

Given a chaotic dynamical system  $(A, T)$  and a nice observable  $f: A \rightarrow \mathbb{R}$ , the sequence  $(f \circ T^n)_{n \geq 0}$ , under nice  $T$ -invariant probability measure  $\mu$  on  $A$ , behaves like a sequence of i.i.d. random variables. While  $f \circ T^n$  is typically not independent from  $f \circ T^m$  with  $n \neq m$ , if  $n$  and  $m$  are far enough apart, information on one gives little information on the other.

This similarity can be exploited to show that  $(f \circ T^n)_{n \geq 0}$  has random-like properties:

- \* central limit theorem;
- \* local central limit theorem;
- \* large deviations...

In this mini-course, my goal is to explain how to adapt probabilistic potential theory, which is classical when it comes to random walks, to a class of chaotic dynamical systems. The plan is as follows:

- I- Limit theorems for random walks
- II- Spectral theory of the transfer operator
- III- Probabilistic potential theory for dynamical systems.

Part I and II delve into the background; new results will be discussed in part III (Thursday morning?).

## I- Limit theorems for random walks

For this section: let  $(X_k)_{k \geq 0}$  be a sequence of i.i.d.,  $\mathbb{Z}^d$ -valued random variables. I shall assume:

- \*  $X_0 \in \mathbb{L}^2$ ;
- \*  $\mathbb{E}(X_0) = 0$  (centered).

Set  $S_n := \sum_{k=0}^{n-1} X_k$ . The process  $(S_n)_{n \geq 0}$  is a Markov chain on  $\mathbb{Z}^d$ .

### 1- The central limit theorem

What are the typical values of  $S_n$ ? By the law of large numbers,  $S_n/n \xrightarrow[n \rightarrow \infty]{a.s.} 0$ ; by the central limit theorem, the typical values are of order  $\sqrt{n}$ .

### Central limit theorem

$\lim_{n \rightarrow \infty} \frac{S_n}{\sqrt{n}} = \mathcal{N}^d$ , where the limit is in distribution and  $\mathcal{N}^d \sim \mathcal{N}(0, \Sigma^d)$  has a Gaussian distribution of

variance  $\Sigma^d: \langle \Sigma^d u, \Sigma^d v \rangle = \mathbb{E}(\langle X_0, u \rangle \langle X_0, v \rangle)$ . It has density  $\frac{1}{2\pi \det \Sigma^d} e^{-\frac{\langle \Sigma^d \xi, \xi \rangle}{2}}$  if  $\Sigma$  is non-degenerate.

### Proof

By Lévy's criterion, it is enough to prove that the Fourier transform of  $S_n/\sqrt{n}$  converges to  $\mathbb{E}(e^{i \langle \xi, \mathcal{N}^d \rangle}) = e^{-\frac{\langle \Sigma^d \xi, \xi \rangle}{2}}$ .

Let  $\varphi(\xi) := \mathbb{E}(e^{i \langle \xi, X_0 \rangle})$  be the characteristic function of  $X_0$ .

Then  $\mathbb{E}(e^{i \langle \xi, S_n/\sqrt{n} \rangle}) = \mathbb{E}(e^{i \sum_{k=0}^{n-1} \langle \xi, X_k \rangle}) = \mathbb{E}(e^{i \langle \xi/\sqrt{n}, X_0 \rangle})^n = \varphi(\xi/\sqrt{n})^n$ .

Since  $X_0 \in \mathbb{L}^2$ ,  $\varphi$  is  $\mathcal{C}^2$ , with  $\varphi(0) = 1$ ,  $\varphi'(0) = i \mathbb{E}(X_0) = 0$  and  $\varphi''(0) = -\mathbb{E}(X_0 \otimes X_0) = -\Sigma^d$ .

Then  $\varphi(\xi/\sqrt{n})^n = \left(1 - \frac{\langle \Sigma^d \xi, \xi \rangle}{2n} + o(1/n)\right)^n \xrightarrow[n \rightarrow \infty]{} e^{-\frac{\langle \Sigma^d \xi, \xi \rangle}{2}} \square$ .

## 2- The local central limit theorem

Thanks to the CLT, we know for instance that:

$$\mathbb{P}(S_n \in [-\varepsilon\sqrt{n}, \varepsilon\sqrt{n}]^c) \xrightarrow{n \rightarrow \infty} \int_{[-\varepsilon\sqrt{n}, \varepsilon\sqrt{n}]^c} \frac{1}{(2\pi)^d} e^{-\frac{\langle \xi, S_n \rangle^2}{2n}} d\xi = \frac{2\varepsilon^d}{\pi^d} (1 + o(\varepsilon)).$$

What is the probability that  $S_n$  is in a given site, say, 0? If all the sites in the box  $[-\varepsilon\sqrt{n}, \varepsilon\sqrt{n}]^c$  are roughly equally likely, this probability would behave like

$$\frac{1}{(2\varepsilon\sqrt{n})^d} \cdot \frac{2\varepsilon^d}{\pi^d} (1 + o(\varepsilon)) \sim \frac{1}{2\pi^d (\mathbb{Z})_n}$$

To make it rigorous, we shall again use the Fourier transform.

