ON THE DEGREE OF CAUSTICS BY REFLECTION

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Abstract. Given $\mathcal{S} \in \mathbb{P}^2$ and an algebraic curve $\mathcal{C}$ of $\mathbb{P}^2$ (with any type of singularities), we consider the lines $\mathcal{R}_m$ got by reflection of lines $(\mathcal{S}m)$ $(m \in \mathcal{C})$ on $\mathcal{C}$. The caustic by reflection $\Sigma_{\mathcal{S}}(\mathcal{C})$ is defined as the Zariski closure of the envelope of the reflected lines $\mathcal{R}_m$. We identify this caustic with the Zariski closure of $\Phi(\mathcal{C})$, where Φ is some rational map. We use this approach to give general and explicit formulas for the degree (with multiplicity) of every caustic by reflection. Our formulas are expressed in terms of intersection numbers of the initial curve $\mathcal{C}$ (or of its branches). Our method is based on a fundamental lemma for rational map thanks to the notion of Φ-polar and on computation of intersection numbers. In particular, we use precise estimates related to the intersection numbers of $\mathcal{C}$ with its polar at any point and to the intersection numbers of $\mathcal{C}$ with its hessian determinant. These computations are linked with generalized Plücker formulas for the class and for the number of inflection points of $\mathcal{C}$.

Introduction

Von Tschirnhausen was the first to consider the caustic by reflection as the envelope of reflected rays from a point $\mathcal{S}$ on a mirror curve $\mathcal{C}$ (see [18]). Many mathematicians have studied individually different caustics. In [14, 8], when $\mathcal{S}$ is at finite distance, Quetelet and Dandelin showed that the caustic is the evolute of the $\mathcal{S}$-centered homothety (with ratio 2) of the pedal from $\mathcal{S}$ of $\mathcal{C}$, i.e. the evolute of the orthotomic of $\mathcal{C}$ with respect to $\mathcal{S}$. This decomposition has also been used in a modern approach by [2, 3, 4] to study the source genericity (in the real case).

In [6], Chasles got a formula (in generic but restrictive cases) for the class of the caustic in terms of the degree and of the class of $\mathcal{C}$. In [15, p. 137, 154], Salmon and Cayley establish formulas, at a more general level, for the class and the degree of the evolute and pedal curves. The formulas of Salmon and Cayley apply only to curves having no singularities other than ordinary nodes and cups [15, p. 10].

Apparently thanks to these last results, in [1], Brocard and Lemoyne gave, without any proof, formulas for the degree and the class of caustics by reflection, for $\mathcal{S}$ is at finite distance and for algebraic curve $\mathcal{C}$ admitting no other singularities than ordinary nodes and cusps. The formulas of Brocard and Lemoyne are not satisfactory. First no proof is given. Second, the direct composition of the formulas got by Salmon and Cayley for evolute and pedal curves is not correct since the pedal curve of a curve having no singularities other than ordinary nodes and cups is not necessarily a curve satisfying the same properties. For example, the pedal curve of the rational cubic $V(y^2z - x^3)$ from $[4 : 0 : 1]$ is a quartic curve with a triple ordinary point.

More recently, a study of the evolute has been done by Fantechi in [9], including necessary and sufficient condition for the birationality of the evolute map and a description of the number and type of the singularities of the general evolute. This work has been extended in higher dimension

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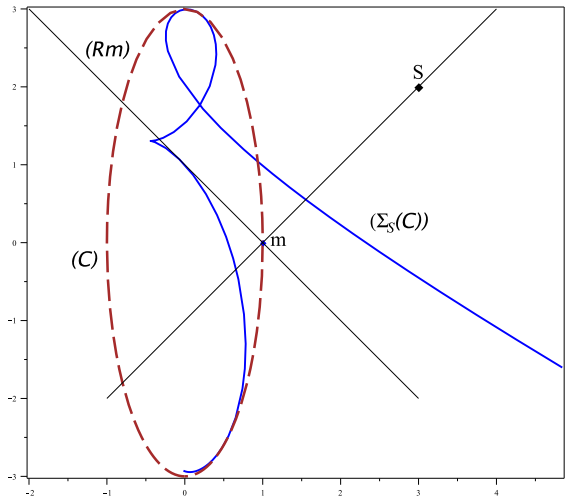
by Trifogli [17], Catanese and Trifogli [5] giving, in particular, formulas for degrees of focal loci of smooth algebraic curves.

Our aim is here to give formulas for the degree (with multiplicity) of the caustic by reflection for any light point $\mathcal{S}$ (including the case when $\mathcal{S}$ is on the infinite line) and any algebraic curve $\mathcal{C}$ (without any restriction neither on the singularity points nor on the flex points). We express the degree (with multiplicity) of $\Sigma_{\mathcal{S}}(\mathcal{C})$ in terms of intersection numbers of the initial curve $\mathcal{C}$. Our proofs use the notion of pro-branches (also called partial branches) considered by Halphen [12] and more recently by Wall [19, 20].

Given an algebraic curve $\mathcal{C}$ in the euclidean affine plane $\mathcal{E}_2$ ($\mathcal{C}$ is called mirror curve) and given a light position $\mathcal{S}$ (in $\mathcal{E}_2$ or at infinity), the caustic by reflection $\Sigma_{\mathcal{S}}(\mathcal{C})$ is the Zariski closure of the envelope of the reflected lines $\{R_m; m \in \mathcal{C}\}$ where, for every $m \in \mathcal{C}$, the reflected line R_m at m is the line containing m and such that the tangent line $\mathcal{T}_m\mathcal{C}$ to $\mathcal{C}$ at m is the bissectrix of the incident line (Sm) and of R_m.



The notion of caustic by reflection $\Sigma_{\mathcal{S}}(\mathcal{C})$ is easily extendible to the complex projective case for an irreducible algebraic curve $\mathcal{C} = V(F)$ of $\mathbb{P}^2 := \mathbb{P}^2(\mathbb{C})$ with $F \in \mathbb{C}[x, y, z]$ a homogeneous polynomial of degree d and a light position $\mathcal{S} = [x_0 : y_0 : z_0] \in \mathbb{P}^2$. It will be also useful to consider $S := (x_0, y_0, z_0) \in \mathbb{C}^3 \setminus \{0\}$.

Plan of the paper. The paper is organized as follows.

In section 1, we present our main results and illustrate them with an example.

In section 2, we introduce the notion of reflected lines and use it to define the caustic by reflection.

In section 3, we study the properties of our rational map $\Phi_{F,S}$ (link with the caustic $\Sigma_{\mathcal{S}}(\mathcal{C})$, base points,etc.).

In section 4, for any rational map $\varphi : \mathbb{P}^p \to \mathbb{P}^q$ and any irreducible algebraic curve $\mathcal{C}$ of $\mathbb{P}^p$, we introduce the notion of φ-polar $\mathcal{P}_{\varphi,a}$ and give a general fundamental lemma expressing the degree of $\overline{\varphi(\mathcal{C})}$ in terms of intersection numbers of $\mathcal{C}$ with $\mathcal{P}_{\varphi,a}$ at the Base points of φ on $\mathcal{C}$. Thanks to this result, the computation of the degree of the caustic by reflection $\Sigma_{\mathcal{S}}(\mathcal{C})$ is related to the intersection numbers of the curve $\mathcal{C}$ with its polar curves $\delta_{\mathcal{P}}F$ with respect to $\mathcal{P}$ and with $V(H_F)$ (H_F being the Hessian determinant of F).

In section 5 completed with the appendix, we recall some facts on intersection numbers and use it to establish the precise computations we need. This section contains also generalized Plücker formulas for the class and for the number of inflection points of $\mathcal{C}$.

In section 6, we prove our general Theorems 2 and 3 on the degree of caustics by reflection. In section 7, we prove our Corollary 5.

1. Main results

Throughout the paper, we will write ℓ_∞ the infinite line of $\mathbb{P}^2$ and $\Pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$ the canonical projection. As usual, we denote by F_x, F_y, F_z the partial derivatives of F and by $F_{xx}, F_{xy}, F_{xz}, F_{yy}, F_{yz}, F_{zz}$ its second order partial derivatives.

We recall that when $d = 1$, $\Sigma_{\mathcal{S}}(\mathcal{C})$ is well defined as soon as $\mathcal{C}$ contains neither $\mathcal{S}$, nor $\mathcal{I}$, nor $\mathcal{J}$ and we have $\Sigma_{\mathcal{S}}(\mathcal{C}) = \{\mathcal{S}_1\}$ (where $\mathcal{S}_1$ corresponds to the euclidean symetric point[1] of $\mathcal{S}$ with respect to line $\mathcal{C}$).

The aim of this paper is to give an effective way to compute the degree of $\Sigma_{\mathcal{S}}(\mathcal{C})$ when $d \geq 2$. To this end, we define a rational map $\Phi_{F,S}$ (with $S = (x_0, y_0, z_0)$). This maps is given by $\tilde{\Phi}_{F,S} : \mathbb{C}^3 \to \mathbb{C}^3$ defined by :

$$\tilde{\Phi}_{F,S} := -\frac{2H_F N_S}{(d-1)^2} \cdot \mathrm{Id} + \Delta_S F \cdot \begin{pmatrix} F_y^2 x_0 - F_x^2 x_0 - 2F_x F_y y_0 - 2F_x F_z z_0 \\ F_x^2 y_0 - F_y^2 y_0 - 2F_x F_y x_0 - 2F_y F_z z_0 \\ z_0(F_x^2 + F_y^2) \end{pmatrix}, \tag{1}$$

with H_F the hessian determinant of F, i.e.

$$H_F := F_{xx}F_{yy}F_{zz} - F_{xx}F_{yz}^2 + 2F_{xy}F_{yz}F_{xz} - F_{xy}^2 F_{zz} - F_{yy}F_{xz}^2,$$

with

$$N_S(x, y, z) := (xz_0 - x_0 z)^2 + (yz_0 - y_0 z)^2,$$

and with

$$\forall P = (x_P, y_P, z_P) \in \mathbb{C}^3 \setminus \{0\}, \quad \Delta_P F := DF(\cdot)(P) = x_P F_x + y_P F_y + z_P F_z.$$

Let us recall that $V(\Delta_P F)$ is the polar $\delta_{\Pi(P)}(\mathcal{C})$ of $\mathcal{C}$ with respect to $\Pi(P)$.

Theorem 1. *If $d \geq 2$,*

$$\Sigma_{\mathcal{S}}(\mathcal{C}) = \overline{\Phi_{F,S}(\mathcal{C})}, \tag{2}$$

where the closure is in the sense of Zariski.

Moreover, $\Phi_{F,S}$ maps generic $m \in \mathcal{C}$ to the corresponding point of $\Sigma_{\mathcal{S}}(\mathcal{C})$.

Let $\mathcal{I} := [1 : i : 0]$ and $\mathcal{J} := [1 : -i : 0]$ be the two cyclic points of $\mathbb{P}^2$. We also define $I := (1, i, 0)$ and $J := (1, -i, 0)$. Points $\mathcal{I}$ and $\mathcal{J}$ will play a particular role in our study (see theorems below). They will be crucial in the construction of the reflected lines $\mathcal{R}_m$. Moreover, we will see that

$$\Sigma_{\mathcal{I}}(\mathcal{C}) = \{\mathcal{J}\} \text{ and } \Sigma_{\mathcal{J}}(\mathcal{C}) = \{\mathcal{I}\}. \tag{3}$$

We will use Theorem 1 and a general fundamental lemma to express the degree of $\Sigma_{\mathcal{S}}(\mathcal{C})$ in terms of some intersection numbers (computed in Proposition 30). Before giving our formulas, let us introduce some notations.

We write $Sing(\mathcal{C})$ the set of singular points of $\mathcal{C}$ (i.e. the set of points $m = [x : y : z] \in \mathcal{C}$ such that $DF(x, y, z) = 0$) and $Reg(\mathcal{C}) := \mathcal{C} \setminus Sing(\mathcal{C})$. We denote by $\mathcal{T}_{m_1}\mathcal{C}$ the tangent line to

[1] If $F(x, y, z) = ax + by + cz$, we have $\mathcal{S}_1 = \Pi\left((a^2 + b^2)S - 2(x_0 a + y_0 b + cz_0)\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}\right)$.

$\mathcal{C}$ at m_1 when m_1 is non-singular. We also write $i_{m_1}(\cdot,\cdot)$ the intersection numbers. For every $m_1 \in \mathcal{C}$, we write $Branch_{m_1}(\mathcal{C})$ the set of branches of $\mathcal{C}$ at m_1. Now, for every $m_1 \in \mathcal{C}$ and every $\mathcal{B} \in Branch_{m_1}(\mathcal{C})$, $\mathcal{T}_{m_1}\mathcal{B}$ denotes the tangent line to $\mathcal{B}$ at m_1 and $e_{(\mathcal{B})}$ the multiplicity of $\mathcal{B}$. If $m_1 \in Sing(\mathcal{C})$, such a line $\mathcal{T}_{m_1}\mathcal{B}$ will be called a singular tangent line to $\mathcal{C}$ at m_1. The quantity $\mathbf{1}_{\mathcal{P}}$ is equal to 1 if property $\mathcal{P}$ is true and 0 otherwise.

We will denote by $\mathrm{mdeg}(\Sigma_{\mathcal{S}}(\mathcal{C}))$ the degree with multiplicity of $\Sigma_{\mathcal{S}}(\mathcal{C})$. We have

$$\mathrm{mdeg}(\Sigma_{\mathcal{S}}(\mathcal{C})) = \delta_1(\mathcal{S},\mathcal{C}) \times \deg(\Sigma_{\mathcal{S}}(\mathcal{C}))$$

(with convention $\infty \times 0 = 0$), where $\deg(\Sigma_{\mathcal{S}}(\mathcal{C}))$ is the degree of the algebraic curve $\Sigma_{\mathcal{S}}(\mathcal{C})$ and where $\delta_1(\mathcal{S},\mathcal{C})$ is the degree of $\Phi_{F,S}$. We recall that $\delta_1(\mathcal{S},\mathcal{C})$ corresponds to the number of preimages on $\mathcal{C}$ of a generic point of $\Sigma_{\mathcal{S}}(\mathcal{C})$ by $\Phi_{F,S}$. The fact that $\mathrm{mdeg}(\Sigma_{\mathcal{S}}(\mathcal{C})) = 0$ means that $\Sigma_{\mathcal{S}}(\mathcal{C})$ contains a single point.

We start with the generic and simple case when $\mathcal{S}$, $\mathcal{I}$ and $\mathcal{J}$ do not belong to a singular tangent line to $\mathcal{C}$.

Theorem 2. *Assume that $d \geq 2$ and that $\mathcal{S}$ is equal neither to $\mathcal{I}$ nor to $\mathcal{J}$. Assume moreover that $\mathcal{S}$ is not contained in a singular tangent line to $\mathcal{C}$ and that ℓ_∞ is not a singular tangent line to $\mathcal{C}$. Then*

$$mdeg(\Sigma_{\mathcal{S}}(\mathcal{C})) = 3d^\vee - v_1 - v_2 - v_2' - v_3\mathbf{1}_{\mathcal{S}\in\mathcal{C}} - v_4\mathbf{1}_{\mathcal{S}\notin\ell_\infty},$$

where $d^\vee$ is the class of $\mathcal{C}$ (i.e. the degree of its dual curve) and with

$$v_1 := \sum_{m_1\in Sing(\mathcal{C})\setminus((\mathcal{I}\mathcal{S})\cup(\mathcal{J}\mathcal{S}))} \sum_{\mathcal{B}\in Branch_{m_1}(\mathcal{C})} \min(i_{m_1}(\mathcal{B},\mathcal{T}_{m_1}\mathcal{B}) - 2e_{(\mathcal{B})}, 0),$$

$$v_2 := \sum_{m_1\in Reg(\mathcal{C})\setminus\{\mathcal{S}\}:\mathcal{S}\in\mathcal{T}_{m_1}\mathcal{C}, i_{m_1}(\mathcal{C},\mathcal{T}_{m_1}\mathcal{C})\neq 2} [i_{m_1}(\mathcal{C},\mathcal{T}_{m_1}\mathcal{C})(1 + \mathbf{1}_{\mathcal{I}\in\mathcal{T}_{m_1}\mathcal{C}} + \mathbf{1}_{\mathcal{J}\in\mathcal{T}_{m_1}\mathcal{C}}) - 2 - \mathbf{1}_{\mathcal{I},\mathcal{J}\in\mathcal{T}_{m_1}\mathcal{C}}];$$

$$v_2' := \sum_{m_1\in Reg(\mathcal{C})\setminus\{\mathcal{S}\}:\mathcal{S}\in\mathcal{T}_{m_1}\mathcal{C}, i_{m_1}(\mathcal{C},\mathcal{T}_{m_1}\mathcal{C})=2} 3 \times \mathbf{1}_{\{\mathcal{I},\mathcal{J}\}\cap\mathcal{T}_{m_1}\mathcal{C}\neq\emptyset};$$

$$v_3 := \begin{cases} i_{\mathcal{S}}(\mathcal{C},\mathcal{T}_{\mathcal{S}}\mathcal{C}) + (i_{\mathcal{S}}(\mathcal{C},\mathcal{T}_{\mathcal{S}}\mathcal{C}) - 1)(\mathbf{1}_{\mathcal{I}\in\mathcal{T}_{\mathcal{S}}\mathcal{C}} + \mathbf{1}_{\mathcal{J}\in\mathcal{T}_{\mathcal{S}}\mathcal{C}}) & \text{if } i_{\mathcal{S}}(\mathcal{C},\mathcal{T}_{\mathcal{S}}\mathcal{C}) \neq 2 \\ 2 + \mathbf{1}_{\mathcal{I}\in\mathcal{T}_{\mathcal{S}}\mathcal{C}} + \mathbf{1}_{\mathcal{J}\in\mathcal{T}_{\mathcal{S}}\mathcal{C}} + 2 \times \mathbf{1}_{\mathcal{I},\mathcal{J}\in\mathcal{T}_{\mathcal{S}}\mathcal{C}} & \text{if } i_{\mathcal{S}}(\mathcal{C},\mathcal{T}_{\mathcal{S}}\mathcal{C}) = 2; \end{cases}$$

$$v_4 := \sum_{m_1\in Reg(\mathcal{C}):\mathcal{T}_{m_1}\mathcal{C}=\ell_\infty} [i_{m_1}(\mathcal{C},\mathcal{T}_{m_1}\mathcal{C}) - 2 + \mathbf{1}_{m_1\in\{\mathcal{I},\mathcal{J}\}}].$$

Now let us give the more general but also more technical result. In this result we do not distinguish singular and non-singular points of $\mathcal{C}$. We recall that, at a non-singular point m_1, $\mathcal{C}$ admits a single branch and that the multiplicity of this branch is equal to 1. Let us consider the set $\mathcal{E}$ of all possible couples $(m_1,\mathcal{B})$ with $m_1 \in \mathcal{C}$ and $\mathcal{B}$ a branch of $\mathcal{C}$ at m_1. Given $(m_1,\mathcal{B}) \in \mathcal{E}$, if $i_{m_1}(\mathcal{B},\mathcal{T}_{m_1}\mathcal{B}) = 2e_{(\mathcal{B})}$, we define

$$\gamma_1(m_1,\mathcal{B}) := \min(3e_{(\mathcal{B})}, i_{m_1}(\mathcal{B},\mathcal{C}')),$$

where $\mathcal{C}'$ is any curve non-singular at m_1 such that $i_{m_1}(\mathcal{B},\mathcal{C}') > 2e_{(\mathcal{B})}$ (for example, one can take for $\mathcal{C}'$ the osculating "circle" $\mathcal{O}_{m_1}(\mathcal{B})$ of any pro-branch of $\mathcal{B}$, see section 5 for the definition of pro-branches) and

$$\gamma_2(m_1,\mathcal{B}) := \min(\gamma_1(m_1,\mathcal{B}) - 2e_{(\mathcal{B})}, 2e_{(\mathcal{B})}) \text{ if } \gamma_1(m_1,\mathcal{B}) \neq 3e_{(\mathcal{B})}$$

and

$$\gamma_2(m_1,\mathcal{B}) := \min(i_{m_1}(\mathcal{B},\mathcal{C}'') - 2e_{(\mathcal{B})}, 2e_{(\mathcal{B})}) \text{ if } \gamma_1(m_1,\mathcal{B}) = 3e_{(\mathcal{B})},$$

where $\mathcal{C}''$ is any algebraic curve non-singular at m_1 such that $i_{m_1}(\mathcal{B},\mathcal{C}'') > 3e_{(\mathcal{B})}$.

