Théorèmes limites pour des marches aléatoires affines conditionnées à rester positives

Ion Grama[1], Ronan Lauvergnat[1], Émile Le Page[1]

[1] *Laboratoire de Mathématiques de Bretagne Atlantique*
(LMBA, UMR CNRS 6205)
Université de Bretagne Sud, France

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Plan

1. Motivation.
2. Conditions, fonction harmonique, approximation par martingales.
3. Les résultats principaux.
4. Éléments de preuves, approximation KMT pour les chaines de Markov.
5. Ouvertures.

Affine Markov walk

1. We study the sum $S_n = \sum_{k=1}^{n} X_k$, $n \geqslant 1$.
2. $(X_n)_{n \geqslant 0}$ is a Markov chain defined by the stochastic recursion

$$X_n = a_n X_{n-1} + b_n, \qquad n \geqslant 1, \quad X_0 = x \in \mathbb{R}, \tag{1}$$

   where $(a_k, b_k)_{k \geqslant 1}$ are i.i.d. of the same law as the pair (a, b).

Notations:

- $\mathbb{P}(x, \cdot)$ - the transition prob. of $(X_n)_{n \geqslant 0}$
- $\mathbb{P}f(x) = \int f(x')\mathbb{P}(x, dx')$ - the transition operator:
- $\mathbb{P}_x$ and $\mathbb{E}_x$ - generated by $(X_n)_{n \geqslant 0}$ with $X_0 = x$.

CLT

Conditions of Guivarc'h and Le Page

- H1: $\mathbb{E}\,|\log|a|| < +\infty, \quad \mathbb{E}\,|\log|b|| < +\infty$

  There exists $\alpha > 2$ such that

- H2: $\phi(\alpha) = \mathbb{E}\,|a|^{\alpha} = 1.$
- H3: $\mathbb{E}\,|a|^{\alpha}\,|\log|a|| < +\infty$ and $\mathbb{E}\,|b|^{\alpha} < +\infty.$

- H4: No fixed point: $\mathbb{P}(ax + b = x) = 0$ for any x.

CLT

1. TLC (Guivarc'h and Le Page (2008)): Under H1-H4 there exist constants μ and $\sigma > 0$ such that, for any $t \in \mathbb{R}$,

$$\mathbb{P}_x\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leqslant t\right) \to \Phi(t) \quad \text{as} \quad n \to +\infty. \tag{2}$$

   There are easy expressions of μ and σ in terms of law of the pair (a, b):

$$\mu = \frac{\mathbb{E}b}{1 - \mathbb{E}a}, \qquad \sigma^2 = \frac{\mathbb{E}b^2}{1 - \mathbb{E}a^2}\frac{1 + \mathbb{E}a}{1 - \mathbb{E}a}.$$

2. In the sequel we consider that:

$$\mathbb{E}b = 0 \qquad \text{so that} \qquad \mu = 0.$$

The exit time

1. Consider the affine Markov walk $y + S_n$ with starting point $y > 0$.
2. The exit time from $\mathbb{R}_+^*$ is defined by

$$\tau_y = \min\{k \geqslant 1 : y + S_k \leqslant 0\},$$

where

$$\{\tau_y > n\} = \{y + S_1 > 0, \ldots, y + S_n > 0\}.$$

The problem is twofold:

1. Determine the asymptotic of the probability $\mathbf{Pr}(\tau_y > n)$.
2. Determine the asymptotic of the conditional distribution of $\frac{1}{\sigma\sqrt{n}}(y + S_n)$, given the event $\{\tau_y > n\}$.

Previous results:

- For sums of i.i.d. r.v.'s in $\mathbb{R}^1$: Bolthausen (1972), Iglehart (1974), Spitzer (1976), Doney (1985), Bertoin and Doney (1994), Borovkov (2004), Vatutin and Wachtel (2009); (by Wiener-Hopf factorization).
- Markov chains: Varapoulos (1999) - upper and lower bounds for $\mathbf{Pr}(\tau_y > n)$.
- I.i.d. in $\mathbb{R}^d$: Eischelsbacher and Konig (2008), Denisov and Wachtel (2009, 2011).

