-1}$. We call it the *Anderson Gaussian free field*. It is denoted by ϕ and

defined by the formula

$$\phi := \sum_{n \geq 0} \gamma_n (c + \lambda_n(\widehat{\xi}))^{-1/2} u_n, \quad (5.1)$$

where the γ_n are independent, identically distributed, real-valued random variables with law $\mathcal{N}(0, 1)$, defined on a probability space Ω' with expectation operator \mathbb{E}' . The random variable ϕ is defined on the product probability space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \otimes \mathbb{P}')$, so it has two independent layers of randomness, one coming from H , that is ξ , and the other coming from the γ_n . A notation emphasizing that fact would be

$$\phi(\omega, \omega') = \sum_{n \geq 0} \gamma_n(\omega') \frac{u_n(\omega)}{(c(\omega) + \lambda_n(\omega))^{1/2}},$$

for two chances elements (ω, ω') in the product space $\Omega \times \Omega'$. The environment, or chance element ω , is fixed from now on until Corollary 39. We do not keep track of the dependence on c in the notation for ϕ . We start by giving an almost sure regularity estimate for the Anderson Gaussian free field. As the classical Gaussian free field in dimension 2, it turns out to have regularity 0^- . Then we construct the Wick square of ϕ in Theorem 38 and prove in Theorem 39 that the law of the spectrum of H is encoded in the law of the random partition function of the Wick square of ϕ .

We first show that the random field ϕ is (ω, ω') almost surely essentially 0^- regular. Recall one can think of $\alpha' < 1$ as arbitrarily close to 1.

Theorem 36 – *The Anderson Gaussian free field is almost surely in $H^{-\nu}(\mathcal{S})$, for every $\nu > 1 - \alpha'$.*

Proof – We use the fact that the L^2 trace does not depend on the choice of an orthonormal basis of $L^2(\mathcal{S})$ to write

$$\begin{aligned} \mathbb{E}' \left[\left\| (\Delta + 1)^{-\frac{\nu}{2}} \phi \right\|_{L^2}^2 \right] &= \sum_{n \geq 0} \frac{1}{c + \lambda_n(\omega)} \left\langle u_n, (\Delta + 1)^{-\nu} u_n \right\rangle_{L^2} = \sum_{n \geq 0} \frac{1}{c + \lambda_n(\omega)} \|u_n\|_{H^{-\nu}}^2 \\ &= \text{tr}_{L^2} \left((\Delta + 1)^{-\nu} (H + c)^{-1} \right). \end{aligned}$$

We check that the operator $(\Delta + 1)^{-\nu} (H + c)^{-1}$ is indeed trace class. Denote by K its Schwartz kernel. Note that the Schwartz kernel of $(H + c)^{-1}$ is positive since it is defined by the convergent integral $\int_0^\infty e^{-t(H+c)} dt$ where $e^{-t(H+c)}$ has non-negative kernel and that $(\Delta + 1)^{-\nu}$ also has a non-negative kernel by the Hadamard-Schwinger-Fock formula $\frac{1}{\Gamma(\nu)} \int_0^\infty e^{-t(\Delta+1)} t^{\nu-1} dt$ where $\Gamma(\nu) > 0$ and again the heat kernel $e^{-t(\Delta+1)}$ is positive. Therefore the composite Schwartz kernel K is also non-negative. The decomposition

$$(H + c)^{-1} = (c + \Delta)^{-1} + \int_0^1 e^{-tc} A(t) dt + \int_1^\infty (e^{-t(H+c)} - e^{-t(\Delta+c)}) dt$$

and the properties of $A(t)$ proved in item (1) of Theorem 19 ensure that the kernel K of the operator $(\Delta + 1)^{-\nu} (H + c)^{-1}$ is continuous, so we have

$$\int_{\mathcal{S}} K(x, x) \mu(dx) < \infty.$$

since \mathcal{S} is compact. It follows from the Lemma at the bottom of p.65 in [47], Section XI.4, that the operator $(\Delta + 1)^{-\nu} (H + c)^{-1}$ is trace class, with trace equal to $\int_{\mathcal{S}} K(x, x) \mu(dx)$. \triangleright

The above statement gives both the well-defined character of ϕ and its regularity. The usual proof of this result for the Gaussian free field uses the fact that the operator $(\Delta + 1)^{-1}$ increases regularity by 2, so one can use the fact that an operator that increases regularity by 2^+ in the Sobolev scale is trace class. We cannot resort to that mechanism here as $(H + c)^{-1}$ only sends $L^2(\mathcal{S})$ into $H^\alpha(\mathcal{S})$, so the usual reasoning only gives regularity -1^- for ϕ . As $\alpha' < 1$ can be chosen arbitrarily close to 1 we see that ϕ is almost surely in all the spaces $H^{-\nu}(\mathcal{S})$, for $\nu > 0$.

We note from the fact that the operator H is not conformally invariant (in law) that one cannot expect the random field ϕ to be conformally invariant.

The Cameron-Martin space of the Gaussian law of ϕ is the random Hilbert space

$$\text{CM} := \left\{ h_a := \sum_{n \geq 0} \frac{a_n}{(\lambda_n(\omega) + c)^{1/2}} u_n ; (a_n)_n \in \ell^2(\mathbb{N}) \right\},$$

with norm

$$\|h_a\|_{\text{CM}} := \|a\|_{\ell^2}.$$

For $s \in (0, 1)$ define the operator $(H + c)^s$ by its spectral action and the operator $(H + c)^{-s}$ on $L^2(\mathcal{S})$ by functional calculus

$$(H + c)^{-s} = \frac{1}{\Gamma(\frac{s-1}{2})} \int_0^\infty e^{-t(H+c)} t^{s-1} ds.$$

For $h_a \in \text{CM}$ one has

$$\|(H + c)^s h_a\|_{L^2}^2 = \sum_{n \geq 0} \frac{a_n^2}{(c + \lambda_n(\omega))^{1-2s}} < \infty$$

for all $0 < s < 1/2$ from the fact that $|\lambda_n(\widehat{\xi})|$ goes to ∞ . One thus has the continuous inclusion

$$\text{CM} \subset (H + c)^{-s}(L^2(\mathcal{S})).$$

The maps

$$e^{-t(H+c)} : L^2(\mathcal{S}) \rightarrow H^\nu(\mathcal{S})$$

have norms bounded above by a $\widehat{\xi}$ -dependent constant multiple of $t^{-\nu/2}$, for $0 < t \leq 1$ – a bound given by $e^{-t(\Delta+c)}$ that dominates the bound on $e^{-t(H+c)} - e^{-t(\Delta+c)}$ obtained from Theorem 19.

By decomposing the integral giving $(H + c)^{-s}$ into an integral over $(0, 1]$ and an integral over $(1, \infty)$, and using in the analysis of this second integral the same regularizing effect of $e^{-\frac{1}{2}(H+c)}$ as in the proof Theorem 35, one sees that $(H + c)^{-s}$ sends $L^2(\mathcal{S})$ into $H^{1-\chi}(\mathcal{S})$, for all $\chi > 0$. So we have the continuous inclusion

$$\text{CM} \subset H^{1-\chi}(\mathcal{S}). \quad (5.2)$$

We prove below that the Wick square $:\phi^2:$ of ϕ can be defined as a random element of $H^{-2\nu}(\mathcal{S})$. Its distribution depends on the enhanced noise $\widehat{\xi}$ since H does, so it is random. Theorem 38 below shows that the law of the spectrum of H is characterized by the law of the random law of $:\phi^2:$. We need an intermediate result before stating and proving Theorem 38. We choose below the letter ‘ G ’ for ‘Green function’. It is a direct application of the integral representation (??) of $(H + c)^{-1}$ and the lower and upper Gaussian bounds (4.6) on the heat kernel of H .

Lemma 37 – *The operator $(H + c)^{-1}$ has a Schwartz kernel $G(x, y)$ that is continuous outside the diagonal and such that*

$$m^{-1} |\log d(x, y)| \leq G(x, y) \leq m |\log d(x, y)|,$$

for a constant $m > 0$ independent of $x, y \in \mathcal{S}$.

For $n \geq 2$ set

$$a_n := \int \prod_{i=1}^n G(x_i, x_{i+1}) dx_1 \dots dx_n,$$

with the convention that $x_{n+1} = x_1$ in the integral. Lemma 37 ensures that all the a_n are well-defined for $n \geq 2$. One has actually

$$a_n = \text{tr}_{L^2}((H + c)^{-n}).$$

Here again, it is not the (poor) regularizing property of $(H + c)^{-1}$ that ensures that $(H + c)^{-n}$ is trace class but rather Weyl estimates from Corollary 31. The quantity a_n is purely spectral as we have from Lidskii’s theorem

$$a_n = \sum_{k \geq 0} (\lambda_k(\widehat{\xi}) + c)^{-n}. \quad (5.3)$$

Given a positive regularization parameter, r denote by

$$\phi_r = e^{-r\Delta}(\phi)$$

the heat regularized Anderson Gaussian free field. We define the regularized Wick square $:\phi_r^2:$ of ϕ_r setting

$$:\phi_r^2: := \phi_r^2 - \mathbb{E}'[\phi_r^2].$$

(Recall the enhanced noise $\widehat{\xi}$ is fixed and \mathbb{E}' stands for the expectation operator on the probability space where the γ_n are defined.) It will be crucial in the proof of the next statement that while $(H+c)^{-1}$ is not trace class, the Weyl law stated in Corollary 31 ensures that $(H+c)^{-1}$ is Hilbert-Schmidt.

Theorem 38 – *The regularized Wick square $:\phi_r^2:$ of Anderson Gaussian free field converges in law as $r > 0$ goes to 0, as a random variable on Ω' with values in $H^{-2\nu}(\mathcal{S})$, to a limit random variable denoted by $:\phi^2:$. One has for all $\lambda \in \mathbb{C}$ sufficiently small*

$$Z(\lambda) := \mathbb{E}' \left[e^{-\lambda : \phi^2 : (1)} \right] = \det_2 \left(\text{Id} + \lambda (H+c)^{-1} \right)^{-1/2} = \exp \left(\sum_{n \geq 2} \frac{(-\lambda)^n a_n}{2n} \right). \quad (5.4)$$

This function of λ has an analytic extension to all of \mathbb{C} .

Proof – We first take care of the probabilistic convergence of $:\phi_r^2:$ before looking at the partition function.

• Fix a large integer p . We first prove the convergence in $L^2(\Omega', \mathbb{E}')$ of $:\phi_r^2:$ as a random variable with values in $B_{2p, 2p}^{-2\nu}(\mathcal{S})$; we conclude with Besov embedding and the fact that $\nu > 1 - \alpha'$ can actually be chosen arbitrarily close to $1 - \alpha'$.