Theorem 3. *Assume that $d \geq 2$ and that $\mathcal{S}$ is equal neither to $\mathcal{I}$ nor to $\mathcal{J}$. Then*

$$mdeg(\Sigma_{\mathcal{S}}(\mathcal{C})) = 3d^{\vee} - v_1 - v_2 - v_2' - (v_3 + v_3')\mathbf{1}_{\mathcal{S}\in\mathcal{C}} - v_4\mathbf{1}_{\mathcal{S}\notin\ell_\infty},$$

where $d^{\vee}$ is the class of $\mathcal{C}$ (i.e. the degree of its dual curve) and with

$$v_1 := \sum_{(m_1,\mathcal{B})\in\mathcal{E}:m_1\in Sing(\mathcal{C})\setminus[(\mathcal{IS})\cup(\mathcal{JS})],S\notin\mathcal{T}_{m_1}\mathcal{B},\mathcal{T}_{m_1}\mathcal{B}\neq\ell_\infty} \min(i_{m_1}(\mathcal{B},\mathcal{T}_{m_1}\mathcal{B}) - 2e_{(\mathcal{B})}, 0),$$

$$v_2 := \sum_{(m_1,\mathcal{B})\in\mathcal{E}:m_1\neq\mathcal{S},\mathcal{S}\in\mathcal{T}_{m_1}\mathcal{B},i_{m_1}(\mathcal{B},\mathcal{T}_{m_1}\mathcal{B})\neq 2e_{(\mathcal{B})}} [i_{m_1}(\mathcal{B},\mathcal{T}_{m_1}\mathcal{B})(1 + \mathbf{1}_{\mathcal{I}\in\mathcal{T}_{m_1}\mathcal{B}} + \mathbf{1}_{\mathcal{J}\in\mathcal{T}_{m_1}\mathcal{B}}) - (2 + \mathbf{1}_{\mathcal{I},\mathcal{J}\in\mathcal{T}_{m_1}\mathcal{B}})e_{(\mathcal{B})}];$$

$$v_2' := \sum_{(m_1,\mathcal{B})\in\mathcal{E}:m_1\neq\mathcal{S},\mathcal{S}\in\mathcal{T}_{m_1}\mathcal{B},i_{m_1}(\mathcal{B},\mathcal{T}_{m_1}\mathcal{B})=2e_{(\mathcal{B})}} [\gamma_1(m_1,\mathcal{B})\mathbf{1}_{\#(\{\mathcal{I},\mathcal{J}\}\cap\mathcal{T}_{m_1}\mathcal{B})=1} + 3e_{(\mathcal{B})}\mathbf{1}_{\mathcal{T}_{m_1}\mathcal{B}=\ell_\infty}];$$

$$v_3 := \sum_{(\mathcal{S},\mathcal{B})\in\mathcal{E}:i_{\mathcal{S}}(\mathcal{B},\mathcal{T}_{\mathcal{S}}\mathcal{B})\neq 2e_{(\mathcal{B})}} \left[i_{\mathcal{S}}(\mathcal{B},\mathcal{T}_{\mathcal{S}}\mathcal{B}) + (i_{\mathcal{S}}(\mathcal{B},\mathcal{T}_{\mathcal{S}}\mathcal{B}) - e_{(\mathcal{B})})(\mathbf{1}_{\mathcal{I}\in\mathcal{T}_{\mathcal{S}}\mathcal{B}} + \mathbf{1}_{\mathcal{J}\in\mathcal{T}_{\mathcal{S}}\mathcal{B}})\right];$$

$$v_3' := \sum_{(\mathcal{S},\mathcal{B})\in\mathcal{E}:i_{\mathcal{S}}(\mathcal{B},\mathcal{T}_{\mathcal{S}}\mathcal{B})=2e_{(\mathcal{B})}} \left[(2 + \mathbf{1}_{\mathcal{I}\in\mathcal{T}_{\mathcal{S}}\mathcal{B}} + \mathbf{1}_{\mathcal{J}\in\mathcal{T}_{\mathcal{S}}\mathcal{B}})e_{(\mathcal{B})} + \gamma_2(m_1,\mathcal{B})\mathbf{1}_{\mathcal{I},\mathcal{J}\in\mathcal{T}_{\mathcal{S}}\mathcal{B}}\right];$$

$$v_4 := \sum_{(m_1,\mathcal{B})\in\mathcal{E}:\mathcal{T}_{m_1}\mathcal{B}=\ell_\infty} [i_{m_1}(\mathcal{B},\mathcal{T}_{m_1}\mathcal{B}) + (\mathbf{1}_{m_1\in\{\mathcal{I},\mathcal{J}\}} - 2)e_{(\mathcal{B})}].$$

Remark 4. *Denote by $Flex(\mathcal{C})$ the set of inflection points of $\mathcal{C}$, i.e. the set of non-singular points of $\mathcal{C}$ such that $i_{m_1}(\mathcal{C},\mathcal{T}_{m_1}(\mathcal{C})) > 2$. Recall that, since $\mathcal{C}$ is irreducible and if $d \geq 2$, (by the Bezout theorem and Corollary 26 below) we have*

$$3d(d-2) = \sum_{m_1\in\mathcal{C}} i_{m_1}(\mathcal{C}, V(H_F))$$

$$= 3[d(d-1) - d^{\vee}] + \sum_{m_1\in Sing(\mathcal{C})} \sum_{\mathcal{B}\in Branch_{m_1}(\mathcal{C})} [i_{m_1}(\mathcal{B},\mathcal{T}_{m_1}\mathcal{B}) - 2e_{(\mathcal{B})}] + \sum_{m_1\in Flex(\mathcal{C})} (i_{m_1}(\mathcal{C},\mathcal{T}_{m_1}\mathcal{C}) - 2)$$

and so

$$3d^{\vee} - \sum_{(m_1,\mathcal{B})\in\mathcal{E},m_1\in Sing(\mathcal{C})} (i_{m_1}(\mathcal{B},\mathcal{T}_{m_1}\mathcal{B}) - 2e_{(\mathcal{B})}) = 3d + \sum_{m_1\in Flex(\mathcal{C})} (i_{m_1}(\mathcal{C},\mathcal{T}_{m_1}\mathcal{C}) - 2).$$

Hence, under assumptions of Theorem 2, if we suppose moreover that $Sing(\mathcal{C})\cap((\mathcal{IS})\cup(\mathcal{JS})) = \emptyset$ and that, for every $m_1 \in Sing(\mathcal{C})$ and every branch $\mathcal{B}$ of $\mathcal{C}$ at m_1, we have $i_{m_1}(\mathcal{B},\mathcal{T}_{m_1}\mathcal{B}) \leq 2e_{(\mathcal{B})}$ (this is true for instance, if all the singular points of $\mathcal{C}$ are ordinary cusps and nodes), we get that

$$3d^{\vee} - v_1 = 3d + \sum_{m_1\in Flex(\mathcal{C})} (i_{m_1}(\mathcal{C},\mathcal{T}_{m_1}\mathcal{C}) - 2).$$

Corollary 5. *Assume that $d \geq 2$ and that $\mathcal{S}$ is equal neither to $\mathcal{I}$ nor to $\mathcal{J}$. Assume that $\mathcal{S}$ is not contained in a singular tangent line to $\mathcal{C}$ and that ℓ_∞ is not a singular tangent to $\mathcal{C}$.*

Assume moreover that $Sing(\mathcal{C}) \cap ((\mathcal{IS}) \cup (\mathcal{JS})) = \emptyset$ and that, for every $m_1 \in Sing(\mathcal{C})$ and every branch $\mathcal{B}$ of $\mathcal{C}$ at m_1, we have $i_{m_1}(\mathcal{B},\mathcal{T}_{m_1}\mathcal{B}) \leq 2e_{(\mathcal{B})}$.

If $\mathcal{S} \notin \ell_\infty$, then

$$mdeg(\Sigma_{\mathcal{S}}(\mathcal{C})) = 3d + i_0 - t_0 - n_0 - 2\times\mathbf{1}_{\mathcal{S}\in\mathcal{C}} + \mathbf{1}_{\mathcal{S}\in\mathcal{C};\{\mathcal{I},\mathcal{J}\}\cap\mathcal{T}_{\mathcal{S}}\mathcal{C}\neq\emptyset} - \mathbf{1}_{\mathcal{I}\in\mathcal{C},\mathcal{T}_{\mathcal{I}}\mathcal{C}=\ell_\infty} - \mathbf{1}_{\mathcal{J}\in\mathcal{C},\mathcal{T}_{\mathcal{J}}\mathcal{C}=\ell_\infty},$$

where

- *i_0 is the number of inflection points m_1 of $\mathcal{C}$ such that $\mathcal{T}_{m_1}\mathcal{C}$ does not contain $\mathcal{S}$ and is not equal to ℓ_∞ :*

$$i_0 := \sum_{m_1 \in Flex(\mathcal{C}), \mathcal{S} \notin \mathcal{T}_{m_1}\mathcal{C}, \mathcal{T}_{m_1}\mathcal{C} \neq \ell_\infty} (i_{m_1}(\mathcal{C}, \mathcal{T}_{m_1}\mathcal{C}) - 2)$$

- *t_0 is the number of tangencies of $\mathcal{C}$ with $(\mathcal{IS})$ or $(\mathcal{JS})$:*

$$t_0 := \sum_{m_1 \in Reg(\mathcal{C}) : \mathcal{T}_{m_1}\mathcal{C} \subseteq (\mathcal{IS}) \cup (\mathcal{JS})} i_{m_1}(\mathcal{C}, \mathcal{T}_{m_1}\mathcal{C});$$

- *n_0 is the cardinality of the set of non-singular $m_1 \in \mathcal{C} \setminus (Flex(\mathcal{C}) \cup \{\mathcal{S}\})$ such that $\mathcal{T}_{m_1}\mathcal{C}$ is $(\mathcal{IS})$ or $(\mathcal{JS})$.*

If $\mathcal{S} \in \ell_\infty$, then

$$mdeg(\Sigma_{\mathcal{S}}(\mathcal{C})) = 3d + i_0' - t_0' - 3 \times \mathbf{1}_{\mathcal{S} \in \mathcal{C}} + 2 \times \mathbf{1}_{\mathcal{S} \in \mathcal{C}, \mathcal{T}_{\mathcal{S}}\mathcal{C} = \ell_\infty, i_{\mathcal{S}} \neq 2},$$

where

- *i_0' is the number of inflection points m_1 of $\mathcal{C}$ such that $\mathcal{T}_{m_1}\mathcal{C}$ does not contain $\mathcal{S}$:*

$$i_0' := \sum_{m_1 \in Flex(\mathcal{C}), \mathcal{S} \notin \mathcal{T}_{m_1}\mathcal{C}} (i_{m_1}(\mathcal{C}, \mathcal{T}_{m_1}\mathcal{C}) - 2);$$

- *t_0' is given by*

$$t_0' := \sum_{m_1 \in Reg(\mathcal{C}) : \mathcal{T}_{m_1}\mathcal{C} = \ell_\infty} (2 \times i_{m_1}(\mathcal{C}, \mathcal{T}_{m_1}\mathcal{C}) - 1).$$

An example. We give now an example in order to show how our formula can be used in practice.

We consider the quintic curve $\mathcal{C} = V(F)$ with $F(x, y, z) = y^2z^3 - x^5$. This curve admits two singular points: $A_1 := [0 : 0 : 1]$ and $A_2 := [0 : 1 : 0]$, we have $d = 5$.

In the chart $z = 1$, at A_1, $\mathcal{C}$ has a single branch $\mathcal{B}_{A_1}$, which has equation $y^2 - x^5 = 0$ and multiplicity 2. Hence, $\mathcal{C}$ admits two pro-branches at A_1 of equations ($y = g_i(x)$, $i = 1, 2$) with $g_1(x) := x^{5/2}$ and $g_2(x) := -x^{5/2}$. The tangential intersection number i_{A_1} of the branch $\mathcal{B}_{A_1}$ is equal to

$$i_{A_1} = \sum_{j=1}^{2} \mathrm{val}(g_i) = 5.$$

In the chart $y = 1$, at A_2, $\mathcal{C}$ has a single branch $\mathcal{B}_{A_2}$, which has equation $z^3 - x^5 = 0$ and multiplicity 3. Hence, $\mathcal{C}$ admits three pro-branches at A_2 of equations ($z = h_i(x)$, $i = 1, 2, 3$) with $h_1(x) := x^{5/3}$, $h_2(x) := jx^{5/3}$ and $h_3(x) := j^2x^{5/3}$ (where j is a fixed complex number satisfying $1 + j + j^2 = 0$). The tangential intersection number i_{A_2} of the branch $\mathcal{B}_{A_2}$ is equal to

$$i_{A_2} = \sum_{j=1}^{3} \mathrm{val}(h_i) = 5.$$

According to Corollary 26, the class $d^\vee$ of $\mathcal{C}$ is given by

$$d^\vee = d(d-1) - \sum_{i,j \in \{1,2\} : i \neq j} val(g_i - g_j) - \sum_{i,j \in \{1,2,3\} : i \neq j} val(h_i - h_j) = 5 \times 4 - 2 \times \frac{5}{2} - 6 \times \frac{5}{3} = 5.$$

Using again Corollary 26, we know that the number of inflection points of $\mathcal{C}$ (computed with multiplicity) is equal to

$$3d(d-2) - 3\left[\sum_{i,j\in\{1,2\}:i\neq j} val(g_i - g_j) + \sum_{i,j\in\{1,2,3\}:i\neq j} val(h_i - h_j)\right] = 0.$$

Therefore, $\mathcal{C}$ has no inflection points.

The curve $\mathcal{C}$ admits six isotropic non-singular tangent lines (three for $\mathcal{I}$ and three for $\mathcal{J}$), which are pairwise distinct.

We consider a light point $\mathcal{S} = [x_0 : y_0 : z_0] \in \mathbb{P}^2 \setminus \{\mathcal{I}, \mathcal{J}\}$. We will see that

$$\begin{aligned}\mathrm{mdeg}(\Sigma_{\mathcal{S}}(\mathcal{C})) \quad = \quad & 15 - \mathbf{1}_{y_0=0,x_0\neq 0} - 6\times \mathbf{1}_{z_0=0,x_0\neq 0} - 3\times n_0 - 5\times \mathbf{1}_{\mathcal{S}=A_1} - 9\times \mathbf{1}_{\mathcal{S}=A_2} + \mathbf{1}_{z_0\neq 0} \\ & -2\times \mathbf{1}_{\mathcal{S}\in Reg(\mathcal{C}),\mathcal{T}_{\mathcal{S}}\mathcal{C}\not\subseteq(\mathcal{I}\mathcal{S})\cup(\mathcal{J}\mathcal{S})},\end{aligned}$$

where n_0 is the number of non-singular isotropic tangent lines to $\mathcal{C}$ containing $\mathcal{S}$. Hence, we have

Condition on $\mathcal{S}$	$\mathrm{mdeg}(\Sigma_{\mathcal{S}}(\mathcal{C})) =$
for generic $\mathcal{S} \in \mathbb{P}^2$	16
for generic $\mathcal{S} \in \mathcal{C}$	14
for generic $\mathcal{S} \in \mathcal{T}_{A_1}\mathcal{B}_{A_1}$	15
for generic $\mathcal{S} \in \mathcal{T}_{A_2}\mathcal{B}_{A_2} = \ell_\infty$	9
$\mathcal{S} \in \mathcal{T}_{A_1}\mathcal{B}_{A_1} \cap \mathcal{T}_{A_2}\mathcal{B}_{A_2}$	8
$\mathcal{S} = A_1$ (double point)	11
$\mathcal{S} = A_2$ (triple point)	6
for generic $\mathcal{S}$ on a single isotropic tangent	13
$\mathcal{S}$ on two isotropic tangents	10
$\mathcal{S} = \mathcal{I}$ or $\mathcal{S} = \mathcal{J}$	0

Let us prove the above formula. According to Theorem 3, we have

$$\mathrm{mdeg}(\Sigma_{\mathcal{S}}(\mathcal{C})) = 3d^\vee - v_1 - v_2 - v_2' - (v_3 + v_3')\mathbf{1}_{\mathcal{S}\in\mathcal{C}} - v_4\mathbf{1}_{z_0\neq 0}.$$

- $d^\vee = 5$ (see above).
- Since A_1 is the single singular point of $\mathcal{C}$ outside ℓ_∞, we have
$$v_1 = \min(i_{A_1} - 2\times 2, 0)\mathbf{1}_{\mathcal{S}\notin(A_1\mathcal{I})\cup(A_1\mathcal{J}),y_0\neq 0} = 0.$$
- The couples $(m_1, \mathcal{B}) \in \mathcal{E}$ that may contribute to v_2 corresponds to inflection points or to singular tangent. Since $\mathcal{C}$ admits no inflection points, since the singular tangent line at A_1 contains neither $\mathcal{I}$ nor $\mathcal{J}$ and since the singular tangent line at A_2 contains $\mathcal{I}$ and $\mathcal{J}$, we get
$$v_2 = (i_{A_1} - 2\times 2)\mathbf{1}_{y_0=0,x_0\neq 0} + (3i_{A_2} - 3\times 3)\mathbf{1}_{z_0=0,x_0\neq 0}$$
- Since $\mathcal{C}$ has no point on ℓ_∞ except A_2, we have
$$v_2' = 3\#\{m_1 \in Reg(\mathcal{C}) \setminus \{\mathcal{S}\} : \mathcal{S} \in (m_1\mathcal{I}) \cup (m_1\mathcal{J})\}.$$
- Since $\mathcal{C}$ admits no inflection points, the only points that possibly contributes to v_3 are the singular points. We have
$$v_3 = i_{A_1}\times \mathbf{1}_{\mathcal{S}=A_1} + (3i_{A_2} - 2\times 3)\mathbf{1}_{\mathcal{S}=A_2}$$
- Since the non singular point of $\mathcal{C}$ with isotropic tangent are not in ℓ_∞, we have
$$v_3' = 3\times \mathbf{1}_{\mathcal{S}\in Reg(\mathcal{C}),\mathcal{T}_{\mathcal{S}}\mathcal{C}\subseteq(\mathcal{I}\mathcal{S})\cup(\mathcal{J}\mathcal{S})} + 2\times \mathbf{1}_{\mathcal{S}\in Reg(\mathcal{C}),\mathcal{T}_{\mathcal{S}}\mathcal{C}\not\subseteq(\mathcal{I}\mathcal{S})\cup(\mathcal{J}\mathcal{S})}.$$
- $v_4 = i_{A_2} - 2\times 3$.

2. Caustic by reflection and reflected lines

Let us consider a light position $\mathcal{S} = [x_0 : y_0 : z_0] \in \mathbb{P}^2$ and an irreducible algebraic (mirror) curve $\mathcal{C} = V(F)$ of $\mathbb{P}^2$ given by a homogeneous polynomial F of degree $d \geq 2$.

Definition 6. *The* **caustic by reflection** $\Sigma_{\mathcal{S}}(\mathcal{C})$ *is the Zariski closure of the envelope of the reflected lines* $\{\mathcal{R}_m; m \in Reg(\mathcal{C}) \setminus (\{\mathcal{S}\} \cup \ell_\infty)\}$*, where* $\mathcal{R}_m$ *is the reflected line at* m *of an incident line coming from* $\mathcal{S}$ *after reflection on* $\mathcal{C}$*.*

Let us define the reflected lines $\mathcal{R}_m$. Since our problem is euclidean, we endow $\mathbb{P}^2$ with an angular structure for which $\mathcal{I} = [1 : i : 0]$ and $\mathcal{J} = [1 : -i : 0]$ play a particular role. To this end, let us recall the definition of the cross-ratio β of 4 points of ℓ_∞. Given four points $(P_i = [a_i : b_i : 0])_{i=1,\dots,4}$ such that each point appears at most 2 times, we define the cross-ratio $\beta(P_1, P_2, P_3, P_4)$ of these four points as follows :

$$\beta(P_1, P_2, P_3, P_4) = \frac{(b_3a_1 - b_1a_3)(b_4a_2 - b_2a_4)}{(b_3a_2 - b_2a_3)(b_4a_1 - b_1a_4)},$$

with convention $\frac{1}{0} = \infty$.

For any distinct lines A and B not equal to ℓ_∞, containing neither $\mathcal{I}$ nor $\mathcal{J}$, we define the oriented angular measure between A and B by $\theta \in [0; \pi[$ such that $e^{2i\theta} = \beta(a, b, \mathcal{I}, \mathcal{J})$ (where a is the point of A at infinity and where b is the point of B at infinity).

For every $m = [x : y : z] \in Reg(\mathcal{C}) \setminus \{\mathcal{S}\}$ with $z \neq 0$, we define the reflected line $\mathcal{R}_m$ at m as follows. Let $\mathcal{T}_m\mathcal{C}$ be the tangent line to $\mathcal{C}$ at m (with equation $F_xX + F_yY + F_zZ = 0$ and with point $t_m = [F_y : -F_x : 0]$ at infinity). The incident line at m is line $(\mathcal{S}m)$. We denote by $s_m := [\bar{x} : \bar{y} : 0]$ its point at infinity. We have $s_m := [x_0z - z_0x : y_0z - z_0y : 0]$ if $\mathcal{S} \notin \ell_\infty$ and $s_m = [x_0 : y_0 : 0]$ if $\mathcal{S} \in \ell_\infty$. When s_m and t_m are equal neither to $\mathcal{I}$ nor $\mathcal{J}$, we define the reflected line $\mathcal{R}_m$ at $m \in \mathcal{C}$ as the line (mr_m) with point r_m at infinity given by the Snell-Descartes reflection law $Angle((\mathcal{S}m), \mathcal{T}_m) = Angle(\mathcal{T}_m, \mathcal{R}_m)$, i.e.

$$\beta(s_m, t_m, \mathcal{I}, \mathcal{J}) = \beta(t_m, r_m, \mathcal{I}, \mathcal{J}). \tag{4}$$

Observe that r_m is well defined by this formula as soon as $t_m \notin \{\mathcal{I}, \mathcal{J}\}$. In particular, if $(\mathcal{S}m) = \mathcal{T}_m\mathcal{C}$, then the reflected line at m is $(\mathcal{S}m)$. Moreover, with definition (4), we have $(s_m = \mathcal{I} \ \Rightarrow \ r_m = \mathcal{J})$ and $(s_m = \mathcal{J} \ \Rightarrow \ r_m = \mathcal{I})$.

According to (4), we have

$$r_m := [\bar{x}(F_x^2 - F_y^2) + 2\bar{y}F_xF_y : -\bar{y}(F_x^2 - F_y^2) + 2\bar{x}F_xF_y : 0].$$

Hence $\mathcal{R}_m$ is the set of $[X : Y : Z] \in \mathbb{P}^2$ such that

$$(F_x^2 - F_y^2)(-z\bar{y}X - z\bar{x}Y + Z(\bar{y}x + y\bar{x})) + 2F_xF_y(z\bar{x}X - z\bar{y}Y + Z(-\bar{x}x + \bar{y}y)) = 0.$$

Let us define $\langle (X_1, Y_1, Z_1), (X_2, Y_2, Z_2) \rangle := X_1X_2 + Y_1Y_2 + Z_1Z_2$.