### Definition

$(S_n)_{n \geq 0}$  is ~~non-arithmetic~~ <sup>non-arithmetic</sup> if it does not take its values in a proper sub-lattice of  $\mathbb{Z}^d$  (e.g. is not always even, or in  $\mathbb{Z} \times \{0\}$ ).

$(S_n)_{n \geq 0}$  is aperiodic if it does not take its values in a translate of a proper sub-lattice of  $\mathbb{Z}^d$ .

### Example

The simple random walk on  $\mathbb{Z}^d$  is ~~aperiodic~~ <sup>non-arithmetic</sup> but not aperiodic (it takes its values in  $\{(x, y) : x+y \text{ is odd}\}$ ).

These properties can be read on the characteristic function:

### Proposition

$(S_n)_{n \geq 0}$  is ~~aperiodic~~ <sup>non-arithmetic</sup> if and only if  $\varphi(\xi) \neq 1$  for  $\xi \neq 0$  (otherwise, we would have a non-trivial invariant function).

$(S_n)_{n \geq 0}$  is aperiodic if and only if  $|\varphi(\xi)| \neq 1$  for  $\xi \neq 0$ .

### Example

For the simple random walk on  $\mathbb{Z}^d$ ,  $\varphi(\pi, \pi) = -1$ .

## Local central limit theorem

Assume that  $(S_n)_{n \geq 0}$  is aperiodic. Then:

$$\mathbb{P}(S_n = 0) \sim \frac{1}{2\pi^d (\mathbb{Z})_n}$$

### Proof.

Using again the Fourier transform,

$$\begin{aligned} \mathbb{P}(S_n = 0) &= \mathbb{E}(\mathbb{1}_0(S_n)) = \mathbb{E} \left( \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{i\langle \xi, S_n \rangle} d\xi \right) \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \varphi(\xi)^n d\xi. \end{aligned}$$

Other notation:

$$\begin{aligned} \mathbb{P}(S_n = 0) &= \mathbb{E}(\mathbb{1}_0 \cdot \widehat{P}^n(\mathbb{1}_0)) \\ &= \mathbb{E}(\widehat{P}^n(\mathbb{1}_0)) \end{aligned}$$

$\widehat{P}$  acts by multiplication by  $\varphi(\xi)$  in Fourier domain.

$$\mathbb{P}(S_n = 0) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \varphi(\xi)^n d\xi.$$

Let  $\varepsilon > 0$ . On  $B(0, \varepsilon)^c$ ,  $|\varphi(\xi)| < 1$  by aperiodicity, so  $\frac{1}{2\pi^d} \int_{B(0, \varepsilon)^c} \varphi(\xi)^n d\xi = o(n^{-d})$ .

Let  $\varepsilon > 0$ . On  $B(0, \varepsilon)^c$ ,  $|\varphi(\xi)| < 1$  by aperiodicity, so  $\frac{1}{(2\pi)^d} \int_{B(0, \varepsilon)^c} \varphi(\xi)^n d\xi = o(n^{-d})$  (exponential decay),  $|\ln(\varphi(\xi))| < 1$ .

On  $B(0, \epsilon)$ ,  $\psi(\xi) = 1 - \frac{\|\Sigma \xi\|^2}{2} + o(\|\xi\|^2)$ ,  $\leftarrow$  ~~comparable at fixed  $\epsilon$~~   $\rightarrow$  uniformly small when  $\epsilon \rightarrow 0$

$$\begin{aligned} \frac{1}{(2\pi)^d} \int_{B(0, \epsilon)} \left(1 - \frac{\|\Sigma \xi\|^2}{2} + o(\|\xi\|^2)\right)^n d\xi &\approx \frac{1}{(2\pi)^d} \int_{B(0, \epsilon)} e^{-\frac{\|\Sigma \xi\|^2}{2} + o(\|\xi\|^2)} d\xi \\ &\approx \frac{1}{(2\pi)^d} \int_{B(0, \epsilon/n)} e^{-\frac{\|\Sigma \xi\|^2}{2} + o(\|\xi\|^2)} d\xi \quad \text{uniformly small when } \epsilon \rightarrow 0 \\ &\approx \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-\frac{\|\Sigma \xi\|^2}{2}} d\xi = \frac{1}{\det(\Sigma)} \pi^{-d/2} \end{aligned}$$

### Remark

For the simple random walk on  $\mathbb{Z}^d$ ,

$$P(S_n = 0) \underset{n \rightarrow \infty}{\sim} \frac{1}{\det(\Sigma)^{1/2} n^{d/2}} \cdot \begin{cases} 2 \\ 1 \end{cases} \quad \text{oscillation due to the contribution of } \psi(\xi)^n \text{ for } \xi \sim (0, \dots, \pi/n).$$

### Remark

Note that, if  $(S_n)_{n \geq 0}$  is ergodic (and centered, with  $\mathbb{Z}^d$  increments), then

$$E(\#\{0 \leq k \leq n-1 : S_k = 0\}) \sim \sum_{k=0}^{n-1} \frac{1}{\det(\Sigma)^{1/2} (k+1)^{d/2}} \sim \frac{h(n)}{\det(\Sigma)^{1/2}}$$

On average, the number of time the walk goes back to 0 is unbounded. A stronger result is true:

$$\lim_{n \rightarrow \infty} \#\{0 \leq k \leq n-1 : S_k = a\} = +\infty \quad \text{a.s. } \forall a \in \mathbb{Z}^d.$$

The random walk almost surely travels infinitely many times through any site  $a \in \mathbb{Z}^d$ : it is recurrent.