For $0 < r_1, r_2 \leq 1$ hypercontractivity ensures that we have

$$\mathbb{E}' \left[\left\| :\phi_{r_1}^2: - :\phi_{r_2}^2: \right\|_{B_{2p, 2p}^{-2\nu}}^{2p} \right] \lesssim \sum_{j \geq -1} 2^{2pj(-2\nu)} \left(\int_{\mathcal{S}} \mathbb{E}' \left[P_j \left(:\phi_{r_1}^2: - :\phi_{r_2}^2: \right) (x)^2 \right] dx \right)^p,$$

so it suffices to see that one has an x -uniform bound

$$\mathbb{E}' \left[P_j \left(:\phi_{r_1}^2: - :\phi_{r_2}^2: \right) (x)^2 \right] = o_{r_1, r_2}(1), \quad (5.5)$$

as $r_1 > 0$ and $r_2 > 0$ go to 0. Using the definition of Littlewood-Paley blocks from Appendix B, we get

$$\begin{aligned} & \mathbb{E}' \left[P_j \left(:\phi_{r_1}^2: - :\phi_{r_2}^2: \right) (x)^2 \right] \\ &= \int_{\mathcal{S} \times \mathcal{S}} \left\{ 2(e^{-r_1 \Delta} (H+c)^{-1} e^{-r_1 \Delta} (z_1, z_2))^2 + 2(e^{-r_2 \Delta} (H+c)^{-1} e^{-r_2 \Delta} (z_1, z_2))^2 \right. \\ & \quad \left. - 2(e^{-r_1 \Delta} (H+c)^{-1} e^{-r_2 \Delta} (z_1, z_2))^2 - 2(e^{-r_2 \Delta} (H+c)^{-1} e^{-r_1 \Delta} (z_1, z_2))^2 \right\} \\ & \quad \times P_j(x, z_1) P_j(x, z_2) dz_1 dz_2. \end{aligned}$$

We first start with the decomposition

$$e^{-r_1 \Delta} (H+c)^{-1} e^{-r_2 \Delta} (x, y) = e^{-r_1 \Delta} \left(\int_0^1 e^{-t(H+c)} dt \right) e^{-r_2 \Delta} + e^{-r_1 \Delta} \left(\int_1^\infty e^{-t(H+c)} dt \right) e^{-r_2 \Delta}.$$

Writing

$$\int_1^\infty e^{-t(H+c)} dt = e^{-\frac{1}{4}(H+c)} \left(\int_1^\infty e^{-(t-\frac{1}{2})(H+c)} dt \right) e^{-\frac{1}{4}(H+c)}$$

with

$$e^{-(t-\frac{1}{2})(H+c)} : L^2(\mathcal{S}) \rightarrow L^2(\mathcal{S})$$

with operator norm bounded by $e^{-(t-\frac{1}{2})k}$ for $k > 0$, we see that

$$\int_1^\infty e^{-(t-\frac{1}{2})(H+c)} dt = O_{\mathcal{B}(L^2, L^2)}(1).$$

Since the operator $e^{-\frac{1}{4}(H+c)}$ has continuous positive kernel the map

$$x \in \mathcal{S} \mapsto e^{-\frac{1}{4}(H+c)}(x, \cdot) \in L^2(\mathcal{S})$$

is continuous therefore we deduce that the composite operator

$$e^{-\frac{1}{4}(H+c)} \left(\int_1^\infty e^{-(t-\frac{1}{2})(H+c)} dt \right) e^{-\frac{1}{4}(H+c)}$$

has a continuous Schwartz kernel. This means that one has the convergence

$$e^{-r_1\Delta} \left(\int_1^\infty e^{-t(H+c)} dt \right) e^{-r_2\Delta} \xrightarrow{r_1, r_2 \rightarrow 0} \int_1^\infty e^{-t(H+c)} dt \in C^0(\mathcal{S} \times \mathcal{S}).$$

Consider now the term $\int_0^1 e^{-t(H+c)} dt$ which decomposes as

$$\int_0^1 e^{-t(H+c)} dt = \int_0^1 \left(e^{-t(\Delta+c)} + A(t)e^{-tc} \right) dt.$$

Since $A(t, x, y) = O(t^{-1+1/q-\frac{\varepsilon_0+\eta}{2}})$ and $\frac{\varepsilon_0+\eta}{2} < 1/q$, the function $\int_0^1 A(t)e^{-tc} dt \in C^0(\mathcal{S} \times \mathcal{S})$ converges with a continuous kernel and

$$e^{-r_1\Delta} \left(\int_0^1 A(t)e^{-tc} dt \right) e^{-r_2\Delta} \xrightarrow{r_1, r_2 \rightarrow 0} \int_0^1 A(t)e^{-tc} dt \in C^0(\mathcal{S} \times \mathcal{S}).$$

It remains to observe that since the only ‘singular’ term in

$$\begin{aligned} B_{r_1, r_2}(z_1, z_2) &:= 2(e^{-r_1\Delta}(H+c)^{-1}e^{-r_1\Delta}(z_1, z_2))^2 + 2(e^{-r_2\Delta}(H+c)^{-1}e^{-r_2\Delta}(z_1, z_2))^2 \\ &\quad - 2(e^{-r_1\Delta}(H+c)^{-1}e^{-r_2\Delta}(z_1, z_2))^2 - 2(e^{-r_2\Delta}(H+c)^{-1}e^{-r_1\Delta}(z_1, z_2))^2 \end{aligned}$$

is of the form $\int_0^1 e^{-(t+r_1+r_2)\Delta}(z_1, z_2)dt$, we have the convergence

$$\lim_{r_1, r_2 \rightarrow 0} B_{r_1, r_2}(z_1, z_2) = 0$$

in $C^0(\mathcal{S} \times \mathcal{S})$. We recall in identity (B.3) of Appendix B that the kernels P_j satisfy identities of the form

$$P_j(x, y) = 2^{j(\frac{d}{2}-1)} K_j(x, 2^{\frac{j}{2}}(x-y))$$

in well-chosen charts $U \times U$, where K_j is a bounded family of smooth functions. It follows that one has

$$\left| \int_{U \times U} B_{r_1, r_2}(z_1, z_2) P_j(x, z_1) P_j(x, z_2) d^2 z_1 d^2 z_2 \right| \leq C 2^{-2j} \|B_{r_1, r_2}\|_{C^0(\mathcal{S} \times \mathcal{S})} \xrightarrow{r_1, r_2 \rightarrow 0} 0$$

where a positive constant C independent of j, r_1, r_2 . This concludes the proof of the bound (5.5).

- Define the joint variable

$$\mathbf{X}(\phi) := (\phi, : \phi^2 :) \in H^{-\nu}(\mathcal{S}) \times H^{-2\nu}(\mathcal{S}),$$

and equip the product space $H^{-\nu}(\mathcal{S}) \times H^{-2\nu}(\mathcal{S})$ with the norm

$$\langle (a, b) \rangle := \|a\|_{H^{-\nu}} + \|b\|_{H^{-2\nu}}^{1/2}.$$

We consider \mathbf{X} as a measurable function of ϕ . The Cameron-Martin embedding (5.2) implies that almost surely one has for all $h \in \text{CM}$

$$\mathbf{X}(\phi + h) = \mathbf{X}(\phi) + 2h\phi + h^2,$$

with a well-defined product $h\phi$. The function $\langle \mathbf{X}(\cdot) \rangle$ satisfies then ϕ -almost surely the estimate

$$\langle \mathbf{X}(\phi) \rangle \lesssim \langle \mathbf{X}(\phi - h) \rangle + \|h\|_{\text{CM}} \quad (5.6)$$

for all $h \in \text{CM}$, for an absolute implicit multiplicative constant in the inequality. One then gets from Friz & Oberhauser generalized Fernique’s theorem [26] that the random variable $\langle \mathbf{X}(\phi) \rangle$ has a Gaussian tail. The random variable $\exp(-\lambda : \phi^2 : (\mathbf{1}))$ is thus integrable for $\lambda \in \mathbb{C}$ small enough. If one defines similarly

$$\mathbf{X}_r(\phi) := (\phi_r, : \phi_r^2 :) \in H^{-\nu}(\mathcal{S}) \times H^{-2\nu}(\mathcal{S}),$$

then the function $\langle \mathbf{X}_r(\cdot) \rangle$ also satisfies the estimate

$$\langle \mathbf{X}_r(\phi) \rangle \lesssim \langle \mathbf{X}_r(\phi - h) \rangle + \|h\|_{\text{CM}}$$

with the same implicit constant as in (5.6). The conclusion of Fernique's generalized theorem is quantitative and can be written in terms of the $\overline{\text{erf}}$ function

$$\overline{\text{erf}}(z) = 1 - \text{erf}(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-a^2/2} da.$$

If one sets

$$\mu_{a,r} := \mathbb{P}'(\|\mathbf{X}_r(\phi)\| \leq a), \quad a'_r := \text{erf}^{-1}(\mu_{a,r}),$$

for a fixed $a > 0$ such that $0 < \mu_{a,r} < 1$, then

$$\mathbb{P}'(\|\mathbf{X}_r(\phi)\| > m) \leq \overline{\text{erf}}(a'_r + \sigma m),$$

for a positive constant σ that depends only on a and the implicit constant in (5.6). As $\|\mathbf{X}_r(\cdot)\|$ is converging in $L^2(\Omega', \mathbb{E}')$ to $\|\mathbf{X}(\cdot)\|$ one can choose a constant a such that $\mathbb{P}'(\|\mathbf{X}(\cdot)\| \leq a)$ is also in $(0, 1)$. It is thus possible to find an a' such that one has

$$\sup_{0 < r \leq 1} \mathbb{P}'(\|\mathbf{X}_r(\phi)\| > m) \leq \overline{\text{erf}}(a' + \sigma m).$$

It follows from that estimate that the family of random variables $\exp(-\lambda : \phi_r^2 : (\mathbf{1}))$, for $0 < r \leq 1$ and λ in a small ball of \mathbb{C} , is uniformly integrable; so it converges in $L^1(\Omega', \mathbb{E}')$ to $\exp(-\lambda : \phi^2 : (\mathbf{1}))$.

• Denote by $\|\cdot\|_{\text{HS}}$ the Hilbert-Schmidt norm. One knows from Proposition 9.3.1 in Glimm & Jaffe's book [27] and the elementary properties of the Gohberg-Krein \det_2 determinant on the space of Hilbert-Schmidt operators that one has the equality of analytic functions

$$\mathbb{E}'[\exp(-\lambda : \phi_r^2 : (\mathbf{1}))] = \det_2(\text{Id} + \lambda e^{-2r\Delta}(H+c)^{-1})^{-1/2} \quad (5.7)$$

on the disc $|\lambda| < \|e^{-2r\Delta}H^{-1}\|_{\text{HS}}$ of the complex plane. For $r > 0$ fixed the analytic continuation property of the Gohberg-Krein determinant tells us that both sides of the equation extend as a meromorphic function over all of \mathbb{C} .

We see the convergence of $e^{-2r\Delta}(H+c)^{-1}$ to $(H+c)^{-1}$ in the space of Hilbert-Schmidt operators by noting first that the operators $(H+c)^{-1}e^{-r\Delta}(H+c)^{-1}$ are indeed trace class for all $s \in [0, 1]$ as they are symmetric non-negative and their kernels $K_r(x, y)$ satisfy the estimate

$$\int_{\mathcal{S}} K_s(x, x) \mu(dx) < \infty$$

uniformly in $s \in [0, 1]$, from the log estimate on G in Lemma 37. As in the proof of Theorem 36, it follows that

$$\begin{aligned} & \text{tr}_{L^2} \left((H+c)^{-1} (e^{-2r\Delta} - 1) (e^{-2r\Delta} - 1) (H+c)^{-1} \right) \\ &= \text{tr}_{L^2} \left((H+c)^{-1} e^{-4r\Delta} H^{-1} \right) - 2 \text{tr}_{L^2} \left(H^{-1} e^{-2r\Delta} (H+c)^{-1} \right) + \text{tr}_{L^2} \left((H+c)^{-2} \right) \\ &= \int_{\mathcal{S}} G(x, y) p_{4r}^\Delta(y, z) G(z, x) dz dy dx - 2 \int_{\mathcal{S}} G(x, y) p_{2r}^\Delta(y, z) G(z, x) dz dy dx + \int_{\mathcal{S}} G(x, y)^2 dx \end{aligned}$$

is converging to 0.

The continuity of the \det_2 function on the ideal of Hilbert-Schmidt operators on $L^2(\mathcal{S})$ implies then the equality

$$\mathbb{E}'[\exp(-\lambda : \phi^2 : (\mathbf{1}))] = \lim_{r, 0^+} \mathbb{E}'[\exp(-\lambda : \phi_r^2 : (\mathbf{1}))] = \det_2(\text{Id} + \lambda(H+c)^{-1})^{-1/2}.$$

Since the analytic continuation to all of \mathbb{C} of the locally defined function $\lambda \mapsto \det_2(\text{Id} + \lambda(H+c)^{-1})$ has its zero set equal to $\{-z^{-1}; z \in \sigma((H+c)^{-1})\}$ we see that the partition function $Z(\cdot)$ determines the spectrum of $H+c$, hence the spectrum of H . The formula involving the a_n in (5.4) comes from identity (5.7) and the general identity

$$\det_2(1 + \lambda A) = \exp \left(- \sum_{n \geq 2} \frac{(-\lambda)^n}{n} \text{tr}(A^n) \right),$$

valid for any Hilbert-Schmidt operator A on $L^2(\mathcal{S})$. \triangleright

The proof of Theorem 38 actually tells us that for every non-negative function f in $B_{p,\infty}^{1/p}(\mathcal{S})$ with $1/p > 2\nu$, one has the formula

$$Z(f) := \mathbb{E}' \left[e^{-:\phi^2:(f)} \right] = \det_2 \left(\text{Id} + M_{f^{1/2}}(H + c)^{-1} M_{f^{1/2}} \right)^{-1/2}. \quad (5.8)$$

Indicators of subsets of \mathcal{S} with finite perimeter are elements of the spaces $B_{p,\infty}^{1/p}(\mathcal{S})$ with $1/p > 2\nu$ – see e.g. Theorem 2 in Sickel’s survey [50]. To emphasize that the real-valued quantities $Z(\lambda)$ and a_n are random and their laws depend on the Riemannian metric space (\mathcal{S}, g) we write $Z(\lambda)(\mathcal{S}, g)$ and $a_n(\mathcal{S}, g)$. The next statement gives a characterization of the law of the spectrum of H , a function of (\mathcal{S}, g) , in terms of the law of the $a_n(\mathcal{S}, g)$. Write here $H(\mathcal{S}, g)$ to emphasize this dependence.