Definition 7. *Let* $m = [x : y : z] \in Reg(\mathcal{C}) \setminus (\{\mathcal{S}\} \cup \ell_\infty)$*, an equation of the reflected line* $\mathcal{R}_m$ *at* m *is given by* $\langle \tilde{\rho}(x, y, z), (X, Y, Z) \rangle = 0$ *with*

$$\tilde{\rho}(x, y, z) = \begin{pmatrix} z(z_0y - zy_0)(F_x^2 - F_y^2) + 2z(zx_0 - z_0x)F_xF_y \\ z(z_0x - zx_0)(F_x^2 - F_y^2) + 2z(z_0y - zy_0)F_xF_y \\ (xzy_0 + yzx_0 - 2z_0xy)(F_x^2 - F_y^2) + 2(yzy_0 - xzx_0 + z_0x^2 - z_0y^2)F_xF_y \end{pmatrix} \text{ if } z_0 \neq 0$$

or with

$$\tilde{\rho}(x, y, z) = \begin{pmatrix} (-zy_0)(F_x^2 - F_y^2) + 2(zx_0)F_xF_y \\ (-zx_0)(F_x^2 - F_y^2) + 2(-zy_0)F_xF_y \\ (xy_0 + yx_0)(F_x^2 - F_y^2) + 2(yy_0 - xx_0)F_xF_y \end{pmatrix} \text{ if } z_0 = 0.$$

3. Rational map $\Phi_{F,S}$

3.1. Proof of Theorem 1.

Theorem 8. *Let $\mathcal{C} = V(F)$ be an algebraic curve of $\mathbb{P}^2$ given by some irreducible homogeneous polynomial F of degree $d \geq 2$ and let $S = (x_0, y_0, z_0) \in \mathbb{C}^3 \setminus \{0\}$.*

For every $m = [x : y : 1] \in Reg(\mathcal{C}) \setminus V(F_x^2 + F_y^2)$ which is not a Base point of $\Phi_{F,S}$, the reflected line $\mathcal{R}_m$ is well defined and is tangent to $\mathcal{C}' := \overline{\Phi_{F,S}(\mathcal{C})}$ at $\Phi_{F,S}(m)$.

Moreover the set of base points of $(\Phi_{F,S})_{|\mathcal{C}}$ is finite.

$\mathcal{C}' := \overline{\Phi_{F,S}(\mathcal{C})}$ is the caustic by reflection $\Sigma_{\mathcal{S}}(\mathcal{C})$ of $\mathcal{C}$ with source point $\mathcal{S} = [x_0 : y_0 : z_0]$.

Before proving this theorem, we explain how the expression $\tilde{\Phi}(x, y, 1)$ can be simplified when $(x, y, 1)$ is in $\mathcal{C}$. Let us recall that, since F is homogeneous with degree d, we have

$$d\,F = xF_x + yF_y + zF_z$$

and therefore

$$(d-1)F_z = xF_{xz}+yF_{yz}+zF_{zz},\ (d-1)F_x = xF_{xx}+yF_{xy}+zF_{xz} \text{ and } (d-1)F_y = xF_{xy}+yF_{yy}+zF_{yz}.$$

Remark 9. *Thanks to the expression of F_{zz}, F_{xz}, F_{yz}, F_z, we have*

$$z^2 H_F = (d-1)^2 h_F \quad \text{on } \mathcal{C},$$

with

$$h_F := 2F_{xy}F_yF_x - F_{xx}F_y^2 - F_{yy}F_x^2 \quad \text{and} \quad z\Delta_S F(x, y, 1) = (zx_0 - x)F_x + (zy_0 - y)F_y.$$

Therefore, for any $m = [x : y : 1] \in \mathcal{C}$, we have

$$\tilde{\Phi}_{F,S}(x, y, 1) = \begin{pmatrix} -2xh_F N_S + (F_y^2 x_0 - F_x^2(x_0 - 2xz_0) - 2F_xF_y(y_0 - yz_0)\Delta_S F \\ -2yh_F N_S + (F_x^2 y_0 z - F_y^2(y_0 - 2yz_0) - 2F_xF_y(x_0 - xz_0))\Delta_S F \\ -2h_F N_S + z_0(F_x^2 + F_y^2)\Delta_S F \end{pmatrix}.$$

Proof of Theorem 8. Let us first observe that, since F is irreducible of degree $d \geq 2$, we have $\mathcal{C} \not\subseteq \{F_x^2 + F_y^2 = 0\} \cup \ell_\infty$.

Let us write $\tilde{\rho}_1$, $\tilde{\rho}_2$ and $\tilde{\rho}_3$ the coordinates of $\tilde{\rho}$. Observe that $z\tilde{\rho}_3(m) = -x\tilde{\rho}_1(m) - y\tilde{\rho}_2(m)$.

We will prove that, on $z = 1$, we have $(\tilde{\rho} \wedge W) = (F_x^2 + F_y^2)\tilde{\Phi}$, for some W.

Consider now $\tilde{m} := (x, y, 1) \in \mathbb{C}^3$ be such that $m = [x : y : 1]$ is in $Reg(\mathcal{C}) \setminus (\{\mathcal{S}\} \cup \ell_\infty \cup V(F_x^2 + F_y^2))$ and is not a base point of $\Phi_{F,S}$. Hence $\tilde{\rho}(x, y, 1) \neq 0$ and the reflected line $\mathcal{R}_m$ is well defined.

To simplify notations, we ommit indices F, S in $\tilde{\Phi}$ and in Φ.

To prove that $\Phi(m)$ belongs to $\mathcal{R}_m$, it is enough to prove that

$$\langle \tilde{\Phi}(\tilde{m}), \rho(\tilde{m}) \rangle = 0, \tag{5}$$

If this is true, to prove that $\mathcal{R}_m$ is tangent to $\mathcal{C}'$ at $\Phi(m)$, it is enough to prove that

$$\langle \tilde{\Phi}(\tilde{m}), W(m) \rangle = 0, \quad \text{with } W := \begin{pmatrix} W_1 := (\tilde{\rho}_1)_x(-F_y) + (\tilde{\rho}_1)_y F_x \\ W_2 := (\tilde{\rho}_2)_x(-F_y) + (\tilde{\rho}_2)_y F_x \\ W_3 := -xW_1 - yW_2 - \tilde{\rho}_1(-F_y) - \tilde{\rho}_2(F_x) \end{pmatrix}. \tag{6}$$

Indeed, let us consider a parametrization $M(t) = [x(t) : y(t) : 1]$ of $\mathcal{C}$ in a neighbourhood of m such that $M(0) = m$ and such that $x'(t) = -F_y$ and $y'(t) = F_x$. This is possible since $DF(m)$

is non-null. Then $\varphi(t) := \Phi(M(t))$ is a parametrization of $\mathcal{C}'$ at a neighbourhood of $\Phi(m)$. Let $r(t) := \rho(M(t))$. The fact that $\mathcal{R}_m$ is tangent to $\mathcal{C}'$ at $\Phi(m)$ means that $\langle r(0), \varphi'(0)\rangle = 0$, i.e. $\langle r'(0), \Phi(m)\rangle = 0$ (since $\langle r(t), \varphi(t)\rangle = 0$, we have $\langle r'(0), \Phi(m)\rangle = -\langle \rho(m), \varphi'(0)\rangle$). We have $r'(0) = W(\tilde{m})$.

Hence, to prove the theorem, it is enough to prove that

$$\boxed{\tilde{\rho}(\tilde{m}) \wedge W(\tilde{m}) = (F_x^2 + F_y^2)\tilde{\Phi}(\tilde{m})},$$

which is the key point of this proof. This can be checked by a fastidious formal computation thanks to the following formulas (and thanks to a symbolic computation software):

$$(\tilde{\rho}_1)_x = 2(z_0y - zy_0)(F_xF_{xx} - F_yF_{xy}) - 2z_0F_xF_y + 2(zx_0 - z_0x)(F_xF_{xy} + F_{xx}F_y),$$
$$(\tilde{\rho}_1)_y := z_0(F_x^2 - F_y^2) + 2(z_0y - zy_0)(F_xF_{xy} - F_yF_{yy}) + 2(zx_0 - z_0x)(F_{xy}F_y + F_xF_{yy}),$$
$$(\tilde{\rho}_2)_x := z_0(F_x^2 - F_y^2) + 2(z_0x - zx_0)(F_xF_{xx} - F_yF_{xy}) + 2(z_0y - zy_0)(F_xF_{xy} + F_{xx}F_y),$$
$$(\tilde{\rho}_2)_y = 2(z_0x - zx_0)(F_xF_{xy} - F_yF_{yy}) + 2z_0F_xF_y + 2(z_0y - zy_0)(F_xF_{yy} + F_{xy}F_y).$$

The fact that the set of base points of Φ is finite on $\mathcal{C}$ comes from the following proposition. The last point follows. □

We can observe that Theorem 8 remains true when $d = 1$ and $\mathcal{S}, \mathcal{I}, \mathcal{J} \notin \mathcal{C}$.

3.2. **Base points of $(\Phi_{F,S})_{|\mathcal{C}}$.**

Remark 10 (Light position at $\mathcal{I}$ or $\mathcal{J}$)**.** *We notice that $\tilde{\Phi}_{F,\alpha I} = -\alpha^2(\Delta_I F)^3 J$ and $\tilde{\Phi}_{F,\alpha J} = -\alpha^2(\Delta_J F)^3 I$. Hence*

$$\Sigma_{\mathcal{I}}(\mathcal{C}) = \{\mathcal{J}\} \quad \text{and} \quad \Sigma_{\mathcal{J}}(\mathcal{C}) = \{\mathcal{I}\}.$$

This is not surprising since, in these cases, we always have $r_m = \mathcal{J}$ and $r_m = \mathcal{I}$ respectively.

Hence, in the sequel, we will suppose that $\mathcal{S} \in \mathbb{P}^2 \setminus \{\mathcal{I}, \mathcal{J}\}$.

Proposition 11. *Let us assume that hypotheses of theorem 8 hold true and that $\mathcal{S} \notin \{\mathcal{I}, \mathcal{J}\}$.*

If $\mathcal{S} \in \ell_\infty$, then $[x : y : z] \in \mathcal{C}$ is a base point of $\Phi_{F,S}$ if and only if

$$h_F(x, y, z) = 0 \quad \text{and} \quad \Delta_S F(x, y, z) = 0.$$

If $\mathcal{S} \notin \ell_\infty$, then $[x : y : z] \in \mathcal{C}$ is a base point of $\tilde{\Phi}$ if and only if

$$[H_F(x, y, z) = 0 \text{ or } N_S(x, y, z) = 0] \quad \text{and} \quad [\Delta_S F(x, y, z) = 0 \text{ or } (z = 0 \text{ and } F_x = F_y = 0)\ )].$$

Remark 12. *This result insures that the set of base points of $\Phi_{F,S}$ is finite on $\mathcal{C} = V(F)$ as soon as F is irreducible and of degree $d \geq 2$.*

Remark 13 (Geometric interpretation)**.** *Let us notice that $N_S(x, y, z) = 0$ means that either $\mathcal{I}, [x : y : z], \mathcal{S}$ lie on a same line or $\mathcal{J}, [x : y : z], \mathcal{S}$ are on a same line.*

If $\mathcal{S} \in \ell_\infty \setminus \{\mathcal{I}, \mathcal{J}\}$, the base points of $\Phi_{F,S}$ on $\mathcal{C}$ are :

- *the singular points of $\mathcal{C}$ (since $DF = 0$ implies $H_F = 0$),*
- *the inflection points m of $\mathcal{C}$ such that $\mathcal{S}$ is in $\mathcal{T}_m\mathcal{C}$,*
- *the points m of $\mathcal{C}$ lying on ℓ_∞ such that $\mathcal{T}_m\mathcal{C}$ is ℓ_∞,*
- *$m = \mathcal{S}$ (if $\mathcal{S}$ belongs to $\mathcal{C}$).*

If $\mathcal{S} \notin \ell_\infty$, the base points of $\Phi_{F,S}$ on $\mathcal{C}$ are :

- *the singular points of $\mathcal{C}$,*
- *the inflection points m of $\mathcal{C}$ such that $\mathcal{S}$ is in $\mathcal{T}_m\mathcal{C}$,*

- *the inflection points m of $\mathcal{C}$ belonging to infinity such that $\mathcal{T}_m\mathcal{C}$ is ℓ_∞,*
- *the points m belongs of $\mathcal{C}$, such that $\mathcal{S}$ belongs to $\mathcal{T}_m\mathcal{C}$ and $m \in (\mathcal{SI}) \cup (\mathcal{SJ})$, i.e.*
  - $m \in \mathcal{C} \cap \{\mathcal{S}\}$,
  - $m \in \mathcal{C} \cap \{\mathcal{I}, \mathcal{J}\}$ *and* $\mathcal{S} \in T_m\mathcal{C}$,
  - $m \in \mathcal{C}$ *such that* $T_m\mathcal{C}$ *equals an isotropic line* $(\mathcal{SI})$ *or* $(\mathcal{SJ})$,
- $m \in \mathcal{C} \cap \{\mathcal{I}, \mathcal{J}\}$ *such that* $\mathcal{T}_m\mathcal{C}$ *is* ℓ_∞.

Proof of Proposition 11. We just prove one implication, the other one being obvious. To simplify notations, we write $\tilde{\Phi}$ instead of $\tilde{\Phi}_{F,S}$.

Let us suppose that $z_0 = 0$.

In this case, we have $N_S = z^2(x_0^2 + y_0^2)$ and so $H_F N_S = (d-1)^2(x_0^2 + y_0^2)h_F$ and

$$\Phi(x, y, z) = \begin{pmatrix} -2x(x_0^2 + y_0^2)h_F + (F_y^2 x_0 - F_x^2 x_0 - 2F_x F_y y_0)\Delta_S F \\ -2y(x_0^2 + y_0^2)h_F + (F_x^2 y_0 - F_y^2 y_0 - 2F_x F_y x_0)\Delta_S F \\ -2z(x_0^2 + y_0^2)h_F \end{pmatrix}.$$

Let $\tilde{m} = (x, y, z)$ be such that $\tilde{\Phi}(\tilde{m}) = 0$. Since $x_0^2 + y_0^2 \neq 0$, thanks to the last equation, we get that $zh_F = 0$, i.e.

$$h_F = 0.$$

According to the two other equations, we get that

$$0 = (F_y^2 x_0 - F_x^2 x_0 - 2F_x F_y y_0)\Delta_S F \text{ and } (F_x^2 y_0 - F_y^2 y_0 - 2F_x F_y x_0)\Delta_S F = 0.$$

If moreover $\Delta_S F \neq 0$, then we must have $a = 0$ and $b = 0$ with

$$a := F_y^2 x_0 - F_x^2 x_0 - 2F_x F_y y_0 \text{ and } b := F_x^2 y_0 - F_y^2 y_0 - 2F_x F_y x_0.$$

Writing successively $x_0 a - y_0 b = 0$ and $y_0 a + x_0 b = 0$ and using the fact that $x_0^2 + y_0^2 \neq 0$, we get that $F_x = F_y = 0$, and so $\Delta_S F = 0$.

Let us suppose that $z_0 \neq 0$.

Let $\tilde{m} = (x, y, z)$ be such that $\tilde{\Phi}(\tilde{m}) = 0$. We observe that we have

$$\tilde{\Phi}(\tilde{m}) = \begin{pmatrix} -\frac{2xH_F N_S}{(d-1)^2} + ((F_x^2 + F_y^2)x_0 - 2F_x\Delta_S F)\Delta_S F \\ -\frac{2yH_F N_S}{(d-1)^2} + ((F_x^2 + F_y^2)y_0 - 2F_y\Delta_S F)\Delta_S F \\ -\frac{2zH_F N_S}{(d-1)^2} + z_0(F_x^2 + F_y^2)\Delta_S F \end{pmatrix}.$$

Hence we have

$$\tilde{\Phi}(\tilde{m}) = -\frac{2H_F N_S}{(d-1)^2}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + (F_x^2 + F_y^2)\Delta_S F \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} - 2(\Delta_S F)^2 \begin{pmatrix} F_x \\ F_y \\ 0 \end{pmatrix} = 0. \tag{7}$$

First, let us consider the case when $H_F N_S = 0$ (which is equivalent to the fact that $H_F = 0$ or $N_S = 0$). In this case, if $\Delta_S F \neq 0$, then we have $F_x^2 + F_y^2 = 0$ (by the third equation) and therefore $F_x = F_y = 0$ (according to the two other equations). Let us notice that, since $zF_z = dF - xF_x - yF_y$, and since $F_z \neq 0$ (since $\Delta_S F \neq 0$), this implies that $z = 0$.

Second, let us consider the case when $H_F N_S \neq 0$. Then $\Delta_S F \neq 0$. Since $xF_x + yF_y + zF_z = F = 0$ and according to the definition of $\Delta_S F$, we get

$$0 = (F_x\ F_y\ F_z) \cdot \tilde{\Phi}(\tilde{m}) = (\Delta_S F)^2(F_x^2 + F_y^2) - 2(\Delta_S F)^2(F_x^2 + F_y^2)$$

and so $F_x^2 + F_y^2 = 0$ and so $z = 0$ (by $\tilde{\Phi}_3 = 0$) and $x^2 + y^2 = 0$ (from $x\tilde{\Phi}_1 + y\tilde{\Phi}_2 = 0$) which contradicts the fact that $N_S \neq 0$. □

3.3. **Degree of $\Phi_{F,S}$.** We recall the definition of the degree of a rational map on an irreducible curve.

Definition 14. *Let $\phi : \mathbb{P}^p \to \mathbb{P}^q$ be a rational map and $\mathcal{C}_1$ an irreducible algebraic curve of $\mathbb{P}^p$. Let $\mathcal{C}_2$ be the Zariski closure $\overline{\phi(\mathcal{C}_1)}$.*

The map $\phi^* : \mathbb{C}(\mathcal{C}_2) \to \mathbb{C}(\mathcal{C}_1)$ defined by $\phi^*(f) = f \circ \phi$ is called the pullback of $\phi_{|\mathcal{C}_1}$.

If $\mathcal{C}_1 \not\subset Base(\phi)$, if $\phi_{|\mathcal{C}_1 \setminus Base(\phi)}$ is not constant, the degree of $\phi_{|\mathcal{C}_1}$ is the degree $[\mathbb{C}(\mathcal{C}_1) : \phi^*(\mathbb{C}(\mathcal{C}_2))]$ of $\mathbb{C}(\mathcal{C}_1)$ as a finite algebraic extension of $\phi^*(\mathbb{C}(\mathcal{C}_2))$.

If $\mathcal{C} \not\subset Base(\phi)$ and if $\phi_{|\mathcal{C} \setminus Base(\phi)}$ is constant, the degree is equal to infinity.

The following interpretation of the degree of a rational map is also useful.

Remark 15. *Let $\phi : \mathbb{P}^p \to \mathbb{P}^q$ be a rational map and $\mathcal{C}$ an irreducible algebraic curve of $\mathbb{P}^p$.*

We recall that, thanks to blowing up ([13, Example II-7-17-3]*) and to a classical morphism result (*[16, Proposition II-2-6], [13, Proposition II-6-8]*), if $\mathcal{C} \not\subset Base(\phi)$, if $\phi_{|\mathcal{C} \setminus Base(\phi)}$ is not constant and has degree δ_1, then there exists a finite set $\mathcal{N}$ such that for every point y of $\phi(\mathcal{C}) \setminus \mathcal{N}$, the number of preimages of y by ϕ is equal to δ_1.*

If $\delta_1 = 1$, then the map $\phi_{|\mathcal{C}}$ is birational onto its image.

When $\#(\phi(\mathcal{C} \setminus Base(\phi))) = 1$, we set $\delta_1 = \infty$.

The question of the degree of the caustic map $\Phi_{F,S}$ is not evident, even if $\mathcal{S} \notin \ell_\infty$. Indeed, when $\mathcal{S} \notin \ell_\infty$, we recall that, as noticed by Quetelet and Dandelin, $\Phi_{F,S}$ is the evolute of the $\mathcal{S}$-centered homothety (with ratio 2) of the pedal of $\mathcal{C}$ from $\mathcal{S}$. It is easy to see that the pedal map is birational on any irreducible curve which is not a line. It is clear that the $\mathcal{S}$-centered homothety (with ratio 2) is an isomorphism of $\mathbb{P}^2$. But, as proved in [9], the degree of the evolute map is not necessarily equal to 1 or to infinity (contrarily to a statement in [7]).

4. About the computation of the degree of the caustic

4.1. **A fundamental lemma.** The idea used in this paper to compute the degree of caustics is is based on the following general lemma giving a way to compute the degree of the image of a curve by a rational map. The proof of the Plücker formula given in [10, p. 91] can be seen as an application of the following lemma.

The following definition extends the notion of polar into a notion of φ-polar.

Definition 16. *Let $p \geq 1$, $q \geq 1$. Given $\varphi : \mathbb{P}^p \to \mathbb{P}^q$ be a rational map given by $\varphi = [\varphi_0 : \cdots : \varphi_q]$ (with homogeneous polynomial functions $\varphi_j : \mathbb{C}^{p+1} \to \mathbb{C}^{q+1}$) and $a = [a_0 : \cdots : a_q] \in \mathbb{P}^q$, we define the φ-polar at a, denoted by $\mathcal{P}_{\varphi,a}$, as follows*

$$\mathcal{P}_{\varphi,a} := V\left(\sum_{j=0}^{q} a_j \varphi_j\right).$$

With this definition, the classical polar of a hypersurface $\mathcal{C} = V(F)$ of $\mathbb{P}^p$ (for some homogeneous polynomial F) at a is the ∇F-polar at a, where $\nabla F(X)$ denotes as usual the vector constitued of the partial derivatives of F at $X \in \mathbb{C}^{p+1} \setminus \{0\}$.

We recall that the set of base points of a rational map $\varphi = [\varphi_0 : ... : \varphi_q] : \mathbb{P}^p \to \mathbb{P}^q$ is the set

$$Base(\varphi) := \bigcap_{j=0}^{q} V(\varphi_j).$$

The cardinality of a set E will be written $\#E$.