Motivation: products of random matrices

G., Le Page and Peigné (2015)

1. Denote by $\mathbb{B}$ the closed unit ball in $\mathbb{R}^d$ and by $\mathbb{B}^c$ its complement. Let v be a starting vector: $v \in \mathbb{B}^c$.
2. Assume that $g_1, \ldots, g_n$ are independent random elements of $\mathbb{G}$ with common distribution μ. Assume that the upper Lyapunov exponent $\gamma = 0$.
3. Define the exit time of the random process $G_n v$ from $\mathbb{B}^c$ by

$$\begin{aligned} \tau_v &= \min\{n \geqslant 1 : g_n \ldots g_1 v \in \mathbb{B}\} \\ &= \min\{n \geqslant 1 : \log \|g_n \ldots g_1 v\| \leqslant 0\}. \end{aligned}$$

Products of r.m.'s

P1: exponential moments for $\|g\|$ and $\|g^{-1}\|$.
P2: irreducibility. **P2**: proximality.

Theorem 1

Assume conditions **P1-P3**. Then, for any starting point $v \in \mathbb{B}^c$,

$$\mathbb{P}(\tau_v > n) = \frac{2V(v)}{\sigma\sqrt{2\pi n}}(1 + o(1)) \quad \text{as } n \to \infty,$$

where V is a positive harmonic function on $\mathbb{B}^c$.
Moreover, for any starting point $v \in \mathbb{B}^c$, and for any $t \geqslant 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\log \|g_n \dots g_1 v\|}{\sigma\sqrt{n}} \leqslant t \,\middle|\, \tau_v > n \right) = \Phi^+(t),$$

where $\Phi^+(t) = 1 - \exp\left(-\frac{t^2}{2}\right)$ is the Rayleigh distribution.

Products of r.m.'s: moments

Denote $N(g) = \max\{\|g\|, \|g^{-1}\|\}$ $(\geqslant 1$, since $\|g^{-1}\| \geqslant \|g\|^{-1}$).
Our first condition requires exponential moments of $\log N(g)$.

P1 (Exponential moments):

There exists $\delta_0 > 0$ such that

$$\int_{\mathbb{G}} \exp(\delta_0 \log N(g))\, \mu(dg) = \int_{\mathbb{G}} N(g)^{\delta_0}\, \mu(dg) < \infty.$$

The CLT under less restrictive moment assumptions (only the second moment of $\log N(g)$) have been obtained only recently.

We refer to the book of Benoist and Quint (2013).

Products of r.m.'s: irreducibility

Definition: a) A subset $\mathbb{T}$ of $\mathbb{G}$ is irreducible if the is no proper linear subspace $\mathbb{S}$ of $\mathbb{V}$ such that, for any $g \in \mathbb{T}$,

$$g(\mathbb{S}) = \mathbb{S}.$$

b) A subset $\mathbb{T}$ of $\mathbb{G}$ is strongly irreducible if there is no finite family of proper linear subspaces $\mathbb{S}_1, \ldots, \mathbb{S}_m$ of $\mathbb{V}$ such that, for any $g \in \mathbb{T}$,

$$g(\mathbb{S}_1 \cup \cdots \cup \mathbb{S}_m) = \mathbb{S}_1 \cup \cdots \cup \mathbb{S}_m.$$

Example: $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is irreducible but not strongly irreducible.

P2 (Strong irreducibility):

The support $\mathrm{supp}\mu$ *of* μ *acts strongly irreducibly on* $\mathbb{V}$.

This condition requires, roughly speaking, that the dimension of $\mathrm{supp}\mu$ cannot be reduced.