Corollary 39 – *Let (\mathcal{S}_1, g_1) and (\mathcal{S}_2, g_2) be two Riemannian closed surfaces. Then the spectra of the operators $H(\mathcal{S}_1, g_1)$ and $H(\mathcal{S}_2, g_2)$ have the same law iff the sequences $(a_n(\mathcal{S}_1, g_1))_{n \geq 2}$ and $(a_n(\mathcal{S}_2, g_2))_{n \geq 2}$ have the same law.*

Either condition is equivalent to the fact that the functions $Z(\cdot)(\mathcal{S}_1, g_1)$ and $Z(\cdot)(\mathcal{S}_2, g_2)$ have the same law.

Proof – Use Skorohod representation theorem to turn equality in law into almost sure equality on a different probability space.

If the two sequences $(c_n(\mathcal{S}_1, g_1))_{n \geq 2}$ and $(c_n(\mathcal{S}_2, g_2))_{n \geq 2}$ are equal the two functions $Z(\cdot)(\mathcal{S}_1, g_1)$ and $Z(\cdot)(\mathcal{S}_2, g_2)$ are equal, and the functions $\det_2(1 + \lambda H(\mathcal{S}_1, g_1))$ and $\det_2(1 + \lambda H(\mathcal{S}_2, g_2))$ of λ coincide on a small disk, hence on all of \mathbb{C} . Given the relation between the zero set of these functions and the spectrum of the operators $H(\mathcal{S}_1, g_1)$ and $H(\mathcal{S}_2, g_2)$ these spectra need to coincide. The function Z is determined by the spectrum of H since the a_n has that property from (5.3). \triangleright

Corollary 39 somehow says that the law of the partition function of $:\phi^2:$ determines the law of the spectrum of H .

Remark – *The Anderson Gaussian free field introduced in this section is a new object. It echoes some other works that somewhat share a similar spirit. In Caravenna, Sun & Zygouras’ work [17] and Bowditch & Sun’s work [13] the authors consider the scaling limit of an Ising model on \mathbb{Z}^2 at the critical temperature subject to some random singular magnetic field modelled by white noise. From a constructive quantum field theory viewpoint this is similar to studying some ϕ_2^4 measure with source term*

$$\mathbb{E} \left(e^{-\int_{\mathcal{S}} :\phi^4: - \lambda_c :\phi^2: \mu + \int_{\mathcal{S}} \phi \xi \mu} \right)$$

where $\lambda_c \geq 0$ is chosen to be the critical parameter of the ϕ_2^4 measure – it plays the role of the critical temperature in the Ising model, with a white noise source term ξ , and where the expectation is taken with respect to a particular massive Gaussian free field measure. The existence of the critical value λ_c follows from the work of Glimm, Jaffe & Spencer [27]. In our case, we study a free field where white noise plays the role of a random singular potential instead of a random magnetic field.

6 – A polymer measure and Anderson diffusion

We construct a polymer measure which describes the evolution of a Brownian particle in a space white noise environment. Section 6.1 is dedicated to the construction of the polymer measure and the proof of some of its properties. The Anderson diffusion is another Markovian dynamics that is associated with the Anderson operator. We prove in Section 6.2 some large deviation results for this measure on path space and its bridges stated as Theorem 5 in Section 1. We relate in Section 6.3 the occupation measure of a Poisson point process of diffusion loops with the Wick square of the Anderson Gaussian free field.

The polymer measure on path space over the 2-dimensional torus was first constructed by Cannizzaro & Chouk in [15]. Their approach consists in building the polymer measure on a fixed time interval $[0, T]$ as the law of the solution to a stochastic differential equation of the form

$$dX_t = \nabla k(T - t, X_t) dt + dB_t$$

with B a Brownian motion and k a solution of a KPZ-type singular stochastic partial differential equation

$$(\partial_t + \Delta)k = |\nabla k|^2 + \xi$$

with ξ a space white noise. Note that the drift in the dynamics of X is a time-dependent distribution so this dynamics is non-classical. They develop a paracontrolled approach to the study of such (partial and/or stochastic) equations in the setting of a 2 or 3 dimensional torus. They further proved that the law of the polymer measure is singular with respect to the law of Brownian motion; we get back that property in our setting in Proposition 40.

Our construction relies on the fact that conditionnaly on its environment ξ , the paths of the polymer measure form a Markov process with probability transitions essentially given by the semigroup of H . A similar approach was used by Alberts, Khanin & Quastel in [1] to construct a polymer measure in one-dimensional space environment given by a time-varying spacetime white noise. Note the important fact that our polymer measure differs from theirs in the fact that our environment does not change randomly with respect to the time evolution.

The Anderson diffusion is another diffusion that one can associate to the Anderson operator. It is formally given by the Markov process with generator $\Delta - 2(\nabla \ln u_0)\nabla$. It is introduced in Section 6.2, where we prove some large deviation results for it.

We work throughout this section with a coupling function h in (1.1) identically equals to 1.

6.1 Construction and properties of a polymer measure

We construct the polymer measure in Section 6.1.1 from the semigroup generated by H . We show in Section 6.1.2 that the polymer diffusion has a deterministic quadratic variation process and reprove in Section 6.1.3 that the polymer measure is singular with respect to the Wiener measure.

6.1.1 – Construction of the polymer measure. For a finite positive horizon time T , we construct the polymer measure as the Markov process on \mathcal{S} with transition probability

$$\mathbb{P}_x(X_{t_i} \in dx_i, (1 \leq i \leq n)) = \frac{p_{T-t_n}(x_n)}{p_T(x)} \prod_{i=0}^{n-1} p_{t_{i+1}-t_i}(x_{i+1}, x_i) dx_{i+1}$$

with $x_0 = x, t_0 = 0$ and

$$p_t(x) := \int_{\mathcal{S}} p_t(x, y) \mu(dy).$$

This expression can be formally understood from the formal computation

$$\begin{aligned} \mathbb{E}_x[f(X_t)] &= \mathbb{E}_x \left[f(B_t) \frac{1}{Z_T} e^{\int_0^T \xi(B_s) ds} \right] \\ &= \mathbb{E}_x \left[f(B_t) \frac{1}{Z_T} e^{\int_0^t \xi(B_s) ds} \mathbb{E} \left[e^{\int_t^T \xi(B_s) ds} \mid B_u, u \leq t \right] \right] \\ &= \frac{1}{Z_T} \mathbb{E}_x \left[f(B_t) e^{\int_0^t \xi(B_s) ds} p_{T-t}(B_t) \right] \\ &= \int_{\mathcal{S}} \frac{p_{T-t}(y)}{p_T(x)} p_t(x, y) f(y) \mu(dy) \end{aligned}$$

with X a polymer path starting at $x \in \mathcal{S}$ using the Feynman-Kac formula. In particular, the transition probabilities depend on the final time T . It follows from the Gaussian upper bound (4.6) on the heat kernel of H and Kolmogorov regularity criterion that this Markov process has a modification that takes values in the space of γ -Hölder paths, for any $\gamma < 1/2$. We denote by Q_x^T

the polymer measure on $C^\gamma([0, T], \mathcal{S})$, for $0 < \gamma < 1/2$, corresponding to an initial starting point x for the (doubly) random process. It is a random measure that depends on the enhancement $\widehat{\xi}$ of the white noise ξ used in the definition of the Anderson operator and its heat kernel.

6.1.2 – Quadratic variation process. We prove here that the quadratic variation of the canonical process on path space is a well-defined random variable under Q_x^T . This means that

$$\sum_{i=0}^n d(w_{t_{i+1}}, w_{t_i})^2$$

converges in $L^2(Q_x^T)$ to (the constant random variable) t , for each t when the mesh of a partition $0 < t_1 < \dots < t_n < t$ of an interval $[0, t]$, with $t_0 := 0$ and $t_{n+1} := 1$, goes to 0. (Do not mingle the fact for a process to have a finite quadratic variation process and the property of its sample paths to be almost surely of finite 2-variation. Brownian motion has for instance a finite quadratic variation process on any finite interval but has almost surely an infinite 2-variation on any finite interval.) To prove the preceding convergence in probability it suffices to notice that the fine asymptotic from Theorem 19 for the heat kernel of H gives

$$\mathbb{E}_x [d(w_{t_{i+1}}, w_{t_i})^2] = t_{i+1} - t_i + O(t_{i+1} - t_i)^b \quad (6.1)$$

for a constant $b > 1$, and that

$$\mathbb{E}_x [d(w_{t_{i+1}}, w_{t_i})^4] = O(t_{i+1} - t_i)^b,$$

from the ‘scaling’ bound (4.9) – or the Gaussian upper bound (4.6). Chebychev inequality then gives the result. We note here for later purposes that for each t there is a sequence of partitions of the interval $[0, t]$ such that the corresponding sum of squared increments converges almost surely to t . The quadratic variation process thus depends only on the equivalence class of a finite non-negative measure on path space under the equivalence relation given by reciprocal absolute continuity.

Note that the Gaussian lower and upper estimates on the heat kernel p_t proved in Proposition 26 are not sufficient to get back the exact scaling relation (6.1). One really needs the result of item (1) Theorem 19 for that purpose.

6.1.3 – Singularity with respect to Wiener measure. The Wiener measure $P_{\mathcal{W}, x}$ on \mathcal{S} is the law of the Brownian motion started from x . Given a positive time horizon T and $\gamma < 1/2$ it is convenient to denote by Q_x^T and $P_{\mathcal{W}, x}^T$ the restrictions to $C^\gamma([0, T], \mathcal{S})$ of the measures Q_x and $P_{\mathcal{W}, x}$. We denote by \mathbb{E}_x^T and $\mathbb{E}_{\mathcal{W}, x}^T$ their associated expectation operators. We can follow the analysis of Cannizzaro & Chouk in Section 7.3 of [15] to prove the following result. We define the measure $Q_{r, x}^T$ by its density

$$D_r(w) := \frac{dQ_{r, x}^T}{dP_{\mathcal{W}, x}^T}(w) := \exp \left(- \int_0^T \left(\xi_r + \frac{\log r}{4\pi} \right) (w_t) dt \right)$$

with respect to $P_{\mathcal{W}, x}$ – it is associated with the renormalized regularized Anderson operator $\Delta + \xi_r + \frac{\log r}{4\pi}$.

Proposition 40 – Pick $x \in \mathcal{S}$. The polymer measure Q_x^T is \mathbb{P} -almost surely singular with respect to the Wiener measure $P_{\mathcal{W}, x}^T$.

Proof – The proof proceeds as in the proof of Theorem 1.4 of [15] given in Section 7.2 of this work; we recall the main points of the details for the reader’s convenience. Pick a sequence $(r_n)_{n \geq 0}$ decreasing to 0 and look at the event $\limsup_n \{Y_{r_n} < 1\}$. We show that it has $P_{\mathcal{W}, x}^T$ -probability 1 and Q_x^T -probability 0.