Lemma 17 (Fundamental lemma)**.** *Let $\mathcal{C}$ be an irreducible algebraic curve of $\mathbb{P}^p$. Let $p \geq 1$, $q \geq 1$ be two integers and $\varphi : \mathbb{P}^p \to \mathbb{P}^q$ be a rational map given by $\varphi = [\varphi_0 : \cdots : \varphi_q]$, with $\varphi_0, ..., \varphi_q \in \mathbb{C}[x_0, ..., x_p]$ some homogeneous polynomials of degree δ. Assume that $\mathcal{C} \not\subseteq Base(\varphi)$ and that $\varphi_{|\mathcal{C}}$ has degree $\delta_1 \in \mathbb{N} \cup \{\infty\}$.*

Then, for generic $a = [a_0 : \cdots : a_q] \in \mathbb{P}^q$, the following formula holds true

$$\delta_1.deg\left(\overline{\varphi(\mathcal{C})}\right) = \delta.deg(\mathcal{C}) - \sum_{p \in Base(\varphi_{|\mathcal{C}})} i_p\left(\mathcal{C}, \mathcal{P}_{\varphi,a}\right),$$

with convention $0.\infty = 0$ and $deg(\overline{\varphi(\mathcal{C})}) = 0$ if $\#\overline{\varphi(\mathcal{C})} < \infty$.

Before proving this result, we make some observations.

Lemma 18. *Let $\varphi_0, ..., \varphi_q \in \mathbb{C}[x_0, ..., x_p]$ be homogeneous polynomials of degree δ. Consider $\Phi := (\varphi_0, ..., \varphi_q)$ and the rational map $\varphi : \mathbb{P}^p \to \mathbb{P}^q$ defined by $\varphi = [\varphi_0 : \cdots : \varphi_q]$. Let $\mathcal{K}$ be the cone surface associated to an algebraic irreducible curve $\mathcal{C}$ of $\mathbb{P}^p$. If $\#(\varphi(\mathcal{C} \setminus Base(\varphi))) > 1$ and if that $\varphi_{|\mathcal{C}}$ has degree δ_1, then the set of regular points $m = [x_0 : ... : x_p] \in \mathcal{C}$ such that*

$$D\Phi(\tilde{m})(\mathcal{T}_{\tilde{m}}\mathcal{K}) \subset Vect(\Phi(\tilde{m})), \quad with \;\; \tilde{m} := (x_0, ..., x_p)$$

(where $\mathcal{T}_{\tilde{m}}\mathcal{K}$ is the vector tangent plane to $\mathcal{K}$ at $\tilde{m}$) is finite.

Proof. Let $\tilde{m} = (y_0, ..., y_p) \in \mathbb{C}^{p+1}$ be such that $m := \Pi(\tilde{m})$ is a non-singular point of $\mathcal{C}$. We know that there exist an integer $s \geq p-1$ and s homogeneous polynomials $F^{(1)}, ..., F^{(s)} \in \mathbb{C}[x_0, ..., x_p]$ such that $\mathcal{C} = \cap_{i=1}^s V(F^{(i)})$. We also know that $\mathcal{T}_{\tilde{m}}\mathcal{K}$ is $\cap_{i=1}^s V(\sum_{j=0}^p X_j F^{(i)}_{x_j}(\tilde{m}))$. Thanks to classical methods of resolution of linear systems, we know that there exist an integer $r \geq 1$ and $(G^{(u)}_k; k = 0, ..., p; u = 1, ..., r)$ a family of homogeneous polynomials of $(F^{(j)}_{x_i}; i = 0, ..., p; j = 1, ..., s)$ such that

$$\mathcal{T}_{\tilde{m}}\mathcal{K} = Vect(V_u(\tilde{m});\; u = 1, ..., r)$$
$$\text{with} \;\; V_u(\tilde{m}) := (G^{(u)}_k(F^{(j)}_{x_i}(\tilde{m})); i = 0, ..., p; j = 1, ..., s)_{k=0,...,p}.$$

Now the fact that

$$D\Phi(\tilde{m})(\mathcal{T}_{\tilde{m}}\mathcal{K}) \subset Vect(\Phi(\tilde{m})) \tag{8}$$

is equivalent to

$$\forall u = 1, ..., r, \quad \forall i, j = 0, ..., p, \quad [D\Phi(\tilde{m}) \cdot V_u(\tilde{m})]_i \varphi_j(\tilde{m}) - [D\Phi(\tilde{m}) \cdot V_u(\tilde{m})]_j \varphi_i(\tilde{m}) = 0.$$

and so to the fact that m belongs to some algebraic variety. Since $\mathcal{C}$ is irreducible, we conclude that either the set of such m is finite or is equal to $\mathcal{C}$. The fact that this set is $\mathcal{C}$ would mean that for every point $\tilde{m}$ such that $m := \Pi(\tilde{m})$ is a non-singular point of $\mathcal{C}$ and is not a base point of φ, for every $t \mapsto X(t) = (X_0(t), ..., X_p(t))$ such that $t \mapsto \Pi(X(t))$ is a local parametrization of $\mathcal{C}$ satisfying $X(0) = m$, we have $(\Phi \circ X)'(t) = D\Phi(X(t)) \cdot X'(t)$ with $X'(t) \in \mathcal{T}_{X(t)}\mathcal{K}$ and so

$$(\varphi_i \circ X)'(t)\varphi_j(X(t)) - (\varphi_j \circ X)'(t)\varphi_i(X(t)) = 0.$$

This implies that $\Phi(X(t)) = \frac{\varphi_i(X(t))}{\varphi_i(m)}\Phi(m)$ if i is such that $\varphi_i(m) \neq 0$. This means that $\Pi(\Phi(X(t))) = \Pi(\Phi(m))$ for every t. This is impossible by hypothesis. □

Proof of Lemma 17. Let us consider a polynomial map $\Phi : \mathbb{C}^{p+1} \to \mathbb{C}^{q+1}$ associated to φ. For every $a = [a_0 : \cdots : a_q] \in \mathbb{P}^q$, we consider the hyperplane $V(\psi_a) := \{[y_0 : \cdots : y_q] \in \mathbb{P}^q : \sum_{i=0}^q a_i y_i = 0\}$ (with $\psi_a(y_0, \cdots, y_q) := \sum_{i=0}^q a_i y_i$ for some choice of representant of a). Let $B := Base(\varphi_{|\mathcal{C}})$ (i.e. the set of $m = \Pi(\tilde{m}) \in \mathcal{C}$ such that $\Phi(\tilde{m}) = 0$). This set clearly belongs to $\mathcal{C} \cap V(\psi_a \circ \Phi)$.

• If $\#(\varphi(\mathcal{C} \setminus Base(\varphi))) = 1$, we consider the point M of $\mathbb{P}^q$ such that $\varphi(\mathcal{C} \setminus Base(\varphi)) = \{M\}$ and we observe that, for every $a \in \mathbb{P}^q \setminus V(\psi_M)$, we have $\mathcal{C} \cap V(\psi_a \circ \Phi) = B$ and so, by Bezout theorem (using the fact that $V(\psi_a \circ \Phi)$ is an hypersurface that does not contain $\mathcal{C}$)

$$\begin{aligned}
\deg(\mathcal{C})\delta &= \deg(\mathcal{C})\deg(\psi_a \circ \Phi) \\
&= \sum_{m \in \mathcal{C} \cap V(\psi_a \circ \Phi)} i_m(\mathcal{C}, V(\psi_a \circ \Phi)) \\
&= \sum_{m \in B} i_m(\mathcal{C}, V(\psi_a \circ \Phi)) \\
&= \sum_{m \in B} i_m(\mathcal{C}, V(\psi_a \circ \Phi)) + \delta_1.\deg\left(\overline{\varphi(\mathcal{C})}\right),
\end{aligned}$$

since $\deg\left(\overline{\varphi(\mathcal{C})}\right) = 0$.

• Assume now that $\#(\varphi(\mathcal{C} \setminus Base(\varphi))) > 1$.

Let $a = [a_0 : \cdots : a_q] \in \mathbb{P}^q$ be such that

(1) for every $y \in \overline{\varphi(\mathcal{C} \setminus \mathcal{B})} \cap V(\psi_a)$ is in $\varphi(\mathcal{C})$ and satisfies $\#\varphi^{-1}(\{y\}) = \delta_1$,
(2) $\deg\left(\sum_{i=0}^q a_i\varphi_i\right) = \delta$,
(3) for every $y \in \overline{\varphi(\mathcal{C} \setminus \mathcal{B})} \cap V(\psi_a)$, we have $i_y(\overline{\varphi(\mathcal{C} \setminus \mathcal{B})}, V(\psi_a)) = 1$,
(4) $V(\psi_a)$ contains no point $\Pi(\tilde{m})$ of $\mathcal{C}$ such that $D\phi(\tilde{m}) \cdot (\mathcal{T}_{\tilde{m}}\mathcal{K}) \subseteq Vect(\Phi(\tilde{m}))$, where $\mathcal{K}$ is the cone surface associated to $\mathcal{C}$ and where $\mathcal{T}_{\tilde{m}}\mathcal{K}$ is its tangent plane at m,
(5) $\#(\overline{\varphi(\mathcal{C} \setminus \mathcal{B})} \cap V(\psi_a)) < \infty$,
(6) $V(\psi_a) \cap \varphi(Sing(\mathcal{C})) = \emptyset$, with $Sing(\mathcal{C}) = \{m = [x : y : z] \in \mathcal{C} \ : \ DF(x, y, z) = 0\}$.

Now let us write $V(\psi_a \circ \Phi) := \{[y_0 : \cdots : y_p] \in \mathbb{P}^p \ : \ \sum_{i=0}^q a_i\varphi_i(y_0, ..., y_p) = 0\}$.

Point 5 insures that $\#(\mathcal{C} \cap V(\psi_a \circ \Phi)) < \infty$. Hence, $\mathcal{C}$ and $V(\psi_a \circ \Phi)$ have no common component and since $\mathcal{C}$ is a curve and $V(\psi_a \circ \Phi)$ is an hypersurface, according to Bezout theorem, since $V(\psi_a \circ \Phi)$ does not contain $\mathcal{C}$, we have

$$\begin{aligned}
\deg(\mathcal{C})\delta &= \deg(\mathcal{C})\deg(\psi_a \circ \Phi) \\
&= \sum_{m \in \mathcal{C} \cap V(\psi_a \circ \Phi)} i_m(\mathcal{C}, V(\psi_a \circ \Phi)) \\
&= \sum_{m \in B} i_m(\mathcal{C}, V(\psi_a \circ \Phi)) + \sum_{m \in (\mathcal{C} \setminus B) \cap V(\psi_a \circ \Phi)} i_m(\mathcal{C}, V(\psi_a \circ \Phi)).
\end{aligned}$$

Let us now consider any $m = [x_0 : \cdots : x_p] \in (\mathcal{C} \setminus B) \cap V(\psi_a \circ \Phi)$. We have $\varphi(m) \in \varphi(\mathcal{C} \setminus B) \cap V(\psi_a)$. According to point 6, $\mathcal{C}$ admits a tangent line $\mathcal{T}_m$ at m. Set $\tilde{m} := (x_0, ..., x_p)$. According to point 4, we consider a tangent vector $v = (v_0, \ldots, v_p) \in \mathbb{C}^{p+1}$ to $\mathcal{K}$ such that $D\Phi(\tilde{m})(v)$ and $\Phi(\tilde{m})$ are linearly independent.

Therefore, according to point 3, with the notation $a := (a_0, ..., a_q)$ we have

$$\langle a, (D\Phi)(\tilde{m})(v)\rangle \neq 0 \ \text{ and so } \ \langle v, {}^t(D\Phi)(\tilde{m})(a)\rangle \neq 0.$$

Since ${}^t(D\Phi)(\tilde{m}) \cdot a = \nabla(\psi_a \circ \Phi)(\tilde{m})$, we get that $i_m(\mathcal{C}, V(\psi_a \circ \Phi)) = 1$. Hence, we have

$$\begin{aligned}
\sum_{m \in (\mathcal{C}\setminus B) \cap V(\psi_a \circ \Phi)} i_m(\mathcal{C}, V(\psi_a \circ \Phi)) &= \#\left((\mathcal{C} \setminus B) \cap V(\psi_a \circ \Phi)\right) \\
&= \delta_1 \#\left(\varphi(\mathcal{C} \setminus B)) \cap V(\psi_a)\right) \\
&= \delta_1 \#\left(\overline{\varphi(\mathcal{C} \setminus B))} \cap V(\psi_a)\right), \text{ according to Point 1} \\
&= \delta_1 \sum_{y \in \overline{\varphi(\mathcal{C})} \cap V(\psi_a)} i_y(\overline{\varphi(\mathcal{C})}, V(\psi_a)), \text{ according to Point 3} \\
&= \delta_1 . \deg(\overline{\varphi(\mathcal{C})}).
\end{aligned}$$

□

4.2. **About intersection numbers at the base points of $\Phi_{F,S}$.** Theorems 2 and 3 will be a direct consequence of the fundamental lemma (Lemma 17) and of the computation of $i_{m_1}(\mathcal{C}, \mathcal{P}_{\Phi,a})$ at every base point. To compute these intersection numbers, we will compute intersection numbers of branches thanks to the notion of pro-branches. It will be important to observe that, since $F_x^2 + F_y^2 = \Delta_I F . \Delta_J F$, $\tilde{\Phi}_{F,S}$ can be rewritten

$$\tilde{\Phi}_{F,S} = -\frac{2H_F . f_{I,S} . f_{J,S}}{(d-1)^2} . Id + \Delta_I F . \Delta_J F . \Delta_S F . S - (\Delta_S F)^2 [\Delta_I F . J + \Delta_J F . I], \tag{9}$$

where $f_{A,B}(C) = \det(A|B|C)$, for every $A, B, C \in \mathbb{C}^3$. We observe that $V(f_{A,B})$ is the line $(\Pi(A)\Pi(B))$. In the proof of Theorems 2 and 3, thanks to formula (9), we will easily see that, for a generic $a \in \mathbb{P}^2$,

$$i_{m_1}\left(\mathcal{C}, \mathcal{P}_{\Phi_{F,S},a}\right) \geq \min\left(i_{m_1}\left(\mathcal{C}, V(\Psi_1)\right), i_{m_1}\left(\mathcal{C}, V(\Psi_2)\right), i_{m_1}\left(\mathcal{C}, V(\Psi_3)\right), i_{m_1}\left(\mathcal{C}, V(\Psi_4)\right)\right), \tag{10}$$

with

$$\Psi_1 := -\frac{2H_F . f_{I,S} . f_{J,S}}{(d-1)^2}, \qquad \Psi_2 := \Delta_I F . \Delta_J F . \Delta_S F,$$
$$\Psi_3 := (\Delta_S F)^2 \Delta_I F \quad \text{and} \quad \Psi_4 := (\Delta_S F)^2 \Delta_J F.$$

More precisely, we will prove the same inequality for branches of $\mathcal{C}$ at m_1 instead of $\mathcal{C}$, this inequality being an equality in most of the cases but not in every case. This will be detailed in section 6. Before going on in the proof of Theorems 2 and 3, let us give some general results on intersection numbers including general formulas for the following intersection numbers

$$i_{m_1}(\mathcal{C}, V(H_F)) \quad \text{and} \quad i_{m_1}(\mathcal{C}, V(\Delta_P F)).$$

5. About computation of intersection number : classical results and extensions

As in [10], we write $\mathbb{C}[[x]]$ the ring of formal power series and $\mathbb{C}[[x^*]] := \bigcup_{N \geq 1} \mathbb{C}[[x^{\frac{1}{N}}]]$ the ring of formal fractional power series.

Definition 19. *Let $g \in \mathbb{C}[[x^*]]$.*

- *If $g(x) = ax^q$ with $a \in \mathbb{C}^*$ and $q \in \mathbb{Q}_+$, we say that the* **degree** *of g is equal to q.*
- *We denote by $LM(g)$ the* **lowest degree monomial term** *of g and we call (rational)* **valuation** *of g, also denoted $val(g)$ or $val_x(g(x))$ the degree of $LM(g)$.*

Definition 20. *Let $F \in \mathbb{C}[X, Y, Z]$ be a homogeneous polynomial such that $F(0, 0, 1) = 0$. The* **tangent cone** *of $V(F)$ at $[0 : 0 : 1]$ is $V(LM(F))$ where $LM(F)$ is the sum of terms of lowest degree in $F(x, y, 1)$.*

We recall that the degree in $\{x, y\}$ of $LM(F)$ is the multiplicity of $[0 : 0 : 1]$ in $V(F)$.

We recall now the classical use of the Weierstrass preparation theorem combined with the Puiseux expansions (see [10], pages 107, 137 and 142). For every $N \in \mathbb{N}^*$, we set $\mathbb{C}\langle x^{\frac{1}{N}}\rangle$ and $\mathbb{C}\langle x^{\frac{1}{N}}, y\rangle$ the rings of convergent power series of $x^{\frac{1}{N}}$ and of $x^{\frac{1}{N}}, y$. Let $\mathbb{C}\langle x^*\rangle := \bigcup_{N\geq 1}\mathbb{C}\langle x^{\frac{1}{N}}\rangle$ and $\mathbb{C}\langle x^*, y\rangle := \bigcup_{N\geq 1}\mathbb{C}\langle x^{\frac{1}{N}}, y\rangle$.

Let $F(x,y,z) \in \mathbb{C}[x,y,z]$ be a homogeneous polynomial. We suppose that $[0:0:1]$ is a point of $V(F)$ with multiplicity q. This means that $F(x,y,1)$ has valuation q in (x,y). Suppose that $\{X=0\}$ is not contained in the tangent cone of $V(F)$ at $[0:0:1]$. This implies that q is also the valuation of $F(0,y,1)$ in y.

Now, the Weierstrass theorem insures the existence of a unit U of $\mathbb{C}\langle x,y\rangle$ and of $\Gamma(x,y) \in \mathbb{C}\langle x\rangle[y]$ monic, such that $\Gamma(0,y)$ has degree q in y and

$$F(x,y,1) = U(x,y)\Gamma(x,y)$$

There exists an integer $b \geq 1$ and $\Gamma_1(x,y), ..., \Gamma_b(x,y) \in \mathbb{C}\langle x\rangle[y]$ monic irreducible such that

$$\Gamma(x,y) = \prod_{\beta=1}^{b} \Gamma_\beta(x,y).$$

We recall that the $\mathcal{B}_\beta$'s with equation $\Gamma_\beta = 0$ on $z=1$ are the **branches** of $V(F)$ at $[0:0:1]$. The **tangent line $\mathcal{T}_\beta$ to branch $\mathcal{B}_\beta$ at** $[0:0:1]$ is the reduced tangent cone of $\mathcal{B}_\beta$ at $[0:0:1]$ (the notion of tangent cone is well defined with the same definition as for polynomials, see [10, P. 148]). Line $\mathcal{T}_\beta$ is given by $(\Gamma_\beta)_x X + (\Gamma_\beta)_y Y = 0$.

The degree e_β in y of $\Gamma_\beta(x,y)$ is called the **multiplicity of the branch $\mathcal{B}_\beta$**.

Thanks to the Puiseux theorem, for every $\beta \in \{1, ..., b\}$, there exists $\varphi_\beta(t) \in \mathbb{C}\langle t\rangle$ such that

$$\Gamma_\beta(x,y) = \prod_{k=1}^{e_\beta} \left(y - \varphi_{\beta,k}(x)\right), \quad \text{with} \quad \varphi_{\beta,k}(x) := \varphi_\beta\left(e^{\frac{2ik\pi}{e_\beta}} x^{\frac{1}{e_\beta}}\right) \in \mathbb{C}\langle x^*\rangle.$$

Of course we have $q = \sum_{\beta=1}^{b} e_\beta$.

We recall that $\left(y = \varphi_\beta\left(e^{\frac{2ik\pi}{e_\beta}} x^{\frac{1}{e_\beta}}\right),\ k = 1, ..., e_\beta\right)$ are equations of the **pro-branches** of branch $V(H_\beta)$ (this notion can be found in [12, 19]).

This is summarized in the following theorem in which the pro-branches are numbered by $i \in \mathcal{I} = \{1, ..., q\}$ and are denoted by g_i.

Theorem 21. *Let $F(x,y,z) \in \mathbb{C}[x,y,z]$ be a homogeneous polynomial. We suppose that $[0:0:1]$ is a point of $V(F)$ with multiplicity q and such that $\{x=0\}$ is not contained in the tangent cone of $V(F)$ at $[0:0:1]$. Then there exists $U(x,y)$ being a unit of $\mathbb{C}\langle x,y\rangle$ and $g_1, ..., g_q \in \mathbb{C}\langle x^*\rangle$ such that*

$$F(x,y,1) = U(x,y)\prod_{i=1}^{q}(y - g_i(x)) \quad \text{in} \quad \mathbb{C}\langle x^*, y\rangle.$$

Now, let us introduce now the notions of intersection numbers for pro-branches and for branches (see [12, 19]).

Definition 22. *Let $F(x,y,z)$ and $G(x,y,z)$ in $\mathbb{C}[x,y,z]$ be two homogeneous polynomials. Let $m_1 \in V(F) \cap V(G)$ be a point of multiplicity q of $V(F)$. Let $M \in GL(\mathbb{C}^3)$ be such that $m_1 = \Pi(M(0,0,1))$ and such that $\{x=0\}$ is not contained in the tangent cone of $V(F \circ M)$ at $[0:0:1]$.*

According to theorem 21, we have

$$F(M(x,y,1)) = U(x,y)\prod_{i=1}^{q}(y-g_i(x)),\ \text{ with } g_i(x)\in\mathbb{C}[[x^*]],$$

$U(x,y)$ being a unit in the ring of convergent power series $\mathbb{C}\langle x,y\rangle$. We also use notations $\mathcal{B}_\beta$ for branches of $V(F\circ M)$ and $\mathcal{T}_\beta$ for tangent line to $\mathcal{B}_\beta$ at $[0:0:1]$.

