On the martingale-coboundary decomposition for random fields

Joint work with Davide GIRAUDO

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- $\mathcal{M} \subset \mathcal{F}$ is a σ-algebra such that $\mathcal{M} \subset T^{-1}\mathcal{M}$;
- for $p \in [1, \infty]$ and $\mathcal{B} \subset \mathcal{A}$ two sub-σ-algebras of $\mathcal{F}$,

$$\mathbb{L}^p(\Omega, \mathcal{A}, \mu) \ominus \mathbb{L}^p(\Omega, \mathcal{B}, \mu) = \{f \in \mathbb{L}^p, f \text{ is } \mathcal{A}\text{-measurable and } \mathbb{E}(f \mid \mathcal{B}) = 0\}.$$

Theorem (Gordin (1969))

Let $f \in \mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, \cap_{i\in\mathbb{Z}} T^{-i}\mathcal{M}, \mu)$ *such that*

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$(m \circ T^i)_{i\in\mathbb{Z}}$ is a martingale-difference sequence with respect to the filtration $(T^{-i}\mathcal{M})_{i\in\mathbb{Z}}$.

For any $h\colon \Omega \to \mathbb{R}$ measurable, we denote $S_n(h) := \sum_{i=0}^{n-1} h \circ T^i$.

Central Limit Theorem

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- From the Billingsley-Ibragimov central limit theorem, we have

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If $f : \Omega \to \mathbb{R}$ satisfies (PC) then

$$n^{-1/2} S_n(f,.) \to \sqrt{\mathbb{E}[m^2]} \cdot W \text{ in distribution in } C[0,1]$$

Weak Invariance Principle

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This result reduces to the classical Donsker WIP when $(f \circ T^k)_{k \in \mathbb{Z}}$ are iid.

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For any $A\in\mathcal{A}$ and any $n\in\mathbb{N}^*$,

$$S_n(A)=\sum_{i\in\{1,\ldots,n\}^d}\lambda(nA\cap R_i)X_i$$

where $R_i=]i_1-1,i_1]\times\ldots\times]i_d-1,i_d]$ and λ is the Lebesgue measure on $\mathbb{R}^d$.

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For any $t \in [0,1]$,

$$S_n(t) = \sum_{i=1}^{[nt]} X_i + (nt - [nt])X_{[nt]+1} = \sum_{i=1}^{n} \lambda\left(n[0,t] \cap [i-1,i]\right) X_i.$$

So,

$$\{n^{-1/2} S_n(t)\,;\ t \in [0,1]\} = \{n^{-1/2} S_n(A)\,;\ A \in \mathcal{Q}_1\}$$

where $\mathcal{Q}_1 = \{[0,t]\,;\ t \in [0,1]\}$.

Definition (Metric entropy)

$H(\mathcal{A}, \rho, \epsilon)$ is the logarithm of the smallest number of open balls of radius ϵ with respect to ρ which form a covering of $\mathcal{A}$.

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Assume that $\mathcal{A}$ is totally bounded with inclusion i.e. for each positive ϵ there exists a finite collection $\mathcal{A}(\epsilon)$ of Borel subsets of $[0,1]^d$ such that for any $A \in \mathcal{A}$, there exist A_1 and A_2 in $\mathcal{A}(\epsilon)$ with $A_1 \subseteq A \subseteq A_2$ and $\rho(A_1, A_2) \leq \epsilon$.

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Definition (Bracketing entropy)

$\mathbb{H}(\mathcal{A}, \rho, \epsilon)$ is the logarithm of the cardinality of the smallest collection $\mathcal{A}(\epsilon)$.

Definition (Brownian motion indexed by $\mathcal{A}$)

A standard Brownian motion indexed by $\mathcal{A}$ is a mean zero Gaussian process W with sample paths in $C(\mathcal{A})$ and Cov(W(A),W(B))$= \lambda(A \cap B)$.

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From Dudley (1973), we know that such a process exists if

$$\int_0^1 \sqrt{H(\mathcal{A}, \rho, \epsilon)}\, d\epsilon < \infty.$$

Definition

We say that the weak invariance principle (WIP) holds if the sequence $\{n^{-d/2}S_n(A)\,;\,A\in\mathcal{A}\}$ converges in distribution to a mixture of $\mathcal{A}$-indexed Brownian motions in the space $C(\mathcal{A})$.