Products of r.m.'s: contraction property

Let $\mathbb{T}_\mu$ be the closed semigroup generated by $\operatorname{supp}\mu$.

P3 (Proximality):

$\mathbb{T}_\mu$ contains a contracting sequence for the projective space $\mathbb{P}(\mathbb{V})$.

- Consider a sequence $(G_n)_{n\geqslant 1}$ in $\mathbb{G}$. Any $G_n \in \mathbb{G}$ admits a polar decomposition: $G_n = H_n^1 A_n H_n^2$, where H_n^1, H_n^2 are orthogonal and A_n is diagonal with diagonal entries $A_n(1) \geqslant \ldots \geqslant A_n(d) > 0$.
- **Def. (Contracting sequence):** The sequence $(G_n)_{n\geqslant 1}$ is contracting if $\lim_{n\to\infty}(\log A_n(1) - \log A_n(2)) = \infty$.

Example: Let $G_n = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}^n$, $|\lambda| < 1$. Then $G_n \cdot \overline{v} = \overline{\begin{pmatrix} 1 & 0 \\ 0 & \lambda^n \end{pmatrix} v} \to \overline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ as $n \to \infty$.

- For example **P3** is satisfied if $\mathbb{T}_\mu$ contains a matrix with a simple eigenvalue of maximal modulus.

The idea of the proof for products of r.m.

- Consider the homogenous Markov chain $(X_n)_{n\geqslant 0}$ with values in the product space $\mathbb{X} = \mathbb{G} \times \mathbb{P}(\mathbb{V})$ and initial value $X_0 = (g, \overline{v}) \in \mathbb{X}$ by setting:

$$X_{n+1} = (g_{n+1}, g_n ... g_1 g \cdot \overline{v}), \quad n \geqslant 0.$$

- Recall that the norm cocycle is: $\rho(g, \overline{v}) := \log \frac{\|gv\|}{\|v\|}$, for $(g, \overline{v}) \in \mathbb{G} \times \mathbb{P}(\mathbb{V})$.

1. **Markov walk** representation: iterating the cocycle property $\rho(g''g', \overline{v}) = \rho(g'', g' \cdot \overline{v}) + \rho(g', \overline{v})$

$$\log \|g_n \dots g_1 g v\| = y + \sum_{k=1}^{n} \rho(X_k) = y + S_n, \quad y = \log \|gv\|$$

2. **Martingale** approximation:

$$\mathbb{P}_x \left(\sup_{n\geqslant 0} |S_n - M_n| \leqslant c \right) = 1.$$

Retourn to affine Markov walks: Conditions

Condition 1 (which implies $\exists$ of the spectral gap).

- There exists $\alpha > 2$ s.t. $\phi(\alpha) = \mathbb{E}(e^{\alpha \log|a|}) = \mathbb{E}(|a|^{\alpha}) < 1$ and $\mathbb{E}(|b|^{\alpha}) < +\infty$.
- b is non-degenerated ($\mathbb{P}(b \neq 0) > 0$) and $\mathbb{E}(b) = 0$.

C.1 is less restrictive that the conditions H1-H4 of Guivarc'h and Le Page:

Refined CLT: Under Condition 1, for any $\varepsilon > 0$ and any $x \in \mathbb{R}$,

$$\sup_t \left| \mathbb{P}_x \left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leqslant t \right) - \Phi(t) \right| \leqslant \frac{C_{p,\varepsilon}}{n^{\varepsilon}} (1 + |x|^p). \tag{3}$$

The CLT above is a consequence of more general results for the associated Markov walk $(y + S_n)_{n \geqslant 0}$.

Further conditions C2 and C3 are related to the harmonic function.

Harmonic function

- $(X_n, y + S_n)$ est une chaine de Markov.
- $\mathbb{Q}(x, y, dx', dy')$ - the transition probability of $(X_n, y + S_n)$.
- $\mathbb{Q}_+(x, y, dx', dy')$ - the restriction on $\mathbb{R} \times \mathbb{R}_+^*$.