We define

- *the* **intersection numbers** $i_{m_1}^{(i,j)}$ **of pro-branches** *of $V(F\circ M)$ of equations $y=g_i(x)$ and $y=g_j(x)$ are given by*
$$i_{m_1}^{(i,j)} := val(g_i-g_j),$$
- *the* **tangential intersection number** $i_{m_1}^{(i)}$ **of pro-branch** *of $V(F\circ M)$ of equation $y=g_i(x)$ are given by the following formula* [2]
$$i_{m_1}^{(i)} = val_x(g_i(x)-g_i'(0)x)$$
($i_{m_1}^{(i)}$ corresponds to the intersection number of the pro-branch of equation $y=g_i(x)$ with the tangent line $\mathcal{D}_{m_1}^{(i)}=\mathcal{T}_\beta$ where $\mathcal{B}_\beta$ is the branch associated to this pro-branch ; this tangent line has equation $y=g_i'(0)x$).
- *the* **intersection number** $i_{[0:0:1]}(V(G\circ M),\mathcal{B}_\beta)$ **of a branch** $\mathcal{B}_\beta$ **of** $V(F\circ M)$ **with** $V(G\circ M)$ *is defined by*
$$i_{[0:0:1]}(V(G\circ M),\mathcal{B}_\beta) := \sum_{i\in\mathcal{I}_\beta} val_x(G(M(x,g_i(x),1))),$$
where $\mathcal{I}_\beta$ is the set of indices $i\in\{1,...,q\}$ such that the pro-branch of $V(F\circ M)$ of equation $y=g_i(x)$ is associated to branch $\mathcal{B}_\beta$.

We recall that, under hypotheses of this definition, the intersection number $i_{m_1}(V(F),V(G))$ is given by

$$i_{m_1}(V(F),V(G)) = \sum_{i=1}^{q}\mathrm{val}_x\left(G(x,g_i(x),1)\right) = \sum_{\beta=1}^{b} i_{[0:0:1]}(V(G\circ M),\mathcal{B}_\beta). \tag{11}$$

This observation will be crucial here in computations of intersection numbers.

Remark 23. *Quantity $i_{m_1}^{(i)}$ corresponds to the degree of the smallest degree term of g_i of degree greater than or equal to 1 (i.e. $g_i(x)=\alpha_1 x+\alpha x^{i_{m_1}^{(i)}}+...$ with $\alpha\neq 0$). It is not difficult to see that*

$$i_{m_1}^{(i)} = val_x(g_i(x)-g_i'(0)x) = val_x(xg_i'(x)-g_i(x)) = val_x(g_i'(0)-g_i'(x))+1 = valg_i''+2.$$

Another interesting observation is that, in some sense, the notion of branches as well as their intersection numbers do not depend on the choice of matrix M. This is the oject of the following proposition, the proof of which is postponed in appendix A.

Proposition 24. *Let $F(x,y,z)$ in $\mathbb{C}[x,y,z]$ be a homogeneous polynomial. Let m_1 be a point of multiplicity q of $V(F)$. Let $M,\hat{M}\in GL(\mathbb{C}^3)$ be such that $m_1=\Pi(M(0,0,1))=\Pi(\hat{M}(0,0,1))$ and such that $\{x=0\}$ is not in the tangent cones of $V(F\circ M)$ or of $V(F\circ\hat{M})$ at $[0:0:1]$. Then*

[2]The fact that $X=0$ is not contained in the tangent cone of $V(F\circ M)$ implies that $valg_i\geq 1$.

- *the multiplicities of* $[0:0:1]$ *on* $V(F\circ M)$ *and on* $V(F\circ \hat{M})$ *are equal,*
- $V(F\circ M)$ *and* $V(F\circ \hat{M})$ *have the same number* b *of branches at* $[0:0:1]$, $\mathcal{B}_1,...,\mathcal{B}_b$ *and* $\hat{\mathcal{B}}_1,...,\hat{\mathcal{B}}_b$ *respectively.*
- *There exists a permutation* σ *of* $\{1,...,b\}$ *such that, for every* $\beta\in\{1,...,b\}$,
  - $\mathcal{B}_\beta$ *and* $\hat{\mathcal{B}}_{\sigma(\beta)}$ *have the same multiplicity* e_β,
  - *If* $\mathcal{T}_\beta$ *and* $\hat{\mathcal{T}}_{\sigma(\beta)}$ *of* $\mathcal{B}_\beta$ *and* $\hat{\mathcal{B}}_{\sigma(\beta)}$ *at* $[0:0:1]$, *then* $M(\mathcal{T}_\beta)=\hat{M}(\hat{\mathcal{T}}_{\sigma(\beta)})$,
  - *for every homogeneous polynomial* $G(x,y,z)\in\mathbb{C}[x,y,z]$, *we have*
  $$i_{[0:0:1]}(V(G\circ M),\mathcal{B}_\beta)=i_{[0:0:1]}(V(G\circ \hat{M}),\hat{\mathcal{B}}_{\sigma(\beta)})$$
  - *In a neighbourhood of* $[0:0:1]$, *if* $\mathcal{B}_\beta$ *has equation* $\Gamma_\beta(x,y)=0$ *on* $z=1$, $\hat{\mathcal{B}}_{\sigma(\beta)}$ *has equation* $\Gamma_\beta\left(\frac{X(x,y,1)}{Z(x,y,1)},\frac{Y(x,y,1)}{Z(x,y,1)}\right)=0$ *where* X, Y, Z *are the coordinates of map* $M^{-1}\circ\hat{M}$.

Proposition 25. *Let* $\mathcal{C}=V(F)$ *with* F *a homogeneous polynomial of degree* $d\geq 2$. *Let* m_1 *be a point of* $\mathcal{C}$ *of multiplicity* q *and let* $P\in\mathbb{C}^3\setminus\{0\}$ *be such that* $\Delta_P F(m_1)=0$. *Assume assumptions of Definition 22. We have*

$$i_{m_1}(\mathcal{C},V(\Delta_P F))=\left[\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right]+\sum_{i\in\mathcal{I}}\left[(i_{m_1}^{(i)}-1)\mathbf{1}_{\Pi(P)\in\mathcal{D}_{m_1}^{(i)}}+\mathbf{1}_{\Pi(P)=m_1}\right],$$

and

$$i_{m_1}(\mathcal{C},V(H_F))=\left(3\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right)+\sum_{i\in\mathcal{I}}[i_{m_1}^{(i)}-2].$$

We will see in Remark 29 that, with the notations of Definition 22, the values of

$$V_{m_1}:=\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\quad\text{and}\quad I_{m_1}:=\sum_{i\in\mathcal{I}}[i_{m_1}^{(i)}-2]$$

do not depend on the choice of M.

Corollary 26. *Proposition 25 combined with* [10, p. 91-92] *(one can also use our fundamental lemma 17) can be used to get precised Plücker formulas for the class and for the number of inflection points for a general plane algebraic curve. Indeed, for generic* $P\in\mathbb{P}^2$, *we have*

$$\boxed{d^\vee=d(d-1)-\sum_{m_1\in Sing(\mathcal{C})} i_{m_1}(\mathcal{C},V(\Delta_P F))=d(d-1)-\sum_{m_1\in Sing(\mathcal{C})} V_{m_1}}$$

and

$$\boxed{3d(d-2)-\sum_{m_1\in Sing(\mathcal{C})} i_{m_1}(\mathcal{C},V(H_F))=\sum_{m_1\in Reg(\mathcal{C})} I_{m_1}}$$

which corresponds to the number of inflection points. Moreover, we have

$$\sum_{m_1\in Sing(\mathcal{C})} i_{m_1}(\mathcal{C},V(H_F))=3\sum_{m_1\in Sing(\mathcal{C})} V_{m_1}+\sum_{m_1\in Sing(\mathcal{C})} I_{m_1}.$$

Applying Proposition 25 to non-singular points (including flexes), nodes and cusps, we obtain directly:

Corollary 27. *Under assumption of proposition 25,*

- *If $\mathcal{C}$ is smooth at m_1 with $i_{m_1}(\mathcal{C}, \mathcal{T}_{m_1}\mathcal{C}) = p$ (for some $p \geq 2$), then we have*
$$i_{m_1}(\mathcal{C}, V(H_F)) = p-2 \quad \textit{and} \quad i_{m_1}(\mathcal{C}, V(\Delta_P F)) = (p-1) + \mathbf{1}_{m_1=\Pi(P)}.$$
- *If F admits at m_1 an ordinary node, we have $q=2$, $b=2$, $e_1=e_2=1$, $i^{(1)}_{m_1} = i^{(2)}_{m_1} = 2$, $i^{(1,2)}_{m_1} = 1$, and so*
$$i_{m_1}(\mathcal{C}, V(H_F)) = 6 \quad \textit{and} \quad i_{m_1}(\mathcal{C}, V(\Delta_P F)) = \begin{cases} 2 & \textit{if } \Pi(P) \notin (\mathcal{D}^{(1)}_{m_1} \cup \mathcal{D}^{(2)}_{m_1}) \\ 3 & \textit{if } \Pi(P) \in (\mathcal{D}^{(1)}_{m_1} \cup \mathcal{D}^{(2)}_{m_1}) \setminus \{m_1\} \\ 5 & \textit{if } \Pi(P) = m_1 \end{cases}.$$
- *If F admits at m_1 an ordinary cusp, we have $q=2$, $b=1$, $e_1=2$, $\mathcal{D}^{(1)}_{m_1} = \mathcal{D}^{(2)}_{m_1}$, $i^{(1)}_{m_1} = i^{(2)}_{m_1} = 3/2$, $i^{(1,2)}_{m_1} = 3/2$, and so*
$$i_{m_1}(\mathcal{C}, V(H_F)) = 8 \quad \textit{and} \quad i_{m_1}(\mathcal{C}, V(\Delta_P F)) = \begin{cases} 3 & \textit{if } \Pi(P) \notin \mathcal{D}^{(1)}_{m_1} \\ 4 & \textit{if } \Pi(P) \in \mathcal{D}^{(1)}_{m_1} \setminus \{m_1\} \\ 6 & \textit{if } \Pi(P) = m_1 \end{cases}.$$

In this corollary, we recognize the terms appearing in the classical Plücker formulas (see [11, p. 278-279]). Proposition 25 is a direct consequence of (11) and of the following lemma.

Lemma 28. *Under assumptions of Proposition 25, for every $i = 1, ..., q$, using notations $R_i(x) := U(x, g_i(x)) \prod_{j \in \mathcal{I}: j \neq i}(g_i(x) - g_j(x))$ and $(x_P, y_P, z_P) = M^{-1}(P)$, we have*
$$\Delta_P F(M(x, g_i(x), 1)) = R_i(x) \left[y_P - x_P g_i'(0) + x_P(g_i'(0) - g_i'(x)) + z_P(x g_i'(x) - g_i(x)) \right]$$
and
$$val_x(\Delta_P F(M(x, g_i(x), 1)) = \left[\sum_{j \in \mathcal{I}: j \neq i} i^{(i,j)}_{m_1} \right] + (i^{(i)}_{m_1} - 1)\mathbf{1}_{\Pi(P) \in \mathcal{D}^{(i)}_{m_1}} + \mathbf{1}_{\Pi(P)=m_1}.$$
Moreover, we have
$$H_F(M(x, g_i(x), 1)) = (detM)^{-2}(d-1)^2 (R_i(x))^3 g_i''(x)$$
and
$$val_x(H_F(M(x, g_i(x), 1)) = 3 \left[\sum_{j \in \mathcal{I}: j \neq i} i^{(i,j)}_{m_1} \right] + (i^{(i)}_{m_1} - 2).$$

Proof. We have, in $\mathbb{C}[[x^*]][y]$,
$$F(M(x,y,1)) = U(x,y)G(x,y) \quad \text{with} \quad G(x,y) := \prod_{i=1}^{q}(y - g_i(x)), \quad \text{and} \quad U(0,0) \neq 0,$$
with $g_i(x) \in \mathbb{C}[[x^*]]$. Let $i = 1, ..., q$. Let us set $F_i(x) := (x, g_i(x), 1)$. We get[3]
$$(F \circ M)_x(F_i(x)) = -U(x, g_i(x)) g_i'(x) \prod_{j \in \mathcal{I}: j \neq i}(g_i(x) - g_j(x)),$$
$$(F \circ M)_y(F_i(x)) = U(x, g_i(x)) \prod_{j \in \mathcal{I}: j \neq i}(g_i(x) - g_j(x)).$$

[3]On $\{z = 1\}$, we have $(F \circ M)_x = U_x G + U G_x$, $(F \circ M)_y = U_y G + U G_y$, $G_x(x,y,1) = -\sum_{i \in \mathcal{I}} g_i'(x) \prod_{j \in \mathcal{I}: j \neq i}(y - g_j(x))$, $G_y(x,y,1) = \sum_{i \in \mathcal{I}} \prod_{j \in \mathcal{I}: j \neq i}(y - g_j(x))$. We conclude by using the fact that $G(x, g_i(x)) = 0$.

- Since $\Delta_P F(M(x,y,1)) = \Delta_{M^{-1}P}(F\circ M)(x,y,1) = W_P(x,y,1) + dF(M(x,y,1))$ with $W_P(x,y,z) := (x_P z - x z_P)(F\circ M)_x + (y_P z - y z_P)(F\circ M)_y$. So, we have

$$\begin{aligned}\Delta_P F(F_i(x)) &= W_P(F_i(x))\\ &= U(x,g_i(x))\left[(y_P - g_i(x)z_P) - (x_P - x z_P)g_i'(x)\right]\prod_{j\in\mathcal{I}:j\neq i}(g_i(x)-g_j(x)),\end{aligned}$$

which gives the first formula.

  - If $\Pi(P)\notin\mathcal{D}^{(i)}_{m_1}$, i.e. $y_P - x_P g_i'(0)\neq 0$, then

$$LM\left(\Delta_P F(M(F_i(x)))\right) = LM\left(U(0,0)\left[y_P - x_P g_i'(0)\right]\prod_{j\in\mathcal{I}:j\neq i}(g_i(x)-g_j(x))\right); \tag{12}$$

  - if $\Pi(P)\in\mathcal{D}^{(i)}_{m_1}$, i.e. $y_P - x_P g_i'(0) = 0$, quantity $(y_P - g_i(x)z_P) - (x_P - x z_P)g_i'(x)$ can be rewritten

$$x_P(g_i'(0) - g_i'(x)) + z_P(g_i'(x)x - g_i(x)).$$

    We distinguish now the cases $\Pi(P)\in\mathcal{D}^{(i)}_{m_1}$ and $\Pi(P) = m_1$.
  - if $\Pi(P)\in\mathcal{D}^{(i)}_{m_1}\setminus\{m_1\}$, i.e. $y_P = x_P g_i'(0)$ and $x_P\neq 0$, then

$$LM\left(\Delta_P F(M(F_i(x)))\right) = LM\left(U(0,0)\left[x_P(g_i'(0) - g_i'(x))\right]\prod_{j\in\mathcal{I}:j\neq i}(g_i(x)-g_j(x))\right). \tag{13}$$

  - if $\Pi(P) = m_1$, then

$$LM\left(\Delta_P F(M(F_i(x)))\right) = LM\left(U(0,0)\left[z_P(x g_i'(x) - g_i(x))\right]\prod_{j\in\mathcal{I}:j\neq i}(g_i(x)-g_j(x))\right). \tag{14}$$

  We conclude thanks to Remark 23.
- We have

$$\begin{aligned}H_F(M(F_i(x))) &= (\det M)^{-2}H_{F\circ M}(F_i(x))\\ &= (\det M)^{-2}(d-1)^2 h_{F\circ M}(F_i(x))\\ &= (\det M)^{-2}(d-1)^2 U^3(x,g_i(x))h_G(x,g_i(x)),\end{aligned}$$

with $h_G := 2G_{x,y} + U_x G_y + U_y G_x + U G_{xy}$. [4] Now, let us compute $h_G(x,g_i(x))$. We have[5]

$$G_x(x,g_i(x)) = -g_i'(x)\prod_{j\in\mathcal{I}:j\neq i}(g_i(x)-g_j(x)),\quad G_y(x,g_i(x)) = \prod_{j\in\mathcal{I}:j\neq i}(g_i(x)-g_j(x)),$$

$$G_{xx}(x,g_i(x)) = 2\sum_{j\in\mathcal{I}:j\neq i} g_i'(x)g_j'(x)\prod_{k\neq i,j}(g_i(x)-g_k(x)) + (-g_i''(x))\prod_{k\neq i}(g_i(x)-g_k(x)),$$

$$G_{yy}(x,g_i(x)) = 2\sum_{j\in\mathcal{I}:j\neq i}\prod_{k\neq i,j}(g_i(x)-g_k(x)),$$

$$G_{xy}(x,g_i(x)) = \sum_{j\in\mathcal{I}:j\neq i}(-g_i'(x) - g_j'(x))\prod_{k\neq i,j}(g_i(x)-g_k(x)).$$

[4] Since, on $\{z=1\}$, $F\circ M = UG$, so $(F\circ M)_x = U_x G + U G_x$, $(F\circ M)_y = U_y G + U G_y$ and $(F\circ M)_{xx} = U_{xx}G + 2U_x G_x + U G_{xx}$, $(F\circ M)_{yy} = U_{yy}G + 2U_y G_y + U G_{yy}$ and $(F\circ M)_{xy} = U_{xy}G + U_x G_y + U_y G_x + U G_{xy}$. Now the fact that $h_{F\circ M}(F_i(x)) = U(x,g_i(x))h_G(x,g_i(x))$ comes from $G(x,g_i(x)) = 0$.

[5] Indeed we have $G_x(x,y) = -\sum_{i\in\mathcal{I}} g_i'(x)\prod_{j\in\mathcal{I}:j\neq i}(y-g_j(x))$, $G_y(x,y) = \sum_{i\in\mathcal{I}}\prod_{j\in\mathcal{I}:j\neq i}(y-g_j(x))$, $G_{xx}(x,y) = \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{I}:j\neq i} g_i'(x)g_j'(y)\prod_{k\neq i,j}(y-g_k(x)) - \sum_{i\in\mathcal{I}} g_i''(x)\prod_{j\in\mathcal{I}:j\neq i}(y-g_j(x))$, $G_{yy}(x,y) = \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{I}:j\neq i}\prod_{k\neq i,j}(y-g_k(x))$ and $G_{xy}(x,y) = -\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{I}:j\neq i} g_i'(x)\prod_{k\neq i,j}(y-g_k(x))$.

A direct computation gives

$$h_G(\cdot, g_i(\cdot), 1) = \left(\prod_{j\in\mathcal{I}:j\neq i} (g_i - g_j) \right)^3 g_i'',$$

and so the two last results since $\mathrm{val}(g_i'') = i_{m_1} - 2$.

□

Let us observe that we always have $3\sum_{j\in\mathcal{I}:j\neq i} \mathrm{val}(g_i - g_j) + i_{m_1} - 2 \geq 0$ (if $y = g_i(x)$ is a pro-branch of a branch $\mathcal{B}_\beta$ with $e_\beta = 1$, then $i_{m_1} \geq 2$, otherwise $\mathcal{B}_\beta$ admits at least another pro-branch $y = g_j(x)$ and $\mathrm{val}(g_i - g_j) + val(g_i'') \geq 2i_{m_1}^{(i)} - 2 \geq 0$).

Now, according to Proposition 24 and to Lemma 28, we have

Remark 29. *Under hypotheses of Proposition 24, the following quantities are equal for M and for $\hat{M}$:*

$$\sum_{i\in\mathcal{I}_\beta} i_{m_1}^{(i)} = i_{m_1}(\mathcal{T}_\beta, \mathcal{B}_\beta),$$

$$\sum_{i\in\mathcal{I}_\beta} \sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)} = \frac{1}{3}\left[i_{m_1}(V(H_F), \mathcal{B}_\beta) - i_{m_1}(\mathcal{T}_\beta, \mathcal{B}_\beta) + 2e_\beta\right] = i_{m_1}(\mathcal{B}_\beta, \Delta_P),$$

where $\mathcal{I}_\beta$ is the set of indices $i \in \mathcal{I} = \{1, ..., q\}$ such that $y = g_i(x)$ is a pro-branch of $\mathcal{B}_\beta$ and for any $P \in \mathbb{P}^2 \setminus \mathcal{T}_\beta$.

6. Proof of Theorems 2 and 3

According to the fundamental lemma and to (11), to prove Theorems 2 and 3, we have to compute intersection numbers of branches of $V(F \circ M)$ at $[0 : 0 : 1]$ with $M^{-1}(\mathcal{P}_{\Phi,a})$ for some suitable $M \in GL(\mathbb{C}^3)$ and for generic $a \in \mathbb{P}^2$. This is the aim of the following result.

Proposition 30. *We suppose that $\mathcal{C} = V(F)$ is irreducible with degree $d \geq 2$ and that $\mathcal{S} \notin \{\mathcal{I}, \mathcal{J}\}$. Let m_1 be a point of multiplicity q of $\mathcal{C}$, that is a base point of $\Phi_{F,S}$. Let $M \in GL(\mathbb{C}^3)$ be such that $\Pi(M(0, 0, 1)) = m_1$, such that the tangent cone of $V(F \circ M)$ at $[0 : 0 : 1]$ does not contain $X = 0$. We write $\mathcal{B}_1, ..., \mathcal{B}_b$ the branches of $V(F \circ M)$ at $[0 : 0 : 1]$; $\mathcal{T}_1, ..., \mathcal{T}_b$ their respective tangent lines. Let also $\mathcal{I} := \{1, ..., q\}$ and $y = g_i(x); i \in \mathcal{I}$ be the equations of the pro-branches of $V(F \circ M)$ at $[0 : 0 : 1]$.*

Let $\beta \in \{1, ..., b\}$. Let $\mathcal{I}_\beta$ be the set of indices $i \in \mathcal{I}$ such that $y = g_i(x)$ are the equations of the pro-branches ot $V(F \circ M)$ associated to branch $\mathcal{B}_\beta$ at m_1 (for $i \in \mathcal{I}_\beta$, we have $\mathcal{D}_{m_1}^{(i)} = \mathcal{T}_\beta$).