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Theorem (Bass (1985), Alexander and Pyke (1986))

$(X_k)_{k\in\mathbb{Z}^d}$ *centered i.i.d. random field such that* $X_0\in L^2$ *and*

$$\int_0^1 \sqrt{\mathbb{H}(\mathcal{A},\rho,\varepsilon)}\,d\varepsilon<\infty$$

then the WIP holds.

Weak Invariance Principle

Theorem (E.M., Ouchti (2006))

For any positive real number p, there exists a stationary field $(X_k)_{k\in\mathbb{Z}^d}$ *of independent, symmetric and p-integrable real random variables and a collection* $\mathcal{A}$ *of Borel subsets of* $[0,1]^d$ *which satisfies the condition*

$$\int_0^1 \sqrt{H(\mathcal{A},\rho,\varepsilon)}\, d\varepsilon < \infty$$

such that $\{n^{-d/2}S_n(A)\,;\,A\in\mathcal{A}\}$ *do not be tight in the space* $C(\mathcal{A})$.

Theorem (Dedecker (2001))

$(X_k)_{k\in\mathbb{Z}^d}$ a centered stationary random field such that $X_0 \in L^\infty$ and

$$\sum_{k<_{lex}0} \|X_k E(X_0|\mathcal{F}_k)\|_\infty < \infty$$

with $\mathcal{F}_k = \sigma(X_j\,;\, j <_{lex} 0\,;\, |j| \geq |k|)$ and such that

$$\int_0^1 \sqrt{H(\mathcal{A},\rho,\varepsilon)}\,d\varepsilon < \infty$$

then the WIP holds.

Theorem (Dedecker (2001))

$(X_k)_{k\in\mathbb{Z}^d}$ a centered stationary random field such that $X_0 \in L^p$ for $p > 2$ and

$$\sum_{k<_{lex}0} \|X_k E(X_0|\mathcal{F}_k)\|_{\frac{p}{2}} < \infty$$

with $\mathcal{F}_k = \sigma(X_j\,;\, j <_{lex} 0\,;\, |j| \geq |k|)$ then the WIP holds for $\mathcal{A} = \mathcal{Q}_d$.

Let $\psi : \mathbb{R}_+ \to \mathbb{R}$ be a Young function.

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We consider the Orlicz space $\mathbb{L}_\psi$ defined by

$$\mathbb{L}_\psi = \{Z \mid \exists c > 0 \quad \mathbb{E}\left(\psi(|Z|/c)\right) < \infty\}$$

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If $\psi(x) = x^p$ then $\mathbb{L}_\psi = \mathbb{L}_p$ and $\|\,.\,\|_\psi = \|\,.\,\|_p$.

For any $x \geq 0$,

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For any $\beta > 0$ and any $x \geq 0$,

$$\psi_\beta(x) = \exp((x + h_\beta)^\beta) - \exp(h_\beta^\beta)$$

where $h_\beta = ((1-\beta)/\beta)^{1/\beta}\, \mathbb{1}_{\{0<\beta<1\}}$.

Theorem (E.M. (2002))

$(X_k)_{k\in\mathbb{Z}^d}$ a centered and stationary real random field. If there exist $0<q<2$ and $\theta>0$ such that $E[\exp(\theta|X_0|^{\beta(q)})]<\infty$ and

$$\sum_{k<_{lex}0}\left\|\sqrt{|X_k E(X_0|\mathcal{F}_k)|}\right\|_{\psi_{\beta(q)}}^2<\infty$$

where $\beta(q)=2q/(2-q)$ and

$$\int_0^1 (H(\mathcal{A},\rho,\varepsilon))^{1/q}\,d\varepsilon<\infty$$

then the WIP holds.

Theorem (E.M., Volný (2002))

For any nonnegative real p, there exist a p-integrable stationary real random field $(X_k)_{k\in\mathbb{Z}^d}$ *and a collection* $\mathcal{A}$ *of regular Borel subsets of* $[0,1]^d$ *such that*

- *For any k in* $\mathbb{Z}^d$, $\mathbb{E}\left(X_k|\sigma(X_i\,;\,i\neq k)\right)=0$. *We say that the random field* $(X_k)_{k\in\mathbb{Z}^d}$ *is a strong martingale-difference random field.*
- *The collection* $\mathcal{A}$ *satisfies the bracketing entropy condition*

$$\int_0^1\sqrt{\mathbb{H}(\mathcal{A},\rho,\varepsilon)}\,d\varepsilon<\infty.$$

- *The partial sum process* $\{n^{-d/2}S_n(A)\,;\,A\in\mathcal{A}\}$ *is not tight in the space* $C(\mathcal{A})$.