• First, we have

$$\mathbb{E}_{\mathcal{W}, x}^T [D_{r_n}^{1/2}] = \mathbb{E}_{\mathcal{W}, x}^T \left[e^{-\frac{1}{2} \int_0^T (\xi_{r_n} + \log r / (4\pi)) (w_t) dt} \right] = \left(e^{-T(\Delta + \xi_{r_n}/2 + (\log r_n)/(8\pi))} \mathbf{1} \right) (x).$$

One has

$$\left(e^{-T(\Delta + \xi_{r_n}/2 + (\log r_n)/(8\pi))} \mathbf{1} \right) (x) = e^{-T(\log r_n)/(16\pi)} \left(e^{-T(\Delta + \xi_{r_n}/2 + (\log r_n)/(16\pi))} \mathbf{1} \right) (x)$$

where the last term converges as n goes to infinity as it involves the semigroup of the Anderson operator with noise $\xi/2$ – recall the quadratic dependence of the renormalization constant on the coupling constant. So

$$\lim_{n \rightarrow \infty} \mathbb{E}_{\mathcal{W},x}^T \left[D_{r_n}^{1/2} \right] = 0$$

and $\mathbb{P}_{\mathcal{W},x}^T(D_{r_n} > 1)$ tends to 0 from Chebychev inequality. One has as a consequence

$$\mathbb{P}_{\mathcal{W},x}^T \left(\limsup_n \{D_{r_n} < 1\} \right) \geq \limsup_n \mathbb{P}_{\mathcal{W},x}^T(D_{r_n} < 1) = 1.$$

• Now for a fixed $k \geq 1$ we have

$$Q_x^T(D_{r_k} < 1) \leq \liminf_n Q_{r_n,x}^T(D_{r_k} < 1),$$

and

$$\begin{aligned} Q_{r_n,x}^T(D_{r_k} < 1) &= \mathbb{E}_{\mathcal{W},x}^T \left[e^{-\int_0^T (\xi_{r_n} + (\log r_n)/(4\pi))(w_t) dt} D_{r_k}^{1/2-1/2} \mathbf{1}_{D_{r_k} < 1} \right] \\ &\leq \mathbb{E}_{\mathcal{W},x}^T \left[e^{-\int_0^T [\xi_{r_n} + (\log r_n)/(4\pi) - 1/2(\xi_{r_k} + (\log r_k)/(4\pi))](w_t) dt} \right] \\ &\leq e^{-T(\log r_k)/(16\pi)} \mathbb{E}_{\mathcal{W},x}^T \left[e^{-\int_0^T [\xi_{r_n} + (\log r_n)/(4\pi) - (1/2\xi_{r_k} + (\log r_k)/(16\pi))](w_t) dt} \right]. \end{aligned}$$

As

$$\Pi \left(X_{r_n} + \frac{1}{2} X_{r_k}, \xi_{r_n} + \frac{1}{2} \xi_{r_k} \right) - \frac{\log r_n}{4\pi} + \frac{5}{4} \frac{\log r_k}{4\pi}$$

is converging in probability in $C^{2\alpha-2}(\mathcal{S})$, under $\mathbb{P}_{\mathcal{W},x}^T$, as n goes to ∞ then k goes to ∞ , one sees that the quantity

$$\mathbb{E}_{\mathcal{W},x}^T \left[e^{-\int_0^T [\xi_{r_n} + (\log r_n)/(4\pi) - (1/2\xi_{r_k} + (\log r_k)/(16\pi)) - (\log r_k)/(4\pi)](w_t) dt} \right]$$

is converging as n goes first to ∞ then k goes to ∞ . It follows that

$$Q_{r_n,x}^T(D_{r_k} < 1) \lesssim e^{\frac{3T}{4} \frac{\log r_k}{4\pi}}$$

uniformly in n and k , so

$$Q_x^T(D_{r_k} < 1) \lesssim e^{\frac{3T}{4} \frac{\log r_k}{4\pi}}.$$

Choosing a sequence r_k that decreases sufficiently fast to 0 provides then an upper bound for $Q_{r_n,x}^T(D_{r_k} < 1)$ that allows to conclude with Borel-Cantelli lemma that

$$Q_x^T \left(\limsup_k \{D_{r_k} < 1\} \right) = 0.$$

(The speed of convergence of r_k to 0 will depend on T .) ▷

6.2 Anderson diffusion

The Anderson diffusion is the time homogeneous conservative Markov process on \mathcal{S} with transition kernel

$$\frac{e^{t\lambda_0(\hat{\xi})} p_t(x, y) u_0(y)}{u_0(x)}.$$

We denote by Q_x the law on path space of the Markov process with initial condition $x \in \mathcal{S}$. We used its regularized renormalized version in Section 4.4 to get the Gaussian bounds on p_t and the spectral gap estimates. We prove in this section that the Anderson diffusion on free-end paths and bridges satisfies the same large deviation results as the Wiener measure and its induced bridge measures. These results were stated as Theorem 5 in the introduction. The effect on these measures of the white noise environment is thus evanescent as the travelling time goes to 0. This is not surprising if one considers that this diffusion has generator $\Delta - 2(\nabla \ln u_0)\nabla$, even though u_0 is only $(1 - \varepsilon)$ -Hölder regular. On a technical level, one can trace this fact back to Theorem 19. This statement implies in particular that the effect of the random environment is contained in the correction term to the Riemannian heat kernel. The conclusion will follow from the fact that large deviation results are essentially driven by the dominant term in the small-time heat kernel expansion – the proof below will make that point clear. A reader interested only on the relation

between the Anderson diffusion and the Wick square of the Anderson Gaussian free field can skip the remainder of this section.

We note here for use in Section 6.3 that one shows as in Section 6.1.2 that one can associate that the diffusion a quadratic variation process.

Our proof of the large deviation results of Theorem 5 follows partly the proofs of the analogue statements for the Wiener measure on \mathcal{S} and its bridges. We give some details on the large deviation result for Q_x , as we give a non-classical proof, and give the essential ingredients of the proof of the corresponding result for the bridges of polymer paths. Pick $0 < \gamma < 1/2$. Given $0 < r \leq 1$ let $Q_x^{(r)}$ be the image measure of the restriction to $C^\gamma([0, r], \mathcal{S})$ by the time change map $s \in [0, 1] \mapsto sr$ – this is a non-negative finite measure on $C^\gamma([0, 1], \mathcal{S})$ for all $0 < r \leq 1$.

6.3.1 – Large deviation principle for $Q_x^{(r)}$. Pick $x \in \mathcal{S}$. Most proofs of the large deviation principle for the Wiener measure $P_{\mathcal{W}, x}$ use its dynamical description as the law of a diffusion process solution of a stochastic differential equation, for which one can resort to Freidlin & Wentzell theory of large deviations. (Rough paths theory provides an economical way of understanding the large deviation principles obtained in this way from a unique large deviation principle satisfied by the Brownian rough path.) We cannot proceed similarly here as stochastic differential equations cannot be used to describe the typical dynamics of a polymer path.

We use a different way of proving a large deviation principle, by proving a large deviation principle for the finite-dimensional time marginals of the process and proving that the family of measures is (\mathbb{P} -almost surely) exponentially tight. One can then resort to the general theory, such as exposed for instance in Section 4.7 of Feng & Kurtz textbook [24], to conclude. The identification of the (good) rate function as the function $\mathcal{I}(\cdot)$ from (1.3) comes from the fact that the finite-dimensional large deviation principle involves the squared geodesic distance, a consequence of the asymptotic behaviour of the heat kernel of H stated in item (i) of Theorem 19.

Proposition 41 – *Fix $0 < s_1 < \dots < s_n \leq 1$ and subsets A_1, \dots, A_n of \mathcal{S} . One has*

$$\begin{aligned} \limsup_{r \rightarrow 0^+} r \log Q_x^{(r)}(w_{s_1} \in \overset{\circ}{A}_1, \dots, w_{s_n} \in \overset{\circ}{A}_n) \\ \geq \inf \left\{ \sum_{i=0}^n (s_{i+1} - s_i) d(x_{i+1}, x_i)^2; x_0 = x, x_1 \in \overset{\circ}{A}_1, \dots, x_n \in \overset{\circ}{A}_n \right\}, \end{aligned}$$

and

$$\begin{aligned} \liminf_{r \rightarrow 0^+} r \log Q_x^{(r)}(w_{t_1} \in \overline{A}_1, \dots, w_{t_n} \in \overline{A}_n) \\ \leq \inf \left\{ \sum_{i=0}^n (s_{i+1} - s_i) d(x_{i+1}, x_i)^2; x_0 = x, x_1 \in \overline{A}_1, \dots, x_n \in \overline{A}_n \right\}. \end{aligned}$$

We recognize in the infimum the rate function satisfied by the finite-dimensional marginals of Brownian motion on \mathcal{S} .

Proof – This is a direct consequence of the exact formula

$$\int p_{s_1 r}(x, x_1) \mathbf{1}_{x_1 \in B_1} p_{(s_2 - s_1)r}(x_1, x_2) \cdots p_{(s_n - s_{n-1})r}(x_{n-1}, x_n) \mathbf{1}_{x_n \in B_n} dx_n \cdots dx_1$$

for

$$Q_x^{(r)}(w_{s_1} \in B_1, \dots, w_{s_n} \in B_n),$$

valid for any subsets B_1, \dots, B_n of \mathcal{S} , the sharp Gaussian asymptotic giving p_t as a $O(t^{-\beta'/2})$ perturbation of the heat kernel of the Laplace operator, and an elementary change of variable. \triangleright

Here again, we note that lower and upper Gaussian estimates on p_t would not be sufficient to prove Proposition 41. Pick $0 < \gamma < \frac{1}{2}$. We obtain the exponential tightness of the family $(Q_x^{(r)})_{0 < r \leq 1}$ by proving that the γ -Hölder norm $\|w\|_\gamma$ of a typical polymer path has a Gaussian moment. We denote by $E_x^{(r)}$ the expectation operator associated with the finite non-negative measure $Q_x^{(r)}$.

Proposition 42 – *There is a positive constant c_0 such that one has*

$$E_x^{(r)}[\exp(c_0\|w\|_\gamma^2)] < \infty, \quad (6.2)$$

uniformly in $0 < r \leq 1$ and $x \in \mathcal{S}$.

Proof – We use Besov inequality

$$\|w\|_\gamma^{2k} \lesssim_\gamma \int_0^1 \int_0^1 \left(\frac{d(w_t, w_s)}{|t-s|^{1/2}} \right)^{2k} ds dt,$$

valid for any continuous path w , any integer $k \geq 1$ and $0 < a < 1/2$, to get from the scaling bound (4.9) and a time change of variable the bound

$$E_x^{(r)}[\|w\|_\gamma^{2k}] \lesssim_\gamma \int_0^1 \int_0^1 |t-s|^{-k} E_x^{(r)}[d(w_t, w_s)^{2k}] ds dt \lesssim_\gamma r^k 2^k k! \quad (6.3)$$

– the conclusion follows. \triangleright

The exponential tightness of the family $(Q_x^{(r)})_{0 < r \leq 1}$ of finite non-negative measures on the space $C^\gamma([0, 1], \mathcal{S})$ and the identification of the large deviation principle satisfied by its finite-dimensional marginals entail that the family $(Q_x^{(r)})_{0 < r \leq 1}$ satisfies itself a large deviation principle in $C([0, 1], \mathcal{S})$ with rate function determined by the rate function of the finite-dimensional large deviation principle – see for instance Theorem 4.30 in [24]. As the latter rate function is identical to the rate function of the large deviation principle satisfied by the finite-dimensional marginals of Brownian motion, this leads to the identification of the rate function $\mathcal{I}(\cdot)$ as the functional (1.3). This is a good rate function. This proves the first item of Theorem 5.

6.3.2 – Large deviation principle for the bridge probability measures $Q_{x,x}^{(r)}$. The proof of the large deviation result for the bridges of polymers follows from the large deviation result for $Q_x^{(r)}$ proved in Section 6.2.1 and the following two analytic estimates that are consequences of our estimates on the heat kernel of H . One has

$$\lim_{r \searrow 0} r \log p_r(x, y) = -\frac{d(x, y)^2}{2}, \quad (6.4)$$

uniformly in $x, y \in \mathcal{S}$, and

$$p_r(x, y) \leq cr^{-1}, \quad (6.5)$$

for a positive constant c and all $x, y \in \mathcal{S}$ and $0 < r \leq 1$. The pattern of proof was devised in [33] by E. P. Hsu in his study of the large deviation principle for the bridges of Brownian motion. As it works almost verbatim here we will only sketch the lines of the reasoning, referring to [33] for the details. We fix for the remainder of this section two distinct points x, y of \mathcal{S} . Recall the notations of Section 1.