Then, for a generic point $a \in \mathbb{P}^2$, we have

$$i_{[0:0:1]}\left(\mathcal{B}_\beta, M^{-1}(\mathcal{P}_{\Phi_{F,S},a})\right) = \sum_{i\in\mathcal{I}_\beta} \left(\alpha_i + 3 \sum_{j\in\mathcal{I}:j\neq i} val(g_i - g_j) \right).$$

with

- **Generic cases**

(S1) $\alpha_i := i_{m_1}^{(i)} - 2$ *if* $i_{m_1}^{(i)} < 2$ *and* $m_1 \notin (\mathcal{I}\mathcal{S}) \cup (\mathcal{J}\mathcal{S})$,
(S2) $\alpha_i := 0$ *if* $i_{m_1}^{(i)} \geq 2$ *and* $\mathcal{I}, \mathcal{J}, \mathcal{S} \notin \mathcal{D}_{m_1}^{(i)}$
(S3) $\alpha_i := 0$ *if* $i_{m_1}^{(i)} < 2$, $m_1 \in (\mathcal{I}\mathcal{S}) \cup (\mathcal{J}\mathcal{S})$ *and* $\mathcal{I}, \mathcal{J}, \mathcal{S} \notin \mathcal{D}_{m_1}^{(i)}$.

- **If $\mathcal{I}$ is in $\mathcal{D}^{(i)}_{m_1}$**

(S4) $\alpha_i := 0$ *if* $\mathcal{I} \in \mathcal{D}^{(i)}_{m_1} \setminus \{m_1\}$, $i^{(i)}_{m_1} \geq 2$ *and* $\mathcal{J}, \mathcal{S} \notin \mathcal{D}^{(i)}_{m_1}$.
(S5) $\alpha_i := 2(i^{(i)}_{m_1} - 1)$ *if* $\mathcal{S} \in \mathcal{D}^{(i)}_{m_1} \setminus \{m_1\}$, $\mathcal{I} \in \mathcal{D}^{(i)}_{m_1}$, $\mathcal{J} \notin \mathcal{D}^{(i)}_{m_1}$ *and* $i^{(i)}_{m_1} \neq 2$.
$\alpha_i := \min(3, \beta_1)$ *if* $\mathcal{S} \in \mathcal{D}^{(i)}_{m_1} \setminus \{m_1\}$, $\mathcal{I} \in \mathcal{D}^{(i)}_{m_1}$, $\mathcal{J} \notin \mathcal{D}^{(i)}_{m_1}$, $i^{(i)}_{m_1} = 2$ *and if* β_1 *is the degree of the lowest degree term of* $g_i(x)$ *of degree strictly larger than 2.*
(S6) $\alpha_i := i^{(i)}_{m_1} - 2$ *if* $\mathcal{I}, \mathcal{J} \in \mathcal{D}^{(i)}_{m_1} \setminus \{m_1\}$, *and* $\mathcal{S} \notin \mathcal{D}^{(i)}_{m_1}$.
(S7) $\alpha_i := 3i^{(i)}_{m_1} - 3$ *if* $\mathcal{I}, \mathcal{J}, \mathcal{S} \in \mathcal{D}^{(i)}_{m_1} \setminus \{m_1\}$.
(S8) $\alpha_i := 0$ *if* $\mathcal{I} = m_1$, *and* $\mathcal{J}, \mathcal{S} \notin \mathcal{D}^{(i)}_{m_1}$.
(S9) $\alpha_i := i^{(i)}_{m_1} - 1$ *if* $\mathcal{I} = m_1$, $\mathcal{J} \in \mathcal{D}^{(i)}_{m_1}$, $\mathcal{S} \notin \mathcal{D}^{(i)}_{m_1}$.
(S10) $\alpha_i = 3i^{(i)}_{m_1} - 3$ *if* $\mathcal{I} = m_1$, $\mathcal{S}, \mathcal{J} \in \mathcal{D}^{(i)}_{m_1}$,

- **Other cases when $\mathcal{S}$ is in $\mathcal{D}^{(i)}_{m_1}$**

(S11) $\alpha_i := i^{(i)}_{m_1} - 2$ *if* $\mathcal{S} \in \mathcal{D}^{(i)}_{m_1} \setminus \{m_1\}$ *and* $\mathcal{I}, \mathcal{J} \notin \mathcal{D}^{(i)}_{m_1}$.
(S12) $\alpha_i := i^{(i)}_{m_1}$ *if* $\mathcal{S} = m_1$ *and* $\mathcal{I}, \mathcal{J} \notin \mathcal{D}^{(i)}_{m_1}$.
(S13) $\alpha_i := 2i^{(i)}_{m_1} - 1$ *if* $\mathcal{S} = m_1$, $\mathcal{I} \in \mathcal{D}^{(i)}_{m_1}$ *and* $\mathcal{J} \notin \mathcal{D}^{(i)}_{m_1}$.
(S14) $\alpha_i := 3i^{(i)}_{m_1} - 2$ *if* $\mathcal{S} = m_1$, $\mathcal{I}, \mathcal{J} \in \mathcal{D}^{(i)}_{m_1}$ *and* $i^{(i)}_{m_1} \neq 2$.
$\alpha_i := \min(\beta_2 + 2, 6)$ *if* $\mathcal{S} = m_1$, $\mathcal{I}, \mathcal{J} \in \mathcal{D}^{(i)}_{m_1}$ *and* $i^{(i)}_{m_1} = 2$ *and if* β_2 *is the degree of the lowest degree term of* $g_i(x)$ *of degree* $\notin \{1, 2, 3\}$.

For symetry reasons, once this will be proven, same formulas will also hold true if we exchange $\mathcal{I}$ and $\mathcal{J}$.

Scheme of the proofs of Theorems 2 and 3. Assume assumptions of Theorems 2 or 3 hold true. According to the fundamental lemma and to (11), we have, for generic $a \in \mathbb{P}^2$,

$$\mathrm{mdeg}(\Sigma_{\mathcal{S}}(\mathcal{C})) = \delta_1.\deg(\Sigma_{\mathcal{S}}(\mathcal{C})) = 3d(d-1) - \sum_{m_1 \in Base((\Phi_{F,\mathcal{S}})_{|\mathcal{C}})} i_{m_1}\left(\mathcal{C}, \mathcal{P}_{\Phi_{F,\mathcal{S}},a}\right),$$

with δ_1 the degree of the rational curve $\Phi_{F,\mathcal{S}}$. With the notations of Proposition 30, let us write, for every m_1 as in Proposition 30,

$$\alpha(m_1) := \sum_{i \in \mathcal{I}} \alpha_i \quad \text{and} \quad V_{m_1} := \sum_{i,j \in \mathcal{I}: i \neq j} i^{(i,j)}_{m_1}.$$

According to Proposition 30, we get

$$\mathrm{mdeg}(\Sigma_{\mathcal{S}}(\mathcal{C})) = 3d(d-1) - 3 \sum_{m_1 \in Sing(\mathcal{C})} V_{m_1} - \sum_{m_1 \in Base((\Phi_{F,\mathcal{S}})_{|\mathcal{C}})} \alpha(m_1)$$

Now, using Corollary 26, we get that

$$\mathrm{mdeg}(\Sigma_{\mathcal{S}}(\mathcal{C})) = 3d^{\vee} - \sum_{m_1 \in Base((\Phi_{F,\mathcal{S}})_{|\mathcal{C}})} \alpha(m_1).$$

We conclude the proofs by using the expressions of α_i given in Proposition 30. □

We will use the following technical lemma concerning changes of coordinates. For any $A, B, S', P \in \mathbb{C}^3 \setminus \{0\}$ and any homogeneous polynomial $F \in \mathbb{C}[X, Y, Z]$, we define

$$\tilde{\Phi}^{(A,B)}_{F,S'} = -\frac{2H_F.f_{A,S'}.f_{B,S'}}{(d-1)^2}.Id + \Delta_A F.\Delta_B F.\Delta_{S'} F.S' - (\Delta_{S'} F)^2[\Delta_A F.B + \Delta_B F.A], \qquad (15)$$

where $f_{A,B}(C) = \det(A|B|C)$, for every $A, B, C \in \mathbb{C}^3$. We recall that $V(f_{A,B})$ is the line $(\Pi(A)\Pi(B))$. We have already observed in (9) that

$$\tilde{\Phi}_{F,S} = \tilde{\Phi}_{F,S}^{(I,J)}.$$

Lemma 31. *For any M in $GL(\mathbb{C}^3)$, any A, B, S, P in $\mathbb{C}^3$ and any homogeneous polynomial F, we have*

$$\tilde{\Phi}_{F\circ M^{-1},M(S)}^{(M(A),M(B))}(M(P)) = M(\tilde{\Phi}_{F,S}^{(A,B)}(P)).$$

Proof. The lemma is a direct consequence of the following facts :

$$H_{F\circ M^{-1}}(M(P)) = (\det M)^{-2} H_F(P),$$

$$f_{M(A),M(B)}(M(P)) = \det(M) f_{A,B}(P),$$

$$\Delta_{M(A)}(F \circ M^{-1})(M(P)) = \Delta_A F(P).$$

□

Proof of Proposition 30. Let m_1 be a point of $\mathcal{C}$ with multiplicity q. We use notations of Proposition 30, in particular $\mathcal{I} := \{1, ..., q\}$. Let us write $M_1 := (0, 0, 1)$. Now, proposition 24 and Remark 29 allow us to consider each branch separately and to adapt our change of variable to each of them.

Consider a branch $\mathcal{B}_\beta$ of $\mathcal{C}$ at m_1. Let $i \in \mathcal{I}_\beta$. We suppose that our change of variable is such that $g_i'(0) = 0$ (i.e. $\mathcal{T}_\beta$ has equation $Y = 0$). We define $A := M^{-1}(I)$, $B := M^{-1}(J)$, $S' := M^{-1}(S)$ and $\hat{F} := F \circ M$. According to Lemma 31, we have

$$\tilde{\Phi}_{F,S}^{(I,J)}(M(P)) = M\tilde{\Phi}^{(A,B)}(P), \quad \text{with} \quad \tilde{\Phi}^{(A,B)} := \tilde{\Phi}_{\hat{F},S'}^{(A,B)}.$$

To simplify notations, we write

$$F_i(x) := (x, g_i(x), 1) \quad \text{and} \quad R_i(x) = U(x, g_i(x)) \prod_{j\in\mathcal{I}:j\neq i} (g_i(x) - g_j(x)).$$

We know that, for every $a = [a_1 : a_2 : a_3]$

$$i_{[0:0:1]}\left(\mathcal{B}_\beta, M^{-1}(\mathcal{P}_{\Phi_{F,S},a})\right) = \sum_{i\in\mathcal{I}_\beta} \text{val}\left(\sum_{j=1}^{3} ({}^tM \cdot a)_j \tilde{\Phi}_j^{(A,B)} \circ F_i\right).$$

We notive that, for a generic $a \in \mathbb{P}^2$, we have

$$\text{val}\left(\sum_{j=1}^{3} a_j \tilde{\Phi}_j^{(A,B)} \circ F_i\right) = \min\left(\text{val}\left(\tilde{\Phi}_j^{(A,B)} \circ F_i\right) \ , \ j = 1, 2, 3\right).$$

Let us rewrite formula (15):

$$\tilde{\Phi}^{(A,B)} = \psi_1.Id + \psi_2.S' - \psi_3.B - \psi_4.A, \tag{16}$$

with

$$\psi_1 := -\frac{2H_{\hat{F}}.f_{A,S'}.f_{B,S'}}{(d-1)^2}, \quad \psi_2 := \Delta_A\hat{F}.\Delta_B\hat{F}.\Delta_{S'}\hat{F},$$

$$\psi_3 := (\Delta_{S'}\hat{F})^2\Delta_A\hat{F} \quad \text{and} \quad \psi_4 := (\Delta_{S'}\hat{F})^2\Delta_B\hat{F}.$$

Therefore, $\mathrm{val}\left(\sum_{j=1}^{3} a_j \tilde{\Phi}_j^{(A,B)} \circ F_i\right)$ is greater than or equal to the minimum of the four following quantities (the computation of which comes directly from lemma 28):

$$\mathrm{val}\,(\psi_1 \circ F_i) = \left(3 \sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + (i_{m_1}^{(i)} - 2) + \mathbf{1}_{m_1\in(\mathcal{IS})} + \mathbf{1}_{m_1\in(\mathcal{JS})} + (i_{m_1}^{(i)} - 1)(\mathbf{1}_{\mathcal{D}_{m_1}^{(i)}=(\mathcal{IS})} + \mathbf{1}_{\mathcal{D}_{m_1}^{(i)}=(\mathcal{JS})}),$$

$$\mathrm{val}\,(\psi_2 \circ F_i) = \left(3 \sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + (i_{m_1}^{(i)} - 1)\left(\mathbf{1}_{\mathcal{I}\in\mathcal{D}_{m_1}^{(i)}} + \mathbf{1}_{\mathcal{J}\in\mathcal{D}_{m_1}^{(i)}} + \mathbf{1}_{\mathcal{S}\in\mathcal{D}_{m_1}^{(i)}}\right) + \mathbf{1}_{m_1\in\{\mathcal{I},\mathcal{J},\mathcal{S}\}},$$

$$\mathrm{val}\,(\psi_3 \circ F_i) = \left(3 \sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + (i_{m_1}^{(i)} - 1)\left(\mathbf{1}_{\mathcal{I}\in\mathcal{D}_{m_1}^{(i)}} + 2\times\mathbf{1}_{\mathcal{S}\in\mathcal{D}_{m_1}^{(i)}}\right) + \mathbf{1}_{m_1\in\{\mathcal{I},\mathcal{S}\}},$$

$$\mathrm{val}\,(\psi_4 \circ F_i) = \left(3 \sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + (i_{m_1}^{(i)} - 1)\left(\mathbf{1}_{\mathcal{J}\in\mathcal{D}_{m_1}^{(i)}} + 2\times\mathbf{1}_{\mathcal{S}\in\mathcal{D}_{m_1}^{(i)}}\right) + \mathbf{1}_{m_1\in\{\mathcal{J},\mathcal{S}\}}.$$

But it is not clear whether or not it is equal to this minimum. Hence, in some cases, we will need more than the values of these valuations. It will be useful to notice that, according to Lemma 28, we have

$$\Delta_P \hat{F} \circ F_i(x) = R_i(x)\left[y_P - g_i'(x)x_P + z_P(xg_i'(x) - g_i(x))\right]; \tag{17}$$

$$\Delta_P \hat{F} \circ F_i(x) = R_i(x)\left(-g_i'(x)x_P + z_P(xg_i'(x) - g_i(x))\right) \text{ if } \Pi(P) \in \mathcal{D}_{m_1}^{(i)}; \tag{18}$$

$$\Delta_P \hat{F} \circ F_i(x) = z_P R_i(x)(xg_i'(x) - g_i(x)) \text{ if } \Pi(P) = m_1; \tag{19}$$

$$H_{\hat{F}} \circ F_i(x) = (d-1)^2 (R_i(x))^3 g_i''(x). \tag{20}$$

(S1) <u>Suppose that $i_{m_1}^{(i)} < 2$, that m_1 does not belong to lines $(\mathcal{IS})$ and $(\mathcal{JS})$.</u>

Thanks to lemma 31 and to formulas (17), (18), (19), (20) of the proof of Proposition 25, we have

$$\mathrm{val}\left(-\frac{2H_{\hat{F}}.f_{A,S'}.f_{B,S'}}{(d-1)^2} \circ F_i\right) = \left(3 \sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + (i_{m_1}^{(i)} - 2)$$

is strictly less than the valuation of the three others terms. Therefore

$$\left(3 \sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + (i_{m_1}^{(i)} - 2) \leq \min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)} \circ F_i\right)\ ,\ j = 1,2,3\right)$$

and

$$\mathrm{val}\left(\tilde{\Phi}_3^{(A,B)} \circ F_i\right) = \left(3 \sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + (i_{m_1}^{(i)} - 2).$$

So

$$\min\left(\mathrm{val}\left(\tilde{\Phi}_j^{(A,B)} \circ F_i\right)\ ,\ j = 1,2,3\right) = \left(3 \sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + (i_{m_1}^{(i)} - 2).$$

(S2) Suppose that $i_{m_1}^{(i)} \geq 2$ and if $\mathcal{I}, \mathcal{J}, \mathcal{S} \notin \mathcal{D}_{m_1}^{(i)}$.

Thanks to lemma 31 and to formulas (17), (18), (19) and (20), we have

$$3 \sum_{j \in \mathcal{I}: j \neq i} i_{m_1}^{(i,j)} \leq \min \left(\mathrm{val}_x \left(\tilde{\Phi}_j^{(A,B)}(x, g_i(x), 1) \right) \ , \ j = 1, 2, 3 \right).$$

Adapting our change of variable, we can suppose that $S' = (0, 1, 0)$ and that $g_i'(0) = 0$. This implies that $y_A \neq 0$ and $y_B \neq 0$. We have

$$LM(\Delta_A \hat{F} \circ F_i) = y_A LM(R_i), \ LM(\Delta_B \hat{F} \circ F_i) = y_B LM(R_i),$$

$$LM(\Delta_{S'} \hat{F} \circ F_i) = LM(R_i) \ \text{ and } \ \mathrm{val}(H_{\hat{F}} \circ F_i) \geq 3 \sum_{j \in \mathcal{I}: j \neq i} i_{m_1}^{(i,j)}.$$

So

$$\min \left(\mathrm{val} \left(\tilde{\Phi}_j^{(A,B)} \circ F_i \right) \ , \ j = 1, 2, 3 \right) \geq 3 \sum_{j \in \mathcal{I}: j \neq i} i_{m_1}^{(i,j)}$$

$$LM((\Delta_A \hat{F} \Delta_B \hat{F} \Delta_{S'} \hat{F}) \circ F_i) = y_A y_B LM((R_i)^3)$$

and

$$LM(((\Delta_{S'} \hat{F})^2 (\Delta_A \hat{F} y_B + \Delta_B \hat{F} y_A)) \circ F_i) = -2 y_A y_B LM((R_i)^3).$$

Therefore

$$LM(\tilde{\Phi}_2^{(A,B)} \circ F_i) = -y_A y_B LM((R_i)^3)$$

and finally

$$\min \left(\mathrm{val}_x \left(\tilde{\Phi}_j^{(A,B)} \circ F_i \right) \ , \ j = 1, 2, 3 \right) = 3 \sum_{j \in \mathcal{I}: j \neq i} i_{m_1}^{(i,j)}.$$

(S3) We suppose that $i_{m_1}^{(i)} < 2$, that m_1 is in line $(\mathcal{I}\mathcal{S})$ and that $\mathcal{I}, \mathcal{J}, \mathcal{S} \notin \mathcal{D}_{m_1}^{(i)}$.

Again, we suppose that $S' = (0, 1, 0)$ and that $g_i'(0) = 0$. Since $i_{m_1}^{(i)} > 1$, we observe that

$$\mathrm{val}([H_{\hat{F}} f_{A,S'} f_{B,S'}] \circ F_i) > 3 \sum_{j \in \mathcal{I}: j \neq i} i_{m_1}^{(i,j)},$$

$$LM((\Delta_A \hat{F} \Delta_B \hat{F} \Delta_{S'} \hat{F}) \circ F_i) = y_A y_B LM((R_i)^3)$$

$$LM([(\Delta_{S'} \hat{F})^2 (\Delta_A \hat{F} y_B + \Delta_B \hat{F} y_A)] \circ F_i) = -2 y_A y_B LM((R_i)^3).$$

As in the previous point, we get that

$$\min \left(\mathrm{val} \left(\tilde{\Phi}_j^{(A,B)} \circ F_i \right) \ , \ j = 1, 2, 3 \right) = 3 \sum_{j \in \mathcal{I}: j \neq i} i_{m_1}^{(i,j)}.$$

(S4) Suppose that $\mathcal{I} \in \mathcal{D}_{m_1}^{(i)} \setminus \{m_1\}$, $i_{m_1}^{(i)} \geq 2$, $\mathcal{J}, \mathcal{S} \notin \mathcal{D}_{m_1}^{(i)}$.