Theorem (E.M., Ouchti (2006))

Let $(X_k)_{k\in\mathbb{Z}^d}$ be a stationary field of martingale-difference random variables with finite variance such that $\mathbb{E}\left(X_0^2|\sigma(X_i\,;\,i<_{lex}0)\right)$ is bounded almost surely and assume that

$$\int_0^1 \sqrt{\mathbb{H}(\mathcal{A},\rho,\varepsilon)}\,d\varepsilon<\infty,$$

then the WIP holds.

Let $(X_i)_{i\in\mathbb{Z}^d}$ be centered and defined by

$$X_i = g(\varepsilon_{i-s}\,;\, s\in\mathbb{Z}^d).$$

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- g is a measurable function.

Let $(\varepsilon_j^{'})_{j\in\mathbb{Z}^d}$ be an i.i.d. copy of $(\varepsilon_j)_{j\in\mathbb{Z}^d}$.

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We define the coupled version X_i^* of X_i by

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where for any j in $\mathbb{Z}^d$,

$$\varepsilon_j^* = \begin{cases} \varepsilon_0^{'} & \text{if } j = 0 \\ \varepsilon_j & \text{if } j \neq 0 \end{cases}$$

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The random field X defined by

$$X_i = g\left(\varepsilon_{i-s}\,;\, s \in \mathbb{Z}^d\right)$$

is said to be ψ-*stable* if

$$\Delta_\psi := \sum_{i\in\mathbb{Z}^d} \delta_{i,\psi} < \infty.$$

Example : $(\varepsilon_i)_{i\in\mathbb{Z}^d}$ i.i.d with $\varepsilon_i \in \mathbb{L}^p$, $p \geq 2$. The linear random field X defined for any i in $\mathbb{Z}^d$ by

$$X_i = \sum_{s\in\mathbb{Z}^d} a_s \varepsilon_{i-s}$$

is of the form $X_i = g(\varepsilon_{i-s}, s \in \mathbb{Z}^d)$ with a linear function g.

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If K is a Lipschitz function then $K(X_i)$ is also p-stable under the above condition.

Proposition (E.M., Volný, Wu (2013))

Let Γ *be a finite subset of* $\mathbb{Z}^d$ *and let* $(a_i)_{i\in\Gamma}$ *be a family of real numbers. For any* $p \geq 2$*, we have*

$$\left\| \sum_{i\in\Gamma} a_i X_i \right\|_p \leq \left(2p \sum_{i\in\Gamma} a_i^2 \right)^{\frac{1}{2}} \Delta_p$$

where $\Delta_p = \sum_{i\in\mathbb{Z}^d} \delta_{i,p}$.

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- $\mathcal{A}$ *shatters* E if each of its subsets can be picked out in this manner.
- The *VC*-index $V(\mathcal{A})$ of $\mathcal{A}$ is the smallest n for which no set of size n is shattered by $\mathcal{A}$.

VC-classes of sets:

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- If $\mathcal{A}$ is a *VC*-class then for any $\varepsilon > 0$,

$$N(\mathcal{A}, \rho, \varepsilon) \leq KV(\mathcal{A})(4e)^{V(\mathcal{A})}\left(\frac{1}{\varepsilon}\right)^{2(V(\mathcal{A})-1)}.$$

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(A3) $X_0 \in \mathbb{L}^\infty$, $\Delta_\infty := \sum_{i \in \mathbb{Z}^d} \delta_{i,\infty} < \infty$ and

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Theorem (E.M., Volný, Wu (2013))

Let $\mathcal{A} \subset \mathcal{B}\left([0,1]^d\right)$ *and assume that* $(A1)$, $(A2)$ *or* $(A3)$ *holds. Let* $(X_i)_{i\in\mathbb{Z}^d}$ *be a centered random field of the form*

$$X_i = g(\varepsilon_{i-s}\,;\, s \in \mathbb{Z}^d)$$

Then $\{n^{-d/2}S_n(A)\,;\, A \in \mathcal{A}\}$ *converge in distribution in* $\mathcal{C}(\mathcal{A})$ *to* σW *where* W *is an* $\mathcal{A}$*-indexed standard Brownian motion and* $\sigma^2 = \sum_{k\in\mathbb{Z}^d} \mathbb{E}\,(X_0 X_k)$.