1. Definition:
   The function $V : \mathbb{R} \times \mathbb{R}_+^* \to \mathbb{R}$ is positive $\mathbb{Q}_+$-harmonic if
   $$\mathbb{Q}_+ V(x, y) = V(x, y), \qquad x, y \in \mathbb{R} \times \mathbb{R}_+^*.$$
2. Equivalent formulation: Doob's transform
   The function $V : \mathbb{R} \times \mathbb{R}_+^* \to \mathbb{R}$ is positive $\mathbb{Q}_+$-harmonic if
   $$\mathbb{Q} V(X_1, y + S_1; \tau_y > 1) = V(x, y), \qquad x, y \in \mathbb{R} \times \mathbb{R}_+^*$$
   or, by iteration,
   $$\mathbb{Q} V(X_n, y + S_n; \tau_y > n) = V(x, y), \qquad x, y \in \mathbb{R} \times \mathbb{R}_+^*, n \geqslant 1.$$

More conditions:
Positivity of the harmonic function

Conditions for the existence of a positive harmonic function:

Condition 2. (si $\mathbb{E}(a) \geqslant 0$) for any $x \in \mathbb{R}$ and $y > 0$:

- $\mathbb{P}_x(\tau_y > 1) = \mathbb{P}_x(y + X_1 > 0) = \mathbb{P}(ax + b > -y) > 0.$

Condition 3. (si $\mathbb{E}(a) < 0$) for any $x \in \mathbb{R}$ and $y > 0$ $\exists n_0 \geqslant 1$ s.t.

- $\mathbb{P}_x(y + S_{n_0} > C(1 + |X_{n_0}|^p, \tau_y > n_0) > 0.$

Martingale approximation

- Consider the Poisson equation: $u - \mathbb{P}u = Id$
  $\theta(x) = \sum_{k=0}^{\infty} \mathbb{P}^k Id(x) = \frac{x}{1-\mathbb{E}a}$.

  Define Gordin's $\mathbb{P}_x$-martingale:

$$M_n = \sum_{k=0}^{\infty} \left(\theta(X_k) - \mathbb{P}\theta(X_{k-1}) \right).$$

- In another form:

$$M_0 = 0, \qquad M_n = S_n + \frac{\mathbb{E}a}{1 - \mathbb{E}a} (X_n - x).$$

- With the notation $\rho = \frac{\mathbb{E}a}{1-\mathbb{E}a}$ we have:

$$(y + S_n) - (z_{x,y} + M_n) = -\rho X_n, \quad z_{x,y} = y + \rho x, \quad \mathbb{P}_x\text{-a.s.}$$

Existence of the harmonic function

Theorem 1

Assume $\mathbb{E}a \geqslant 0, C1, C2$ or $\mathbb{E}a < 0, C1, C3$. Then:

1. for any $x \in \mathbb{R}$ and $y > 0$, it holds $\mathbb{E}_x \left(\left| M_{\tau_y} \right| \right) < +\infty$ and therefore the function $V(x, y) = -\mathbb{E}_x M_{\tau_y}$ is well defined.
2. the function V is positive and $\mathbb{Q}_+$-harmonic:

$$\mathbb{Q}_+ V(x, y) = V(x, y), \qquad x \in \mathbb{R}, y > 0.$$

3. properties:
   - for any $x \in \mathbb{R}$: $V(x, \cdot)$ is non-decreasing
   - for any $x \in \mathbb{R}$: $\lim_{n \to \infty} \frac{V(x,y)}{y} = 1$
   - $V(x, y) \geqslant \max 0, (1 + \delta)y - c_{\delta,p}(1 + |x|^p)$
   - $V(x, y) \leqslant \left(1 + \delta(1 + |x|^{p-1})\right) y + c_{\delta,p}(1 + |x|^p)$,

   for any $\delta > 0, p \in (2, \alpha)$.

