# ON CAUSTICS BY REFLECTION OF ALGEBRAIC SURFACES

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Abstract. Given a point $S$ (the light position) in $\mathbb{P}^3$ and an algebraic surface $\mathcal{Z}$ (the mirror) of $\mathbb{P}^3$, the caustic by reflection $\Sigma_S(\mathcal{Z})$ of $\mathcal{Z}$ from $S$ is the Zariski closure of the envelope of the reflected lines $\mathcal{R}_m$ got by reflection of $(Sm)$ on $\mathcal{Z}$ at $m \in \mathcal{Z}$. We use the ramification method to identify $\Sigma_S(\mathcal{Z})$ with the Zariski closure of the image, by a rational map, of an algebraic 2-covering space of $\mathcal{Z}$. We also give a general formula for the degree (with multiplicity) of caustics (by reflection) of algebraic surfaces of $\mathbb{P}^3$.

## Introduction

Let $S[x_0 : y_0 : z_0 : t_0] \in \mathbb{P}^3 := \mathbb{P}(\mathbf{W})$ (with $\mathbf{W}$ a 4-dimensional complex vector space) and let $\mathcal{Z} = V(F)$ be a surface of $\mathbb{P}^3$ given by some $F \in Sym^d(\mathbf{W}^\vee)$ (i.e. $F$ corresponds to a polynomial of degree $d$ in $\mathbb{C}[x, y, z, t]$). The **caustic by reflection** $\Sigma_S(\mathcal{Z})$ of $\mathcal{Z}$ from $S \in \mathbb{P}^3$ is the Zariski closure of the envelope of the reflected lines $\mathcal{R}_m$ of the lines $(mS)$ after reflection at $m$ on the mirror surface $\mathcal{Z}$.

Since the seminal work of Tschirnhausen [14], caustics by reflection of planar curves have been studied namely by Chasles [6], Quetelet [12] and Dandelin [7]. Let us also mention the work of Bruce, Giblin and Gibson [3, 1, 2] in the real case. A precise computation of the degree and class of caustics by reflection of planar algebraic curves has been done in [9, 10, 11]. The idea was based on the fact that the caustic by reflection of an irreducible algebraic curve $\mathcal{C}$ of $\mathbb{P}^2$ from source $S_0 \in \mathbb{P}^2$ is the Zariski closure of a rational map, called caustic map, defined on $\mathcal{C}$. Moreover, in the planar case, the generic birationality of the caustic map has been established in [11, 4]. The study of caustics by reflection of algebraic surfaces is more delicate. We will see that a generic point $m$ of $\mathcal{Z}$ is associated to two (instead of a single one) points on $\Sigma_S(\mathcal{Z})$. This is precisely described here.

A classical way to study envelopes is the ramification theory. Let us mention that this approach has been used namely by Trifogli in [15] and by Catanese and Trifogli in [5] for focal loci (which generalize the notion of evolute to higher dimension). We use here the ramification theory to construct the caustic by reflection $\Sigma_S(\mathcal{Z})$ and to identify it with the Zariski closure of the image, by some rational map $\Phi$, of an algebraic 2-covering space $\hat{Z}$ of $\mathcal{Z}$. We will see that, contrary to the case of caustics by reflection of planar curves, **the set of base points of $\Phi_{|\hat{Z}}$ is never empty**. We give a general formula expressing the degree (with multiplicity) of $\Sigma_S(\mathcal{Z})$ in terms of intersection numbers of $\mathcal{Z}$ with a particular curve (called reflected polar curve) computed at the projection on $\mathcal{Z}$ of the base points of $\Phi_{|\hat{Z}}$.

The paper is organized as follows. Section 1 is devoted to the (complex) projectivization of orthogonality in the real euclidean affine 3-space (which plays a crucial role in the present work) and its link with the ombilical curve. In Section 2, we construct the reflected lines. In Section

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3, we use the reflected lines and the ramification method to define the caustic by reflection. In Section 4, we define the appropriate 2-covering $\hat{Z}$ of $\mathcal{Z}$ and the rational map $\Phi$. In Section 5, we determine precisely the base points of $\Phi_{|\hat{Z}}$. We define the reflected polar in section 6 and use it in Section 7 to establish a formula for the degree of the caustic by reflection. Finally, in Section 8, we precise a significative difference between the caustic by reflection studied in this paper and the focal loci of generic varieties considered in [15, 5].

We denote by $\mathcal{H}^\infty$ the plane at infinity of $\mathbb{P}^3$: $\mathcal{H}^\infty = \{[x:y:z:t] \in \mathbb{P}^3 \ : \ t=0\}$.

## 1. Affine and projective perpendicularity, link with ombilical conjugation

Consider the real euclidean affine 3-space $E_3$ of direction the 3-vector space $\mathbf{E}_3$ (endowed with some fixed basis). Let $\mathbf{W} := (\mathbf{E}_3 \oplus \mathbb{R}) \otimes \mathbb{C}$ (endowed with the induced basis). Let $j : E_3 \hookrightarrow \mathbb{P}^3 := \mathbb{P}(\mathbf{W})$ be the natural map defined on coordinates by $j(x,y,z) := [x:y:z:1]$ for every $\underline{m}(x,y,z) \in E_3$. We are interested in the interpretation in the plane at infinity of $\mathbb{P}^3$ of perpendicularity at a point of two affine subvarieties of $E_3$. Consider the two following quadratic forms

$$q(x,y,z) = x^2+y^2+z^2 \text{ on } \mathbf{E}_3 \otimes \mathbb{C} \quad \text{and} \quad Q(x,y,z,t) = x^2+y^2+z^2 \text{ on } \mathbf{W}.$$

**Definition 1.** *The* ***ombilical curve*** *of $\mathbb{P}^3$ is the irreducible conic $\mathcal{C}_\infty := V(Q_{|\mathcal{H}^\infty}) \cong V(q) \subset \mathbb{P}(\mathbf{E}_3 \otimes \mathbb{C})$. We call* ***cyclic point*** *any point of $\mathcal{C}_\infty$.*

We recall that every (complex projectivized) sphere contains $\mathcal{C}_\infty$. It is worth noting that, for every $\underline{m} \in E_3$, we have the following classical diagram

$$\begin{array}{ccccc} E_3 & \overset{j}{\hookrightarrow} & \mathbb{P}(\mathbf{W}) & \overset{\Pi}{\longleftarrow} & \mathbf{W}\backslash\{0\} \\ & \overset{\xi_{\underline{m}}}{\searrow} & & & \\ & & \mathcal{H}^\infty & & \end{array}$$

where $\Pi$ is the canonical projection and with $\xi_{\underline{m}}$ is defined on coordinates by $\xi_{\underline{m}}(\underline{m}+(x,y,z)) = [x:y:z:0]$. Given any vector subspace $\mathbf{V} \subset \mathbf{E}_3$, the projective subspace $\mathcal{V} := \overline{j(\underline{m}+\mathbf{V})}$ of $\mathbb{P}^3$ (where $\overline{K}$ denotes the Zariski closure of $K$) is the complex projectivization of the affine subspace $V = \underline{m} + \mathbf{V}$ of $E_3$. We observe that $\xi_{\underline{m}}(V)$ is $\mathcal{V}_\infty := \mathcal{V} \cap \mathcal{H}^\infty$.

An affine line $L$ (resp. an affine plane $H$) containing $\underline{m} \in E_3$ is defined by $\underline{m} + V_1$ (resp. $\underline{m} + V_2$) with $V_i$ an $i$-dimensional subspace of $\mathbf{E}_3$. Recall that the (complex) projectivization $\mathcal{L}$ of $L$ (resp. $\mathcal{H}$ of $H$) is the projective line (resp. plane) of $\mathbb{P}^3$ of equations obtained by homogeneization of the equations of $L$ (resp. $H$).

Hence, two lines $L, L'$ containing $\underline{m}$ are perpendicular at $\underline{m}$ if and only if their points at infinity are conjugated with respect the conic $\mathcal{C}_\infty$ .

A line $L$ and a plane $H$ containing $\underline{m}$ are perpendicular if and only if $\mathcal{H}_\infty$ is the **polar** of $\ell_\infty$ with respect to the conic $\mathcal{C}_\infty$ in $\mathcal{H}^\infty \cong \mathbb{P}^2$. This leads to the following definition of projective normal lines to a plane.

**Definition 2.** *Let $\mathcal{H} = V(h) \subset \mathbb{P}^3$ (with $h \in \mathbf{W}^\vee \setminus \{\mathbf{0}\}$) be a projective plane and $m \in \mathcal{H} \setminus \mathcal{H}^\infty$. The normal line $\mathcal{N}_m(\mathcal{H})$ to $\mathcal{H}$ at $m$ is the line containing $m$ and $n_\infty(\mathcal{H}) := \Pi(\boldsymbol{\kappa}(\nabla h))$ with $\boldsymbol{\kappa} : \mathbf{W} \to \mathbf{W}$ defined on coordinates by $\boldsymbol{\kappa}(a,b,c,d) := (a,b,c,0)$.*

**Remark 3.** *Given a projective plane $\mathcal{H} \subset \mathbb{P}^3$ ($\mathcal{H} \neq \mathcal{H}^\infty$), if $n_\infty(\mathcal{H}) = [u:v:w:0]$ lies on the ombilical (i.e. $(u,v,w)$ lies on the* ***isotropic cone*** *$V(q)$ in $\mathbf{E}_3 \otimes \mathbb{C}$), then the line $\mathcal{H}_\infty$ is* ***tangent*** *to $\mathcal{C}_\infty$ at $n_\infty(\mathcal{H})$ in $\mathcal{H}^\infty$. In this case we have $\mathcal{N}_m(\mathcal{H}) \subset \mathcal{H}$.*

Let $m = \Pi(\mathbf{m})$ be a non singular point of $\mathcal{Z} \setminus \mathcal{H}^\infty$. We write $\mathcal{T}_m(\mathcal{Z})$ for the **projective tangent plane** at $m$ to $\mathcal{Z}$. We also define the **projective normal line** $\mathcal{N}_m(\mathcal{Z})$ at $m$ to $\mathcal{Z}$ is the projective normal line to $\mathcal{T}_m(\mathcal{Z})$ at $m$, i.e. $\mathcal{N}_m(\mathcal{Z})$ is the line containing $m$ and $n_{\infty,m}(\mathcal{Z}) = \Pi(\boldsymbol{\kappa}(\nabla F(\mathbf{m})))$.

Observe that the line at infinity $\mathcal{T}_{\infty,m}(\mathcal{Z})$ of $\mathcal{T}_m(\mathcal{Z})$ is the **polar** of the point at infinity $n_{\infty,m}(\mathcal{Z})$ of $\mathcal{N}_m(\mathcal{Z})$ with respect the conic $\mathcal{C}_\infty$.

Later, we will see that the base points of the reflected map can be seen on the geometry on the normals at infinity with respect to the ombilical. In particular isotropic tangent plane to $\mathcal{Z}$ containing $S$ will play some role.

**Definition 4.** *A plane $\mathcal{H} = V(h)$ (with $h \in \mathbf{W}^\vee \setminus \{\mathbf{0}\}$) is said to be* **isotropic** *if $\nabla h$ is an isotropic vector for $Q$.*

**Remark 5.** *A plane $\mathcal{H} \subset \mathbb{P}^3$ is isotropic if and only if either it is the plane at infinity $\mathcal{H}^\infty$ or if $n_\infty(\mathcal{H})$ is in $\mathcal{C}_\infty$ (i.e. $\mathcal{H}$ contains its normal lines).*

In particular, the surface $\mathcal{Z}$ admits an isotropic tangent plane at one of its nonsingular point $m[x : y : z : 1]$ if and only if $m$ belongs to $V(Q(\nabla F), F)$. We note that the whole curve $\mathcal{C}_\infty$ is contained in every complex projectivized sphere $\mathcal{S}_r$ and that we have $\mathcal{N}_m(\mathcal{S}_r) \subset \mathcal{T}_m(\mathcal{S}_r)$ for all $m \in \mathcal{S}_r \setminus \mathcal{H}^\infty$. This is also true for tori.