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Biermé and Durieu (2015) extended this result by considering

$$S_n(A) = \sum_{i\in\{1,\dots,n\}^d} \mu(nA \cap R_i)X_i$$

where μ is a σ-finite measure on $\mathbb{R}^d$ absolutely continuous with respect to the Lebesgue measure and such that $\mu(nA) = n^\beta \mu(A)$ for some $\beta > 0$. In this case, the limit process is a centered Gaussian process $(W(A))_{A\in\mathcal{A}}$ such that $\mathrm{Cov}(W(A), W(B)) = \mu(A \cap B)$.

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Question : Is it possible to obtain a WIP for $\mathcal{A} = \mathcal{Q}_d$ when $p = 2$?

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- For any $s = (s_1, \ldots, s_d)$ and any $t = (t_1, \ldots, t_d)$ in $\mathbb{Z}^d$, we write $s \preceq t$ (resp. $s \prec t$, $s \succeq t$ and $s \succ t$) if and only if $s_k \leqslant t_k$ (resp. $s_k < t_k$, $s_k \geqslant t_k$ and $s_k > t_k$) for any k in $\langle d \rangle$.

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Definition (Cairoli (1969))

Let $(\Omega,\mathcal{F},\mu)$ be a probability space. A family $(\mathcal{G}_i)_{i\in\mathbb{Z}^d}$ of σ-algebras is a commuting filtration if $\mathcal{G}_i\subset\mathcal{G}_j\subset\mathcal{F}$ for any i and j in $\mathbb{Z}^d$ such that $i\preceq j$ and

$$\mathbb{E}\left(\mathbb{E}\left(Z\mid\mathcal{G}_s\right)\mid\mathcal{G}_t\right)=\mathbb{E}\left(Z\mid\mathcal{G}_{s\wedge t}\right)\quad a.s.$$

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Example: if $(\varepsilon_j)_{j \in \mathbb{Z}^d}$ is an independent random field and $\mathcal{F}_i := \sigma(\varepsilon_j, j \prec i)$, then $(\mathcal{F}_i)_{i \in \mathbb{Z}^d}$ is a commuting filtration.

Orthomartingale-difference random fields

Definition (Cairoli (1969))

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. A random field $(X_k)_{k\in\mathbb{Z}^d}$ is an orthomartingale-difference (OMD) random field if there exists a commuting filtration $(\mathcal{G}_i)_{i\in\mathbb{Z}^d}$ such that X_k belongs to $\mathbb{L}^1(\Omega, \mathcal{G}_k, \mu) \ominus \mathbb{L}^1(\Omega, \mathcal{G}_l, \mu)$ a.s. for any $l \not\succeq k$ and k in $\mathbb{Z}^d$.

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Let $p \geqslant 1$ *and let* $\mathcal{M} \subset \mathcal{F}$ *be a* σ*-algebra such that* $(T^{-i}\mathcal{M})_{i\in\mathbb{Z}^d}$ *is a commuting filtration. If* f *belongs to* $\mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, \cap_{k\in\mathbb{N}^d} T^k\mathcal{M}, \mu)$ *and*

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- *$(U^i m)_{i\in\mathbb{Z}^d}$ and $\left(U^i_{J^c} m_J\right)_{i\in\mathbb{Z}^{d-|J|}}$ are OMD random fields.*

Main result

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$$f = m + (I - U_1)m_1 + (I - U_2)m_2 + (I - U_1)(I - U_2)g,$$

where $(U^i m)_{i\in\mathbb{Z}^2}$ is an OMD random field and $(U_2^k m_1)_{k\in\mathbb{Z}}$ and $(U_1^k m_2)_{k\in\mathbb{Z}}$ are MD sequences.