Main results

Theorem 2

Assume $\mathbb{E}a \geqslant 0, C1, C2$ or $\mathbb{E}a < 0, C1, C3$. Then:

1. for any $x \in \mathbb{R}$ and $y > 0$, it holds

$$\sqrt{n}\mathbb{P}_x\left(\tau_y > n\right) \leqslant c_p(1 + y + |x|^p), \quad \text{with } p > 2.$$

2. for any fixed $x \in \mathbb{R}$ and fixed $y > 0$, it holds

$$\mathbb{P}_x\left(\tau_y > n\right) \sim \frac{2V(x,y)}{\sqrt{2\pi n}\sigma}.$$

- Corollary: for any $\gamma < 1$ and $p > 2$,

$$\mathbb{E}_x\left(\tau_y^\gamma\right) \leqslant c_{p,\gamma}(1 + y + |x|^p).$$

Main results

Theorem 3

Assume $\mathbb{E}a \geqslant 0$, $C1$, $C2$ or $\mathbb{E}a < 0$, $C1$, $C3$. Then, for any $x \in \mathbb{R}$ and $y > 0$, it holds

$$\mathbb{P}_x \left(\frac{y + S_n}{\sigma\sqrt{n}} \leqslant t \,\middle|\, \tau_y > n \right) \to \Phi^+(t),$$

where $\Phi^+(t) = 1 - e^{-\frac{t^2}{2}}$ is the Rayleigh d.f.

Extension for y<0:

Define $\mathcal{D}^+ = \{(x, y) \in \mathbb{R} \times \mathbb{R}_- : \mathbb{P}_x(ax + b > -y) > 0\}$. Then

1. V is Q_+-harmonic on $\mathcal{D}^+ \cup \mathbb{R} \times \mathbb{R}_+^*$

   and

2. Theorem 2 and 3 hold true for $(x, y) \in \mathcal{D}^+$.

Discussion on Conditions 2 and 3

The case $\mathbb{E}(a) \geqslant 0$ ($\mathbb{E}b = 0$, b non-degenerated)

- C2. $\mathbb{P}_x(\tau_y > 1) = \mathbb{P}_x(y + X_1 > 0) = \mathbb{P}(ax + b > -y) > 0.$
- C2a. a, b dependent, $\mathbb{P}(b \geqslant C\,|a|) > 0$, for any $C > 0$.
- C2b. a, b independent, b no conditions (for example: b - Rademaher), $\mathbb{P}(|a| \leqslant \varepsilon) > 0$ for any $\varepsilon > 0$ (for instance $a = 0$).
- C2c. a, b independent, a no conditions, $\mathbb{P}(b > A) > 0$ for any $A > 0$.

The case $\mathbb{E}(a) < 0$ ($\mathbb{E}b = 0$, b non-degenerated)

- C3. $\mathbb{P}(y + S_{n_0} > C(1 + |X_{n_0}|^p, \tau_y > n_0) > 0.$
- C3a. a, b independent, b no conditions, $\mathbb{P}(a \in (-1, 0)) > 0$ and $\mathbb{P}(a \in (0, 1)) > 0$.
- C3b. a, b dependent, $\mathbb{P}((a, b) \in (-1, 0) \times [0, c]) > 0$ and $\mathbb{P}((a, b) \in (0, 1) \times [0, c]) > 0$, for some $c > 0$.