Assume that $M_1 = (0, 0, 1)$, $S' = (0, 1, 0)$, that $A = (1, 0, 0)$. We have $g_i'(0) = 0$. Using (17) for B and S', (18) for A, we get

$$\mathrm{val}([H_{\hat{F}} f_{A,S'} f_{B,S'}] \circ F_i) \geq 3 \sum_{j \in \mathcal{I}: j \neq i} i_{m_1}^{(i,j)},$$

$$\mathrm{val}((\Delta_A \hat{F} \Delta_B \hat{F} \Delta_{S'} \hat{F}) \circ F_i) = \left(3 \sum_{j \in \mathcal{I}: j \neq i} i_{m_1}^{(i,j)} \right) + i_{m_1}^{(i)} - 1 > 3 \sum_{j \in \mathcal{I}: j \neq i} i_{m_1}^{(i,j)},$$

$$\mathrm{val}([(\Delta_{S'} \hat{F})^2 \Delta_B \hat{F}] \circ F_i) = 3 \sum_{j \in \mathcal{I}: j \neq i} i_{m_1}^{(i,j)},$$

$$\mathrm{val}([(\Delta_{S'}\hat{F})^2\Delta_A\hat{F}]\circ F_i) = \left(3\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + i_{m_1}^{(i)} - 1.$$

So

$$\min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)}\circ F_i\right)\ ,\ j=1,2,3\right) \geq 3\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}.$$

Moreover $\mathrm{val}(\tilde{\Phi}_1^{(A,B)}\circ F_i) = \mathrm{val}([(\Delta_{S'}\hat{F})^2\Delta_B\hat{F}]\circ F_i)$. So

$$\min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)}\circ F_i\right)\ ,\ j=1,2,3\right) = 3\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}.$$

(S5) <u>Suppose that $\mathcal{S}\in\mathcal{D}_{m_1}^{(i)}\setminus\{m_1\}$, $\mathcal{I}\in\mathcal{D}_{m_1}^{(i)}$ and $\mathcal{J}\notin\mathcal{D}_{m_1}^{(i)}$.</u>

Assume that $M_1=(0,0,1)$, $S'=(1,0,0)$, $B=(0,1,0)$, $g_i'(0)=0$. We have $y_A=0$, $z_A\neq 0$, $f_{A,S'}(x,y,1)=z_Ay-y_A=z_Ay$ and $f_{B,S'}(x,y,1)=-1$. Using (17) for B, (18) for A and S' and (20), we get

$$-\frac{2[H_{\hat{F}}f_{A,S'}f_{B,S'}](F_i(x))}{(d-1)^2} = 2z_A(R_i(x))^3g_i''(x)g_i(x),$$

$$\Delta_{S'}\hat{F}(F_i(x)) = -R_i(x)g_i'(x),\quad \Delta_B\hat{F}(F_i(x)) = R_i(x),$$

$$\Delta_A\hat{F}(F_i(x)) = R_i(x)[-g_i'(x)x_A + z_A(xg_i'(x)-g_i(x))]$$

and so

$$(\Delta_A\hat{F}\Delta_B\hat{F}\Delta_{S'}\hat{F})(F_i(x)) = R_i(x)^3(-g_i'(x))[-g_i'(x)x_A + z_A(xg_i'(x)-g_i(x))],$$

$$[(\Delta_{S'}\hat{F})^2\Delta_B\hat{F}](F_i(x)) = R_i(x)^3(g_i'(x))^2,$$

$$[(\Delta_{S'}\hat{F})^2\Delta_A\hat{F}](F_i(x)) = R_i(x)^3(g_i'(x))^2[-g_i'(x)x_A + z_A(xg_i'(x)-g_i(x))].$$

We cannot conclude since three terms have the smallest valuation. We will see that if $i_{m_1}^{(i)}=2$, the smallest degree terms are cancelled. The situation here requires some precise estimate. Therefore we have

$$\tilde{\Phi}_1^{(A,B)}(F_i(x)) = z_A(R_i(x))^3[2g_i''(x)g_i(x)x - g_i'(x)(xg_i'(x)-g_i(x))],$$

$$\tilde{\Phi}_2^{(A,B)}(F_i(x)) = (R_i(x))^3[2z_Ag_i''(x)(g_i(x))^2 - (g_i'(x))^2[-g_i'(x)x_A + z_A(xg_i'(x)-g_i(x))]$$

$$\tilde{\Phi}_3^{(A,B)}(F_i(x)) = z_A(R_i(x))^3[2g_i''(x)g_i(x) - (g_i'(x))^2].$$

We use the fact that there exists $\alpha\neq 0$ and $\beta = i_{m_1}^{(i)} > 1$ such that

$$LM(g_i(x)) = \alpha x^\beta,\ LM(g_i'(x)) = \alpha\beta x^{\beta-1}\ \text{ and }\ LM(g_i''(x)) = \alpha\beta(\beta-1)x^{\beta-2}.$$

We get that

$$LM(\tilde{\Phi}_1^{(A,B)}(F_i(x))) = LM(z_A(R_i(x))^3[\alpha^2\beta(\beta-1)x^{2\beta-1}].$$

So

$$\mathrm{val}(\tilde{\Phi}_1^{(A,B)}\circ F_i) = \left(3\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + 2i_{m_1}^{(i)} - 1.$$

Moreover

$$\mathrm{val}(\tilde{\Phi}_2^{(A,B)}\circ F_i) \geq \left(3\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + 3(i_{m_1}^{(i)} - 1).$$

Oberve now that

$$\mathrm{val}(\tilde{\Phi}_3^{(A,B)} \circ F_i) \geq \left(3 \sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + 2i_{m_1}^{(i)} - 2$$

and that term of degree $2i_{m_1}^{(i)} - 2$ of $2g_i''(x)g_i(x) - (g_i'(x))^2$ is

$$\alpha^2(2\beta(\beta-1) - \beta^2)x^{2\beta-2} = \alpha^2\beta(\beta-2)x^{2\beta-2}.$$

Therefore

$$\boxed{\min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)} \circ F_i\right)\ ,\ j=1,2,3\right) = \left(3 \sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + 2i_{m_1}^{(i)} - 2 \quad \text{if} \quad i_{m_1}^{(i)} \neq 2}.$$

Now, if $\beta = 2$ and if $LM(g_i(x) - \alpha x^2) = \alpha_1 x^{\beta_1}$ (with $\alpha_1 \neq 0$ and $\beta_1 > 2$), we get that

$$val(\tilde{\Phi}_1^{(A,B)} \circ F_i) = \left(3 \sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + 3, \quad val(\tilde{\Phi}_2^{(A,B)} \circ F_i) \geq \left(3 \sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + 3$$

and

$$LM(2g_i''(x)g_i(x) - (g_i'(x))^2) = 2\alpha\alpha_1(\beta_1-1)(\beta_1-2)x^{\beta_1}$$

and so

$$\boxed{\min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)} \circ F_i\right)\ ,\ j=1,2,3\right) = 3\left(\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + \min(3,\beta_1) \quad \text{if} \quad i_{m_1}^{(i)} = 2}.$$

(S6) <u>Suppose that $\mathcal{I}, \mathcal{J} \in \mathcal{D}_{m_1}^{(i)} \setminus \{m_1\}$ and that $\mathcal{S} \notin \mathcal{D}_{m_1}^{(i)}$.</u>

We suppose that $M_1 = (0,0,1)$, $S' = (0,1,0)$, $A = (1,0,0)$. We have $g_i'(0) = 0$, $y_B = 0$, $x_B \neq 0$, $z_B \neq 0$, $f_{B,S'}(x,y,1) = x_B - xz_B$ and $f_{A,S'}(x,y,1) = 1$.

We have

$$LM\left(-\frac{2[H_{\hat{F}} f_{A,S'} f_{B,S'}](F_i(x))}{(d-1)^2}\right) = LM\left(-2x_B(R_i(x))^3 g_i''(x)\right),$$

$$LM(\Delta_A\hat{F}\Delta_B\hat{F}\Delta_{S'}\hat{F})(F_i(x))) = LM(R_i(x)^3 x_B(g_i'(x))^2),$$

$$LM([(\Delta_{S'}\hat{F})^2\Delta_B\hat{F}](F_i(x))) = LM(R_i(x)^3(-g_i'(x))x_B),$$

$$LM([(\Delta_{S'}\hat{F})^2\Delta_A\hat{F}](F_i(x))) = LM(R_i(x)^3(-g_i'(x))).$$

Hence

$$\min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)} \circ F_i\right)\ ,\ j=1,2,3\right) \geq \left(3 \sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + i_{m_1}^{(i)} - 2.$$

Moreover

$$LM(\tilde{\Phi}_3^{(A,B)}(F_i(x))) = LM(-2(R_i(x))^3 g_i''(x)x_B).$$

Therefore

$$\min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)} \circ F_i\right)\ ,\ j=1,2,3\right) \geq \left(3 \sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + i_{m_1}^{(i)} - 2.$$

(S7) Suppose that $\mathcal{I}, \mathcal{J}, \mathcal{S} \in \mathcal{D}^{(i)}_{m_1} \setminus \{m_1\}$.

We suppose that $M_1 = (0,0,1)$, $S' = (1,0,0)$. We have $g_i'(0) = 0$, $y_A = y_B = 0$, $x_A \neq 0$, $z_A \neq 0$, $x_B \neq 0$, $z_B \neq 0$, $f_{A,S'}(x,y,1) = z_A y$ $f_{B,S'}(x,y,1) = z_B y$.

We have

$$-\frac{2[H_{\hat{F}} f_{A,S'} f_{B,S'}](F_i(x))}{(d-1)^2} = -2 z_A z_B (R_i(x))^3 g_i''(x)(g_i(x))^2,$$

$$LM(\Delta_A \hat{F} \Delta_B \hat{F} \Delta_{S'} \hat{F})(F_i(x))) = LM(R_i(x)^3 x_A x_B (-g_i'(x))^3),$$
$$LM([(\Delta_{S'} \hat{F})^2 \Delta_B \hat{F}](F_i(x))) = LM(R_i(x)^3 (-g_i'(x))^3 x_B),$$
$$LM([(\Delta_{S'} \hat{F})^2 \Delta_A \hat{F}](F_i(x))) = LM(R_i(x)^3 (-g_i'(x))^3 x_A).$$

Hence

$$\min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)} \circ F_i\right)\ ,\ j = 1,2,3\right) \geq \left(3 \sum_{j \in \mathcal{I}: j \neq i} i_{m_1}^{(i,j)}\right) + 3(i_{m_1}^{(i)} - 1).$$

Since

$$\mathrm{val}\left(\tilde{\Phi}_1^{(A,B)} \circ F_i\right) = \mathrm{val}\left((R_i(x))^3 (g_i'(x))^3 x_A x_B\right);$$

Hence

$$\min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)} \circ F_i\right)\ ,\ j = 1,2,3\right) = \left(3 \sum_{j \in \mathcal{I}: j \neq i} i_{m_1}^{(i,j)}\right) + 3(i_{m_1}^{(i)} - 1).$$

(S8) Suppose that $\mathcal{I} = m_1$ and $\mathcal{J}, \mathcal{S} \notin \mathcal{D}^{(i)}_{m_1}$.

We suppose that $M_1 = A = (0,0,1)$, $S' = (0,1,0)$. We have $g_i'(0) = 0$, $y_B \neq 0$, $f_{B,S'}(x,y,1) = x_B - x z_B$, $f_{A,S'}(x,y,1) = x_A - x z_A = -x$.

We have

$$-\frac{2[H_{\hat{F}} f_{A,S'} f_{B,S'}](F_i(x))}{(d-1)^2} = 2(R_i(x))^3 g_i''(x) x (x_B - x z_B),$$

$$LM(\Delta_A \hat{F} \Delta_B \hat{F} \Delta_{S'} \hat{F})(F_i(x))) = LM(R_i(x)^3 y_B (x g_i'(x) - g_i(x))),$$
$$LM([(\Delta_{S'} \hat{F})^2 \Delta_B \hat{F}](F_i(x))) = LM(R_i(x)^3 y_B),$$
$$LM([(\Delta_{S'} \hat{F})^2 \Delta_A \hat{F}](F_i(x))) = LM(R_i(x)^3 (x g_i'(x) - g_i(x))).$$

Hence

$$\min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)} \circ F_i\right)\ ,\ j = 1,2,3\right) \geq 3 \sum_{j \in \mathcal{I}: j \neq i} i_{m_1}^{(i,j)}.$$

Since

$$\mathrm{val}\left(\tilde{\Phi}_3^{(A,B)} \circ F_i\right) = \mathrm{val}\left([(\Delta_{S'} \hat{F})^2 \Delta_B \hat{F}] \circ F_i\right),$$

we have

$$\min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)} \circ F_i\right)\ ,\ j = 1,2,3\right) = 3 \sum_{j \in \mathcal{I}: j \neq i} i_{m_1}^{(i,j)}.$$

(S9) Suppose that $\mathcal{I} = m_1$, $\mathcal{J} \in \mathcal{D}^{(i)}_{m_1}$, $\mathcal{S} \notin \mathcal{D}^{(i)}_{m_1}$.

We suppose that $M_1 = A = (0,0,1)$, $S' = (0,1,0)$, $B = (1,0,0)$. We have $g_i'(0) = 0$, $f_{B,S'}(x,y,1) = x_B - x z_B = x_B$, $f_{A,S'}(x,y,1) = x_A - x z_A = -x$.

We have

$$-\frac{2[H_{\hat{F}} f_{A,S'} f_{B,S'}](F_i(x))}{(d-1)^2} = 2 x_B (R_i(x))^3 g_i''(x) x,$$

$$LM(\Delta_A \hat{F} \Delta_B \hat{F} \Delta_{S'} \hat{F})(F_i(x))) = LM(R_i(x)^3 (x g_i'(x) - g_i(x)) x_B (-g_i'(x))),$$
$$LM([(\Delta_{S'} \hat{F})^2 \Delta_B \hat{F}](F_i(x))) = LM(R_i(x)^3 (-g_i'(x)) x_B),$$

$$LM([(\Delta_{S'}\hat{F})^2\Delta_A\hat{F}](F_i(x))) = LM(R_i(x)^3(xg_i'(x) - g_i(x))).$$

Hence

$$\min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)} \circ F_i\right)\ ,\ j = 1,2,3\right) \geq \left(3\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + i_{m_1}^{(i)} - 1.$$

Observe that

$$\mathrm{val}\left(\tilde{\Phi}_3^{(A,B)} \circ F_i\right) = \mathrm{val}_x\left((R_i(x))^3(2xg_i''(x) + g_i'(x))x_B\right).$$

Moreover, if $LM(g_i) = \alpha x^\beta$ for some $\alpha \neq 0$ and some $\beta = i_{m_1}^{(i)} > 1$, we get that $LM(2xg_i''(x) + g_i'(x)) = \alpha\beta(2\beta - 1)x^{\beta-1}$. Therefore

$$\min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)} \circ F_i\right)\ ,\ j = 1,2,3\right) = \left(3\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + i_{m_1}^{(i)} - 1.$$

(S10) <u>Suppose that $\mathcal{I} = m_1$, $\mathcal{S}, \mathcal{J} \in \mathcal{D}_{m_1}^{(i)}$.</u>

We suppose that $M_1 = A = (0,0,1)$, $S' = (1,0,0)$. We have $g_i'(0) = 0$, $y_B = 0$, $x_B \neq 0$, $z_B \neq 0$, $f_{B,S'}(x,y,1) = z_By - y_B = z_By$, $f_{A,S'}(x,y,1) = z_Ay - y_A = y$.

We have

$$-\frac{2[H_{\hat{F}}f_{A,S'}f_{B,S'}](F_i(x))}{(d-1)^2} = -2(R_i(x))^3g_i''(x)(g_i(x))^2z_B,$$

$$LM((\Delta_A\hat{F}\Delta_B\hat{F}\Delta_{S'}\hat{F})(F_i(x))) = LM(R_i(x)^3x_B(xg_i'(x) - g_i(x))(-g_i'(x))^2),$$
$$LM([(\Delta_{S'}\hat{F})^2\Delta_B\hat{F}](F_i(x))) = LM(R_i(x)^3x_B(-g_i'(x))^3),$$
$$LM([(\Delta_{S'}\hat{F})^2\Delta_A\hat{F}](F_i(x))) = LM(R_i(x)^3(-g_i'(x))^2(xg_i'(x) - g_i(x)).$$

Hence

$$\min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)} \circ F_i\right)\ ,\ j = 1,2,3\right) \geq \left(3\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + 3i_{m_1}^{(i)} - 3.$$

Moreover

$$LM\left(\tilde{\Phi}_3^{(A,B)}(F_i(x))\right) = LM(-[(\Delta_{S'}\hat{F})^2\Delta_B\hat{F}](F_i(x))),$$

we have

$$\min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)} \circ F_i\right)\ ,\ j = 1,2,3\right) = \left(3\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + 3i_{m_1}^{(i)} - 3.$$

(S11) <u>Suppose that $\mathcal{S} \in \mathcal{D}_{m_1}^{(i)} \setminus \{m_1\}$, that $\mathcal{I}, \mathcal{J} \notin \mathcal{D}_{m_1}^{(i)}$.</u>

We suppose that $S' = (1,0,0)$ and that $g_i'(0) = 0$. We have $y_A \neq 0$ and $y_B \neq 0$, $f_{A,S'}(M_1) \neq 0$ and $f_{B,S'}(M_1) \neq 0$. Thanks to (17) for $\Delta_A\hat{F}$ and $\Delta_B\hat{F}$, (18) for $\Delta_{S'}\hat{F}$ and (20), we have

$$\mathrm{val}([H_{\hat{F}}f_{A,S'}f_{B,S'}] \circ F_i) = \left(3\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right) + i_{m_1}^{(i)} - 2,$$

$$LM([\Delta_A\hat{F}\Delta_B\hat{F}\Delta_{S'}\hat{F}] \circ F_i(x)) = y_Ay_BLM((R_i)^3(-g_i'(x))),$$
$$LM(((\Delta_{S'}\hat{F})^2\Delta_A\hat{F}) \circ F_i(x)) = y_ALM((R_i)^3(-g_i'(x))^2),$$
$$LM(((\Delta_{S'}\hat{F})^2\Delta_B\hat{F}) \circ F_i(x)) = y_BLM((R_i)^3(-g_i'(x))^2).$$

Hence,

$$\min\left(\mathrm{val}\left(\tilde{\Phi}_j^{(A,B)}\circ F_i\right)\ ,\ j=1,2,3\right)\geq\left(3\sum_{j\in\mathcal{I}:j\neq i}i_{m_1}^{(i,j)}\right)+i_{m_1}^{(i)}-2$$

and

$$\mathrm{val}(\tilde{\Phi}_3^{(A,B)}\circ F_i)=\mathrm{val}([H_{\hat{F}}f_{A,S'}f_{B,S'}]\circ F_i).$$

So

$$\min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)}\circ F_i\right)\ ,\ j=1,2,3\right)=\left(3\sum_{j\in\mathcal{I}:j\neq i}i_{m_1}^{(i,j)}\right)+i_{m_1}^{(i)}-2.$$

(S12) Suppose that $\mathcal{S}=m_1$ but that $\mathcal{I}$ and $\mathcal{J}$ do not belong to $\mathcal{D}_{m_1}^{(i)}$.

We suppose that $g_i'(0)=0$, that $S'=(0,0,1)$, $y_A\neq 0$ and $y_B\neq 0$, $f_{A,S'}(x,y,1)=y_Ax-yx_A$ and $f_{B,S'}(x,y,1)=y_Bx-yx_B$. Thanks to (17) for $\Delta_A\hat{F}$ and $\Delta_B\hat{F}$, (19) for $\Delta_{S'}\hat{F}$ and (20), we have

$$LM\left(\left[-\frac{2H_{\hat{F}}f_{A,S'}f_{B,S'}}{(d-1)^2}\right]\circ F_i(x)\right)=-2LM(R_i(x)^3g_i''(x))y_Ay_Bx^2,$$

$$LM([\Delta_A\hat{F}\Delta_B\hat{F}\Delta_{S'}\hat{F}](F_i(x)))=y_Ay_BLM((R_i)^3(xg_i'(x)-g_i(x))),$$
$$LM(((\Delta_{S'}\hat{F})^2\Delta_A\hat{F})(F_i(x))=y_ALM((R_i)^3(xg_i'(x)-g_i(x))^2),$$
$$LM(((\Delta_{S'}\hat{F})^2\Delta_B\hat{F})(F_i(x))=y_BLM((R_i)^3(xg_i'(x)-g_i(x))^2).$$

Valuations of the two first terms are in $\left(3\sum_{j\in\mathcal{I}:j\neq i}i_{m_1}^{(i,j)}\right)+i_{m_1}^{(i)}$, valuations of the two last terms are in $\left(3\sum_{j\in\mathcal{I}:j\neq i}i_{m_1}^{(i,j)}\right)+2i_{m_1}^{(i)}$. Hence

$$\min\left(\mathrm{val}\left(\tilde{\Phi}_j^{(A,B)}\circ F_i\right)\ ,\ j=1,2,3\right)\geq\left(3\sum_{j\in\mathcal{I}:j\neq i}i_{m_1}^{(i,j)}\right)+i_{m_1}^{(i)}.$$

Suppose that $LM(g_i)=\alpha x^\beta$ with $\beta>1$, then $LM(xg_i'(x))=\alpha\beta x^\beta$ and $LM(x^2g_i''(x))=\alpha\beta(\beta-1)x^\beta$. Therefore

$$LM(\tilde{\Phi}_3^{(A,B)}(F_i(x)))=LM(R_i(x))^3y_Ay_B\alpha[-2\beta(\beta-1)+\beta-1]x^\beta.$$

So

$$\min\left(\mathrm{val}\left(\tilde{\Phi}_j^{(A,B)}\circ F_i\right)\ ,\ j=1,2,3\right)=\left(3\sum_{j\in\mathcal{I}:j\neq i}i_{m_1}^{(i,j)}\right)+i_{m_1}^{(i)}.$$

(S13) Suppose that $\mathcal{S}=m_1$, $\mathcal{I}\in\mathcal{D}_{m_1}^{(i)}$ and $\mathcal{J}\notin\mathcal{D}_{m_1}^{(i)}$.

We suppose that $M_1=S'=(0,0,1)$, $A=(1,0,0)$, $B=(0,1,0)$. We have $g_i'(0)=0$, $f_{B,S'}(x,y,1)=y_Bx-yx_B=x$, $f_{A,S'}(x,y,1)=y_Ax-yx_A=-y$.

We have

$$-\frac{2[H_{\hat{F}}f_{A,S'}f_{B,S'}](F_i(x))}{(d-1)^2}=-2(R_i(x))^3g_i''(x)(-g_i(x))x,$$

$$LM(\Delta_A\hat{F}\Delta_B\hat{F}\Delta_{S'}\hat{F})(F_i(x)))=LM(R_i(x)^3(-g_i'(x))(xg_i'(x)-g_i(x)),$$
$$LM([(\Delta_{S'}\hat{F})^2\Delta_B\hat{F}](F_i(x)))=LM(R_i(x)^3(xg_i'(x)-g_i(x))^2),$$
$$LM([(\Delta_{S'}\hat{F})^2\Delta_A\hat{F}](F_i(x)))=LM(R_i(x)^3(xg_i'(x)-g_i(x))^2(-g_i'(x)).$$

Hence

$$\min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)}\circ F_i\right)\ ,\ j=1,2,3\right)\geq\left(3\sum_{j\in\mathcal{I}:j\neq i}i_{m_1}^{(i,j)}\right)+2i_{m_1}^{(i)}-1.$$

Moreover, if $LM(g_i(x))=\alpha x^\beta$ with $\alpha\neq 0$ and $\beta=i_{m_1}^{(i)}>1$, we have

$$LM\left(\tilde{\Phi}_3^{(A,B)}\circ F_i)\right)=LM((R_i(x))^3)\alpha^2\beta(\beta-1)x^{2\beta-1},$$

and so

$$\min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)}\circ F_i\right)\ ,\ j=1,2,3\right)=\left(3\sum_{j\in\mathcal{I}:j\neq i}i_{m_1}^{(i,j)}\right)+2i_{m_1}^{(i)}-1.$$

(S14) Suppose that $\mathcal{S}=m_1$ and $\mathcal{I},\mathcal{J}\in\mathcal{D}_{m_1}^{(i)}$.