Consider some particular points on $\mathcal{Z}$, playing a particular role in the construction of the caustic map. Let $\mathcal{B}_0 := V(F, \Delta_{\mathbf{S}} F, Q(\nabla F))$ in $\mathbb{P}^3$, the interpretation in the plane at infinity is the following one. Let $m$ be a nonsingular point of $\mathcal{Z} \setminus \mathcal{H}^\infty$ then

$$m \in \mathcal{B}_0 \Longleftrightarrow \left\{ \begin{array}{c} S \in \mathcal{T}_m(\mathcal{Z}) \\ n_{\infty,m}(\mathcal{Z}) \in \mathcal{C}_\infty \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{c} (mS) \subset \mathcal{T}_m(\mathcal{Z}) \\ n_{\infty,m}(\mathcal{Z}) \in \mathcal{C}_\infty \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{c} (mS)_\infty \in \mathcal{T}_{\infty,m}(\mathcal{Z}) \\ \mathcal{T}_{\infty,m}(\mathcal{Z}) = \mathcal{T}_{n_{\infty,m}(\mathcal{Z})}(\mathcal{C}_\infty) \end{array} \right. . \tag{1}$$

We observe that $\mathcal{B}_0$ is in general a finite set, but that, for the unit sphere, $\mathcal{B}_0$ is a curve (the circle apparent contour of $\mathcal{Z}$ seen from $S$).

Let us now specify some additional notations used in this paper. We write $\mathbf{S}(x_0, y_0, z_0, t_0) \in \mathbf{W} \setminus \{0\}$. For any $m[x : y : z : t] \in \mathbb{P}^3$, we will write $\mathbf{m}(x, y, z, t) \in \mathbf{W} \setminus \{0\}$. For any $d' \geq 1$ and any $G \in Sym^{d'}(\mathbf{W}^\vee)$, we write as usual $G_x, G_y, G_z, G_t \in Sym^{d'-1}(\mathbf{W}^\vee)$ for the partial derivatives of in $x$, $y$, $z$ and $t$ respectively.

## 2. Reflected lines

The incident lines are the lines $(S\, m)$ with $m \in \mathcal{Z}$. We will define the reflected line $\mathcal{R}_m$ as the orthogonal symmetric of $(S\, m)$ with respect to the tangent plane to $\mathcal{Z}$ at $m$. To this end, we will define the orthogonal symmetric $\sigma(m)$ of $S$ with respect to the tangent plane to $\mathcal{Z}$ at $m$. Let us first explain how one can give a sense to the notion of orthogonal symmetries in $\mathbb{P}^3$ by complex projectivization of the euclidean affine situation.

2.1. **Orthogonal symmetric and map $\sigma$.** To every injective linear map $\mathbf{W} \stackrel{f}{\to} \mathbf{W}$, corresponds a unique morphism $\mathbb{P}(\mathbf{W}) \stackrel{\mathbb{P}(f)}{\to} \mathbb{P}(\mathbf{W})$. Therefore, to every injective affine map $E_3 \stackrel{g}{\to} E_3$, corresponds a unique algebraic map $\mathbb{P}(\mathbf{W}) \stackrel{\iota(g)}{\to} \mathbb{P}(\mathbf{W})$. This defines an injective groups homomorphism $\iota : Aff(E_3) \cong \mathbf{E}_3 \rtimes Gl(\mathbf{E}_3) \to \mathbb{P}(Gl(\mathbf{W}))$ such that $\iota(Is(E_3)) = \iota(\mathbf{E}_3 \rtimes O(\mathbf{E}_3)) \subset \mathbb{P}(O(\hat{Q}))$, with $\hat{Q} = x^2 + y^2 + z^2 + t^2$ on $\mathbf{W}$. We apply this to the orthogonal symmetry $s_H$ with respect to some affine plane $H = V(\tilde{h}) \subseteq E_3$ with $\tilde{h} = ax + by + cz + d$. Recall that $s_H$ is defined by

$s_H(P) = P - 2\,\tilde{h}(P)\frac{\nabla\tilde{h}}{q(\nabla\tilde{h}))}$. This leads to the morphism $s_{\mathcal{H}} := \iota(s_h) : \mathbb{P}^3 \to \mathbb{P}^3$ defined by $\mathbb{P}(\mathbf{s}_h)$ with

$$\forall \mathbf{P} \in \mathbf{W}, \quad \mathbf{s}_h(\mathbf{P}) := Q(\nabla h) \cdot \mathbf{P} - 2h(\mathbf{P}) \cdot \boldsymbol{\kappa}(\nabla h) \in \mathbf{W},$$

with $\mathcal{H} = V(h) \subset \mathbb{P}^3$ and with $h = ax + by + cz + dt$ the homogeneized of $\tilde{h}$. Now we extend this definition to any projective plane $\mathcal{H} \subset \mathbb{P}^3$ as follows.

**Definition 6.** *Consider a plane $\mathcal{H} = V(h) \subseteq \mathbb{P}^3$ (with $h \in \mathbf{W}^\vee \setminus \{\mathbf{0}\}$). We define the orthogonal symmetry $s_{\mathcal{H}}$ with respect to $\mathcal{H}$ as the rational map given by $s_{\mathcal{H}} = \mathbb{P}(\mathbf{s}_h)$ with*

$$\forall \mathbf{P} \in \mathbf{W}, \quad \mathbf{s}_h(\mathbf{P}) := Q(\nabla h) \cdot \mathbf{P} - 2h(\mathbf{P}) \cdot \boldsymbol{\kappa}(\nabla h) \in \mathbf{W}.$$

We can notice that, when $\mathcal{H} \neq \mathcal{H}^\infty$, $s_{\mathcal{H}}(P)$ is well defined in $\mathbb{P}^3$ except if $\mathcal{H}$ is an isotropic plane containing $P$ (see Proposition 7). For any non singular $m[x : y : z : t] \in \mathcal{Z}$, we define $\sigma(m) := s_{\mathcal{T}_m\mathcal{Z}}(S) = \mathbb{P}(\boldsymbol{\sigma})(m)$ with

$$\boldsymbol{\sigma} := Q(\nabla F) \cdot \mathbf{S} - 2\Delta_{\mathbf{S}} F \cdot \boldsymbol{\kappa}(\nabla F) \in \mathbf{W} \;\text{ on }\; \Pi^{-1}(\mathcal{Z}),$$

with $\Delta_{\mathbf{S}} F$ the equation of the polar hypersurface of $\mathcal{P}_S(\mathcal{Z})$ given by $\Delta_{\mathbf{S}} F := DF \cdot \mathbf{S}$ (where $DF$ is the differential of $F$). We extend the definition of $\boldsymbol{\sigma}(\mathbf{m})$ to any $\mathbf{m} \in \mathbf{W} \setminus \{0\}$. Observe that $\boldsymbol{\sigma}$ defines a unique rational map $\sigma : \mathbb{P}^3 \to \mathbb{P}^3$.

**Proposition 7.** *The base points of the rational map $\sigma_{|\mathcal{Z}}$ are the singular points of $\mathcal{Z}$, the points of tangency of $\mathcal{Z}$ with $\mathcal{H}^\infty$ and the points at which $\mathcal{Z}$ has an isotropic tangent plane containing $S$.*

*Proof.* We prove that the base points of $\sigma$ are the points of $\mathbb{P}^3$ such that $F_x = F_y = F_z = 0$ or such that $Q(\nabla F) = 0$ and $\Delta_{\mathbf{S}} F = 0$. It is easy to see that these points are base points of $\sigma$. Now let $m = [x : y : z : t]$ be a point of $\mathbb{P}^3$ such that $\sigma(m) = 0$.

- If $\Delta_{\mathbf{S}}(F) = 0$, then, since $\mathbf{S} \neq 0$, we get that $Q(\nabla F) = 0$.
- If $Q(\nabla F) = 0$, then either $\Delta_{\mathbf{S}} F = 0$ or $\boldsymbol{\kappa}(\nabla F) = 0$.
- Assume now that $Q(\nabla F) \neq 0$. We have $Q(\nabla F) \cdot \mathbf{S} = 2\Delta_{\mathbf{S}} F \cdot \boldsymbol{\kappa}(\nabla F)$. This implies that $\boldsymbol{\kappa}(\nabla F)$ is non zero and proportional to $\mathbf{S}$ (which is also non zero), so that $t_0 = 0$ and $0 = y_0 F_x - x_0 F_y = z_0 F_y - y_0 F_z = x_0 F_z - z_0 F_x$. Therefore, writing $\sigma^{(i)}$ for the $i$th coordinate of $\boldsymbol{\sigma}$, we have

  $$\begin{aligned} 0 &= \sigma^{(1)} = Q(\nabla F)x_0 - 2(x_0 F_x^2 + y_0 F_x F_y + z_0 F_x F_z) \\ &= Q(\nabla F)x_0 - 2(x_0 F_x^2 + x_0 F_y^2 + x_0 F_z^2) = -Q(\nabla F)x_0. \end{aligned}$$

  In the same way, we get $0 = \sigma^{(2)} = -Q(\nabla F)y_0$ and $0 = \sigma^{(3)} = -Q(\nabla F)z_0$. This contradicts the fact that $Q(\nabla F) \neq 0$ (since $\mathbf{S} \neq 0$).

$\square$

**Remark 8.** *Each $\sigma^{(i)}$ belongs to $Sym^{2(d-1)}(\mathbf{W}^\vee)$. Moreover, for a general $(\mathcal{Z}, S)$, the set $V(F, F_x, F_y, F_z)$ is empty and the base points of $\sigma_{|\mathcal{Z}}$ are the $2d(d-1)^2$ points of $V(F, Q(\nabla F), \Delta_{\mathbf{S}} F)$.*

2.2. **Reflected lines.**

**Definition 9.** *For any $m \in \mathcal{Z}$, the* **reflected line** *$\mathcal{R}_m$ on $\mathcal{Z}$ at $m$ is the line $(m\sigma(m))$ when it is well defined.*

**Definition 10.** *We write $\mathcal{M}_{S,\mathcal{Z}}$ for the set of points $m \in \mathbb{P}^3$ such that $\mathbf{m}$ and $\boldsymbol{\sigma}(\mathbf{m})$ are proportional, i.e.*

$$\mathcal{M}_{S,\mathcal{Z}} := \{m \in \mathbb{P}^3 \;:\; \exists[\lambda_0 : \lambda_1] \in \mathbb{P}^1, \quad \lambda_0 \cdot \mathbf{m} + \lambda_1 \cdot \boldsymbol{\sigma}(\mathbf{m}) = 0\}.$$

Observe that $\mathcal{R}_m$ is well defined if $m \in \mathcal{Z} \setminus \mathcal{M}_{S,\mathcal{Z}}$.

**Proposition 11.** *We have* $\mathcal{Z} \cap \mathcal{M}_{S,\mathcal{Z}} = \mathcal{Z} \cap (\mathrm{Base}(\sigma) \cup \{S\} \cup \mathcal{W})$, *with*

$$\mathcal{W} := \{m \in \mathcal{Z} \ : \ m = n_{\infty,m}(\mathcal{Z}), \quad \Delta_{\mathbf{S}} F(m) \neq 0, \ Q(m) = 0\},$$

*with* $n_{\infty,m}(\mathcal{Z}) := \Pi(\boldsymbol{\kappa}(\nabla F(\mathbf{m})))$.