Proofs

Existence of the positive harmonic function (the case $\mathbb{E}(a) \geqslant 0$)

It is important to approximate $y + S_n$ by a martingale:

$$(y+S_n) - (z_{x,y} + M_n) = -\rho X_n, \qquad z_{x,y} = y + \rho x, \quad \rho = \frac{\mathbb{E}a}{1-\mathbb{E}a}.$$

Etape 1. Integrability of $\left|M_{\tau_y}\right|$ (main difficulty)

- $(z_{x,y} + M_n) 1_{\{\tau_y > n\}}$ is a submartingale:
  $u_n = \mathbb{E}_x\left(z_{x,y} + M_n ; \tau_y > n\right)$ is increasing.
- we show that, for any $\varepsilon > 0$

$$u_n \leqslant \left(1 + \frac{c_p}{n^{\varepsilon}}\right) u_{[n^{1-\varepsilon}]} + \frac{1}{n^{\varepsilon}} c_p (1 + y + |x|)(1 + |x|^{p-1}),$$

  which implies that u_n is uniformly bounded in n.
- $\mathbb{E}_x(\left|z_{x,y} + M_{\tau_y}\right| ; \tau_y \leqslant n) = -\mathbb{E}_x(z_{x,y} + M_{\tau_y} ; \tau_y \leqslant n)$
  $= -\mathbb{E}_x(z_{x,y} + M_n ; \tau_y \leqslant n) = -\mathbb{E}_x(z_{x,y} + M_n) + u_n = -z_{x,y} + u_n,$

using the fact that $\rho = \frac{\mathbb{E}a}{1-\mathbb{E}a} \geqslant 0$ implies an ordering among $y + S_n$ and $z_{x,y} + M_n$ on $\{\tau_y = n\}$:

$$(z_{x,y} + M_{\tau_y}) = (y + S_{\tau_y}) + \rho X_{\tau_y} \leqslant (z + S_{\tau_y}) \leqslant 0,$$

where $X_{\tau_y} \leqslant 0$ by the definition of τ_y (and by th. de convergence monotone).

Proofs

Etape 2. Harmonicity: The function $V(x,y) = \mathbb{E}_x M_{\tau_y}$ $\exists$.

- We show that

$$\begin{aligned} V(x,y) &= \lim_{n\to\infty} \mathbb{E}_x \left(z_{x,y} + M_n; \tau_y > n\right) \\ &= \lim_{n\to\infty} \mathbb{E}_x \left(y + S_n; \tau_y > n\right) \end{aligned}$$

- Proof (first): using dominated conv. theorem as $n \to \infty$

$u_n = \mathbb{E}_x(z_{x,y} + M_n; \tau_y > n) = \mathbb{E}_x(z_{x,y} + M_n) - \mathbb{E}_x(z_{x,y} + M_n; \tau_y \leqslant n)$

$= z_{x,y} - \mathbb{E}_x(z_{x,y} + M_{\tau_y}; \tau_y \leqslant n) \to -\mathbb{E}_x(M_{\tau_y}) = V(x,y)$.

- Proof (second): use $(y + S_n) - (z_{x,y} + M_n) \sim X_n$ and moments assumptions.
- Hamonicity is easy: by Markov's property

$$V_n(x,y) = \mathbb{E}_x(y + S_n; \tau_y > n) = \mathbb{E}_x V_{n-1}(X_1, y + S_1; \tau_y > 1)$$

Using an upper bound for V and taking $\lim_{n\to\infty}$ it follows that V is a Doob's transform.

Proofs

Etape 3. Positivity:

- Since V is a Doob transform

$$\begin{aligned} V(x,y) &= \mathbb{E}_x\left(V(X_1, y+S_1); \tau_y > 1\right) \\ &\geqslant \mathbb{E}_x\left(V(X_1, y+S_1); \tau_y > 1, B\right), \end{aligned}$$

where the event B will be chosen below.

- Minoration:

$$V(x,y) = -\mathbb{E}_x(M_{\tau_y}) = z_{x,y} - \mathbb{E}_x(z_{x,y} + M_{\tau_y}) \geqslant z_{x,y} = y + \rho x$$

again we used: $\rho = \frac{\mathbb{E}a}{1-\mathbb{E}a} \geqslant 0$ implies the ordering $(z_{x,y} + M_{\tau_y}) \leqslant (y + S_{\tau_y}) \leqslant 0$

- Positivity: with $B = \left\{X_1 > \frac{-y}{2(1+\rho)}\right\}$,

$$\begin{aligned} V(x,y) &\geqslant \mathbb{E}_x\left(V(X_1, y+S_1); \tau_y > 1, B\right) \\ &\geqslant \mathbb{E}_x\left(y + S_1 + \rho X_1; \tau_y > 1, B\right) \\ &\geqslant \frac{y}{2}\mathbb{P}_x\left(X_1 > -y/2(1+\rho)\right) > 0, \end{aligned}$$

by C2.