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If $d = 3$ then (OMCD) becomes

$$\begin{aligned} f = m &+ (I - U_1)m_1 + (I - U_2)m_2 + (I - U_3)m_3 \\ &+ (I - U_1)(I - U_2)m_{\{1,2\}} + (I - U_1)(I - U_3)m_{\{1,3\}} + (I - U_2)(I - U_3)m_{\{2,3\}} \\ &+ (I - U_1)(I - U_2)(I - U_3)g \end{aligned}$$

where $(U^i m)_{i\in\mathbb{Z}^3}$, $(U^i_{\{2,3\}} m_1)_{i\in\mathbb{Z}^2}$, $(U^i_{\{1,3\}} m_2)_{i\in\mathbb{Z}^2}$ and $(U^i_{\{1,2\}} m_3)_{i\in\mathbb{Z}^2}$ are OMD random fields and $(U_1^k m_{\{2,3\}})_{k\in\mathbb{Z}}$, $(U_2^k m_{\{1,3\}})_{k\in\mathbb{Z}}$ and $(U_3^k m_{\{1,2\}})_{k\in\mathbb{Z}}$ are MD sequences.

Proposition

Let $(\varepsilon_i)_{i\in\mathbb{Z}^d}$ be an iid real random field defined on $(\Omega,\mathcal{F},\mu)$ such that ε_0 has zero mean and belongs to $\mathbb{L}^p(\Omega,\mathcal{F},\mu)$ for some $p\geqslant 2$. Consider the linear random field $(X_k)_{k\in\mathbb{Z}^d}$ defined for any k in $\mathbb{Z}^d$ by $X_k=\sum_{j\in\mathbb{N}^d} a_j\varepsilon_{k-j}$ where $(a_j)_{j\in\mathbb{N}^d}$ is a family of real numbers satisfying $\sum_{j\in\mathbb{N}^d} a_j^2<\infty$. Then the condition (OMPC) *holds if and only if*

$$\sum_{k\in\mathbb{N}^d}\sqrt{\sum_{j\succcurlyeq k} a_j^2}<\infty.$$

Proof of the main result

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Proposition

Let d be a positive integer. Let $p \geqslant 1$ and $\mathcal{M} \subset \mathcal{F}$ such that $(T^{-i}\mathcal{M})_{i\in\mathbb{Z}^{d+1}}$ is a commuting filtration. Assume that f belongs to $\mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, \cap_{k\in\mathbb{N}^{d+1}} T^k\mathcal{M}, \mu)$ and

$$\sum_{k\in\mathbb{N}^{d+1}} \left\| \mathbb{E}\left(f \mid T^k\mathcal{M}\right)\right\|_p < \infty.$$

Then there exist $M \in \mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T_{d+1}\mathcal{M}, \mu)$ and $G \in \mathbb{L}^p(\Omega, T_{d+1}\mathcal{M}, \mu)$ such that

$$f = M + G - G \circ T_{d+1}$$

and

$$\sum_{k\in\mathbb{N}^{d}} \left\| \mathbb{E}\left(M \mid T^{(k,0)}\mathcal{M}\right)\right\|_p + \left\| \mathbb{E}\left(G \mid T^{(k,0)}\mathcal{M}\right)\right\|_p < \infty.$$

Proof of the proposition. From Gordin's result, we know that $f = M + (I - U_{d+1})G$ with

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Similarly, we have also $\sum_{k\in\mathbb{N}^d} \left\|\mathbb{E}\left(G \mid T^{(k,0)}\mathcal{M}\right)\right\|_p < \infty.$ □

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$$G = m'' + \sum_{\emptyset \subsetneq J \subsetneq \langle d \rangle} \prod_{s \in J} (I - U_s) m''_J + \prod_{s=1}^{d} (I - U_s) g''$$

with

- m' and m'' in $\mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T_i\mathcal{M}, \mu)$ for each i in $\langle d \rangle$.
- m'_J and m''_J in $\mathbb{L}^p(\Omega, \prod_{s \in J} T_s\mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T_i \prod_{s \in J} T_s\mathcal{M}, \mu)$ for each i in $\langle d \rangle \setminus J$.
- g' and g'' belong to $\mathbb{L}^p(\Omega, \prod_{s=1}^{d} T_s\mathcal{M}, \mu)$;