*Proof.* We prove $\mathcal{Z} \cap \mathcal{M}_{S,\mathcal{Z}} \subseteq \mathcal{Z} \cap (\mathrm{Base}(\sigma) \cup \{S\} \cup \mathcal{W})$, the inverse inclusion being clear. Let $m \in (\mathcal{Z} \cap \mathcal{M}_{S,\mathcal{Z}}) \setminus \mathrm{Base}(\sigma)$. Observe that, due to the Euler identity, we have $0 = DF(\mathbf{m}) \cdot \mathbf{m}$ and so $0 = DF \cdot \boldsymbol{\sigma} = -\Delta_{\mathbf{S}} F \cdot Q(\nabla F)$. If $\Delta_{\mathbf{S}} F = 0$, then $\boldsymbol{\sigma} = Q(\nabla F) \cdot \mathbf{S}$, so $m = \sigma(m) = S$. If $Q(\nabla F) = 0$, then $\boldsymbol{\sigma} = -2\Delta_{\mathbf{S}} F \cdot \boldsymbol{\kappa}(\nabla F)$. So $m = \sigma(m) = n_{\infty,m}(\mathcal{Z})$; moreover $\Delta_{\mathbf{S}} F \neq 0$ and $Q = 0$. □

## 3. Caustic by reflection

Now, let us introduce some additional notations. We define $N_{\mathbf{S}}(\mathbf{m})$ as the complexified homogenized square euclidean norm of $\mathbf{Sm}$ by

$$N_{\mathbf{S}}(\mathbf{m}) := (xt_0 - x_0 t)^2 + (yt_0 - y_0 t)^2 + (zt_0 - z_0 t)^2.$$

We will also consider the bilinear Hessian form $\mathrm{Hess}_F$ of $F$ and its determinant $H_F$. Let us see how to construct two maps $\psi = \psi^{\pm} : \mathcal{Z} \to \mathbb{P}^3$ such that the surface $\psi(\mathcal{Z})$ is tangent to the reflected line $\mathcal{R}_m$ at $\psi(m)$, for a generic $m \in \mathcal{Z}$. Observe first that $\psi(m)$ is in $\mathcal{R}_m$ implies that $\psi(m)$ can be rewritten

$$\boldsymbol{\psi}(\mathbf{m}) = \lambda_0(\mathbf{m}) \cdot \mathbf{m} + \lambda_1(\mathbf{m}) \cdot \boldsymbol{\sigma}(\mathbf{m}) \in \mathbf{W} \setminus \{0\},$$

with $[\lambda_0(\mathbf{m}) : \lambda_1(\mathbf{m})] \in \mathbb{P}^1$ for every $m \in \mathcal{Z}$. The main result of this section is the next theorem specifying the form of $\lambda_0$ and $\lambda_1$ (belonging to an integral extension of the ring $Sym(\mathbf{W}^\vee)$) which ensures that, for a generic $m \in \mathcal{Z}$, $\mathcal{R}_m$ is tangent to $\psi(\mathcal{Z})$ at $\psi(m)$.

**Theorem 12.** *Let* $\psi : U \to \mathbb{P}^3$ *(with* $U \subseteq \mathcal{Z}$*) be given by*

$$\boldsymbol{\psi}(\mathbf{m}) = \lambda_0(\mathbf{m}) \cdot \mathbf{m} + \lambda_1(\mathbf{m}) \cdot \boldsymbol{\sigma}(\mathbf{m}) \quad \in \mathbf{W},$$

*with* $\lambda_0(\cdot)$ *and* $\lambda_1(\cdot)$ *in an integral extension of* $Sym(\mathbf{W}^\vee)$ *such that*

$$\alpha(\mathbf{m})(\lambda_0(\mathbf{m}))^2 + \beta(\mathbf{m})\lambda_0(\mathbf{m})\lambda_1(\mathbf{m}) + \gamma(\mathbf{m})(\lambda_1(\mathbf{m}))^2 = 0$$

*with* $\alpha, \beta, \gamma \in Sym(\mathbf{W}^\vee)$ *given by*

$$\alpha := \Delta_{\mathbf{S}} F \quad \in Sym^{d-1}(\mathbf{W}^\vee), \tag{2}$$

$$\beta := -2\left[\mathrm{Hess}\, F(\mathbf{S}, \boldsymbol{\sigma}) + (\Delta_{\mathbf{S}} F)^2 (F_{xx} + F_{yy} + F_{zz})\right] \quad \in Sym^{3d-4}(\mathbf{W}^\vee) \tag{3}$$

*and*

$$\gamma := -\frac{4\Delta_{\mathbf{S}} F}{(d-1)^2} N_{\mathbf{S}}\, H_F \quad \in Sym^{5d-7}(\mathbf{W}^\vee). \tag{4}$$

*Then, for every* $m[x : y : z : t] \in \mathcal{Z} \setminus V(tQ(\nabla F))$, *the reflected line* $\mathcal{R}_m$ *is tangent to* $\psi(\mathcal{Z})$ *at* $\psi(m)$.

It will be useful to introduce

$$\forall (\mathbf{m}, \lambda_0, \lambda_1) \in \mathbf{W} \times \mathbb{C}^2, \quad Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = \alpha(\mathbf{m})\lambda_0^2 + \beta(\mathbf{m})\lambda_0\lambda_1 + \gamma(\mathbf{m})\lambda_1^2.$$

One may notice that, for a fixed $\mathbf{m}$, $Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1)$ is a quadratic form in $(\lambda_0, \lambda_1)$. Roughly speaking, Theorem 12 states that the image of $\psi(\cdot) = \lambda_0(\cdot) \cdot id + \lambda_1(\cdot) \cdot \sigma(\cdot)$ (for some $\lambda_0, \lambda_1 \in Sym(\mathbf{W}^\vee)[\sqrt{\beta^2 - 4\alpha\gamma}]$) corresponds to a part of the envelope of the reflected lines $\mathcal{R}_m$. More precisely:

**Definition 13.** *The* **caustic by reflection** $\Sigma_S(\mathcal{Z})$ *of* $\mathcal{Z}$ *from* $S$ *is the Zariski closure of the union on* $\psi$ *of the images of* $\psi(\mathcal{Z})$, *for the* $\psi$ *satisfying the assumptions of Theorem 12, i.e. the Zariski closure of the following set*

$$\{P \in \mathbb{P}^3 \ : \ \exists m \in \mathcal{Z}, \ \exists[\lambda_0 : \lambda_1] \in \mathbb{P}^1, \quad Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0 \ \textit{and} \ \mathbf{P} = \lambda_0 \cdot \mathbf{m} + \lambda_1 \cdot \boldsymbol{\sigma}(\mathbf{m})\}.$$

*Proof of Theorem 12.* Let $m[x : y : z : t] \in \mathcal{Z} \setminus V(t(Q(\nabla F)))$. We will use several times the Euler identity ($xG_x + yG_y + zG_y + tG_t = d_1 G$ if $G$ is in $Sym^{d_1}(W^\vee)$). We use the idea of ramification used for example in [15, 5]. The points of the caustic corresponding to $m$ are the points $\Pi(\lambda_0 \cdot \mathbf{m} + \lambda_1 \cdot \boldsymbol{\sigma}(\mathbf{m}))$ with $[\lambda_0 : \lambda_1] \in \mathbb{P}^1$ such that the rank of the Jacobian matrix $J$ of

$$j : (\mathbf{m}, \lambda_0, \lambda_1) \mapsto (\lambda_0 \cdot \mathbf{m} + \lambda_1 \cdot \boldsymbol{\sigma}(\mathbf{m}), F(\mathbf{m}))$$

is less than 5. We have

$$J := \begin{pmatrix} \lambda_0 + \lambda_1 \sigma_x^{(1)} & \lambda_1 \sigma_y^{(1)} & \lambda_1 \sigma_z^{(1)} & \lambda_1 \sigma_t^{(1)} & x & \sigma^{(1)} \\ \lambda_1 \sigma_x^{(2)} & \lambda_0 + \lambda_1 \sigma_y^{(2)} & \lambda_1 \sigma_z^{(2)} & \lambda_1 \sigma_t^{(2)} & y & \sigma^{(2)} \\ \lambda_1 \sigma_x^{(3)} & \lambda_1 \sigma_y^{(3)} & \lambda_0 + \lambda_1 \sigma_z^{(3)} & \lambda_1 \sigma_t^{(3)} & z & \sigma^{(3)} \\ \lambda_1 \sigma_x^{(4)} & \lambda_1 \sigma_y^{(4)} & \lambda_1 \sigma_z^{(4)} & \lambda_0 + \lambda_1 \sigma_t^{(4)} & t & \sigma^{(4)} \\ F_x & F_y & F_z & F_t & 0 & 0 \end{pmatrix},$$

with $\sigma^{(i)}$ the $i$th coordinates of $\boldsymbol{\sigma}$.

(1) First, let us explain this briefly. Let $\psi(\cdot)$ be of the following form

$$\boldsymbol{\psi}(\mathbf{m}') = \lambda_0(\mathbf{m}') \cdot \mathbf{m}' + \lambda_1(\mathbf{m}') \cdot \boldsymbol{\sigma}(\mathbf{m}').$$

We define the following property

$$\text{the line } (m\sigma(m)) \text{ is tangent to } \psi(\mathcal{Z}) \text{ at } \psi(m). \tag{5}$$

Recall that we have assumed $(Q(\nabla F))(\mathbf{m}) \neq 0$. Assume for example $F_x(\mathbf{m}) \neq 0$ (the proof is similar if we replace $F_x$ by $F_y$ or by $F_z$). Now, Property (5) means that there exists $A \in \mathbf{W}^\vee \setminus \{0\}$ such that

$$A(\mathbf{m}) = 0, \ A(\boldsymbol{\sigma}(\mathbf{m})) = 0, \quad A((D\boldsymbol{\psi}(\mathbf{m}) \cdot \begin{pmatrix} F_y(\mathbf{m}) \\ -F_x(\mathbf{m}) \\ 0 \\ 0 \end{pmatrix}) = 0,$$

$$A((D\boldsymbol{\psi}(\mathbf{m})) \cdot \begin{pmatrix} F_z(\mathbf{m}) \\ 0 \\ -F_x(\mathbf{m}) \\ 0 \end{pmatrix}) = 0 \ \text{ and } \ A((D\boldsymbol{\psi}(\mathbf{m})) \cdot \begin{pmatrix} F_t(\mathbf{m}) \\ 0 \\ 0 \\ -F_x(\mathbf{m}) \end{pmatrix}) = 0,$$

and so that

$$A(\mathbf{m}) = 0, \ A(\boldsymbol{\sigma}(\mathbf{m})) = 0, \quad A(\boldsymbol{\psi}_x(\mathbf{m}))F_y(\mathbf{m}) = F_x(\mathbf{m})A(\boldsymbol{\psi}_y(\mathbf{m})),$$

$$A(\boldsymbol{\psi}_x(\mathbf{m}))F_z(\mathbf{m}) = F_x(\mathbf{m})A(\boldsymbol{\psi}_z(\mathbf{m})) \ \text{ and } \ A(\boldsymbol{\psi}_x(\mathbf{m}))F_t(\mathbf{m}) = F_x(\mathbf{m})A(\boldsymbol{\psi}_t(\mathbf{m})).$$

Therefore, by taking $b := A(\boldsymbol{\psi}_x(\mathbf{m}))/F_x(\mathbf{m})$,

$$A(\mathbf{m}) = 0, \quad A(\boldsymbol{\psi}_x(\mathbf{m})) = F_x(\mathbf{m})b, \quad A(\boldsymbol{\psi}_y(\mathbf{m})) = F_y(\mathbf{m})b,$$

$$A(\boldsymbol{\psi}_z(\mathbf{m})) = F_z(\mathbf{m})b \ \text{ and } A(\boldsymbol{\psi}_t(\mathbf{m})) = F_t(\mathbf{m})b$$

and so that the rank of the following matrix is strictly less than 5

$$\hat{J} := \begin{pmatrix} \boldsymbol{\psi}_x(\mathbf{m}) & \boldsymbol{\psi}_y(\mathbf{m}) & \boldsymbol{\psi}_z(\mathbf{m}) & \boldsymbol{\psi}_t(\mathbf{m}) & \mathbf{m} & \boldsymbol{\sigma}(\mathbf{m}) \\ F_x & F_y & F_z & F_t & 0 & 0 \end{pmatrix} \in Mat_{5,6}(\mathbb{C}).$$

Let us write $C_i$ the $i$-th column of $J$. We observe that the four first columns of $\hat{J}$ are respectively equal to $C_1+(\lambda_1)_xC_6+(\lambda_0^\pm)_xC_5$, $C_2+(\lambda_1)_yC_6+(\lambda_0^\pm)_yC_5$, $C_3+(\lambda_1)_zC_6+(\lambda_0^\pm)_zC_5$ and $C_4+(\lambda_1)_tC_6+(\lambda_0^\pm)_tC_5$. Therefore the $J$ and $\hat{J}$ have the same rank and so (5) means that $rank(J)<5$.