### 3- Potential theory.

~~The main~~

Note that  $(S_n)_{n \geq 0}$  can be seen as a Markov chain on  $\mathbb{Z}^d$ , preserving the counting measure  $\tilde{\mu}$ , and with transition kernel

$$\tilde{P}_{ij} = P(X_0 = j - i). \quad \text{It is assumed to be ergodic and recurrent. } \tilde{\mu} \text{ invariant}$$

Recall that, for Markov chains on a general state space  $(\tilde{X}, \tilde{\mu})$  the transition kernel acts on  $\mathcal{L}^\infty(\tilde{X}, \tilde{\mu})$  by

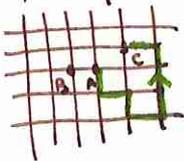
$$(\tilde{P}f)(x) = E(f(M_1) | M_0 = x).$$

### Questions

Given  $I \subset \mathbb{Z}^d$  finite and  $i \in I$ , start a random walk from  $i$ .

~~At what point of~~

what is the next point of  $I$  the random walk will hit?



$$I = \{A, B, C\}.$$

$\rightarrow$  : a path which hits C before A or B.

Related questions:

- \* given  $i, j$ , what is the probability, starting from  $i$ , to hit  $j$  before going back to  $i$  ( $I = \{i, j\}$ )?
- \* given  $i, j, k$ , what is the probability, starting from  $i$ , to hit  $j$  before  $k$  ( $I = \{i, j, k\}$ )?

Example

Let  $\varphi^{(1)} := \inf\{k \geq 1 : S_k \in I\}$  be the first hitting time, ~~By the strong Markov property~~, and  $\varphi^{(n+1)} := \inf\{k \geq \varphi^{(n)} + 1 : S_k \in I\}$ .  
 By the strong Markov property,  $(S_{\varphi^{(n)}})_{n \geq 0}$  is a Markov chain on  $I$ . Let  $P_I$  be its transition matrix.



Question

How can we compute  $P_I$ ?

Proposition Theorem (Kakutani)

Let  $(M_n)$  be a Markov chain on a general state space  $\tilde{X}$ .  
 Let  $\tilde{\mu}$  be a stationary measure (finite). Assume it is recurrent.  
 Let  $I \subset \tilde{X}$  with  $\tilde{\mu}(I) > 0$ .  
 Let  $(M_I)$  be the induced Markov chain. Then  $\tilde{\mu}|_I$  is stationary for  $M_I$ .

Example

In the example above,  $P_I$  preserves the counting measure on  $I$ : it is bi-stochastic.

In particular,  $\mathbb{C}^I = \mathbb{C} \oplus \mathbb{C}_0^I$  is  $P_I$ -invariant. We only need to compute  $P_I \sim \mathbb{C}_0^I$ .  
constants       $\uparrow$        $\uparrow$        $\text{Ker}(\tilde{\mu}|_I)$

Proposition Theorem (Balayage identity)

Same hypotheses as the previous proposition.  
 Let  $f \in L^\infty(\tilde{X}, \tilde{\mu})$  such that  $\text{Supp}(f) \subset I$ .  
 Let  $g \in L^\infty(\tilde{X}, \tilde{\mu})$  such that:

$$(I - \tilde{P})g = f.$$

Then:

$$(I - P_I)g|_I = f|_I.$$

Proof.

Let  $x \in \tilde{X} - I$ . Then  $\mathbb{E}(g | \mathcal{M}_1) | \mathcal{M}_0 = x = \tilde{P}g(x) = g(x) - f(x) = g(x)$ .  
action of  $\tilde{P}$        $\neq \text{Supp}(f)$

Let  $\bar{\varphi} := \inf\{n \geq 0 : M_n \in I\}$ . Then  $(M_{n \wedge \bar{\varphi}})$  is a bounded martingale.

By the optional stopping theorem,  $\mathbb{E}(g | \mathcal{M}_{\bar{\varphi}}) = \mathbb{E}(g | \mathcal{M}_0)$  for all  $x \in \tilde{X}$ ,  $\mathbb{E}(g | \mathcal{M}_{\bar{\varphi}}) | \mathcal{M}_0 = x = \mathbb{E}(g | \mathcal{M}_0) | \mathcal{M}_0 = x = g(x)$ .