\triangleright *Step 1. Exponential tightness of the $Q_{x,y}^{(r)}$ in $C^\gamma([0, 1], \mathcal{S})$.* We describe below how to prove this fact. As the inclusion of $C^\gamma([0, 1], \mathcal{S})$ into $C^0([0, 1], \mathcal{S})$ is continuous it suffices, by the inverse contraction principle, to prove that the probability measures $Q_{x,y}^{(r)}$ satisfy a large deviation principle in $C([0, 1], \mathcal{S})$ with good rate function $\mathcal{I}(\cdot) - d^2(x, y)$, to prove the second point of Theorem 5. This is the object of *Step 2*. Set

$$\Omega_{x,y} := \left\{ \omega \in C([0, 1], \mathcal{S}) ; \omega(0) = x, \omega(1) = y \right\};$$

Given an integer $n \geq 1$ and $k_n \in \mathbb{N} \setminus \{0\}$ to be fixed later, the formula

$$C_{x,y}^n := \left\{ \omega \in \Omega_{x,y} ; \sup_{\substack{s,t \in [0,1] \\ 0 < t-s \leq 1/n}} \frac{|\omega_t - \omega_s|}{|t-s|^\gamma} \leq 1 \right\}$$

defines a compact subset of both $C([0, 1], \mathcal{S})$ and $C^\gamma([0, 1], \mathcal{S})$. We prove that one has the exponential tightness estimate

$$\overline{\lim}_{r \downarrow 0} r \log Q_{x,y}^{(r)}(\Omega_{x,y} \setminus C_{x,y}^n) \leq -n^{1-2\gamma}.$$

It is convenient for that purpose to introduce the two sets

$$C_{x,y}^{n,1} := \left\{ \omega \in \Omega_{x,y}; \sup_{\substack{s,t \in [0,2/3] \\ 0 < t-s \leq 1/n}} \frac{|\omega_t - \omega_s|}{|t-s|^\gamma} \leq 1 \right\}, \quad C_{x,y}^{n,2} := \left\{ \omega \in \Omega_{x,y}; \sup_{\substack{s,t \in [1/3,1] \\ 0 < t-s \leq 1/n}} \frac{|\omega_t - \omega_s|}{|t-s|^\gamma} \leq 1 \right\}$$

and prove separately

$$\overline{\lim}_{r \downarrow 0} r \log Q_{x,y}^{(r)}(\Omega_{x,y} \setminus C_{x,y}^{n,i}) \leq -n^{1-2\gamma}, \quad (6.6)$$

for $i \in \{1, 2\}$. One can concentrate on the $i = 1$ case as one gets the estimate for $i = 2$ from the estimate for $i = 1$ by using the symmetry of H to say that

$$Q_{x,y}^{(r)}(\Omega_{x,y} \setminus C_{x,y}^{n,2}) = Q_{y,x}^{(r)}(\Omega_{y,x} \setminus C_{y,x}^{n,1}).$$

First, the inequality

$$(\star)_r := Q_{x,y}^{(r)}(\Omega_{x,y} \setminus C_{x,y}^{n,1}) \leq \frac{n}{3} \sup_{0 \leq s_0 \leq 2/3} Q_{x,y}^{(r)} \left(\sup_{s_0 \leq t_1 < t_2 \leq s_0 + 2/n} \frac{|\omega_{t_2} - \omega_{t_1}|}{|t_2 - t_1|^\gamma} > 1 \right),$$

guarantees by (6.5) that one has

$$\begin{aligned} (\star)_r &\lesssim n \sup_{0 \leq s'_0 \leq (2r)/3} E_x \left[\frac{p_{r-s'_0-(2r)/n}(\omega_{r(s'_0+(2r)/n)}, y)}{p_r(x, y)}; \sup_{s'_0 \leq t_1 < t_2 \leq s'_0 + (2r)/n} \frac{|\omega_{t_2} - \omega_{t_1}|}{|t_2 - t_1|^\gamma} > 1 \right] \\ &\lesssim \frac{nr^{-1}}{p_r(x, y)} \sup_{z \in \mathcal{S}} P_z \left(\sup_{0 \leq t_1 < t_2 \leq (2r)/n} \frac{|\omega_{t_2} - \omega_{t_1}|}{|t_2 - t_1|^\gamma} > 1 \right). \end{aligned} \quad (6.7)$$

If one rewrites the bound (6.3) under the form

$$E_z^{(r_1)} \left[\exp(c_0 r_1^{-1} \|\omega\|_\gamma^2) \right] \lesssim 1$$

for an implicit multiplicative constant uniform in $0 < r_1 \leq 1$ sufficiently small and $z \in \mathcal{S}$, one can use the exponential form of Chebychev inequality to estimate the term

$$P_z \left(\sup_{0 \leq t_1 < t_2 \leq (2r)/n} \frac{|\omega_{t_2} - \omega_{t_1}|}{|t_2 - t_1|^\gamma} > 1 \right) = P_z^{(2r/n)} \left(\|\omega\|_\gamma > (2r/n)^\gamma \right)$$

in (6.7) and get from (6.4) the estimate

$$\begin{aligned} r \log(\star)_r &\lesssim -r \log p_r(x, y) + r \log(nr^{-1}) + r \log \left(\sup_z P_z(\cdots) \right) \\ &\lesssim \frac{d(x, y)^2}{2} + o_r(1) - (n/r)^{1-2\gamma}. \end{aligned}$$

As $0 < \gamma < 1/2$ this proves (6.6) for $i = 1$. (Remark that the only thing that matters here in the term $r \log p_r$ is the fact that it is uniformly bounded in r on \mathcal{S}^2 . The precise asymptotic has no importance here, while it is fundamental in the details of the proof of the upper and lower bounds in *Step 2*.)

▷ *Step 2. Upper and lower bounds for the large deviation principle.* The proofs of the upper and lower bounds for the large deviation principle satisfied by the $Q_{x,y}^{(r)}$ follow verbatim Hsu's proof [33] of the corresponding principle for the Brownian bridge measure, as the only ingredients he uses are the heat kernel estimates (6.4) and (6.5) and the Brownian equivalent of the exponential tightness result established in the first step. We do not repeat the proof here and refer the reader to Hsu's proof, pp. 109-112. (Hsu works in an unbounded complete Riemannian manifold. The details of [33] were reworked in the simpler setting of a compact manifold, for hypoelliptic diffusions, in Section 2 of [4].)

6.3 Wick square of Anderson Gaussian free field and the Anderson diffusion

The study of the links between some Markov fields and some Poissonian ensembles of Markov loops goes back to Symanzik' seminal work [55]. It was elaborated in a large number of works and we take advantage here of the general result proved by Le Jan in [40], giving a correspondence between the occupation measure of a loop ensemble and Wick square of some Gaussian free field –

see Section 9 therein. It allows at no cost to relate (a measure built from) the Anderson diffusion to the Wick square of the Anderson free field that was the object of Theorem 38. We dress the table before bringing the dishes.

Rather than working with the polymer measure built from the operator $H - \lambda_0(\widehat{\xi})$ we pick a positive constant a and work with the operator built from $H - \lambda_0(\widehat{\xi}) + a$. With the notations of Section 5 one takes here $c = -\lambda_0(\widehat{\xi}) + a$. This choice ensures that the Green function of the corresponding semigroup is finite and has the properties stated and used in Section 5. This amounts to adding killing at a constant rate a for the process built in Section 6.1.1. This does not change its properties and we have in particular that the corresponding diffusion paths have an associated quadratic variation process equal to the travelling time and defined on a random lifetime interval $[0, \zeta)$. Set

$$\frac{e^{t(\lambda_0(\widehat{\xi})-a)} p_t(x, y) u_0(y)}{u_0(x)}$$

and denote by $\overline{P}_{x,x}^t$ the unnormalized excursion measure of duration t started from $x \in \mathcal{S}$. It is characterized by the identity

$$\begin{aligned} \overline{P}_{x,x}^t \left(X_{t_1} \in \mu(dx_1), \dots, X_{t_k} \in \mu(dx_k) \right) &= \overline{p}_{t_1}(x, x_1) \overline{p}_{t_2-t_1}(x_1, x_2) \dots \overline{p}_{t-t_k}(x_k, x) \mu(dx_1) \dots \mu(dx_k) \\ &= p_{t_1}(x, x_1) p_{t_2-t_1}(x_1, x_2) \dots p_{t-t_k}(x_k, x) \mu(dx_1) \dots \mu(dx_k) \end{aligned}$$

for all $0 \leq t_1 \leq \dots \leq t_k \leq t$. Note that these quantities are independent of u_0 . This non-negative measure has a finite mass equal to $\overline{p}_t(x, x)$. A standard argument using the symmetry of $p_t(x, y)$ as a function (x, y) shows that the measure \overline{P} is supported on (rooted) loops of Hölder regularity strictly less than $1/2$. The loop measure is defined as

$$\mathcal{M}(\cdot) := \int_{\mathcal{S}} \int_0^\infty \frac{1}{t} \overline{P}_{x,x}^t(\cdot) dt \mu(dx).$$

It follows from the result of Section 6.1.2 that the factor $1/t$ in this integral accounts for the intrinsic lifetime of the loop – so this non-negative measure is indeed a measure on unrooted loops. Note that it has an infinite mass that comes from the mass of small loops. Denote by $\mathbb{E}_{\mathcal{M}}$ the expectation operator associated with \mathcal{M} and by $\zeta(\ell)$ the lifetime of a loop ℓ . For such a loop we define a measure on \mathcal{S} setting

$$\widehat{\ell}(\cdot) := \int_0^{\zeta(\ell)} \delta_{\ell(s)}(\cdot) ds.$$

One has for any non-negative function f on \mathcal{S} and all $n \geq 1$

$$\mathbb{E}_{\mathcal{M}} [\widehat{\ell}(f)^n] = (n-1)! \int_{\mathcal{S}^n} G(x_1, x_2) f(x_2) G(x_2, x_3) f(x_3) \dots G(x_n, x_1) f(x_1) \mu(dx_1) \dots \mu(dx_n), \quad (6.8)$$

and

$$\mathbb{E}_{\mathcal{M}} \left[e^{-z\widehat{\ell}(f)} + z\widehat{\ell}(f) - 1 \right] = -\log \det_2 \left(\text{Id} + zM_{f^{1/2}} G M_{f^{1/2}} \right),$$

from an elementary series expansion and the preceding equality. We used here the same notation for the Green kernel G of $H + c$ and its associated operator $(H + c)^{-1}$. Le Jan's proof [40] of identity (6.8) applies verbatim here. The quantity that naturally appears in formula (6.8) involves the Green function of the operator $u_0^{-1} e^{-t(H+c)}(u_0 \cdot)$, that is the conjugate of $(H + c)^{-1}$ by the multiplication operator by u_0 . The expression (6.8) being cyclic in (x_1, \dots, x_n) it turns out to be independent of u_0 .

Given $\gamma \geq 0$ denote by Λ_γ a Poisson process on the space of (unrooted) loops over \mathcal{S} with intensity $\gamma \mathcal{M}$. It is characterized by its characteristic function

$$\mathbb{E} [e^{i\Lambda_\gamma(F)}] = \exp \left(\gamma \int (e^{iF(\ell)} - 1) \mathcal{M}(d\ell) \right),$$

for all functions F on loop space that are null on loops of sufficiently small lifetime – so the resulting quantity $\Lambda_\gamma(F)$ is almost surely well-defined. Denote by A_γ the support of Λ_γ , so $\Lambda_\gamma = \sum_{\ell \in A_\gamma} \delta_\ell$. The regularized renormalized occupation measure of Λ_γ is defined for each $r > 0$ as the non-negative

measure on \mathcal{S}

$$\mathcal{O}_\gamma^r(f) := \sum_{\ell \in A_\gamma} \left(\mathbf{1}_{\zeta(\ell) > r} \widehat{\ell}(f) - \gamma \mathbb{E}_{\mathcal{M}} \left[\mathbf{1}_{\zeta(\ell) > r} \widehat{\ell}(f) \right] \right);$$

the expectation is over ℓ' and f is a generic non-negative continuous function on \mathcal{S} . For γ and f fixed the continuous time random process $\gamma \mapsto \mathcal{O}_\gamma^r(f)$ is actually a Lévy process with positive jumps with characteristic function

$$\mathbb{E} \left[e^{-\mathcal{O}_\gamma^r(f)} \right] = \exp \left(-\gamma \mathbb{E}_{\mathcal{M}} \left[\mathbf{1}_{\zeta(\ell) > r} (e^{-\widehat{\ell}(f)} + \widehat{\ell}(f) - 1) \right] \right)$$

converging to its natural limit as r goes to 0. The limit Lévy process is denoted by $(\mathcal{O}_\gamma(f))_{\gamma \geq 0}$. (All this is explained in detail in Le Jan's work [40].) The following result follows from the preceding analysis and formula (5.8) for the partition function of the Wick square of the Anderson Gaussian free field.