(2) Now we observe that, on $\mathcal{Z}$, $xC_1+yC_2+zC_3+tC_4=\lambda_0C_5+\lambda_1C_6$. Since $t\neq 0$, $C_4$ is a linear combination of the other columns and so the rank of $J$ is strictly less than 5 if and only if the following determinant is null:

$$D(\mathbf{m},\lambda_0,\lambda_1):=\begin{vmatrix} \lambda_0+\lambda_1\sigma_x^{(1)} & \lambda_1\sigma_y^{(1)} & \lambda_1\sigma_z^{(1)} & x & \sigma^{(1)} \\ \lambda_1\sigma_x^{(2)} & \lambda_0+\lambda_1\sigma_y^{(2)} & \lambda_1\sigma_z^{(2)} & y & \sigma^{(2)} \\ \lambda_1\sigma_x^{(3)} & \lambda_1\sigma_y^{(3)} & \lambda_0+\lambda_1\sigma_z^{(3)} & z & \sigma^{(3)} \\ \lambda_1\sigma_x^{(4)} & \lambda_1\sigma_y^{(4)} & \lambda_1\sigma_z^{(4)} & t & \sigma^{(4)} \\ F_x & F_y & F_z & 0 & 0 \end{vmatrix}.$$

Now let us define

$$\boldsymbol{\tau}:=Q(\nabla F)\cdot\mathbf{S}+2\frac{(xt_0-x_0t)F_x+(yt_0-y_0t)F_y+(zt_0-z_0t)F_z}{t}\cdot\boldsymbol{\kappa}(\nabla F).$$

Observe that

$$\boldsymbol{\tau}=\boldsymbol{\sigma}+\frac{2t_0dF}{t}\boldsymbol{\kappa}(\nabla F)$$

(due to the Euler identity). Therefore, on $\mathcal{Z}$, we have $\boldsymbol{\sigma}=\boldsymbol{\tau}$. Now we observe that, on $\mathcal{Z}$, we have

$$D(\mathbf{m},\lambda_0,\lambda_1)=\begin{vmatrix} \lambda_0+\lambda_1\tau_x^{(1)} & \lambda_1\tau_y^{(1)} & \lambda_1\tau_z^{(1)} & x & \tau^{(1)} \\ \lambda_1\tau_x^{(2)} & \lambda_0+\lambda_1\tau_y^{(2)} & \lambda_1\tau_z^{(2)} & y & \tau^{(2)} \\ \lambda_1\tau_x^{(3)} & \lambda_1\tau_y^{(3)} & \lambda_0+\lambda_1\tau_z^{(3)} & z & \tau^{(3)} \\ \lambda_1\tau_x^{(4)} & \lambda_1\tau_y^{(4)} & \lambda_1\tau_z^{(4)} & t & \tau^{(4)} \\ F_x & F_y & F_z & 0 & 0 \end{vmatrix}, \tag{6}$$

with $\tau^{(i)}$ the $i$th coordinate of $\boldsymbol{\tau}$. Indeed, if we write $L_i$ the $i$-th line of the matrix (with $\boldsymbol{\sigma}$) used in the definition of $D$ and if we write $\tilde{L}_i$ the $i$-th line of the matrix (with $\boldsymbol{\tau}$) appearing in the above formula, we obtain (due to the Euler identity) that, on $\mathcal{Z}$, we have $\tilde{L}_4=L_4$, $\tilde{L}_5=L_5$ and

$$\tilde{L}_1=L_1+\lambda_1\frac{2t_0d}{t}F_xL_5,\ \tilde{L}_2=L_2+\lambda_1\frac{2t_0d}{t}F_yL_5,\ \ \tilde{L}_3=L_3+\lambda_1\frac{2t_0d}{t}F_zL_5.$$

(3) On $\mathcal{Z}$, we have

$$D(\mathbf{m},\lambda_0,\lambda_1)=\alpha_1(\mathbf{m})\lambda_0^2+\beta_1(\mathbf{m})\lambda_0\lambda_1+\gamma_1(\mathbf{m})\lambda_1^2, \tag{7}$$

where $\alpha_1$, $\beta_1$ and $\gamma_1$ can be expressed as follows (due to Euler's identity ensuring that $-xF_xt_0-yF_yt_0-zF_zt_0+tx_0F_x+ty_0F_y+tz_0F_z=t\Delta_{\mathbf{S}}F$ on $\mathcal{Z}$)

$$\alpha_1:=Q(\nabla F)t\Delta_{\mathbf{S}}F=tQ(\nabla F)\alpha \tag{8}$$

$$\beta_1:=-\frac{2}{t}Q(\nabla F)B \tag{9}$$

$$\gamma_1:=-4t^{-1}N_{\mathbf{S}}.Q(\nabla F).\Delta_{\mathbf{S}}F.h_F \tag{10}$$

with the following definitions of $h_F$ and $B$. First, on $\mathcal{Z}$, we have

$$\begin{aligned} h_F &:= \begin{vmatrix} F_{xx} & F_{xy} & F_{xz} & F_x \\ F_{xy} & F_{yy} & F_{yz} & F_y \\ F_{xz} & F_{yz} & F_{zz} & F_z \\ F_x & F_y & F_z & 0 \end{vmatrix} \\ &= -F_{xx}F_{yy}F_z^2 + 2F_{xx}F_yF_{yz}F_z - F_{xx}F_{zz}F_y^2 + F_{xy}^2F_z^2 \\ &\quad -2F_xF_{xy}F_{yz}F_z - 2F_{xy}F_yF_{xz}F_z + 2F_xF_{xy}F_yF_{zz} + 2F_xF_{xz}F_zF_{yy} + \\ &\quad +F_{xz}^2F_y^2 - 2F_xF_{xz}F_yF_{yz} - F_x^2F_{yy}F_{zz} + F_x^2F_{yz}^2 \\ &= \frac{t^2}{(d-1)^2}H_F, \end{aligned}$$

where $H_F$ is the Hessian determinant of $F$ [1]. Therefore

$$\gamma_1 = -\frac{4t}{(d-1)^2}N_{\mathbf{S}}.Q(\nabla F).\Delta_{\mathbf{S}}F.H_F = tQ(\nabla F)\gamma. \tag{11}$$

Second

$$B := \delta_xF_{xx} + \delta_yF_{yy} + \delta_zF_{zz} + 2(\varepsilon_{x,y}F_{xy} + \varepsilon_{x,z}F_{xz} + \varepsilon_{y,z}F_{yz}),$$

with

$$\begin{aligned} \delta_x &:= (x_0t - xt_0)^2(F_y^2 + F_z^2) + ((t_0y - ty_0)F_y + (t_0z - tz_0)F_z)^2 \\ &= (x_0t - xt_0)^2(F_y^2 + F_z^2) + (t_0(yF_y + zF_z) - t(y_0F_y + z_0F_z))^2 \\ &= (x_0t - xt_0)^2(F_y^2 + F_z^2) + (t_0(xF_x + tF_t) + t(y_0F_y + z_0F_z))^2 \\ &= (x_0t - xt_0)^2(F_y^2 + F_z^2) + (t_0xF_x + t(t_0F_t + y_0F_y + z_0F_z))^2 \\ &= (x_0t - xt_0)^2(F_y^2 + F_z^2) + ((t_0x - x_0t)F_x + t\Delta_{\mathbf{S}}F)^2 \\ &= x^2t_0^2(F_x^2 + F_y^2 + F_z^2) + 2xt_0t[-x_0(F_x^2 + F_y^2 + F_z^2) + F_x\Delta_{\mathbf{S}}F] + \\ &\quad +x_0^2t^2(F_x^2 + F_y^2 + F_z^2) + t^2(\Delta_{\mathbf{S}}F)^2 - 2x_0t^2F_x\Delta_{\mathbf{S}}F \\ &= x^2t_0^2(F_x^2 + F_y^2 + F_z^2) + t\Delta_{\mathbf{S}}F(2xt_0F_x - 2x_0tF_x) + \\ &\quad +t(F_x^2 + F_y^2 + F_z^2)(x_0^2t - 2xx_0t_0) + t^2(\Delta_{\mathbf{S}}F)^2, \end{aligned}$$

[1] Indeed, if we write $\hat{C}_i$ for the $i$-th column of Hess $F$, due to the Euler formula, on $\mathcal{Z}$, we have $\hat{C}_4 = \frac{d-1}{t}\nabla F - (x\hat{C}_1 + y\hat{C}_2 + z\hat{C}_3)/t$ (where $\nabla F$ is the gradient of $F$); therefore

$$H_F := \begin{vmatrix} F_{xx} & F_{xy} & F_{xz} & F_{xt} \\ F_{xy} & F_{yy} & F_{yz} & F_{yt} \\ F_{xz} & F_{yz} & F_{zz} & F_{zt} \\ F_{xt} & F_{yt} & F_{zt} & F_{tt} \end{vmatrix} = \frac{d-1}{t}\begin{vmatrix} F_{xx} & F_{xy} & F_{xz} & F_x \\ F_{xy} & F_{yy} & F_{yz} & F_y \\ F_{xz} & F_{yz} & F_{zz} & F_z \\ F_{xt} & F_{yt} & F_{zt} & F_t \end{vmatrix}.$$

Now, if we write $\hat{L}_i$ the $i$-th line of the above matrix, using again the Euler identity, on $\mathcal{Z}$, we have $\hat{L}_4 = \frac{d-1}{t}(F_x \quad F_y \quad F_z \quad 0) - (x\hat{L}_1 + y\hat{L}_2 + z\hat{L}_3)/t$ and we get $H_F = (d-1)^2h_F/t^2$.