Komlos-Major-Tusnady approximation

- As to the asymptotic properties of the exit time τ_y and of the conditional distribution $\mathbf{Pr}\left(\frac{y+S_n}{\sigma\sqrt{n}} \leqslant t \middle| \tau_y > n \right)$ they are deduced from the respective properties of the continuous time standard Brownian motion (B_s).

Theorem (G - Le Page - Peigné 2014)

Under **C1** there is a construction on the same probability space of the associated Markov process (S_k) and of a standard Brownian motion (B_t) such that for any $x \in \mathbb{X}$ and any $\varepsilon \in (0, \frac{1}{5})$,

$$\mathbb{P}_x \left(n^{-1/2} \sup_{0 \leqslant t \leqslant 1} \left| S_{[tn]} - \sigma B_{tn} \right| > n^{-\varepsilon} \right) \leqslant c_{p,\varepsilon} n^{-\varepsilon} (1 + |x|^p),$$

where $c_{p,\varepsilon}$ is a constant depending only on $p > 2$ and ε.

More general result on KMT approximation result for Markov chains:

$S_n = \sum_{k=1}^n f(X_k)$ and (X_k) is a Markov chain.

Komlos-Major-Tusnady approximation

$S_n = \sum_{k=1}^n f(X_k)$, where (X_k) is a Markov chain with values in $\mathbb{X}$ and f a real function on $\mathbb{X}$.

Theorem (G - Le Page - Peigné 2014)

Assume that the Markov chain $(X_n)_{n\geqslant 0}$ and the function f satisfy the hypotheses **M1**, **M2**, **M3** and **M4**, with $\sigma > 0$. Let $0 < \alpha < \delta$. Then there exists a Markov transition kernel $x \to \widetilde{\mathbb{P}}_x(\cdot)$ from $(\mathbb{X}, \mathcal{X})$ to $(\widetilde{\Omega}, \mathcal{B}(\widetilde{\Omega}))$ such that $\mathcal{L}\left(\left(\widetilde{Y}_i\right)_{i\geqslant 1} | \widetilde{\mathbb{P}}_x\right) \overset{d}{=} \mathcal{L}\left((f(X_i))_{i\geqslant 1} | \mathbb{P}_x\right)$, the $W_i, i \geqslant 1$, are independent standard normal r.v.'s under $\widetilde{\mathbb{P}}_x$ and for any $0 < \rho < \frac{\alpha}{2(1+2\alpha)}$,

$$\widetilde{\mathbb{P}}_x\left(N^{-\frac{1}{2}} \sup_{k\leqslant N}\left|\sum_{i=1}^{k}\left(\widetilde{Y}_i - \mu - \sigma W_i\right)\right| > 6N^{-\rho}\right) \leqslant C(x) N^{-\alpha\frac{1+\alpha}{1+2\alpha}+\rho(2+2\alpha)},$$

with $C(x) = C_1\left(1 + \mu_\delta(x) + \|\delta_x\|_{\mathcal{B}'}\right)^{2+2\delta}$, where C_1 is a constant depending only on $\delta, \alpha, \kappa, C_{\mathbf{P}}, C_Q, \|e\|_{\mathcal{B}}$ and $\|\nu\|_{\mathcal{B}'}$.

Exit time for the Brownian motion

Define the exit time $\tau_y^{bm} = \inf\{t \geqslant 0,\ y + \sigma B_t \leqslant 0\}$.
The following assertion is due to Levy (1954).