Now, start from  $x \in I$  and do one more step.

$$\begin{aligned} (P_I g)(x) &= \mathbb{E}(g | \mathcal{M}_{I,1}) | \mathcal{M}_{I,0} = x = \mathbb{E}(g | \mathcal{M}_{\bar{\varphi}}^{(1)}) | \mathcal{M}_0 = \pi_1 = \mathbb{E}(g | \mathcal{M}_{\bar{\varphi}}) | \mathcal{M}_0 = x \\ &= \mathbb{E}(\mathbb{E}(g | \mathcal{M}_{\bar{\varphi}}^{(1)}) | \mathcal{M}_0 = \pi_1 \text{ and } \mathcal{M}_0 = x) | \mathcal{M}_0 = x \\ &= \mathbb{E}(\tilde{P}g) \end{aligned}$$

$$\begin{aligned}
 (P_Z g)(x) &= \mathbb{E}(g(\pi_{1,1}) | \pi_{1,0} = x) \\
 &= \mathbb{E}(\mathbb{E}(g(\pi_1^m) | M_1) | M_0 = x) \\
 &= \mathbb{E}(g(M_1) | M_0 = x) \\
 &= (\tilde{P}g)(x).
 \end{aligned}$$

Hence,  $(I - P_Z)g|_I = (I - \tilde{P}g)|_I = f|_I$

Remark  
 Since we assumed  $(M_n)$  ergodic,  $I - \tilde{P}$  has 1 as not an eigenvalue of  $\tilde{P}$  (if  $\lambda = 1$ ) the only ~~finite~~ solutions to  $(I - \tilde{P})g = 0$  are constants.

We can thus proceed as follows:

- \* start from  $f|_I \in \mathbb{C}^I$ .
- \* extend  $f|_I$  to  $f: \tilde{A} \rightarrow \mathbb{C}$ , taking it 0 outside  $I$ .
- \* find  $g$  such that  $(I - \tilde{P})g = f$ . How? ~~Minimizing~~ defined up to constants.
- \* restrict  $g$  to  $I$ . Check that  $\tilde{P}|_I(g|_I) = 0$ . Solutions of  $(I - \tilde{P})g = f$  are defined up to constants.

These operations let us compute  $(I - P_Z)^{-1}: \mathbb{C}_0^I \rightarrow \mathbb{C}_0^I$ . Ideally, we need to invert this matrix.

How do we compute  $g$ ?

If  $\tilde{A}$  is finite, linear algebra is enough.

For random walks, Fourier analysis works. Instead of computing  $g$  directly we compute:

$$\begin{aligned}
 S: \mathbb{C}_0^I &\rightarrow \mathbb{C}_0^I \\
 \langle h, Sp \rangle &= \int_I \bar{R} (I - \tilde{P})^{-1}(p) d\tilde{\mu}_I = \int_I \bar{R} (I - P_Z)^{-1}(p) d\tilde{\mu}_I.
 \end{aligned}$$

Well-defined:  $(I - \tilde{P})^{-1}(p)$  is defined up to constants, but  $\bar{R}$  integrates to 0.

$$\text{Here, } \langle h, Sp \rangle = \sum_{n \geq 0} \int_{\mathbb{T}^2} \bar{R} \cdot \tilde{P}^n \left( \frac{p}{z} \right) d\tilde{\mu}_I = \sum_{n \geq 0} \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \bar{R}(\xi) \tilde{P}^n(p) d\xi.$$

But  $\tilde{P}$  acts on Fourier domain by multiplication by  $\psi(\xi)$ :

$$\langle h, Sp \rangle = \sum_{n \geq 0} \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \bar{R}(\xi) \psi(\xi)^n \hat{P}(p) d\xi = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{\bar{R}(\xi) \hat{P}(p)}{1 - \psi(\xi)} d\xi.$$

Example

For the simple random walk on  $\mathbb{Z}^2$ , taking  $\psi = h = \frac{1}{2} - \frac{1}{2}B$ , we get to compute  $\left( \psi(\xi) = \frac{\cos(\xi_1) + \cos(\xi_2)}{2} \right)$

$$\langle h, Sp \rangle = \sum_{i \in \mathbb{Z}} \bar{R}(i) [(I - P_Z)^{-1} p](i) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{(1 - e^{i\xi_1})(1 - e^{-i\xi_2})}{2 - \cos(\xi_1) - \cos(\xi_2)} d\xi = \frac{1}{\pi^2} \int_{\mathbb{T}^2} \frac{1 - \cos(\xi_1)}{2 - \cos(\xi_1) - \cos(\xi_2)} d\xi.$$

In the end, with  $I = \{A, B, C\}$ :

$$P_I = \frac{1}{\pi^2 + 8\pi - 4} \begin{pmatrix} -\frac{1}{2}\pi^2 + 4\pi - 4 & -\frac{1}{2}\pi^2 + 3\pi & \pi \\ -\frac{1}{2}\pi^2 + 3\pi & -\pi^2 + 6\pi - 4 & \frac{1}{2}\pi^2 - \pi \\ \frac{1}{2}\pi^2 - \pi & -\frac{3}{2}\pi^2 + 8\pi - 4 & \end{pmatrix}.$$

Remarks:  $P_I = F(\pi)$  with  $F \in \mathcal{Q}(\pi)$ .  
 $F(0) = I$ ?