$\delta_y$ (resp. $\delta_z$) being obtained from $\delta_x$ by interverting $x$ and $y$ (resp. $x$ and $z$) and

$$\begin{aligned}
\varepsilon_{x,y} \;:=\; & -(t_0x-x_0t)F_y((t_0x-x_0t)F_x+(t_0z-z_0t)F_z) \\
& -(t_0y-y_0t)F_x((t_0y-y_0t)F_y+(t_0z-z_0t)F_z)+(t_0x-x_0t)(t_0y-y_0t)F_z^2 \\
=\; & (t_0x-x_0t)F_y[t_0yF_y+t_0tF_t+x_0tF_x+z_0tF_z]+ \\
& +(t_0y-y_0t)F_x[t_0xF_x+t_0tF_t+y_0tF_y+z_0tF_z]+ \\
& +(t_0x-x_0t)(t_0y-y_0t)F_z^2 \\
=\; & (t_0x-x_0t)F_y[t_0yF_y+t\Delta_{\mathbf{S}}F-ty_0F_y]+(t_0y-y_0t)F_x[t_0xF_x+t\Delta_{\mathbf{S}}F-tx_0F_x]+ \\
& +(t_0x-x_0t)(t_0y-y_0t)F_z^2 \\
=\; & t_0^2xy(F_x^2+F_y^2+F_z^2)+t\Delta_{\mathbf{S}}F((t_0x-x_0t)F_y+(t_0y-y_0t)F_x) \\
& +t(F_x^2+F_y^2+F_z^2)(tx_0y_0-t_0(y_0x+yx_0))
\end{aligned}$$

$\varepsilon_{x,z}$ (resp. $\varepsilon_{y,z}$) being obtained from $\varepsilon_{x,y}$ by interverting $y$ and $z$ (resp. $x$ and $z$). On $\mathcal{Z}$, we have

$$0=xF_x+yF_y+zF_z+tF_t \text{ and } (d-1)F_w=xF_{xw}+yF_{yw}+zF_{zw}+tF_{tw},\ \forall w\in\{x,y,z,t\}.$$

Therefore

$$0=x^2F_{xx}+y^2F_{yy}+z^2F_{zz}+t^2F_{tt}+2(xyF_{xy}+xzF_{xz}+xtF_{xt}+yzF_{yz}+ytF_{yt}+ztF_{zt})$$

and so

$$B=(F_x^2+F_y^2+F_z^2)(b_1+b_2+b_3)+2t\Delta_{\mathbf{S}}Fb_4+t^2(\Delta_{\mathbf{S}}F)^2(F_{xx}+F_{yy}+F_{zz}),$$

with

$$b_1=-t_0^2(t^2F_{tt}+2t(xF_{xt}+yF_{yt}+zF_{zt}))=-t_0^2t(2(d-1)F_t-tF_{tt}),$$

$$b_2=t^2(x_0^2F_{xx}+y_0^2F_{yy}+z_0^2F_{zz}+2x_0y_0F_{xy}+2x_0z_0F_{xz}+2y_0z_0F_{yz}),$$

$$\begin{aligned}
b_3 &= -2tt_0\sum_{w\in\{x,y,z\}}(w_0(xF_{xw}+yF_{yw}+zF_{zw})) \\
&= 2tt_0\sum_{w\in\{x,y,z\}}(w_0(tF_{tw}-(d-1)F_w))
\end{aligned}$$

and

$$\begin{aligned}
b_4 &= \sum_{w\in\{x,y,z\}}F_w((t_0x-tx_0)F_{xw}+(t_0y-ty_0)F_{yw}+(t_0z-tz_0)F_{zw}) \\
&= \sum_{w\in\{x,y,z\}}F_w(t_0(d-1)F_w-t(x_0F_{xw}+y_0F_{yw}+z_0F_{zw}+t_0F_{wt})).
\end{aligned}$$

Putting all these terms together, we get that $B$ is equal to

$$t^2\left[Q(\nabla F)\cdot\mathrm{Hess}_F(\mathbf{S},\mathbf{S})-2\Delta_{\mathbf{S}}F\cdot\mathrm{Hess}_F(\mathbf{S},\boldsymbol{\kappa}(\nabla F))+(\Delta_{\mathbf{S}}F)^2(F_{xx}+F_{yy}+F_{zz})\right].$$

and so

$$B=t^2\left[\mathrm{Hess}_F(\mathbf{S},\boldsymbol{\sigma})+(\Delta_{\mathbf{S}}F)^2(F_{xx}+F_{yy}+F_{zz})\right],$$

which leads to

$$\beta_1=tQ(\nabla F)\beta. \tag{12}$$

Hence the points of the caustic associated to $m$ are the points $\Pi(\lambda_0\cdot\mathbf{m}+\lambda_1\cdot\boldsymbol{\sigma}(\mathbf{m}))$ where $[\lambda_0:\lambda_1]\in\mathbb{P}^1$ satisfies

$$\alpha_1(\mathbf{m})\lambda_0^2+\beta_1(\mathbf{m})\lambda_0\lambda_1+\gamma_1(\mathbf{m})\lambda_1^2=0, \tag{13}$$

with $\alpha_1$, $\beta_1$ and $\gamma_1$ given by (8), (12) and (11). Now, since $tQ(\nabla F) \neq 0$, (13) means that $\alpha(\mathbf{m})\lambda_0^2 + \beta(\mathbf{m})\lambda_0\lambda_1 + \gamma(\mathbf{m})\lambda_1^2 = 0$.

□

## 4. Covering space $\hat{Z}$ and rational map $\Phi$

We consider the algebraic covering space $\hat{Z}$ of $\mathcal{Z}$ given by

$$\hat{Z} := \{(m, [\lambda_0 : \lambda_1]) \in \mathcal{Z} \times \mathbb{P}^1 \ : \ Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0\}.$$

This set is a subvariety of a particular algebraic variety denoted $\mathbb{F}_{(3)}(-2d+3, 0)$ (by extending the notations used by Reid in [13, Chapter 2]) which corresponds to the cartesian product of sets $\mathbb{P}^3 \times \mathbb{P}^1$ endowed with an unusual structure of algebraic variety based on the following definition of multidegree multideg$(P)$ for $P \in Sym(\mathbf{W}^\vee)[\lambda_0, \lambda_1] \cong \mathbb{C}[x, y, z, t, \lambda_0, \lambda_1]$:

$$\text{multideg}(x^{a'} y^{b'} z^{c'} t^{d'} \lambda_0^{e'} \lambda_1^{f'}) = (a' + b' + c' + d' + (2d-3)e', e' + f').$$

With this notion of multidegree, we have

$$Sym(\mathbf{W}^\vee)[\lambda_0, \lambda_1] = \bigoplus_{k,\ell \geq 0} C_{k,\ell},$$

where $C_{k,\ell}$ denotes the homogeneous component of multidegree $(k, \ell)$. Now, we define $\mathbb{F}_{(3)}(-2d+3, 0)$ as the quotient of $\mathbf{W} \times \mathbb{C}^2$ by the equivalence relation $\sim$ given by

$$(x, y, z, t, \lambda_0, \lambda_1) \sim (x', y', z', t', \lambda_0', \lambda_1')$$
$$\Leftrightarrow \ \exists \mu, \nu \in \mathbb{C}^*, \ (x', y', z', t', \lambda_0', \lambda_1') = (\mu x, \mu y, \mu z, \mu t, \mu^{2d-3}\nu\lambda_0, \nu\lambda_1).$$

We observe that $H^0(\mathbb{F}_{(3)}(-2d+3, 0))$ corresponds to the set of $P \in \mathbb{C}[x, y, z, t, \lambda_0, \lambda_1]$ with homogeneous multidegree multideg defined above.

Now, since $F \in Sym^d(\mathbf{W}^\vee)$, $\alpha \in Sym^{d-1}(\mathbf{W}^\vee)$, $\beta \in Sym^{3d-4}(\mathbf{W}^\vee)$ and $\gamma \in Sym^{5d-7}(\mathbf{W}^\vee)$, we get that $F$ and $Q_{\mathbf{S},F}$ are in $H^0(\mathbb{F}_{(3)}(-2d+3, 0))$. Therefore $\hat{Z}$ is a subvariety of $\mathbb{F}_{(3)}(-2d+3, 0)$ since it can be rewritten:

$$\hat{Z} = \{(m, [\lambda_0 : \lambda_1]) \in \mathbb{F}_{(3)}(-2d+3, 0) \ : \ F(\mathbf{m}) = 0 \ \text{ and } \ Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0\}.$$

Since each coordinate of $\sigma$ is in $Sym^{2d-2}(\mathbf{W}^\vee)$, the map $\mathbf{\Phi} : \mathbf{W} \times \mathbb{C}^2 \to \mathbf{W}$ given by

$$\mathbf{\Phi}(\mathbf{m}, \lambda_0, \lambda_1) := \lambda_0 \cdot \mathbf{m} + \lambda_1 \cdot \boldsymbol{\sigma}(\mathbf{m}) \quad \in (C_{2(d-1),1})^4$$

defines a rational map $\Phi : \mathcal{X} \to \mathbb{P}^3$ with

$$\mathcal{X} = \{(m, [\lambda_0 : \lambda_1]) \in \mathbb{F}_{(3)}(-2d+3, 0) \ : \ Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0\}.$$

Let us denote by $B_{\Phi_{|\hat{Z}}}$ the set of base points of the map $\Phi_{|\hat{Z}}$, i.e.

$$B_{\Phi_{|\hat{Z}}} := \{(m, [\lambda_0 : \lambda_1]) \in \hat{Z} \ : \ \mathbf{\Phi}(\mathbf{m}, \lambda_0, \lambda_1) = 0\}.$$

We consider the canonical projection $\pi_1 : \mathbb{F}_{(3)}(-2d+3, 0) \to \mathbb{P}^3$ (given by $\pi_1(m, [\lambda_0 : \lambda_1]) = m$).

**Notation 14.** *We write* $\mathcal{B} := \pi_1(B_{\Phi_{|\hat{Z}}})$.

Observe that, for any $m \in \mathcal{B}$, there exists a unique $[\lambda_0 : \lambda_1] \in \mathbb{P}^1$ such that $\lambda_0 \cdot \mathbf{m} + \lambda_1 \cdot \boldsymbol{\sigma}(\mathbf{m}) = 0$. This gives the following scheme

$$\begin{array}{ccccccc}
B_{\Phi_{|\hat{Z}}} & \hookrightarrow & \hat{Z} & \hookrightarrow & \mathcal{X} & \xrightarrow{\Phi} & \mathbb{P}^3 \\
\pi_{1|B_\Phi} \downarrow 1:1 & & \pi_{1|\hat{Z}} \downarrow 2:1 & & \pi_{1|\mathcal{X}} \downarrow 2:1 & & \\
\mathcal{B} & \hookrightarrow & \mathcal{Z} & \hookrightarrow & \mathbb{P}^3 & &
\end{array}$$

Therefore $\#B_\phi = \#\mathcal{B}$.

**Remark 15.** *The caustic by reflection $\Sigma_S(\mathcal{Z})$ of $\mathcal{Z}$ from $S$ corresponds to the following Zariski closure*

$$\Sigma_S(\mathcal{Z}) = \overline{\Phi(\hat{Z})} \quad \subseteq \mathbb{P}^3.$$

Note that $\mathcal{B} \subseteq \mathcal{M}_{S,\mathcal{Z}}$ (with $\mathcal{M}_{S,\mathcal{Z}}$ defined in Definition 10) Due to the classical blowing-up theorem, we obtain the following result valid in the general case.

**Proposition 16.** *Assume that the set $\mathcal{B}$ is finite and that* $\dim(\mathcal{Z} \cap \mathcal{M}_{S,\mathcal{Z}}) \leq 1$. *Then there exists $\delta \in \mathbb{N}^* \cup \{\infty\}$ such that, for a generic point $P \in \Sigma_S(\mathcal{Z})$, we have $\#[\pi_1(\Phi_{|\hat{Z}}^{-1}(\{P\}))\setminus\mathcal{B}] = \delta$.*

*Proof.* Observe that, by hypothesis, the set $B_{\Phi_{|\hat{Z}}}$ is finite. Now, applying the blowing-up result given in [8, Example II-7.17.3], we get the existence of a variety $\tilde{\hat{Z}}$ and of two morphisms $\pi : \tilde{\hat{Z}} \to \hat{Z}$ and $\tilde{\Phi} : \tilde{\hat{Z}} \to \mathbb{P}^3$ such that

- $\pi$ defines an isomorphism from $\pi^{-1}(\hat{Z} \setminus B_\Phi)$ onto $\hat{Z} \setminus B_\Phi$,
- On $\pi^{-1}(\hat{Z} \setminus B_\Phi)$, we have $\tilde{\Phi} = \Phi \circ \pi$,
- $\tilde{\Phi}(\tilde{\hat{Z}})$ is the Zariski closure of $\Phi(\hat{Z} \setminus B_\Phi)$, i.e. $\tilde{\Phi}(\tilde{\hat{Z}}) = \Sigma_S(\mathcal{Z})$,
- $\dim(\tilde{\hat{Z}}) = 2$,
- $E := \tilde{\hat{Z}} \setminus \pi^{-1}(\hat{Z} \setminus B_\Phi)$ is a variety of dimension at most 1.

Let $\delta$ be the degree of the morphism $\tilde{\Phi}$.

If $\delta = \infty$, then $\dim(\Sigma_S(\mathcal{Z})) \leq \dim(\tilde{\Phi}(\tilde{\hat{Z}})) < 2$.