Assume that $M_1=S'=(0,0,1)$, $A=(1,0,0)$. We have $g_i'(0)=0$, $y_B=0$, $x_B\neq 0$, $z_B\neq 0$, $f_{A,S'}(x,y,1)=y_Ax-yx_A=-y$ and $f_{B,S'}(x,y,1)=y_Bx-yx_B=-yx_B$. Thanks to (20), we have

$$-\frac{2[H_{\hat{F}}f_{A,S'}f_{B,S'}](F_i(x))}{(d-1)^2}=-2(R_i(x))^3g_i''(x)(g_i(x))^2x_B$$

the valuation of which is equal to $\left(3\sum_{j\in\mathcal{I}:j\neq i}i_{m_1}^{(i,j)}\right)+3i_{m_1}^{(i)}-2$. Using (18) for A and B, (19) for S', we get

$$\Delta_{S'}\hat{F}(F_i(x))=R_i(x)(xg_i'(x)-g_i(x)),\quad \Delta_A\hat{F}(F_i(x))=R_i(x)(-g_i'(x)),$$
$$\Delta_B\hat{F}(F_i(x))=R_i(x)[-g_i'(x)x_B+z_B(xg_i'(x)-g_i(x))],$$

and so

$$(\Delta_A\hat{F}\Delta_B\hat{F}\Delta_{S'}\hat{F})(F_i(x))=R_i(x)^3(-g_i'(x))(xg_i'(x)-g_i(x))[-g_i'(x)x_B+z_B(xg_i'(x)-g_i(x))],$$

with valuation $\left(3\sum_{j\in\mathcal{I}:j\neq i}i_{m_1}^{(i,j)}\right)+3i_{m_1}^{(i)}-2$,

$$[(\Delta_{S'}\hat{F})^2\Delta_B\hat{F}](F_i(x))=R_i(x)^3(xg_i'(x)-g_i(x))^2[-g_i'(x)x_B+z_B(xg_i'(x)-g_i(x))],$$

with valuation $\left(3\sum_{j\in\mathcal{I}:j\neq i}i_{m_1}^{(i,j)}\right)+3i_{m_1}^{(i)}-1$

$$[(\Delta_{S'}\hat{F})^2\Delta_A\hat{F}](F_i(x))=R_i(x)^3(xg_i'(x)-g_i(x))^2(-g_i'(x)),$$

with valuation $\left(3\sum_{j\in\mathcal{I}:j\neq i}i_{m_1}^{(i,j)}\right)+3i_{m_1}^{(i)}-1$. We cannot conclude directly. Notice that

$$\tilde{\Phi}_1^{(A,B)}\circ F_i=R_i^3[2[(xg_i'-g_i)^2g_i'-g_i''(g_i)^2x]x_B-z_B(xg_i'-g_i)^3]\tag{21}$$

which has valuation greater than or equal to $\left(3\sum_{j\in\mathcal{I}:j\neq i}i_{m_1}^{(i,j)}\right)+3i_{m_1}^{(i)}-1$. Moreover, we have

$$\tilde{\Phi}_2^{(A,B)}(F_i(x))=-2(R_i(x))^3g_i''(x)(g_i(x))^3x_B,\tag{22}$$

which has valuation $\left(3\sum_{j\in\mathcal{I}:j\neq i}i_{m_1}^{(i,j)}\right)+4i_{m_1}^{(i)}-2$.

Let $\alpha\neq 0$ and $\beta=i_{m_1}^{(i)}>1$ be such that

$$LM(g_i(x))=\alpha x^\beta,\ LM(g_i'(x))=\alpha\beta x^{\beta-1}\ \text{ and }\ LM(g_i''(x))=\alpha\beta(\beta-1)x^{\beta-2}.$$

We have

$$\tilde{\Phi}_3^{(A,B)}(F_i(x))=(R_i(x))^3\left[x_B\{-2g_i''(x)g_i(x)^2+(g_i'(x))^2(xg_i'(x)-g_i(x))\}\right]$$

and so its term of order $\left(3\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right)+3i_{m_1}^{(i)}-2$ is equal to

$$LM((R_i(x))^3\alpha^3\beta(\beta-1)x_B(\beta-2)x^{3\beta-2}.$$

Therefore

$$\boxed{\min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)}\circ F_i\right)\ ,\ j=1,2,3\right)=\left(3\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right)+3i_{m_1}^{(i)}-2\quad\text{if}\ \ i_{m_1}^{(i)}\neq 2}.$$

• Now, we suppose that $\beta=2$ and that $LM(g_i(x)-\alpha x^2)=\alpha_1 x^{\beta_1}$ (with $\alpha_1\neq 0$ and $\beta_1>2$). If $\beta_1\neq 3$, we get that

$$LM(-2g_i''(x)g_i(x)^2+(g_i'(x))^2(xg_i'(x)-g_i(x)))=-2\alpha^2\alpha_1(\beta_1-2)(\beta_1-3)x^{\beta_1+2}$$

and so

$$val(\tilde{\Phi}_3^{(A,B)}\circ F_i)=\left(3\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right)+\beta_1+2.$$

Moreover,

$$val(\tilde{\Phi}_2^{(A,B)}\circ F_i)=\left(3\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right)+6.$$

If $\beta_1<3$, we have

$$LM(\tilde{\Phi}_1^{(A,B)}\circ F_i)=-2\,LM((R_i(x))^3)x_B\alpha^2\alpha_1(\beta_1-2)(\beta_1-4)x^{3+\beta_1};$$

if $\beta_1\geq 3$, we have

$$val(\tilde{\Phi}_1^{(A,B)}\circ F_i)\geq\left(3\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right)+6.$$

Therefore

$$\boxed{\min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)}\circ F_i\right)\ ,\ j=1,2,3\right)=\left(3\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right)+\min(\beta_1+2,6)\ \text{if}\ i_{m_1}^{(i)}=2\ \text{and}\ \beta_1\neq 3}.$$

• Now, assume that $i_{m_1}^{(i)}=2$, that $\beta_1=3$ and that

$$LM(g_i(x)-\alpha x^2-\alpha_1 x^{\beta_1})=\alpha_2 x^{\beta_2},$$

with $\alpha_2\neq 0$ and $\beta_2>\beta_1$.

If $\beta_2<4$, we have

$$LM(\tilde{\Phi}_3^{(A,B)}\circ F_i)=-2\,LM((R_i(x))^3)x_B\alpha^2\alpha_2(\beta_2-2)(\beta_2-3)x^{2+\beta_2};$$

if $\beta_2\geq 4$, we have

$$val(\tilde{\Phi}_3^{(A,B)}\circ F_i)\geq\left(3\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right)+6.$$

Moreover

$$val(\tilde{\Phi}_2^{(A,B)}\circ F_i)=\left(3\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right)+6\leq val(\tilde{\Phi}_1^{(A,B)}\circ F_i).$$

Finally,

$$\boxed{\min\left(\mathrm{val}_x\left(\tilde{\Phi}_j^{(A,B)}\circ F_i\right)\ ,\ j=1,2,3\right)=\left(3\sum_{j\in\mathcal{I}:j\neq i} i_{m_1}^{(i,j)}\right)+\min(\beta_2+2,6)\text{ if } i_{m_1}^{(i)}=2\text{ and }\beta_1=3}.$$

□

7. Proof of Corollary 5

We apply Theorem 2. To simplify notations, let us write i_{m_1} for $i_{m_1}(\mathcal{C},\mathcal{T}_{m_1}\mathcal{C})$ for every non-singular point m_1 of $\mathcal{C}$. According to Remark 4, we have

$$3d^\vee - v_1 = 3d + \sum_{m_1\in Flex(\mathcal{C})}(i_{m_1}-2).$$

<u>Suppose first that $\mathcal{S}\notin\ell_\infty$</u>. We have

$$v_2+v_2' = \sum_{m_1\in Flex(\mathcal{C})\setminus\{\mathcal{S}\},\mathcal{S}\in\mathcal{T}_{m_1}\mathcal{C}}(i_{m_1}-2)+\sum_{m_1\neq\mathcal{S},\mathcal{T}_{m_1}\mathcal{C}\subseteq(\mathcal{I}\mathcal{S})\cup(\mathcal{J}\mathcal{S})} i_{m_1}+\sum_{m_1\in\mathcal{C}\setminus(Flex(\mathcal{C})\cup\{\mathcal{S}\}),\mathcal{T}_{m_1}\mathcal{C}\subseteq(\mathcal{I}\mathcal{S})\cup(\mathcal{J}\mathcal{S})} 1,$$

$$v_3 = i_{\mathcal{S}}\mathbf{1}_{\mathcal{S}\in\mathcal{C}}+(i_{\mathcal{S}}-1)\mathbf{1}_{\mathcal{S}\in\mathcal{C},\mathcal{T}_{\mathcal{S}}\mathcal{C}\subseteq(\mathcal{I}\mathcal{S})\cup(\mathcal{J}\mathcal{S})}.$$

So

$$v_2+v_2'+v_3=\left(\sum_{m_1\in Flex(\mathcal{C}),\mathcal{S}\in\mathcal{T}_{m_1}\mathcal{C}}(i_{m_1}-2)\right)+t_0+n_0+2\times\mathbf{1}_{\mathcal{S}\in\mathcal{C}}-\mathbf{1}_{\mathcal{S}\in\mathcal{C};\{\mathcal{I},\mathcal{J}\}\cap\mathcal{T}_{\mathcal{S}}\mathcal{C}\neq\emptyset}.$$

Finally

$$v_4=\left(\sum_{m_1\in Flex(\mathcal{C}),\mathcal{T}_{m_1}\mathcal{C}=\ell_\infty}(i_{m_1}-2)\right)+\mathbf{1}_{\mathcal{I}\in\mathcal{C},\mathcal{T}_{\mathcal{I}}\mathcal{C}=\ell_\infty}+\mathbf{1}_{\mathcal{J}\in\mathcal{C},\mathcal{T}_{\mathcal{J}}\mathcal{C}=\ell_\infty}.$$

From which we get the first formula.

<u>Suppose now that $\mathcal{S}\in\ell_\infty$</u>. We have

$$v_2+v_2'=\sum_{m_1\in\mathcal{C}\setminus\{\mathcal{S}\},\mathcal{S}\in\mathcal{T}_{m_1}\mathcal{C}}\left[(i_{m_1}-2)+(2i_{m_1}-1)\mathbf{1}_{\mathcal{T}_{m_1}\mathcal{C}=\ell_\infty}\right]$$

and

$$v_3 = i_{\mathcal{S}}\mathbf{1}_{\mathcal{S}\in\mathcal{C}}+4\times\mathbf{1}_{\mathcal{S}\in\mathcal{C},\mathcal{T}_{\mathcal{S}}\mathcal{C}=\ell_\infty,i_{\mathcal{S}}=2}+(2i_{\mathcal{S}}-2)\times\mathbf{1}_{\mathcal{S}\in\mathcal{C},\mathcal{T}_{\mathcal{S}}\mathcal{C}=\ell_\infty,i_{\mathcal{S}}\neq 2}.$$

Therefore, we have

$$v_2+v_2'+v_3=\sum_{m_1\in\mathcal{C},\mathcal{S}\in\mathcal{T}_{m_1}\mathcal{C}}\left[(i_{m_1}-2)+(2i_{m_1}-1)\mathbf{1}_{\mathcal{T}_{m_1}\mathcal{C}=\ell_\infty}\right]+3\times\mathbf{1}_{\mathcal{S}\in\mathcal{C}}-2\times\mathbf{1}_{\mathcal{S}\in\mathcal{C},\mathcal{T}_{\mathcal{S}}\mathcal{C}=\ell_\infty,i_{\mathcal{S}}\neq 2}.$$

The second result follows.

Appendix A. Proof of Proposition 24

Observe that $M^{-1}\hat{M}$ corresponds to the left-multiplication by $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & h \end{pmatrix}$ for some complex numbers a, b, c, d, e, f, h such that $ad - bc \neq 0$ and $h \neq 0$. We define $X(x, y, z) := ax + by$, $Y(x, y, z) := cx + dy$, $Z(x, y, z) := ex + fy + hz$ and

$$\Theta(x, y) := \left(\frac{X(x, y, 1)}{Z(x, y, 1)}, \frac{Y(x, y, 1)}{Z(x, y, 1)} \right).$$

Let d_0 be the degree of polynomial F. By homogeneity of F, we have

$$(F \circ \hat{M})(x, y, 1)) = (h + ex + fy)^{d_0}(F \circ M)(\Theta(x, y), 1).$$

Hence, $F \circ M$ and $F \circ \hat{M}$ have same valuation in x, y so same multiplicity at $[0 : 0 : 1]$.

According to the Weierstrass theorem, there exist $U, \hat{U}$ two units of $\mathbb{C}\langle x, y\rangle$, two integers $b, \hat{b}$ and $b + \hat{b}$ irreducible monic polynomials $\Gamma_1, ..., \Gamma_b, \hat{\Gamma}_1, ..., \hat{\Gamma}_{\hat{b}} \in \mathbb{C}\langle x\rangle[y]$ with respective degrees in y: $e_1, ..., e_b, \hat{e}_1, ..., \hat{e}_{\hat{b}}$ such that

$$F(M(x, y, 1)) = U(x, y) \prod_{\beta=1}^{b} \Gamma_\beta(x, y) \quad \text{and} \quad F(\hat{M}(x, y, 1)) = \hat{U}(x, y) \prod_{\hat{\beta}=1}^{\hat{b}} \hat{\Gamma}_{\hat{\beta}}(x, y).$$

Branches $\mathcal{B}_\beta$ of $V(F \circ M)$ at $[0 : 0 : 1]$ have equations $\Gamma_\beta = 0$ on $z = 1$. Branches $\hat{\mathcal{B}}_{\hat{\beta}}$ of $V(F \circ \hat{M})$ at $[0 : 0 : 1]$ have equations $\hat{\Gamma}_{\hat{\beta}} = 0$ on $z = 1$. We have

$$F(\hat{M}(x, y, 1)) = (h + ex + fy)^{d_0} U(\Theta(x, y)) \prod_{\beta=1}^{b} \Gamma_\beta(\Theta(x, y)).$$

Let $\beta \in \{1, ..., b\}$. Applying the Weierstass theorem to $\Gamma_\beta(\Theta(x, y)) \in \mathbb{C}\langle x, y\rangle$, we get the existence of a unit U_β of $\mathbb{C}\langle x, y\rangle$, an integer $\kappa_\beta \geq 1$ and κ_β irreducible monic polynomials $P_{\beta,1}, ..., P_{\beta,\kappa_\beta} \in \mathbb{C}\langle x\rangle[y]$ such that

$$\Gamma_\beta(\Theta(x, y)) = U_\beta(x, y) \prod_{k=1}^{\kappa_\beta} P_{\beta,k}(x, y).$$

Hence, we have

$$F(\hat{M}(x, y, 1)) = (h + ex + fy)^{d_0} U(\Theta(x, y)) \prod_{\beta=1}^{b} U_\beta(x, y) \prod_{k=1}^{\kappa_\beta} P_{\beta,k}(x, y).$$

By unicity of factorisation, this implies that every $P_{\beta,k}$ is equal to $\hat{\Gamma}_{\hat{\beta}}$ for some $\hat{\beta}$. Hence $b \leq \sum_{\beta=1}^{b} \kappa_\beta = \hat{b}$. Doing the same with exchanging the roles of M and of $\hat{M}$, we finally get that $b = \hat{b}$ and $\kappa_\beta = 1$ for every $\beta \in \{1, ..., b\}$. This implies the existence of a permutation σ of $\{1, ..., b\}$ such that

$$\Gamma_\beta(\Theta(x, y)) = U_\beta(x, y)\hat{\Gamma}_{\sigma(\beta)}(x, y). \tag{23}$$

Let us prove that $\mathcal{T}_\beta = M^{-1}\hat{M}(\hat{\mathcal{T}}_{\sigma(\beta)})$. The tangent line $\mathcal{T}_\beta$ to $\mathcal{B}_\beta$ at $[0 : 0 : 1]$ has equation $(\Gamma_\beta)_x(0, 0)X + (\Gamma_\beta)_y(0, 0)Y = 0$. Hence, $\hat{M}^{-1}(M(\mathcal{T}_\beta))$ has equation

$$(\Gamma_\beta)_x(0, 0)X(x, y, z) + (\Gamma_\beta)_y(0, 0)Y(x, y, z) = 0.$$

Moreover, the tangent line $\hat{\mathcal{T}}_{\sigma(\beta)}$ to $\hat{\mathcal{B}}_{\sigma(\beta)}$ at $[0:0:1]$ has equation $(\hat{\Gamma}_{\sigma(\beta)})_x(0,0)X+(\hat{\Gamma}_{\sigma(\beta)})_y(0,0)Y = 0$. According to (23), this last equation can be rewritten

$$(U_\beta(0,0))^{-1}[(\Gamma_\beta)_x(0,0)(ax+by)+(\Gamma_\beta)_y(0,0)(cx+dy)] = 0,$$

which is an equation of $\hat{M}^{-1}(M(\mathcal{T}_\beta))$.

According to (23), $\Gamma_\beta(\Theta(x,y)) = 0$ is an equation of $\mathcal{B}_{\sigma(\beta)}$. Hence, y-roots $\hat{g}_{\sigma(\beta),1}(x), ..., \hat{g}_{\sigma(\beta),\hat{e}_\beta}(x)$ of $\hat{\Gamma}_{\sigma(\beta)}(x,y)$ coincides with y-roots of $\Gamma_\beta(\Theta(x,y))$.

Let us write e_β and $\hat{e}_\beta$ the respective multiplicities of $\mathcal{B}_\beta$ and $\hat{\mathcal{B}}_{\sigma(\beta)}$.

Since $X = 0$ is not in the tangent cone of $V(F\circ\hat{M})$ and of $V(F\circ M)$, we have $(\Gamma_\beta)_y(0,0) \neq 0$, $(\hat{\Gamma}_{\sigma(\beta)})_y(0,0) \neq 0$, the function $\hat{g}_{\sigma(\beta),k}$ is differentiable at 0 and

$$\hat{g}'_{\sigma(\beta),k}(0) = -\frac{(\hat{\Gamma}_{\sigma(\beta)})_x(0,0)}{(\hat{\Gamma}_{\sigma(\beta)})_y(0,0)}.$$

The map Θ defines a local diffeomorphisms between two neighbourhood of $(0,0)$. There exists a differentiable function $H_{\beta,k}$ such that, for every (x,y) and (X,Y) in a neighourhood of $(0,0)$ satisfying $(X,Y) = \Theta(x,y)$, we have

$$y = \hat{g}_{\sigma(\beta),k}(x) \quad \Leftrightarrow \quad Y = H_{\beta,k}(X), \quad \text{with} \quad H'_{\beta,k}(0) = \frac{\hat{g}'(0)d+c}{\hat{g}'(0)b+a} = -\frac{(\Gamma_\beta)_x(0,0)}{(\Gamma_\beta)_y(0,0)}.$$

We have $\hat{g}'(0) = \frac{aH'_{\beta,k}(0)-c}{d-bH'_{\beta,k}(0)}$. Functions $H_{\beta,1}, ..., H_{\beta,\hat{e}_\beta}$ are y-roots of $\Gamma_\beta(x,y)$. Hence we have $\hat{e}_\beta \leq e_\beta$. For symetry reason, we get that $\hat{e}_\beta = e_\beta$. Let $G \in \mathbb{C}[x,y,z]$ be a homogeneous polynomial of degree d_1. We have

$$i_{[0:0:1]}(V(G\circ M), \mathcal{B}_\beta) = \sum_{k=1}^{e_\beta} \mathrm{val}_x((G\circ M)(x, H_{\beta,k}(x), 1))$$

and

$$\begin{aligned} i_{[0:0:1]}(V(G\circ\hat{M}), \hat{\mathcal{B}}_{\sigma(\beta)}) &= \sum_{k=1}^{e_\beta} \mathrm{val}_x(G(\hat{M}(x, \hat{g}_{\beta,k}(x), 1))) \\ &= \sum_{k=1}^{e_\beta} \mathrm{val}_x\left((G\circ M)(M^{-1}\hat{M}(x, \hat{g}_{\sigma(\beta),k}(x), 1))\right) \\ &= \sum_{k=1}^{e_\beta} \mathrm{val}_x\left((1+ex+f\hat{g}_{\sigma(\beta),k}(x))^{d_1}(G\circ M)(X(x,\hat{g}_{\sigma(\beta),k}(x)), Y(x,\hat{g}_{\sigma(\beta),k}(x)), 1))\right) \\ &= \sum_{k=1}^{e_\beta} \mathrm{val}_x\left((G\circ M)(X(x,\hat{g}_{\sigma(\beta),k}(x)), H_{\beta,k}(X(x,\hat{g}_{\sigma(\beta),k}(x), 1)))\right). \end{aligned}$$

Moreover, since we have

$$LM(X(x,\hat{g}_{\sigma(\beta),k}(x))) = \alpha x, \quad \text{with} \quad \alpha := \frac{a+b\hat{g}'_{\sigma(\beta),k}(0)}{h}x = \frac{ad-bc}{h(d-bH'_{\sigma(\beta),k}(0))}x,$$

we get

$$\mathrm{val}_x((G\circ M)(X(x,\hat{g}_{\sigma(\beta),k}(x)), H_{\beta,k}(X(x,\hat{g}_{\sigma(\beta),k}(x), 1)))) = \mathrm{val}_x\left((G\circ M)\left(x, H_{\beta,k}(x), 1\right)\right).$$

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