Why potential theory?

These techniques have an electrodynamical interpretation. Consider a grid of resistors, with the same resistance  $R$ .



If we fix the potentials at A, B, C, what will be the incoming or outgoing current?

~~Given the incoming currents at A, B, C, which proportion will flow to A, B, C, etc?~~  
~~If we fix the potentials at A, B, C, what will be the incoming or outgoing currents?~~

We invert the question: let  $f: I \setminus \{A, B, C\} \rightarrow \mathbb{R}$  be the incoming currents. To keep the whole circuit neutral, we impose

$\sum_I f = 0$ . Let  $V$  be the potential on the grid.

By the node law, for  $i \notin I$ , we have  $\sum_{j \sim i} V(j) = V(i)$ . For  $i \in I$ , we have  $\sum_{j \sim i} V(j) = -R f(i) + V(i)$ . In other words,

$$(I - \tilde{F})v = \frac{R}{4} f.$$

~~By the later Exam~~

Hence, given the intensities  $f(A), f(B), f(C)$ , we can compute  $V(A), V(B), V(C)$  (up to constant). Inverting this relation

let us compute  $f$  given  $V(A), V(B), V(C)$ .

### $\mathbb{Z}^d$ - and $\mathbb{Z}^2$ -extensions

One way to generalize random walks is by working with extensions of dynamical systems:

\*  $(A, \mu, T)$  measure-preserving dynamical system

\*  $F: A \rightarrow \mathbb{Z}^d$  measurable,  $\mathbb{Z}^d$ -valued

We let:

\*  $\tilde{A} := A \times \mathbb{Z}^d$

\*  $\tilde{\mu} := \mu \otimes$  counting measure

\*  $\tilde{T}: (x, p) \mapsto (T(x), p + F(x))$ .

### Examples

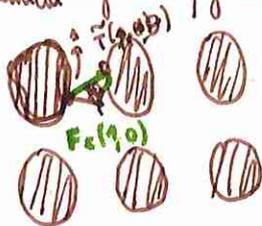
\*  $(A, T)$  is a full shift on  $\mathbb{Z}^d$ ,  $\mu$  corresponds to a i.i.d. sequence of random variables (i.e.  $\mu = \nu^{\otimes \mathbb{N}}$ ).

Then the second coordinate of an orbit is a random walk on  $\mathbb{Z}^d$  (because  $(F \circ T^n)_{n \geq 0}$  is i.i.d.).

\* By taking  $(A, T)$  a subshift on a finite alphabet and  $\mu$  suitable, we get  $\mathbb{Z}^d$ -extensions of finite-state Markov chains

(hidden Markov chains models).

\* There are dynamical systems of geometrical origin:



Lorentz gas: periodic configurations of scatterers with positive curvature boundaries.

$A = \mathbb{S}^1 \times [1/2, \pi/2]$ ,  $\mu = \frac{1}{4\pi} \cos(\theta) d\theta ds$ ,  $T$ : Sinai billiard.

### References

- Principles of random walks, F. Spitzer, Graduate texts in Mathematics, Springer, 1964 (a bit dated...)
- XKCD 356 ([xkcd.com/356/](http://xkcd.com/356/)) and related discussions.

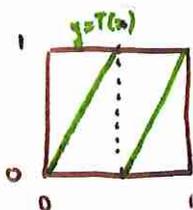
## II - Spectral theory of the transfer operator

See also Roberto's talk

### 1 - Dynamical systems

Anything we can say about  $(\tilde{X}, \tilde{\mu}, \tilde{T})$  depends on  $(A, \mu, T)$  and  $F$ . I'll work with expanding dynamical systems. The simplest example is the doubling map:

$$\begin{cases} A = \mathbb{R}/\mathbb{Z} \\ T: x \mapsto 2x \pmod{1} \\ \mu = \text{Leb} \end{cases}$$



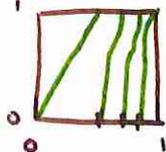
A more general class is given by expanding maps of the circle.

### Definition

A transformation  $T: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$  is a  $C^k$  expanding map if:

$$* T \in C^k(\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z});$$

$$* \lambda := \min |T'| > 1.$$



### Proposition

Given a  $C^k$  expanding map of the circle  $T$ , there exists a unique  $T$ -invariant probability measure  $\mu$  which is absolutely continuous with respect to the Lebesgue measure.

An even larger class is given by Gibbs-Markov maps.