Assume now that $\delta < \infty$. Since $\tilde{\Phi}$ is a morphism, every point of $\tilde{\Phi}(\tilde{\hat{Z}})$ has $\delta$ preimages by $\tilde{\Phi}$ in $\tilde{\Phi}(\tilde{\hat{Z}})$. Now, observe that $\dim(\Sigma_S(\mathcal{Z})) = 2$ and that $\dim(\tilde{\Phi}(E)) < 2$. Therefore, a generic point of $\Sigma_S(\mathcal{Z})$ is in $\Phi(\hat{Z}) \setminus [(\mathcal{Z} \cap \mathcal{M}_{S,\mathcal{Z}}) \cup \tilde{\Phi}(E)]$. Let $P$ in this set. We have

$$\begin{aligned} \delta &= \#\Phi_{|\mathcal{Z}}^{-1}(\{P\}) \\ &= \#\{(m, [\lambda_0 : \lambda_1]) \in \mathcal{Z} \times \mathbb{P}^1 \ : \ Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0 \ \text{ and } \ \Pi(\lambda_0 \cdot \mathbf{m} + \lambda_1 \cdot \boldsymbol{\sigma}(\mathbf{m})) = P\}. \end{aligned}$$

Observe that, for $m \in \mathcal{Z} \cap \mathcal{M}_{S,\mathcal{Z}}$, $\Phi(\pi_1^{-1}(\{m\})) = \{m\}$ by Definition 10. Since $P \notin \mathcal{Z} \cap \mathcal{M}_{S,\mathcal{Z}}$, we know that $\Phi_{|\mathcal{Z}}^{-1}(\{P\}) \cap \pi_1^{-1}(\mathcal{M}_{S,\mathcal{Z}}) = \emptyset$. So, for any $m \in \pi_1(\Phi_{|\mathcal{Z}}^{-1}(\{P\}))$, there exists a unique $[\lambda_0 : \lambda_1] \in \mathbb{P}^1$ such that $(m, [\lambda_0, \lambda_1]) \in \Phi_{|\mathcal{Z}}^{-1}(\{P\})$. Therefore $\delta = \#\pi_1(\Phi_{|\mathcal{Z}}^{-1}(\{P\})) = \#[\pi_1(\Phi_{|\mathcal{Z}}^{-1}(\{P\})) \setminus \mathcal{B}]$ (since $\#[\pi_1(\Phi_{|\mathcal{Z}}^{-1}(\{P\})) \cap \mathcal{B} \subset \pi_1(\Phi_{|\mathcal{Z}}^{-1}(\{P\})) \cap \mathcal{M}_{S,\mathcal{Z}} = \emptyset$. □

## 5. Base points of $\Phi$

**Proposition 17.** *The base points of $\Phi_{|\hat{Z}}$ are the points $(m, [\lambda_0 : \lambda_1]) \in \hat{Z}$ satisfying one of the following conditions:*

1. *$m \in V(F, \Delta_{\mathbf{S}}F, Q(\nabla F))$ (i.e. $m \in Sing(\mathcal{Z})$ or $m$ is a point of tangency of $\mathcal{Z}$ with an isotropic plane containing $S$) and $\lambda_0 = 0$,*
2. *$t = F_x = F_y = F_z = 0$ and $x^2 + y^2 + z^2 = 0$ (i.e. $m$ is a cyclic point with $\mathcal{T}_m\mathcal{Z} = \mathcal{H}^\infty$) and $\lambda_0 = 0$,*
3. *$t = F_x = F_y = F_z = 0$ (i.e. $\mathcal{T}_m\mathcal{Z} = \mathcal{H}^\infty$) and $H_F = 0$ and $\lambda_0 = 0$,*
4. *$m = S \in \mathcal{Z}$ and $[\lambda_0 : \lambda_1]$ is the unique element of $\mathbb{P}^1$ such that $\lambda_0 \cdot \mathbf{m} + \lambda_1 Q(\nabla F) \cdot \mathbf{S} = 0$,*

(5) *$m$ is a cyclic point (i.e. $m \in \mathcal{C}_\infty$), $[\lambda_0 : \lambda_1] = [2\Delta_{\mathbf{S}} F(F_{xx} + F_{yy} + F_{zz}) : 2d - 1] \neq [0 : 1]$ and $(F_{xx} + F_{yy} + F_{zz}) \cdot \mathbf{m} = (2d - 1)\boldsymbol{\kappa}(\nabla F)$.*

*Proof.* Let us prove that any base point $(m, [\lambda_0 : \lambda_1])$ has one of the form announced in the statement of the proposition (the converse being direct). Let $(m, [\lambda_0 : \lambda_1]) \in B_{\Phi_{|\hat{\mathcal{Z}}}}$. By definition of $\Phi$, we have $0 = \lambda_0 \cdot \mathbf{m} + \lambda_1 \cdot \boldsymbol{\sigma}(\mathbf{m})$. So $m \in \mathcal{Z} \cap \mathcal{M}_{S,\mathcal{Z}}$ and these $m$ have been determined in Proposition 11.

- Assume first that $\boldsymbol{\sigma}(\mathbf{m}) = 0$. Then the unique $[\lambda_0 : \lambda_1] \in \mathbb{P}^1$ satisfying $\lambda_0 \cdot \mathbf{m} + \lambda_1 \cdot \boldsymbol{\sigma}(\mathbf{m}) = 0$ is $[\lambda_0 : \lambda_1] = [0 : 1]$. Hence $\lambda_0 = 0$, $\lambda_1 \neq 0$ and so $Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = \gamma(\mathbf{m})\lambda_1^2$.
  If $\Delta_{\mathbf{S}} F(\mathbf{m}) = 0$ and $Q(\nabla F(\mathbf{m})) = 0$, then $\gamma(\mathbf{m}) = 0$.
  Otherwise, according to Proposition 7, we have $F_x = F_y = F_z = 0$ and so $t = 0$. Now, we have $\gamma(\mathbf{m}) = 0$ if and only if $(x^2 + y^2 + z^2)t_0^2 H_F = 0$.
- Assume now that $m = S$ and $\boldsymbol{\sigma}(\mathbf{m}) \neq 0$. Then $\Delta_{\mathbf{S}} F(\mathbf{m}) = 0$ and so $\boldsymbol{\sigma}(\mathbf{m}) = Q(\nabla F(\mathbf{m})) \cdot \mathbf{S}$. We consider the unique $[\lambda_0, \lambda_1]$ such that $\lambda_0 \cdot \mathbf{m} + \lambda_1 \cdot \boldsymbol{\sigma}(\mathbf{m}) = 0$. Observe that $\lambda_0 \neq 0$ and that $\lambda_1 \neq 0$. Since $\Delta_{\mathbf{S}} F(\mathbf{m}) = 0$, we have $Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = \beta(\mathbf{m})\lambda_0\lambda_1$. But $\beta(\mathbf{m}) = -2\,\mathrm{Hess}_{F,\mathbf{m}}(\mathbf{S}, \boldsymbol{\sigma}(\mathbf{m})) = 0$ due to $m = S = \sigma(m)$.
- Assume finally that $m \in \mathcal{W}$. We have $\Delta_{\mathbf{S}} F(\mathbf{m}) \neq 0$, $x^2 + y^2 + z^2 = 0$, $m = [F_x(\mathbf{m}) : F_y(\mathbf{m}) : F_z(\mathbf{m}) : 0]$, $t = 0$ and $\boldsymbol{\sigma}(\mathbf{m}) = -2\Delta_{\mathbf{S}} F(\mathbf{m})\boldsymbol{\kappa}(\nabla F(\mathbf{m}))$. Since $t = 0$, it follows that $N_{\mathbf{S}}(\mathbf{m}) = (x^2 + y^2 + z^2)t_0^2 = 0$ and so that $\gamma(\mathbf{m}) = 0$. Let $[\lambda_0, \lambda_1] \in \mathbb{P}^1$ be such that $\lambda_0 \cdot \mathbf{m} + \lambda_1 \cdot \boldsymbol{\sigma}(\mathbf{m}) = 0$. We observe that $\lambda_1 \neq 0$ and $\lambda_0 \neq 0$. We have $\alpha(\mathbf{m})\lambda_0^2 = \Delta_{\mathbf{S}} F(\mathbf{m})\lambda_0^2$ and
  $$-2 \cdot \mathrm{Hess}_{F,\mathbf{m}}(\mathbf{S}, \boldsymbol{\sigma}(\mathbf{m}))\lambda_0\lambda_1 = 2\,\mathrm{Hess}_{F,\mathbf{m}}(\mathbf{S}, \mathbf{m})\lambda_0^2 = 2(d - 1)\Delta_{\mathbf{S}} F(\mathbf{m}),$$
  since $(d - 1)F_w = xF_{xw} + yF_{yw} + zF_{zw} + tF_{tw}$ for every $w \in \{x, y, z\}$. Hence we have
  $$Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = (2d - 1)\Delta_{\mathbf{S}} F(\mathbf{m})\lambda_0^2 - 2(\Delta_{\mathbf{S}} F(\mathbf{m}))^2(F_{xx}(\mathbf{m}) + F_{yy}(\mathbf{m}) + F_{zz}(\mathbf{m}))\lambda_0\lambda_1.$$
  Hence $Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0$ if and only if $\lambda_0/\lambda_1 = 2\Delta_{\mathbf{S}} F(\mathbf{m})(F_{xx}(\mathbf{m}) + F_{yy}(\mathbf{m}) + F_{zz}(\mathbf{m}))/(2d - 1)$. We conclude by using $\lambda_0 \cdot \mathbf{m} + \lambda_1 \cdot \boldsymbol{\sigma}(\mathbf{m}) = 0$ and the formula obtained for $\boldsymbol{\sigma}(\mathbf{m})$.

□

**Corollary 18.** *A point $m$ is in $\mathcal{B}$ if and only if it satisfies one of the following conditions:*

(1) *$m \in \mathcal{B}_0 = V(F, \Delta_{\mathbf{S}} F, Q(\nabla F))$, i.e. $m$ is a singular point of $\mathcal{Z}$ or $m$ is a point of tangency of $\mathcal{Z}$ with an isotropic plane containing $S$ (see also (1)),*
(2) *$m$ is a point of tangency of $\mathcal{Z}$ with $\mathcal{H}^\infty$ and $m$ lies on the ombilical curve $\mathcal{C}_\infty$,*
(3) *$m$ is a point of tangency of $\mathcal{Z}$ with $\mathcal{H}^\infty$ and $m$ lies in the hessian surface of $\mathcal{Z}$,*
(4) *$m = S \in \mathcal{Z}$,*
(5) *$m$ lies on $\mathcal{C}_\infty$ and $(F_{xx} + F_{yy} + F_{zz}) \cdot \mathbf{m} = (2d - 1)\boldsymbol{\kappa}(\nabla F)$.*

*This can be summarized in the following formula*

$$\mathcal{B} = \mathcal{Z} \cap [V(\Delta_{\mathbf{S}} F, Q(\nabla F)) \cup \{S\} \cup V(H_F \cdot Q, \boldsymbol{\kappa}(\nabla F))_\infty \cup \mathcal{G}_\infty],$$

*with $\mathcal{G}_\infty = \{m \in \mathcal{C}_\infty \ : \ (F_{xx} + F_{yy} + F_{zz}) \cdot \mathbf{m} = (2d - 1)\boldsymbol{\kappa}(\nabla F)\}$.*

**Remark 19.** *The set $\mathcal{B}$ is never empty. Except (iv), the forms of the base points are very similar to the base points of the caustic map of planar curves (see [9]).*

*For a general $(\mathcal{Z}, S)$, the set $\mathcal{B}$ consists of the points at which $\mathcal{Z}$ admits an isotropic tangent plane containing $S$, i.e. $\mathcal{B} = \mathcal{B}_0 = V(F, \Delta_{\mathbf{S}} F, Q(\nabla F))$, and in general $\mathcal{Z}$ has no singular point and $\mathcal{B}_0 \cap \mathcal{H}^\infty = \emptyset$. In this case $\mathcal{B}$ is fully interpreted by (1).*

## 6. Reflected Polar curve

Let $H \in Pic(\mathbb{P}^3)$ be the hyperplane class. We will identify $\pi_{1,*}(\Phi^* H^2) \in A_2(\mathbb{P}^3)$ with the class of sets $\mathcal{P}_{A,B} \subseteq \mathbb{P}^3$ defined as follows.