Since $\mathbb{E}[M \mid T_{d+1}\mathcal{M}] = 0$ and

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Moreover,

$$\begin{aligned}
\mathbb{E}\left[\prod_{s \in J} (I - U_s) m'_J \mid T_{d+1}\mathcal{M}\right] &= \mathbb{E}\left[\sum_{A \subset J} (-1)^{|A|} \prod_{s \in A} U_s m'_J \mid T_{d+1}\mathcal{M}\right] \\
&= \sum_{A \subset J} (-1)^{|A|} \mathbb{E}\left[\prod_{s \in A} U_s m'_J \mid T_{d+1}\mathcal{M}\right] \\
&= \sum_{A \subset J} (-1)^{|A|} \prod_{s \in A} U_s \mathbb{E}\left[m'_J \mid \prod_{s \in A} T_s T_{d+1}\mathcal{M}\right]
\end{aligned}$$

So, we have

$$\mathbb{E}\left[\prod_{s\in J}(I-U_s)m'_J \mid T_{d+1}\mathcal{M}\right]=\sum_{A\subset J}(-1)^{|A|}\prod_{s\in A}U_s\mathbb{E}\left[m'_J \mid \prod_{s\in A}T_s T_{d+1}\mathcal{M}\right].$$

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Similarly, since g' is $\prod_{s=1}^{d}T_s\mathcal{M}$-measurable, we have also

$$\mathbb{E}\left[\prod_{s=1}^{d}(I-U_s)g' \mid T_{d+1}\mathcal{M}\right]=\prod_{s=1}^{d}(I-U_s)\mathbb{E}\left[g' \mid T_{d+1}\mathcal{M}\right].$$

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So, denoting $m := m' - \mathbb{E}[m' \mid T_{d+1}\mathcal{M}]$ and keeping in mind that

$$M = m' + \sum_{\emptyset \subsetneq J \subsetneq \langle d \rangle} \prod_{s \in J}(I - U_s)m'_J + \prod_{s=1}^{d}(I - U_s)g',$$

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where m is $\mathcal{M}$-measurable and $\mathbb{E}[m \mid T_s\mathcal{M}] = 0$ for each s in $\langle d+1\rangle$.

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$$\inf\left\{C>0\ ;\ \Big\|\sum_{0\preccurlyeq k\preccurlyeq n} Z_k\Big\|_p \leqslant C\left(\sum_{0\preccurlyeq k\preccurlyeq n}\|Z_k\|_p^2\right)^{1/2}\ \forall n\in\mathbb{N}^d\right\} \geqslant \kappa p^{d/2}.$$

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where $|n|=\prod_{i=1}^d n_i$ and C_d is a positive constant depending only on d.

Weak Invariance Principle

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Let $(\varepsilon_j)_{j\in\mathbb{Z}^d}$ be an iid real random field defined on a probability space $(\Omega, \mathcal{F}, \mu)$. Consider the commuting filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}^d}$ where $\mathcal{F}_i$ is the σ-algebra generated by ε_j for $j \preceq i$. Let $(X_i)_{i\in\mathbb{Z}^d}$ be a stationary real random field such that X_i is $\mathcal{F}_i$-measurable for each i in $\mathbb{Z}^d$ and

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Then, $\{n^{-d/2} S_n(t)\,;\, t \in [0,1]^d\}$ converges in distribution in $\mathcal{C}([0,1]^d)$ to $\sqrt{\mathbb{E}(X_0^2)}\,W$.

- Under the condition

$$\sum_{k\in\mathbb{N}^d}\frac{\left\|\mathbb{E}\left(X_k|\mathcal{F}_0\right)\right\|_p}{|k|^{1/2}}<\infty$$

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Volný and Wang (2014) obtained the WIP under the weaker condition

$$\sum_{j\in\mathbb{Z}^d}\|P_j(f)\|_2<\infty.$$

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Related results

- Cuny, Dedecker and Volný (2016) obtained recently a WIP for fields of commuting transformations via martingale approximation under a condition in the spirit of Hannan.

- Volný (2016) obtained recently the orthomartingale-coboundary decomposition of a regular and square integrable function f under the condition

$$\sum_{j\in\mathbb{Z}^d} j_1^2 j_2^2 \dots j_d^2 \, \|P_j(f)\|_2^2 < \infty.$$

Thank you !