Theorem 43 – *For every continuous function f on \mathcal{S} that is also in $B_{p,\infty}^{1/p}(\mathcal{S})$, with $1/p < 2\nu$, one has the identity*

$$\mathbb{E} \left[e^{-\mathcal{O}_{1/2}(f)} \right] = \mathbb{E} \left[e^{-\phi^2(f)} \right].$$

One deduces from this identity that the renormalized occupation measure of the loop measure of polymer paths has the same distribution as the Wick square of the Anderson Gaussian free field. It has in particular a version that has almost surely regularity -2ν in the Sobolev scale. This identification does not tell us that $\mathcal{O}_{1/2}$ is a measure, despite its name.

A – Meromorphic Fredholm theory with a parameter

We prove Theorem 9 in this section. As a guide to the subject of this appendix, the reader will find in Appendix D of Zworski's book [57] an elementary account of the usual, parameter-free, meromorphic Fredholm theory.

Proof – Our proof follows closely the proof given by Borthwick in Theorem 6.1 of [12]. It suffices to prove the result near any $z_0 \in U$ which contains only finitely many poles of K . With this assumption, we may decompose

$$K(z, \mathbf{a}) = A(z, \mathbf{a}) + F(z, \mathbf{a}),$$

where $F(z, \mathbf{a})$ is a meromorphic family of finite-rank operators for $z \in U$ and $A(z, \mathbf{a})$ is a holomorphic family of compact operators. Both operators depend continuously on the parameter \mathbf{a} . Using the approximation of the compact operator $A(z_0, \mathbf{a})$ by finite-rank operators, and assuming U is sufficiently small and that we choose a sufficiently small neighbourhood of \mathbf{a}_0 , we can find a fixed finite-rank operator B such that

$$\|A(z, \mathbf{a}) - B\| < 1$$

for all $z \in U$. Note that implies that $\text{Id} - A(z, \mathbf{a}) + B$ is holomorphically invertible for $z \in U$, by the usual Neumann series as

$$(\text{Id} - A(z, \mathbf{a}) + B)^{-1} = \sum_{k=1}^{\infty} (A(z, \mathbf{a}) - B)^k.$$

Since the Neumann series converges absolutely in $\mathcal{B}(\mathcal{H}, \mathcal{H})$ uniformly in (z, \mathbf{a}) in some neighborhood of (z_0, \mathbf{a}_0) and each term $(A(z, \mathbf{a}) - B)^k$ is continuous in u , it follows that the map

$$\mathbf{a} \mapsto (\text{Id} - A(z, \mathbf{a}) + B)^{-1} \in \mathcal{B}(\mathcal{H}, \mathcal{H})$$

is continuous. Thus if we set

$$G(z, \mathbf{a}) := (F(z, \mathbf{a}) + B) (\text{Id} - K(z, \mathbf{a}) + B)^{-1}$$

then we can write

$$\text{Id} - K(z, \mathbf{a}) = (\text{Id} - G(z, \mathbf{a})) (\text{Id} - K(z, \mathbf{a}) + B)^{-1}.$$

It is immediate that $G(z, \mathbf{a})$ has finite rank and depends continuously on \mathbf{a} by its construction involving the finite rank operators $F(z, \mathbf{a}), B$. We already know that $(\text{Id} - K(z, \mathbf{a}) + B)^{-1}$ is holomorphic in z near z_0 and depends continuously on \mathbf{a} , so the problem is reduced to proving

the meromorphic invertibility of $(\text{Id} - G(z, \mathbf{a}))$ and the continuity with respect to the parameter \mathbf{a} . Recall that $G(z, \mathbf{a})$ is meromorphic in z , continuous in \mathbf{a} , with finite rank, so we can always represent it as

$$G(z, \mathbf{a}) = \sum_{1 \leq i, j \leq p} a_{ij}(z, \mathbf{a}) |\varphi_i \rangle \langle \varphi_j|$$

where the coefficients $a_{ij}(z, \mathbf{a})$ are meromorphic in z , continuous in \mathbf{a} and $(\varphi_i)_{i=1}^p$ is a finite family of linearly independent vectors in \mathcal{H} . To solve $(\text{Id} - G(z, \mathbf{a}))v = w$ where w is given, we make the ansatz $v = w + \sum_{i=1}^p b_i \varphi_i$ therefore the equation becomes

$$\begin{aligned} (\text{Id} - G(z, \mathbf{a}))v &= (\text{Id} - G(z, \mathbf{a})) \left(w + \sum_{i=1}^p b_i \varphi_i \right) \\ &= w + \sum_{i=1}^p b_i \varphi_i - \sum_{1 \leq i, j \leq p, k} b_k a_{ij}(z, \mathbf{a}) \varphi_i \langle \varphi_j, \varphi_k \rangle - \sum_{1 \leq i, j \leq p} a_{ij}(z, \mathbf{a}) \varphi_i \langle \varphi_j, w \rangle \end{aligned}$$

that simplifies into the simpler relation

$$\sum_{i=1}^p b_i \varphi_i - \sum_{1 \leq i, j \leq p, k} b_k a_{ij}(z, \mathbf{a}) \varphi_i \langle \varphi_j, \varphi_k \rangle = \sum_{1 \leq i, j \leq p} a_{ij}(z, \mathbf{a}) \varphi_i \langle \varphi_j, w \rangle.$$

By linear algebra, the above equation can be solved on the complement of the zero locus of the polynomial

$$\det \left(\delta_{ik} - \sum_j a_{ij}(z, \mathbf{a}) \langle \varphi_j, \varphi_k \rangle \right)$$

which depends meromorphically on z and continuously on \mathbf{a} . So away from the zero locus of the determinant we can meromorphically invert $\text{Id} - G(z, \mathbf{a})$ hence $\text{Id} - K(z, \mathbf{a})$ and everything depends continuously on the parameter \mathbf{a} . The fact that the poles have finite rank comes from the fact that they only appear through the finite rank operator $G(z, \mathbf{a})$. \triangleright

B – Geometric Littlewood-Paley decomposition

We recall from Klainerman & Rodnianski's work [37] the basics of Littlewood-Paley decomposition in a manifold setting. We use it to provide a self-contained proof of Proposition 10 on the renormalization of $\Pi(h\xi_r, X_{h,r,z_0})$, and Lemma 11 and Lemma 12 – both are used in the construction of the resolvent of H in Section 3.1.

Theorem 44 – *Given $\ell \in \mathbb{N}$ there exists a Schwartz function m such that*

$$\int_0^\infty t^{k_1} \partial_t^{k_2} m(t) dt = 0 \quad (\forall (k_1, k_2), k_1 + k_2 \leq \ell) \quad (\text{B.1})$$

and such that the self-adjoint smoothing operators

$$P_k = \int_0^\infty 2^{2k} m(2^{2k}t) e^{t\Delta} dt \quad (k \in \mathbb{N} \cup \{-1\}) \quad (\text{B.2})$$

enjoy the following properties.

- (a) **Resolution of the identity.** One has $\sum_{k \geq -1} P_k = \text{Id}$.
- (b) **Bessel inequality.** One has

$$\sum_{k \geq 0} \|P_k f\|_{L^2} \lesssim \|f\|_{L^2}.$$

- (c) **Finite band property.** One has

$$\|\Delta P_k f\|_{L^p} \lesssim 2^{2k} \|f\|_{L^p},$$

and

$$\|P_k f\|_{L^p} \lesssim 2^{-2k} \|\Delta f\|_{L^p};$$

also we have the dual estimate $\|P_k \nabla f\|_{L^2} \lesssim 2^k \|f\|_{L^2}$,

- (d) **Flexibility property.** *There exists a function \tilde{m} satisfying (B.1) such that $\Delta P_k = 2^{2k} \tilde{P}_k$ and the family $(\tilde{P}_k)_k$ is a Littlewood–Paley decomposition which might not satisfy the resolution of identity equation.*

We quickly recall the main features of the heat calculus we shall use in the sequel. The heat calculus is a way to encode the salient features of the Euclidean heat kernel $(4\pi t)^{-\frac{d}{2}} e^{-\frac{\|x-y\|^2}{4t}}$ and of the first approximation of the heat kernel on manifolds $K_1(t, x, y) = (4\pi t)^{\frac{d}{2}} e^{-\frac{\|x-y\|_g^2(y)}{4t}}$, which are

- the prefactor $t^{-\frac{d}{2}}$,
- the exponential factor, which is a smooth function of $X = \frac{x-y}{\sqrt{t}}$ and y , exponentially decaying as $\|X\| \rightarrow +\infty$.

This motivates the following definition, in which the notation $C^\infty([0, +\infty)_{\frac{1}{2}})$ stands for the set of functions $f(t)$ which are smooth as functions of \sqrt{t} , for $t \geq 0$.

Definition 45 – *Pick a non-positive index γ . The space Ψ_H^γ is defined to be the set of functions in $C^\infty((0, +\infty) \times S^2)$ satisfying the following axioms*

- *A is smooth, if $x \neq y$ then $A(t, x, y) = O(t^\infty)$,*
- *For any $p \in M$, there exists a chart U containing p and $\tilde{A} \in C^\infty([0, +\infty)_{\frac{1}{2}} \times U \times \mathbb{R}^d)$ such that for $(x, y) \in U^2$ one has*

$$A(t, x, y) = t^{-\frac{d+2}{2}-\gamma} \tilde{A}\left(\sqrt{t}, \frac{x-y}{\sqrt{t}}, y\right)$$

where \tilde{A} has rapid decay in the second variable

$$\|D_{\sqrt{t}, X, y}^\gamma \tilde{A}\| = O(\|X\|^{-\infty})$$

when $\|X\| \rightarrow +\infty$.

The use of the heat calculus gives a familiar form to the operators P_k . Set

$$M(t) := \int_0^t m(s) ds$$

and use the presentation of the heat calculus in the chart from definition 45 to write

$$\int_0^\infty 2^j m(2^j t) e^{t\Delta}(x, y) dt = 2^{-j} \int_0^\infty M(t) 2^{k\frac{d}{2}} t^{-\frac{d}{2}} \tilde{A}\left(2^{-k}t, x, 2^{\frac{k}{2}} \frac{x-y}{\sqrt{t}}\right) dt.$$

Then for any pair of test functions χ_1, χ_2

$$\begin{aligned} (\widehat{P_k \chi_1}) \chi_2(\xi, \eta) &= 2^{-k} \int_{U \times \mathbb{R}^2} \chi_1(x) \chi_2(h) e^{i(\xi \cdot x + h \cdot \eta)} \int_0^\infty M(t) 2^{k\frac{d}{2}} t^{-\frac{d}{2}} \tilde{A}\left(2^{-k}t, x, 2^{\frac{k}{2}} \frac{h}{\sqrt{t}}\right) dt dx 2h \\ &= 2^{-k} \int_{U \times \mathbb{R}^2} \chi_1(x) \chi_2(2^{-\frac{k}{2}} h) e^{i(\xi \cdot x + 2^{-\frac{k}{2}} h \cdot \eta)} \int_0^\infty M(t) t^{-\frac{d}{2}} \tilde{A}\left(2^{-k}t, x, \frac{h}{\sqrt{t}}\right) dt dx dh. \end{aligned}$$

Using the rapid decay in the h variable for all values of $t \in [0, +\infty)$, $x \in U$

$$\sup_{x \in U} \left| \tilde{A}\left(2^{-k}t, x, \frac{h}{\sqrt{t}}\right) \right| \leq C_N (1 + |h|)^{-N}$$

and the fact that $\chi_1(x) \chi_2(2^{-\frac{k}{2}} h) \int_0^\infty M(t) t^{-\frac{d}{2}} \tilde{A}\left(2^{-k}t, x, \frac{h}{\sqrt{t}}\right) dt$ is bounded in $C^\infty(U \times \mathbb{R}^2)$ uniformly in the parameter k , we have an estimate of the form

$$|(\widehat{P_k \chi_1}) \chi_2(\xi, \eta)| \leq C_N 2^{-k} \left(1 + |\xi| + 2^{-\frac{k}{2}} |\eta|\right)^{-N}$$

In position space, in the local chart $U \times U$ from definition 45, the estimate reads

$$P_k(x, y) = 2^{-k} 2^{k\frac{d}{2}} K_k(x, 2^{\frac{k}{2}}(x-y)), \quad (\text{B.3})$$

where the $(K_k)_k$ form a bounded family of smooth functions in $C^\infty(U \times \{|h| \leq 1\})$.