**Definition 20.** *For any $A, B \in \mathbf{W}^\vee$, we define the set $D_{A,B} := V(A,B) \subseteq \mathbb{P}^3$ and the* reflected polar *$\mathcal{P}_{A,B}$ by*

$$\mathcal{P}_{A,B} = \pi_1(\Phi^{-1}(D_{A,B})) \cup \pi_1(Base(\Phi)),$$

*i.e. $\mathcal{P}_{A,B}$ corresponds to the following set:*

$$\{m \in \mathbb{P}^3 : \ \exists[\lambda_0, \lambda_1] \in \mathbb{P}^1, \ A(\lambda_0 \mathbf{m} + \lambda_1 \boldsymbol{\sigma}(\mathbf{m})) = 0, \ B(\lambda_0 \mathbf{m} + \lambda_1 \boldsymbol{\sigma}(\mathbf{m})) = 0, \ Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0\}.$$

Observe that $\dim \mathcal{M}_{S,\mathcal{Z}} \geq 1$ since $\mathrm{Base}(\sigma) \subseteq \mathcal{M}_{S,\mathcal{Z}}$ (see Definition 10 for the definition of $\mathcal{M}_{S,\mathcal{Z}}$).

**Proposition 21.** *Assume that $\dim \mathcal{M}_{S,\mathcal{Z}} = 1$. For generic $A, B$ in $\mathbf{W}^\vee$, $D_{A,B}$ is a line and $\mathcal{P}_{A,B} = V(K_1, K_2, K_3)$, with*

$$K_1(\mathbf{m}) := A(\boldsymbol{\sigma}(\mathbf{m}))B(\mathbf{m}) - A(\mathbf{m})B(\boldsymbol{\sigma}(\mathbf{m})), \quad K_2(\mathbf{m}) := Q_{\mathbf{S},F}(\mathbf{m}, -A(\boldsymbol{\sigma}(\mathbf{m})), A(\mathbf{m})),$$

$$K_3(\mathbf{m}) := Q_{\mathbf{S},F}(\mathbf{m}, -B(\boldsymbol{\sigma}(\mathbf{m})), B(\mathbf{m})).$$

*Proof.* Assume that $D_{A,B}$ is a line that does not correspond to any line $(m\,\sigma(m))$ for $m \in \mathbb{P}^3 \setminus \mathcal{M}_{S,\mathcal{Z}}$ (this is true for a generic $(A,B)$ in $(\mathbf{W}^\vee)^2$) and does not contain any point of $\mathcal{M}_{S,\mathcal{Z}}$ (this is true for generic $(A,B)$ in $(\mathbf{W}^\vee)^2$ since $\dim \mathcal{M}_{S,\mathcal{Z}} = 1$). Hence

$$V(A, B, A \circ \boldsymbol{\sigma}, B \circ \boldsymbol{\sigma}) = \emptyset \ \text{ in } \mathbb{P}^3. \tag{14}$$

Let $m \in \mathcal{P}_{A,B}$ and $[\lambda_0 : \lambda_1] \in \mathbb{P}^1$ be such that $A(\boldsymbol{\Phi}(\mathbf{m}, \lambda_0, \lambda_1)) = 0 = B(\boldsymbol{\Phi}(\mathbf{m}, \lambda_0, \lambda_1)) = 0$ and $Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0$. This implies that

$$\lambda_0 \cdot A(\mathbf{m}) + \lambda_1 \cdot A(\boldsymbol{\sigma}(\mathbf{m})) = 0 = \lambda_0 \cdot B(\mathbf{m}) + \lambda_1 \cdot B(\boldsymbol{\sigma}(\mathbf{m})).$$

Hence $(-A(\boldsymbol{\sigma}(\mathbf{m})), A(\mathbf{m}))$ and $(-B(\boldsymbol{\sigma}(\mathbf{m})), B(\mathbf{m}))$ are proportional to $(\lambda_0, \lambda_1)$. Since $Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0$, we conclude that $m$ is in $V(K_1, K_2, K_3)$.

Conversely, assume now that $m$ is a point of $V(K_1, K_2, K_3)$. According to (14), we have

$$A(\mathbf{m}) \neq 0 \text{ or } B(\mathbf{m}) \neq 0 \text{ or } A(\boldsymbol{\sigma}(\mathbf{m})) \neq 0 \text{ or } B(\boldsymbol{\sigma}(\mathbf{m})) \neq 0.$$

Observe that $(-A(\boldsymbol{\sigma}(\mathbf{m})), A(\mathbf{m}))$ and $(-B(\boldsymbol{\sigma}(\mathbf{m})), B(\mathbf{m}))$ are proportional and at least one is non null. Let $[\lambda_0 : \lambda_1]$ be the corresponding point in $\mathbb{P}^1$. we have

$$A(\lambda_0 \cdot \mathbf{m} + \lambda_1 \cdot \boldsymbol{\sigma}(\mathbf{m})) = 0 \text{ and } B(\lambda_0 \cdot \mathbf{m} + \lambda_1 \cdot \boldsymbol{\sigma}(\mathbf{m})) = 0,$$

$$Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0.$$

□

**Notation 22.** *We write $B_{S,\mathcal{Z}}$ for the set of points $m \in \mathbb{P}^3$ for which $Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0$ in $\mathbb{C}[\lambda_0, \lambda_1]$.*

Observe that, for $m \in \mathbb{P}^3 \setminus B_{S,\mathcal{Z}}$, there are at most two $[\lambda_0 : \lambda_1] \in \mathbb{P}^1$ such that $Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0$, and so $\#(\mathcal{X} \cap \pi_1^{-1}(m)) \leq 2$.

**Remark 23.** *According to the expressions of $\sigma$, $\alpha$, $\beta$ and $\gamma$, we have*

$$B_{S,\mathcal{Z}} = V(\alpha, \beta, \gamma) = V(\Delta_{\mathbf{S}} F, \mathrm{Hess}_F(\mathbf{S}, \mathbf{S}) \cdot Q(\nabla F)).$$

We observe that $\dim B_{S,\mathcal{Z}} \geq 1$. Observe that $(\mathcal{M}_{S,\mathcal{Z}} \cup B_{S,\mathcal{Z}})$ is the set of $m \in \mathbb{P}^3$ such that $\Phi(\pi^{-1}(\{m\})) = \Pi(Vect(\mathbf{m}, \boldsymbol{\sigma}(\mathbf{m})))$. When $m \in \mathcal{M}_{S,\mathcal{Z}}$, $\Phi(\pi^{-1}(\{m\})) = \{m\}$ and when $m \in B_{S,\mathcal{Z}}$, $\Phi(\pi^{-1}(\{m\})) = \mathcal{R}_m$. Recall that $\mathcal{B} = \pi_1(B_{\Phi_{|\hat{\mathcal{Z}}}})$.

**Proposition 24.** *Assume that* $\dim(B_{S,\mathcal{Z}} \cup \mathcal{M}_{S,\mathcal{Z}}) = 1$ *and that* $\#\mathcal{B} < \infty$. *Then, for generic* $A, B \in \mathbf{W}^\vee$, $\dim \mathcal{P}_{A,B} = 1$ *and* $\deg \mathcal{P}_{A,B} = (2d-1)5(d-1)$.

*Proof.* As in the proof of the preceding proposition, we consider generic $(A, B) \in (\mathbf{W}^\vee)^2$ such that (14) holds true. Since $\dim B_{S,\mathcal{Z}} \leq 1$, we also assume that this generic $(A, B) \in (\mathbf{W}^\vee)^2$ satisfies

$$\#\{m \in B_{S,\mathcal{Z}} \ : \ \mathcal{R}_m \cap D_{A,B} \neq \emptyset\} < \infty, \tag{15}$$

where $\mathcal{R}_m$ is the line $(m\,\sigma(m))$. Recall that $\mathcal{P}_{A,B} = V(K_1, K_2, K_3)$. First, we observe that $K_1 \in Sym^{2d-1}(\mathbf{W}^\vee)$ (it is non null for generic $(A, B)$ since $\dim \mathcal{M}_{S,\mathcal{Z}} \leq 1$) whereas $K_2, K_3 \in Sym^{5(d-1)}(\mathbf{W}^\vee)$. Now, if $m$ is a point of $\mathbb{P}^3 \setminus V(A, A \circ \boldsymbol{\sigma})$, then the following equivalence holds true

$$m \in \mathcal{P}_{A,B} \quad \Leftrightarrow \quad m \in V(K_1, K_2)$$

and that, if $m$ is a point of $\mathbb{P}^3 \setminus V(B, B \circ \boldsymbol{\sigma})$, then $m \in \mathcal{P}_{A,B} \Leftrightarrow V \in V(K_1, K_3)$. Therefore $\dim \mathcal{P}_{A,B} \in \{1, 2\}$.

- Let us prove that $\dim \mathcal{P}_{A,B} = 1$. Assume first that $\dim(\Sigma_S(\mathcal{Z})) \leq 1$. Then, for generic $(A, B) \in (\mathbf{W}^\vee)^2$, we have $\Sigma_{\mathcal{S}}(\mathcal{Z}) \cap D_{A,B} = \emptyset$. Therefore, $\pi_1(\Phi^{-1}_{|\hat{\mathcal{Z}}}(D_{A,B})) = \emptyset$ and so $\mathcal{Z} \cap \mathcal{P}_{A,B} = \mathcal{B}$ is a finite set, which implies that $\dim \mathcal{P}_{A,B} \leq 1$.

  Assume now that $\dim(\Sigma_S(\mathcal{Z})) = 2$. Let us consider a generic $(A, B) \in (\mathbf{W}^\vee)^2$ such that $\#(\Sigma_S(\mathcal{Z}) \cap D_{A,B}) < \infty$ and such that, for every $P \in \Sigma_S(\mathcal{Z}) \cap D_{A,B}$, we have

  $$\#[\pi_1(\Phi^{-1}_{|\hat{\mathcal{Z}}}(\{P\})) \setminus \mathcal{B}] = \delta,$$

  (see Proposition 16). This implies that $\#\pi_1(\Phi^{-1}_{|\hat{\mathcal{Z}}}(D_{A,B})) < \infty$ and so $\#(\mathcal{Z} \cap \mathcal{P}_{A,B}) < \infty$ (since $\#\mathcal{B} < \infty$). Hence $\dim \mathcal{P}_{A,B} \leq 1$ since $\dim \mathcal{Z} = 2$.
- Let $(A, B)$ as above. Since $\dim \mathcal{P}_{A,B} = 1$, $\deg \mathcal{P}_{A,B}$ corresponds to $\#(\mathcal{P}_{A,B} \cap \mathcal{H})$ for a generic plane $\mathcal{H}$ in $\mathbb{P}^3$.