- For any $y > 0$, $0 \leqslant a \leqslant b$ and $n \geqslant 1$,

$$\mathbb{P}\left(\tau_y^{bm} > n\right) = \frac{2}{\sqrt{2\pi n}\sigma} \int_0^y e^{-\frac{s^2}{2n\sigma^2}} ds.$$

From this one can deduce easily:

1. For any $y > 0$,
$$\mathbb{P}\left(\tau_y^{bm} > n\right) \leqslant c \frac{y}{\sqrt{n}}.$$
2. For any sequence of real numbers $(\theta_n)_{n \geqslant 0}$ such that $\theta_n \underset{n \to +\infty}{\longrightarrow} 0$,
$$\sup_{y \in [0; \theta_n \sqrt{n}]} \left(\frac{\mathbb{P}\left(\tau_y^{bm} > n\right)}{\frac{2y}{\sqrt{2\pi n}\sigma}} - 1 \right) = O(\theta_n^2).$$

Exit time for $(y + S_n)$

The KMT approximation allows to prove the following intermediate result for y large enough:

Lemma

Let $\varepsilon \in (0, \varepsilon_0)$ *and* $(\theta_n)_{n \geqslant 1}$ *be a sequence of positive numbers such that* $\theta_n \to 0$ *and* $\theta_n n^{\varepsilon/4} \to \infty$ *as* $n \to \infty$. *Then:*

1. *There exists a constant* $c_\varepsilon > 0$ *such that for any* $n \geqslant 1$ *and* $y \geqslant n^{1/2-\varepsilon}$,

$$\sup_{x \in \mathbb{X}} \mathbb{P}_x (\tau_y > n) \leqslant c_\varepsilon \frac{y}{\sqrt{n}}.$$

2. *There exists a constant* $c > 0$ *such that, for* n *sufficiently large,*

$$\sup_{x \in \mathbb{X},\ y \in [n^{1/2-\varepsilon}, \theta_n n^{1/2}]} \left| \frac{\mathbb{P}_x (\tau_y > n)}{\frac{2y}{\sqrt{2\pi n}\sigma}} - 1 \right| \leqslant c\theta_n.$$

Asymptotic for τ_y

- $\mathbb{P}_x(\tau_y > n) = \underset{J_1}{\mathbb{P}_x\left(\tau_y > n; \nu_n \leqslant n^{1-\varepsilon}\right)} + \underset{J_2.}{\mathbb{P}_x\left(\tau_y > n; \nu_n > n^{1-\varepsilon}\right)}$

- By Markov property
  $J_1 = \mathbb{P}_x\left(\tau_y > n; \nu_n \leqslant n^{1-\varepsilon}\right) =$

  $$\int\int \mathbb{P}_x\left(X_{\nu_n} \in dx', y + S_{\nu_n} \in dy'; \tau_y > \nu_n; \nu_n \leqslant n^{1-\varepsilon}\right) \times \mathbb{P}_{x'}\left(\tau_{y'} > n - \nu_n\right)$$

  since $\mathbb{P}_x\left(\tau_{y'} > n - \nu_n\right) = \frac{2y'}{\sqrt{2\pi(n-\nu_n)}} = \frac{2y'}{\sqrt{2\pi n}}(1 + o(1))$

  $$\approx \frac{1}{\sqrt{2\pi n}}\mathbb{E}\left(y + S_{\nu_n}; \tau_y > \nu_n; \nu_n \leqslant n^{1-\varepsilon}\right) \to \frac{1}{\sqrt{2\pi n}} V(x,y).$$

- Rappel: $\mathbb{E}(y + S_n; \tau_y > n) \to V(x,y)$.
  Replace n by ν_n

Future investigations

1. Local theorem: rates $n^{-3/2}$
2. The case $\mathbb{E}(b) < 0$.
3. Matrix affine random walks or more general Markov chains

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