Let P and \tilde{P} be a family of geometric Littlewood-Paley projectors built from functions m and \tilde{m} that vanish at $t = 0$. It will be convenient in the proof of Proposition 10 to control the kernel $\sum_{i,j \geq 0} ((\Delta^\alpha P_i) P_j)(x, y)$ in terms of α . We know from p.140 of [37] that we have the exact identity

$$(\tilde{P}_i P_j)(x, y) = -2^{-2|i-j|} \int_0^\infty \int_0^\infty \int_0^1 e^{-(t_1+st_2)\Delta} \tilde{m}_i(t_1) t_2 m_j(t_2) ds dt_1 dt_2.$$

Using the structure of the heat kernel which follows from the heat calculus we may write in local coordinate chart $x \in U, h \in \mathbb{R}^2$

$$(\tilde{P}_i P_j)(x, x+h) =: 2^{-2|i-j|} K_{ij}(x, h)$$

where

$$\sup_{x \in U} \left| \partial_h^\beta \partial_x^\alpha K_{ij}(x, h) \right| \leq C_{\alpha, \beta} 2^{-i} 2^{i \frac{d}{2}} 2^{i \frac{|\beta|}{2}} \quad (\text{B.4})$$

uniformly in (i, j) . These are the seminorms for the topology of distributions whose wavefront set is concentrated on the conormal bundle of the diagonal.

Now in [37] we also find that $\Delta^\alpha P_i \tilde{P}_j = 2^{2i\alpha} Q_i P_j$, where $(Q_i)_i$ is an admissible family of Littlewood-Paley projectors. We deduce from this observation an estimate of the form

$$(\Delta^\alpha P_i) P_j(x, x+h) = 2^{2i\alpha} 2^{-2|i-j|} K_{ij}(x, h),$$

where the kernel K_{ij} satisfies the same estimate B.4. This is all we need to prove the following technical lemma.

Lemma 46 – *Let the Littlewood-Paley projectors $(P_i)_i$ be constructed from a function m that vanishes at $t = 0$. Fix $k \geq 1$ and $(\alpha_1, \dots, \alpha_k) \in \mathbb{Z}^k$. The series of Schwartz kernels*

$$\sum_{i_1, \dots, i_k, j_1, \dots, j_k} \sum_{|i_1 - i_2| \leq 1, \dots, |i_1 - i_k| \leq 1} ((\Delta^{\alpha_1} P_{i_1}) P_{j_1})(x, y) \dots ((\Delta^{\alpha_k} P_{i_k}) P_{j_k})(x, y) \quad (\text{B.5})$$

converges absolutely in the space of pseudodifferential kernels of order $2(\alpha_1 + \dots + \alpha_k) + (k-1)\frac{d}{2}$.

Proof – Using the above discussion we may rewrite

$$\begin{aligned} & ((\Delta^{\alpha_1} P_{i_1}) P_{j_1})(x, y) \dots ((\Delta^{\alpha_k} P_{i_k}) P_{j_k})(x, y) \\ &= 2^{2(i_1 \alpha_1 + \dots + i_k \alpha_k)} 2^{-2(|i_1 - j_1| + \dots + |i_k - j_k|)} K_{i_1 j_1}(x, y) \dots K_{i_k j_k}(x, y) \end{aligned}$$

where the smooth functions $K_{i_n j_n}(x, y)$ satisfy the estimate (B.4). So one has for all tuples $(i_1, \dots, i_k, j_1, \dots, j_k)$ such that $|i_1 - i_2| \leq 1, \dots, |i_1 - i_k| \leq 1$ an estimate of the form

$$\begin{aligned} & \left| \partial_h^b \partial_x^a K_{i_1, j_1}(x, x+h) \dots K_{i_k, j_k}(x, x+h) \right| \\ & \leq C_{ab} 2^{-(i_1 + \dots + i_k)} 2^{(i_1 + \dots + i_k) \frac{d}{2}} 2^{2 \inf(i_1, \dots, i_k, j_1, \dots, j_k) \frac{|\beta|}{2}} \end{aligned}$$

where the constant C_{ab} does not depend on the indices $(i_1, \dots, i_k, j_1, \dots, j_k)$. This estimate ensures that the sum (B.5) converges in the space of conormal distributions of order $2(\alpha_1 + \dots + \alpha_k) + (k-1)\frac{d}{2}$. \triangleright

We give here the proof of Proposition 10 performing the Wick renormalization of the resonant term $\Pi(h\xi, X_{h, z_0})$.

Proof – *Step 1 – Singular part.* Since the two paraproduct terms in the decomposition of the product $h\xi_r X_{h, r, z_0}$ are converging as r goes to 0 the quantities $\mathbb{E}[\Pi(h\xi_r, X_{h, r, z_0})]$ and $\mathbb{E}[h\xi_r X_{h, r, z_0}]$ differ by a converging quantity. Use now the Markov property of the heat operator and the definition of white noise to see that

$$\mathbb{E} \left[\left((\Delta - z_0)^{-1} h\xi_r \right) (x) h\xi_r(x) \right] = (M_h e^{-2r\Delta} M_h (\Delta - z_0)^{-1})(x, x).$$

where we denoted by M_h the multiplication operator by the smooth function h . Now note that $M_h e^{-2r\Delta} M_h (\Delta - z_0)^{-1} = M_{h^2} e^{-2r\Delta} (\Delta - z_0)^{-1} + M_h [e^{-2r\Delta}, M_h] (\Delta - z_0)^{-1}$ where $[e^{-2r\Delta}, M_h]$ is bounded in $\Psi^{-1}(\mathcal{S})$ uniformly in $r > 0$, by the commutator relation in the pseudodifferential calculus. Hence by the composition Theorem, $M_h [e^{-2r\Delta}, M_h] (\Delta - z_0)^{-1}$ is bounded in $\Psi^{-3}(\mathcal{S})$

uniformly in r and therefore trace class since we work in dimension 2. So the singular part of the above expectation

$$\mathbb{E} \left[\left((\Delta - z_0)^{-1} h\xi_r \right) (x) h\xi_r(x) \right]$$

comes from the term $(M_{h^2} e^{-2r\Delta} (\Delta - z_0)^{-1})(x, x) = h^2(x) \left(e^{-2r\Delta} (\Delta - z_0)^{-1} \right)(x, x)$.

An immediate computation yields

$$e^{-2r\Delta} (\Delta - z_0)^{-1} = \int_{2r}^1 e^{(z_0-2r)s} e^{-s\Delta} (\text{Id} - \pi_0) ds + \int_1^\infty e^{-s\Delta} (\text{Id} - \pi_0) e^{z_0 s} ds$$

where π_0 is the orthogonal projector on the subspace of constant functions. Recall that z_0 is large and negative so the integral over $[1, \infty)$ converges absolutely and defines a smoothing operator; it does not contribute to the singular part of $(e^{-2r\Delta} (\Delta - z_0)^{-1})(x, x)$ when r goes to 0. Now using the asymptotic expansion of the heat kernel yields the identity

$$(e^{-s\Delta} (\text{Id} - \pi_0))(x, x) = \frac{1}{4\pi s} + O(1),$$

with an error term $O(1)$ bounded in s and smooth in the x variable. It follows that

$$(e^{-2r\Delta} (\Delta - z_0)^{-1})(x, x) = \int_{2r}^1 e^{(z_0-2r)s} \frac{1}{4\pi s} ds + O(1) = \frac{|\log(r)|}{4\pi} + O(1).$$

We see here that the singular part of $\mathbb{E}[h\xi_r X_{h,r,z_0}]$ only depends on the point x only through $h(x)$.

Step 2 – Stochastic estimates. Write

$$\mathbb{E} \left[\left(\sum_{|i-j| \leq 1} \Delta^{-s} : (P_i(h\xi_r) \Gamma P_j(h\xi_r)) : \right)^2 \right] = I_1 + I_2,$$

where I_1 equals

$$\sum_{|i_1-j_1| \leq 1, |i_2-j_2| \leq 1} \int \Delta^{-s}(x_1, y_1) \Delta^{-s}(x_1, y_2) (\Delta^{-2} P_{j_1} P_{j_2})(y_1, y_2) (P_{i_1} P_{i_2})(y_1, y_2) h^4(y_2) dy_1 dy_2$$

and I_2 equals

$$\sum_{|i_1-j_1| \leq 1, |i_2-j_2| \leq 1} \int \Delta^{-s}(x_1, y_1) \Delta^{-s}(x_1, y_2) (\Gamma P_{j_1} P_{j_2})(y_1, y_2) (\Gamma P_{i_1} P_{i_2})(y_1, y_2) h^4(y_2) dy_1 dy_2.$$

Lemma 46 shows that the series

$$\sum_{|i_1-j_1| \leq 1, |i_2-j_2| \leq 1} (\Delta^{-2} P_{j_1} P_{j_2})(y_1, y_2) (P_{i_1} P_{i_2})(y_1, y_2)$$

converges to some pseudodifferential kernel in $\Psi^{-2}(\mathcal{S})$, so I_1 is the diagonal restriction of an element in $\Psi^{-2-2s}(\mathcal{S})$, by the composition of pseudodifferential operators, and is therefore bounded in $x_1 \in \mathcal{S}$. Since we are in dimension 2 Lemma 46 shows that

$$\sum_{|i_1-j_1| \leq 1, |i_2-j_2| \leq 1} ((\Gamma P_{j_1}) P_{j_2})^2(y_1, y_2)$$

represents a pseudodifferential kernel in $\Psi^{-2}(\mathcal{S})$ so I_2 is also the diagonal restriction of an element in $\Psi^{-2-2s}(\mathcal{S})$ and is therefore bounded in $x \in \mathcal{S}$.

We conclude using the hypercontractivity property of Gaussian measures and Besov embedding. For every integer $p \in \mathbb{N}$, one has an inequality of the form

$$\begin{aligned} \mathbb{E} \left[\left\| : \Pi(h\xi_r, X_{h,r,z_0}) : \right\|_{B_{2p,2p}^{2p}}^{2p} \right] &= \mathbb{E} \left[\int_{\mathcal{S}} \left((\text{Id} + \Delta)^{\frac{s}{2}} : \Pi(h\xi_r, X_{h,r,z_0}) : \right)^{2p} \right] \\ &\lesssim_p \mathbb{E} \left[\int_{\mathcal{S}} \left((\text{Id} + \Delta)^{\frac{s}{2}} : \Pi(h\xi_r, X_{h,r,z_0}) : \right)^2 \right]^p \\ &\lesssim_p \mathbb{E} \left[\left\| : \Pi(h\xi_r, X_{h,r,z_0}) : \right\|_{H^s}^2 \right]^p. \end{aligned}$$

Sending now r to 0 the upper bound remains bounded. The same computations show moreover that

$$\mathbb{E} \left[\left\| : \Pi(h\xi_r, X_{h,r,z_0}) : - : \Pi(h\xi_{r'}, X_{h,r'}) : \right\|_{B_{2p,2p}^s}^{2p} \right] \lesssim \mathbb{E} \left[\left\| : \Pi(h\xi_r, X_{h,r,z_0}) : - : \Pi(h\xi_{r'}, X_{h,r'}) : \right\|_{H^s}^2 \right]^p$$

with an upper bound that goes to 0 as r and r' go to 0. One can thus define the element $: \Pi(h\xi, \Gamma(h\xi)) :$ as the limit of a Cauchy family in the space $B_{2p,2p}^s(\mathcal{S})$; Besov embedding does the remaining job. \triangleright

The following observation will be useful in the proof of Lemma 11.

Lemma 47 – For any bounded family of smooth functions $(A_j)_{j \in \mathbb{N}}$ in $C^\infty(\mathcal{S} \times \mathcal{S})$, the series

$$\sum_{j=0}^{\infty} (A_j P_j)(x, y)$$

converges in the space of pseudodifferential kernels in $\Psi^\varepsilon(\mathcal{S})$, for all $\varepsilon > 0$, and the partial sums

$$\left(\sum_{j=0}^N (A_j P_j)(x, y) \right)_{N \in \mathbb{N}}$$

are bounded in $\Psi^0(\mathcal{S})$.