### Definition

A Gibbs-Markov map is the data of:

\* a Polish metric space  $(A, d)$ , with  $d$  bounded;

\* a probability measure  $\mu \in \mathcal{S}(A)$ ;

\* ~~a non-singular transformation~~ a measure-preserving transformation  $T: (A, \mu) \rightarrow (A, \mu)$ ;

\* a partition  $\alpha$  of  $A$ ;

with the following properties:

\* ~~it~~ generates the Borel  $\sigma$ -algebra under  $T$  ( $\bigvee_{n=0}^{\infty} T^{-n} \sigma(\alpha) = \mathcal{B} \pmod{\mu}$ )

\* for all  $a \in \alpha$ ,  $T|_{a: a \rightarrow T(a)}$  is a measurable isomorphism.

\*  $\inf_{a \in \alpha} \mu(Ta) > 0$

\* there exists  $\lambda > 1$  such that,  $\forall a \in \alpha, \forall x, y \in a, d(Tx, Ty) \geq \lambda d(x, y)$

\* setting  $\rho_a = \frac{d\mu}{d\mu \circ T|_a}$ , there exists  $C \geq 0$  such that,  $\forall a \in \alpha, \forall x, y \in a, \left| 1 - \frac{\rho_a(x)}{\rho_a(y)} \right| \leq C d(Tx, Ty)$ . Bounded distortion

Markov  
Big image property  
Expansion

## Examples

- Any  $\mathcal{C}^2$  expanding map of the circle  $T$  is Gibbs-Markov:
  - \*  $A: \mathbb{R}/\mathbb{Z}$
  - \*  $T: T$
  - \*  $\mu$ : unique absolutely continuous invariant probability measure.
  - \*  $\alpha$ : partition of  $\mathbb{R}/\mathbb{Z}$  in intervals whose endpoints are in  $T^{-1}(0)$ .
  - \*  $d$ : usual distance.

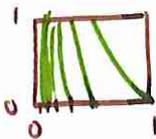
Then  $T(a) \in \mathbb{R}/\mathbb{Z}$  for all  $a \in \alpha$  (Markov), whence  $\mu(T(a)) = 1 \forall a \in \alpha$  (Big image property). The map is expanding (Expansion) and  $\mathcal{C}^2$  (Bounded distortion).

• The Gauss map:

$$A: [0, 1]$$

$$T: x \mapsto \{1/x\}$$

$$\mu: \frac{1}{1+x} \cdot \frac{1}{\ln 2} dx$$



is Gibbs-Markov (exercise; use the distance  $d(x, y) = \left| \int_x^y \frac{1}{1+t} dt \right| = \left| \ln \left( \frac{1+y}{1+x} \right) \right|$ )

• Subshifts of finite type with Gibbs measure are Gibbs-Markov. That said, Gibbs-Markov maps have more flexibility, since  $\pi$  can be countable.

• Full shift on  $\mathbb{Z}^d$  with product measure.  
 • Via Bowen/Ratner coding, subshifts of finite type encode the geodesic flow on compact hyperbolic manifold, using a good Poincaré section.

• Via Young towers, Gibbs-Markov maps encode Siering billiards or low-unimodal maps with the Collet-Eckmann property (although there are additional difficulties, so some of what I will explain does not apply).

## 2- The Koopman operator

~~Let  $A$  and  $T$  be any expanding map of  $\mathbb{R}/\mathbb{Z}$  and  $T$  be the doubling map.~~

A transformation can be seen as a Markov kernel (everything is random if you like Dirac measures):

$$P_x = \delta_{T(x)}$$

If  $(A, \mu, T)$  is measure-preserving, then, for all  $f \in L^\infty(A, \mu)$ ,

$$(Pf)(x) = \mathbb{E}(f(H_1) | \mathcal{H}_0 = x) = \int_A f \cdot d\delta_{T(x)} = f(T(x))$$

i.e.  $Pf = f \circ T$ . The operator of pre-composition by  $T$  is called the Koopman operator, noted  $\mathcal{K}$ .

In the application of potential theory, we had to use the fact that  $(I - P_x)$  is invertible, and invert  $(I - P_x)^{-1}$ . This relies on the fact that  $P_x$  is a matrix. In our dynamical setting, function spaces are infinite-dimensional, so the Koopman operator is not a matrix. But we can hope for the next best thing; is it, for instance, compact?

On  $L^1$ ,  $p \in [1, \infty]$ ,  $\mathcal{K}$  is an isometry; we have no hope for compactness. ~~Let us restrict~~

Let  $T$  be the doubling map, and restrict ourselves to Lipschitz function  $\text{Lip}: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{C}$ .



In general:

$$\| \rho \circ T^{-1} \|_{\infty} = \| \rho \|_{\infty}$$

$$\| \rho \circ T^{-1} \|_{\text{lip}} = 2^n \| \rho \|_{\text{lip}}, \quad \text{with } \| \rho \|_{\text{lip}} = \sup_{y \neq x} \frac{|\rho(x) - \rho(y)|}{d(x, y)}$$

Hence, the Koopman operator has norm 2, but no ~~eigenfunction~~ eigenvalue of modulus larger than 1. This is even worse...

### 3- The transfer operator

The main idea is to use instead the adjoint of  $K$ : the transfer operator  $\mathcal{L}$ .