  Since $\dim \mathcal{M}_{S,\mathcal{Z}} = 1$, for a generic $(A, B) \in (\mathbf{W}^\vee)^2$, we have $\dim V(A, A \circ \boldsymbol{\sigma}) = 1$ and $\dim V(B, B \circ \boldsymbol{\sigma}) = 1$. Since $\dim \mathcal{P}_{A,B} = 1$, we conclude that $\dim V(K_1, K_2) = \dim V(K_1, K_3) = 1$. Moreover, for a generic $(A, B) \in (\mathbf{W}^\vee)^2$, we have

  $$\#E < \infty, \text{ with } E := V(A, A \circ \boldsymbol{\sigma}, K_3) \cup V(B, B \circ \boldsymbol{\sigma}, K_2) < \infty.$$

  Indeed, let us explain how we get $\#V(A, A \circ \boldsymbol{\sigma}, K_3) < \infty$. Let $m \in V(A, A \circ \boldsymbol{\sigma})$. According to (14), $[-B(\boldsymbol{\sigma}(\mathbf{m})) : B(\mathbf{m})] \in \mathbb{P}^1$, therefore $-B(\boldsymbol{\sigma}(\mathbf{m})) \cdot \mathbf{m} + B(\mathbf{m}) \cdot \boldsymbol{\sigma}(\mathbf{m})$ is in $\mathcal{R}_m \cap D_{A,B}$. According to (15), we obtain $\#(V(A, A \circ \boldsymbol{\sigma}, K_3) \cap B_{S,\mathcal{Z}}) < \infty$. Now, for $m \in V(A, A \circ \boldsymbol{\sigma}) \setminus B_{S,\mathcal{Z}}$, there are at most two $[\lambda_0 : \lambda_1] \in \mathbb{P}^1$ such that $Q_{\mathbf{S},F}(m, \lambda_0, \lambda_1) = 0$, therefore, for a generic $B$,

  $$\#\{(m, [\lambda_0 : \lambda_1]) \in \mathcal{X} : \ m \in V(A, A \circ \boldsymbol{\sigma}) \setminus B_{S,\mathcal{Z}}, \ B(\Phi(m, [\lambda_0, \lambda_1])) = 0\} < \infty$$

  and so $\#(V(A, A \circ \boldsymbol{\sigma}, K_3) \setminus B_{S,\mathcal{Z}}) < \infty$. This implies $\#V(A, A \circ \boldsymbol{\sigma}, K_3) < \infty$ for a generic $(A, B) \in (\mathbf{W}^\vee)^2$. In the same way, we get $\#V(B, B \circ \boldsymbol{\sigma}, K_2) < \infty$ for generic $(A, B) \in (\mathbf{W}^\vee)^2$.

  Now, for a generic hyperplane $\mathcal{H}$ (containing no point of $E$ and such that $\#H \cap V(K_1, K_2) = (2d-1)5(d-1)$), we have

  $$\deg \mathcal{P}_{A,B} = \#\mathcal{H} \cap \mathcal{P}_{A,B} = \#\mathcal{H} \cap V(F_1, F_2) = (2d-1)5(d-1).$$

$\square$

## 7. A formula for the degree of the caustic

Recall that $\mathcal{B}$ has been completely described in Corollary 18 (see also Remark 19 for the general case). We refer to Definition 10 and Proposition 11 for $\mathcal{M}_{S,\mathcal{Z}}$ and to Notation 22 and Remark 23 for $\mathcal{B}_{S,\mathcal{Z}}$.

**Proposition 25.** *Assume that* $\dim \mathcal{M}_{S,\mathcal{Z}} = 1$, *that* $\#(\mathcal{B}_{S,\mathcal{Z}} \cap Z) < \infty$ *and that* $\#\mathcal{B} < \infty$.

*If* $\dim(\Sigma_S(\mathcal{Z})) < 2$, *for a generic* $(A,B) \in (\mathbf{W}^\vee)^2$, *we have*

$$0 = 5d(d-1)(2d-1) - \sum_{m\in\mathcal{B}} i_m(\mathcal{Z}, \mathcal{P}_{A,B}).$$

*If* $\dim(\Sigma_S(\mathcal{Z})) = 2$, *for a generic* $(A,B) \in (\mathbf{W}^\vee)^2$, *if* $\delta < \infty$, *we have*

$$mdeg(\Sigma_S(\mathcal{Z})) = 5d(d-1)(2d-1) - \sum_{m\in\mathcal{B}} i_m(\mathcal{Z}, \mathcal{P}_{A,B}),$$

*where* $mdeg(\Sigma_S(\mathcal{Z}))$ *is the degree with multiplicity of* $(\Sigma_S(\mathcal{Z}))$ *(*$mdeg(\Sigma_S(\mathcal{Z})) = \delta \deg(\Sigma_S(\mathcal{Z}))$*, see Proposition 16 for the property satisfied by* $\delta$*), where* $d$ *is the degree of the polynomial* $F$ *such that* $\mathcal{Z} = V(F)$ *and where* $i_m(\mathcal{Z}, \mathcal{P}_{A,B})$ *denotes the intersection number of* $\mathcal{Z}$ *with* $\mathcal{P}_{A,B}$ *at point* $m$.

Let us notice, that in this formula, we can replace $i_m(\mathcal{Z}, \mathcal{P}_{A,B})$ by $i_m(\mathcal{Z}, V(F_1, F_2))$, with the notations of the proof of Proposition 24.

*Proof of Proposition 25.* First observe that for a generic $(A,B)$ in $(\mathbf{W}^\vee)^2$, we have $\deg(\mathcal{P}_{A,B}) = 5(d-1)(2d-1)$.

If $\dim \Sigma_S(\mathcal{Z}) < 2$ (i.e. $\delta = \infty$), taking $(A,B)$ such that $\deg(\mathcal{P}_{A,B}) = 5(d-1)(2d-1)$ and $D_{A,B} \cap \Sigma_S(\mathcal{Z}) = \emptyset$, we have $\mathcal{P}_{A,B} \cap \mathcal{Z} = \mathcal{B}$ and so

$$\begin{aligned} 5d(d-1)(2d-1) &= \deg(\mathcal{Z})\deg(\mathcal{P}_{A,B}) = \sum_{m\in\mathcal{Z}\cap\mathcal{P}_{A,B}} i_m(\mathcal{Z}, \mathcal{P}_{A,B}) \\ &= \sum_{m\in\mathcal{B}} i_m(\mathcal{Z}, \mathcal{P}_{A,B}) = \sum_{m\in\mathcal{B}} i_m(\mathcal{Z}, \mathcal{P}_{A,B}). \end{aligned}$$

Assume now that $\dim \Sigma_S(\mathcal{Z}) = 2$ (i.e. that $\delta$ is finite). We consider $(A,B)$ such that the following conditions hold true:

(0) $D_{A,B}$ is a line containing no reflected line $\mathcal{R}_m = (m\,\sigma(m))$ $(m \in \mathcal{Z})$,
(1) $\deg(\mathcal{P}_{A,B}) = 5(d-1)(2d-1)$,
(2) the points $P \in D_{A,B} \cap \Sigma_S(\mathcal{Z})$ are such that $\#[\pi_1(\Phi^{-1}_{|\hat{\mathcal{Z}}}(\{P\})) \setminus \mathcal{B}] = \delta$ (this is generic according to Proposition 16),
(3) For any $P \in D_{A,B} \cap \Sigma_S(\mathcal{Z})$, we have $i_P(\Sigma_S(\mathcal{Z}), D_{A,B}) = 1$ (this is true for a generic $(A,B)$ since $\Sigma_S(\mathcal{Z})$ is a surface),
(4) the line $D_{A,B}$ intersects no reflected line $\mathcal{R}_m$ with $m \in B_{S,\mathcal{Z}}$ (this is generic since $\#(\mathcal{Z} \cap B_{S,\mathcal{Z}}) < \infty$),
(5) for any $m \in (\mathcal{P}_{A,B} \cap \mathcal{Z}) \setminus \mathcal{B}$, we have $i_m(\mathcal{Z}, \mathcal{P}_{A,B}) = 1$ (this is explained at the end of this proof).

Due to (1), we have

$$\begin{aligned} 5d(d-1)(2d-1) &= \deg(\mathcal{Z})\deg(\mathcal{P}_{A,B}) = \sum_{m\in\mathcal{Z}\cap\mathcal{P}_{A,B}} i_m(\mathcal{Z},\mathcal{P}_{A,B}) \\ &= \sum_{m\in\mathcal{B}} i_m(\mathcal{Z},\mathcal{P}_{A,B}) + \sum_{m\in(\mathcal{Z}\cap\mathcal{P}_{A,B})\setminus\mathcal{B}} i_m(\mathcal{Z},\mathcal{P}_{A,B}). \end{aligned}$$

Now, we have

$$\begin{aligned} \sum_{m\in(\mathcal{Z}\cap\mathcal{P}_{A,B})\setminus\mathcal{B}} i_m(\mathcal{Z},\mathcal{P}_{A,B}) &= \#((\mathcal{Z}\cap\mathcal{P}_{A,B})\setminus\mathcal{B}) \quad \text{due to (5)} \\ &= \#[\pi_1(\Phi^{-1}_{|\hat{\mathcal{Z}}}(D_{A,B}))\setminus\mathcal{B}] \\ &= \delta\#(\Sigma_S(\mathcal{Z})\cap\mathcal{D}_{A,B}) \quad \text{due to (2)} \\ &= \delta\sum_P i_P(\Sigma_S(\mathcal{Z}),\mathcal{D}_{A,B}) = \delta\deg(\Sigma_S(\mathcal{Z})) \quad \text{due to (3)}. \end{aligned}$$

Let us now explain why (5) is true for a generic $(A,B)$. Let $m\in(\mathcal{P}_{A,B}\cap\mathcal{Z})\setminus\mathcal{B}$. Due to (4), $m\in\pi_1(\Phi^{-1}_{|\hat{\mathcal{Z}}}(D_{A,B}))\setminus B_{S,\mathcal{Z}}$. We consider the cone hypersurface $\mathcal{K}_{\mathcal{Z}}$ of $\mathbf{W}$ associated to $\mathcal{Z}$. Since $m\in\mathcal{Z}\setminus B_{S,\mathcal{Z}}$, there exist two maps $\psi^{\pm}:U\to\mathbb{P}^3$ defined on a neighbourhood $U$ of $m$ in $\mathbb{P}^3$ such that, for any $m\in U$, $\Phi(\pi_1^{-1}(\{m\}))=\{\psi^-(m),\psi^+(m)\}$. Let $\varepsilon\in\{+,-\}$ be such that $\Phi(\pi_1^{-1}(\{m\}))\cap D_{A,B}=\{\psi^{\varepsilon}(m)\}$ ($\psi^{\varepsilon}$ is unique for a generic $m\in\mathcal{Z}$ according to (0)) and the tangent space to $\mathcal{P}_{A,B}$ at $m$ is given by

$$V(A\circ D\psi^{\varepsilon}(\mathbf{m}), B\circ D\psi^{\varepsilon}(\mathbf{m})),$$

where $D\psi^{\pm}(\mathbf{m})$ are the jacobian matrices of $\psi^{\pm}$ taken at $\mathbf{m}$. Now, with these notations, for a generic $m$ in $\mathcal{Z}$, $D\psi^{\varepsilon}(\mathbf{m})$ are both invertible. This combined with (3) gives the result. □

## 8. About a reflected bundle

Recall that $O_{\mathcal{Z}}(-1)=\{(m,v)\in\mathcal{Z}\times\mathbf{W}\ :\ v\in m\}$. Observe that the set $\mathbf{R}(-1)$ of $(m,v)$ in the trivial bundle $\mathcal{Z}\times\mathbf{W}$ such that $v$ corresponds to a point of $\mathbb{P}^3$ on the reflected line $\mathcal{R}_m$ is:

$$\mathbf{R}(-1)=O_{\mathcal{Z}}(-1)+\{(m,v)\in\mathcal{Z}\times\mathbf{W}\ :\ v\in\sigma(m)\}.$$

Observe that this sum is direct in the generic case (when $S\notin\mathcal{Z}$ and when $\mathcal{W}=\emptyset$, see Proposition 11). But, contrarily to the normal bundle considered in [15, 5] to study the evolute, $\mathbf{R}(-1)$ does not define a bundle since its rank is not constant. Indeed, the dimension of $Vect(\mathbf{m},\boldsymbol{\sigma}(\mathbf{m}))$ equals 2 in general but not at every point $m\in\mathcal{Z}$ (it is strictly less than 2 when $m$ is a base point of $\sigma_{|\mathcal{Z}}$ and, as seen in Proposition 7, such points always exist).

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