Proof – We would like to show that $\sum_{j=1}^{\infty} P_j(x, y)$ converges in the space of co-normal distributions. The convergence of $\sum_{j=1}^{\infty} P_j(x, y)$ as a distribution is an obvious consequence of the Bessel inequality and the fact that a bounded operator from $L^2(\mathcal{S})$ into itself has a well-defined distributional kernel. We see from the representation (B.3) of the Littlewood-Paley projectors that the series $\sum_{k=1}^{\infty} P_k$ converges absolutely as a co-normal distribution of the diagonal $I(N^* d_2)$ of the form $\int e^{i\xi \cdot (x-y)} a(x; \xi) d\xi$ where the symbol a has order 0. In other words, the series $\sum_{k=1}^{\infty} P_k$ converges as pseudodifferential kernels in $\Psi^0(\mathcal{S})$. \triangleright

We provide now a proof of Lemma 11, which says that for each regularization parameter, $r > 0$ the operator M_r^+ is a pseudodifferential operator of order 0.

Proof – As ξ_r is smooth the resonant part of M_r^+ is smoothing. The paraproduct part is given by

$$f \mapsto \sum_{j_1 \leq j_2 - 2} (P_{j_1} \xi_r) P_{j_2}(hf).$$

Observe that the sequence $\left(\sum_{k \leq j} P_k \xi_r \right)_j$ converges in all Sobolev spaces since ξ_r is smooth. Moreover the family of operators $P_j \circ M_h$ is bounded in $\Psi^0(\mathcal{S})$ and the series $\sum_{j=1}^{\infty} P_j \circ M_h$ converges absolutely in pseudodifferential kernels in $\Psi^a(\mathcal{S})$, for all $a > 0$, therefore the product $((P_j \xi_r)(x) P_j(x, y) h(y))_j$ also forms the general term of a convergent series in $\Psi^a(\mathcal{S})$ for all $a > 0$, by Lemma 47. \triangleright

We finish with the proof of Lemma 12.

Proof – Since one has $Q(z) : H^s(\mathcal{S}) \mapsto H^{s+b}(\mathcal{S}) \subset H^s(\mathcal{S})$ the map $Q(z) : H^s(\mathcal{S}) \mapsto H^s(\mathcal{S})$ is compact and the operators $(\text{Id} + P^{-1}Q(z))^{-1}$ and $(\text{Id} + Q(z)P^{-1})^{-1}$ are well-defined by the meromorphic Fredholm theory. For every compact subset of the complex plane, one can decompose $Q(z)$, for z in the compact set, as a sum

$$Q(z) = \Pi(z) + E(z)$$

of a finite rank part $\Pi(z) : H^s(\mathcal{S}) \mapsto H^s(\mathcal{S})$ that depends holomorphically on z , and a part $E(z) : H^s(\mathcal{S}) \mapsto H^s(\mathcal{S})$ with small operator norm. \triangleright

References

- [1] T. Alberts, K. Khanin and J. Quastel, *The continuum directed random polymer*. J. Stat. Phys., **154**(No. 1–2):305–326, (2015).
- [2] R. Allez and K. Chouk, *The continuous Anderson Hamiltonian in dimension two*. arXiv:1511.02718, (2015).
- [3] H. Bahouri and J.-Y. Chemin and R. Danchin, *Fourier analysis and nonlinear partial differential equations*. Grundlehren der mathematischen Wissenschaften, **343**, Springer, (2011).
- [4] I. Bailleul, *Large deviation principle for bridges of degenerate diffusion processes*. Séminaire Probab., **XLVIII**:189–199, (2016).
- [5] I. Bailleul and F. Bernicot, *Heat semigroup and singular PDEs*. J. Funct. Anal., **270**:3344–3452, (2016).
- [6] I. Bailleul and F. Bernicot and D. Frey, *Spacetime para-products for paracontrolled calculus, 3d-PAM and multiplicative Burgers equations*. Ann. Sc. Éc. Norm. Sup., **51**:1399–1457, 2018.
- [7] I. Bailleul and F. Bernicot, *High order paracontrolled calculus*. Forum Math., Sigma, **7**(e-44):1–93, (2019).
- [8] I. Bailleul, V. Dang, L. Ferdinand and T. D Tô *Global analysis for Φ_3^4 on closed Riemannian manifolds* To appear soon
- [9] D. Bakry and Y. Gentil and M. Ledoux, *Analysis and geometry of diffusion operators*. Grundlehren der mathematischen Wissenschaften, **348**, Springer, (2015).
- [10] Berline, Nicole, Ezra Getzler, and Michèle Vergne. *Heat kernels and Dirac operators*. Springer, 2003.
- [11] M. D. Blair and X. Huang and Y. Sire and C. D. Sogge. *Uniform Sobolev estimates on compact manifolds involving singular potentials*. Revista Matemática Iberoamericana, **37**(1), 2021.
- [12] D. Borthwick, *Spectral theory of infinite-area hyperbolic surfaces*. Basel: Birkhäuser, 2007.
- [13] A. Bowditch and R. Sun, *The two-dimensional continuum random field Ising model*. arXiv:2008.12158, (2020).
- [14] N. Burq and P. Gérard and N. Tzvetkov, *Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds*. Amer. J. Math., **126**(3):569–605, (2004).
- [15] G. Cannizzaro and K. Chouk, *Multidimensional SDEs with singular drift and universal construction of the polymer measure with white noise potential*. Ann. Probab., **46**:1710–1763, (2018).
- [16] G. Cannizzaro and P. Friz and P. Gassiat, *Malliavin calculus for regularity structures: the case of gPAM*. J. Funct. Anal., **272**(1):363–419, (2017).
- [17] F. Caravenna R. Sun and N. Zygouras, *Polynomial chaos and scaling limits of disordered systems*. Journal of the European Mathematical Society, **19**(1):1–65, (2016).
- [18] K. Chouk and W. van Zuijlen, *Asymptotics of the eigenvalues of the Anderson Hamiltonian with white noise potential in two dimensions*. arXiv:1907.01352, (2019).
- [19] A. Dahlqvist and J. Diehl and B. Driver, *The parabolic Anderson model on Riemann surfaces*. Probab. Th. Rel. Fields, **174**:369–444, (2019).
- [20] Davies, E. B., *One-Parameter Semigroups*. Academic Press, London, 1980.
- [21] E. B. Davies, *Heat kernels and spectral theory*. Cambridge tracts in Mathematics, **92**, Cambridge University Press, (1989).
- [22] S. Dyatlov and M. Zworski, *Mathematical theory of scattering resonances*. Graduates Studies in Mathematics, **200**, (2019).
- [23] E. Fabes and D. Stroock, *A new proof of Moser’s parabolic Harnack inequality using old ideas of Nash*. Arch. Rat. Mech. Anal., **96**:327–338, (1986).
- [24] J. Feng and T. Kurtz, *Large deviations for stochastic processes*. Mathematical Surveys and Monographs, **131**, Am. Math. Soc., (2006).
- [25] R.L. Frank and J. Sabin. *Sharp Weyl laws with singular potentials* arXiv preprint arXiv:2007.04284, (2020).
- [26] P. Friz and H. Oberhauser, *A generalized Fernique theorem and applications*. Proc. Amer. Math. Soc., **138**:3679–3688, (2010).
- [27] J. Glimm and A. Jaffe, *Quantum Physics – A Functional Integral Point of View*. Springer, (1987).
- [28] M. Gubinelli and N. Perkowski, *An introduction to singular SPDEs*. arXiv:1702.03195, (2017).
- [29] M. Gubinelli and N. Perkowski, *KPZ reloaded*. Comm. Math. Phys., **349**:165–269, (2017).
- [30] M. Gubinelli and P. Imkeller and N. Perkowski, *Paracontrolled distributions and singular PDEs*. Forum of Math., Pi, **3**-e6:1–75, (2015).
- [31] A. Grigor’yan, *Estimates of heat kernels on Riemannian manifolds*. In ”Spectral Theory and Geometry. ICMS Instructional Conference, Edinburgh, 1998”, ed. B. Davies and Yu. Safarov, Cambridge Univ. Press, London Math. Soc. Lecture Notes 273, 140–225, (1999).
- [32] M. Gubinelli and B. Ugurcan and I. Zachhuber, *Semilinear evolution equations for the Anderson Hamiltonian in two and three dimensions*. Stoch. and Part. Diff. Eq.: Analysis and Computations. **8**:82–149, (2020).
- [33] E. P. Hsu, *Brownian bridges on Riemannian manifolds*. Probab. Th. Rel. Fields, **84**:103–118, (1990).
- [34] X. Huang and C.D. Sogge. *Weyl formulae for Schrödinger operators with critically singular potentials*. Comm. PDE, **46**(11):2088–2133, (2021).
- [35] X. Huang and C.D. Sogge. *Uniform Sobolev Estimates in \mathbb{R}^n Involving Singular Potentials*. J. Geom. Analysis, **32**(2):1–36, (2022).
- [36] O. Kallenberg, *Foundations of Modern Probability Theory*. Probability and its Applications, Springer, (2001).
- [37] S. Klainerman and I. Rodnianski, *A geometric approach to the Littlewood-Paley theory*. Geom. Funct. Anal., **16**(1):126–163, (2006).
- [38] W. König and N. Perkowski and W. van Zuijlen, *Longtime asymptotics of the two-dimensional parabolic Anderson model with white-noise potential*. arXiv:2009.11611, (2020).
- [39] C. Labbé, *The continuous Anderson hamiltonian in $d \leq 3$* . J. Funct. Anal., **227**(9):3187–3225, (2019).

- [40] Y. Le Jan, *Markov loops and renormalization*. Ann. Probab., **38**(3):1280–1319, (2010).
- [41] A. Mouzard, *Weyl law for the Anderson Hamiltonian on a two-dimensional manifold*. arXiv:2009.03549, to appear in Ann. Institut H. Poincaré, (2022).
- [42] A. Mouzard and I. Zachhuber, *Strichartz inequalities with white noise potential on compact surfaces*. arXiv:2104.07940, (2021), to appear in Analysis and PDE.
- [43] T. Oh, T. Robert, N. Tzvetkov and U. Wang, *Stochastic quantization of Liouville conform field theory*. arXiv:2004.04194 .
- [44] E. M. Ouhabaz, *Analysis of heat equations on domains*. London Math. Soc. Monographs, **31**, (2005).
- [45] N. Perkowski and W. van Zuijlen, *Quantitative estimates for diffusions with distributional drift*. arXiv:2009.10786, (2020).
- [46] M. Reed and B. Simon, *Methods of modern mathematical physics I: functional analysis*. Academic Press, (1980).
- [47] M. Reed and B. Simon, *Methods of modern mathematical physics III: scattering theory*. Academic Press, (1979).
- [48] Y. Sawano, *Theory of Besov spaces*. Vol. 56. Singapore: Springer, (2018).
- [49] Schoen, Richard M., and Shing-Tung Yau. *Lectures on differential geometry*. Vol. 2. Cambridge, MA: International press, 1994.
- [50] W. Sickel, *On the regularity of characteristic functions*. Anomalies in partial differential equations, Springer INdAM Ser., 43, Springer, Cham:395–441, (2021).
- [51] B. Simon, *Trace ideals and their applications*. Mathematical surveys and monographs, **120**, Am. Math. Soc., (2005).
- [52] J. Smoller, *Shock waves and reaction—diffusion equations*. Vol. 258. Springer, 2012.
- [53] G. Staffilani and D. Tataru, *Strichartz estimates for a Schrödinger operator with nonsmooth coefficients*. Comm. Part. Diff. Eq., **27**:1337–1372, (2002).
- [54] D. Stroock, *An introduction to partial differential equations for probabilists*. Cambridge studies in advanced mathematics 112, (2008).
- [55] K. Symonzik, *Euclidean quantum field theory*. In Scuola internazionale di Fisica “Enrico Fermi”, XLV Corso:152–223, Academic Press, (1969).
- [56] M. Taylor, *Partial differential equations: basic theory*. Vol. 1. Springer Science, (2013).
- [57] M. Zworski, *Semiclassical analysis*. Graduate Studies in Mathematics, **138**, Am. Math. Soc., (2012).

• I. Bailleul – Univ Brest, CNRS UMR 6205, Laboratoire de Mathématiques de Bretagne Atlantique, France.

E-mail: ismael.bailleul@univ-brest.fr

• V. Dang – Sorbonne Université – Université de Paris, CNRS, UMR 7586, Paris, France. Institut Universitaire de France.

E-mail: nguyen-viet.dang@imj-prg.fr

• A. Mouzard – ENS de Lyon, CNRS, Laboratoire de Physique, F-69342 Lyon, France.

E-mail: antoine.mouzard@ens-lyon.fr