#### Definition

Let  $(X, \mu, T)$  be a ~~probability~~ <sup>measure</sup>-preserving dynamical system, and  $f \in L^1, \infty$ .

The action of  $\mathcal{L}$  on  $L^1(X, \mu)$  is given by:

$$\int_A \mathcal{L}(f) d\mu = \int_{T^{-1}A} f d\mu \quad \forall f \in L^1(X, \mu), \forall A \in \mathcal{B}(X)$$

Remark  
If  $f \in L^1(X, \mu)$ , then  $\mathcal{L}(f) d\mu = T_* (f d\mu)$ : ~~can be seen~~ acts on the push-forward measure on measures with densities.

#### Examples

• If  $T$  is the doubling map and  $\mu$  is the Lebesgue measure,

$$\begin{aligned} \int_0^1 f(x) g(x) dx &= \int_0^{1/2} f(x) g(2x) dx + \int_{1/2}^1 f(x) g(2x-1) dx \\ &= \frac{1}{2} \int_0^1 f(y/2) g(y) dy + \frac{1}{2} \int_0^1 f(y/2+1/2) g(y) dy \end{aligned}$$

$$\text{So } \mathcal{L}f(x) = \frac{f(x/2) + f(x/2+1/2)}{2}$$

• More generally, if  $T$  has countably many branches and  $\mu$  is absolutely continuous,

$$\mathcal{L}f(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|\det DT(y)|}$$

• For Gibbs-Markov maps,

$$\mathcal{L}f(x) = \sum_{y \in T^{-1}(x)} g(y) f(y)$$

#### Markov kernels

By its definition,  $\mathcal{L}$  is the dual Markov kernel of  $K$ .

Its transition kernel is  $P_x = \sum_{y \in T^{-1}(x)} g(y) \delta_y$ .

$\mathcal{L}$  is ergodic if and only if  $T$  is (somewhat) trivial, and recurrent if and only if  $T$  is (non-trivial!).

$\mathcal{L}(\mathbb{1}) = \mathbb{1}$  since  $\mu$  is  $T$ -invariant ( $\int_A g \circ T d\mu = \int_A g d\mu$ );  $\int_A \mathcal{L}(f) d\mu = \int f d\mu$  since  $\mathbb{1} \circ T = \mathbb{1}$ ; and  $\mathcal{L}(\rho \circ T) = \rho$  since  $\mu$  is  $T$ -invariant

## Spectrum of the transfer operator

We restrict ourselves to Lipschitz functions. For the doubling map, given  $f \in \text{Lip}([0,1], \mathbb{C})$ ,

$$|Zf(x) - Zf(y)| = \left| \frac{1}{2} [f(x/2) - f(y/2)] + \frac{1}{2} [f(x/2+1/2) - f(y/2+1/2)] \right|$$

(Make the argument simple)

$$\leq \frac{1}{2} \cdot \|f\|_{\text{Lip}} d(x/2, y/2) + \frac{1}{2} \cdot \|f\|_{\text{Lip}} d(x/2+1/2, y/2+1/2)$$

$$\leq \frac{1}{2} \cdot \frac{\|f\|_{\text{Lip}}}{2} d(x, y) + \frac{1}{2} \cdot \frac{\|f\|_{\text{Lip}}}{2} d(x, y) = \frac{\|f\|_{\text{Lip}}}{2} d(x, y)$$

Hence,  $\|Zf\|_{\text{Lip}} \leq \frac{\|f\|_{\text{Lip}}}{2}$ , and  $\|Z^n f\|_{\text{Lip}} \leq \frac{\|f\|_{\text{Lip}}}{2^n} \forall n \geq 0$ .

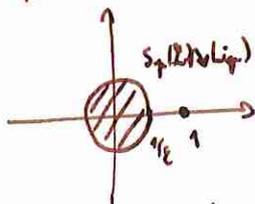
In particular,  $\max(Z^n f) - \min(Z^n f) \leq 2^{-n} \|f\|_{\text{Lip}}$  and  $\int_0^1 Z^n f d\mu = \int_0^1 f \cdot \mathbb{1}_{[0,1]} \circ T^n d\mu = \int_0^1 f d\mu$ , or

$$\ast \|Z^n f - \int_0^1 f d\mu\| \leq 2^{-n} \|f\|_{\text{Lip}}$$

$$\ast \|Z^n f - \int_0^1 f d\mu\|_{\infty} \leq 2^{-n} \|f\|_{\text{Lip}}$$



so that  $Z^{-n}$  has spectral radius at most  $1/2$ , with  $\pi(f) = \int_0^1 f d\mu$ .



$Z$  is not compact, but is the next best thing: quasi-compact.

## Exercise

Using the Fourier transform, given  $f \in B(0, 1/2)$ , prove that  $\lambda$  is an eigenvalue of  $Z \in \text{Lip}$ .

## Quasi-compact operators

Let  $Z \in \mathcal{L}(B)$ , with  $B$  a Banach space. The essential spectral radius of  $Z$ , denoted  $\rho_{\text{ess}}(Z)$ , is equivalently:

- $\ast \inf \{ \alpha > 0 : \text{Sp}(Z) \cap B(0, \alpha)^c \text{ is made of finitely many eigenvalues of finite multiplicity (point spectrum)} \}$
- $\ast \inf \{ \rho(Z \circ K) : K \text{ compact} \}$

An operator is quasi-compact if  $\rho(Z) > \rho_{\text{ess}}(Z)$ . A ~~quasi-compact operator has a spectral gap.~~



## Theorem

Let  $A \in \mathbb{R}/\mathbb{Z}$ ,  $T_A$   $\mathbb{C}^{\infty}$  expanding map of the circle,  $\mu$  the unique a.c.i.p.m. Then:

$$\rho(Z \in \text{Lip}) = 1$$

$$\rho_{\text{ess}}(Z \in \text{Lip}) \leq (\min |T'|)^{-1} < 1.$$

$\rho_{\text{ess}}$

(10) In particular,  $Z \in \text{Lip}$  is quasi-compact.

This stays true for Gibbs-Markov maps if one takes:

$$\| \cdot \|_{\text{Lip}(\alpha_n)} = \sup_{x \in \alpha_n} \sup_{y \notin \alpha_n} \frac{|f(x) - f(y)|}{d(x, y)},$$

where  $\alpha_n$  is the image partition, i.e. the partition generated by  $\{T^n(a) : a \in \alpha\}$ . For instance, for the Gauss map, each branch is surjective, so  $\alpha_n$  is trivial and  $\| \cdot \|_{\text{Lip}(\alpha_n)}$  is the usual Lipschitz seminorm.

#### 4- The central limit theorem

Let  $T$  be an expanding map of the circle,  $\mu$  its unique a.c.i.p.m, and  $F \in \text{Lip}(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ . What can we say about the process  $(S_n F)_{n \geq 0}$ , where  $S_n F = \sum_{k=0}^{n-1} F \circ T^k$ ?

•  $\mu$  is ergodic and  $F$  bounded. By Birkhoff's ergodic theorem,

$$\lim_{n \rightarrow \infty} \frac{S_n F}{n} = \int_A F d\mu \quad \mu - \text{a.o.}$$

Let us assume that  $F$  is centered ( $\int_A F d\mu = 0$ ).

• We try to mimic the proof of the central limit theorem.

$$\mathbb{E} \left( e^{i \xi \frac{S_n}{\sqrt{n}}} \right) = \int_A e^{i \xi \sum_{k=0}^{n-1} F \circ T^k} d\mu = \int_A \prod_{k=0}^{n-1} e^{i \xi F \circ T^k} d\mu.$$

To make the transfer operator appear, we use:

#### Lemma (exercise)

Let  $(p_k)_{0 \leq k < n-1}$  be a sequence of bounded functions. Then:

$$\int_A \prod_{k=0}^{n-1} p_k \circ T^k d\mu = \int_A p_{n-1} \mathcal{L}(p_{n-2} \mathcal{L}(\dots \mathcal{L}(p_0) \dots)) d\mu.$$

Application:  $\int_A \mathbb{E} \left( e^{i \xi \frac{S_n}{\sqrt{n}}} \right) = \int_A \mathcal{L} \left( e^{i \xi F} \mathcal{L} \left( e^{i \xi F} \mathcal{L} \left( \dots \mathcal{L} \left( e^{i \xi F} \right) \dots \right) \right) \right) d\mu.$

Setting  $\mathcal{L}_\xi(p) := \mathcal{L}(e^{i \xi F} p)$ , we get finally:

$$\mathbb{E} \left( e^{i \xi \frac{S_n}{\sqrt{n}}} \right) = \int_A \mathcal{L}_{\xi/\sqrt{n}}^n(p) d\mu.$$

Exact formula.  
Use the transfer operator.

•  $\xi \mapsto \mathcal{L}_\xi$  is a smooth family of operators when acting on Lip. Hence so is  $\xi \mapsto \mathcal{L}_\xi^n$ .

In particular, eigenvalues and eigenfunctions of  $\mathcal{L}_\xi$  depend smoothly on  $\xi$ .



• For simplification, let us assume that  $\mathcal{L}$  is aperiodic: 1 is the only eigenvalue of  $\mathcal{L}$  of modulus 1. Then, for all small enough  $\xi$ , we have an eigen decomposition:

$$\mathcal{L}_\xi = Q_\xi + \lambda_\xi \cdot R_\xi \mu_\xi, \quad \mu_\xi(R_\xi) = 1, \mu(h_\xi) = 1, \mu_0 = \mu, R_0 = \mathbb{1}.$$

$\rho(Q_\xi) < 1$  actually,  $\|Q_\xi\| \ll C n^{-\alpha} \forall \xi$  small enough.

Then :

$$\mathbb{E}(e^{i\xi \frac{S_n F}{\sqrt{n}}}) = \int_A Q_\xi(\mathbb{1}) d\mu + \lambda_\xi \int_A \mu_\xi(\mathbb{1}) \cdot R_\xi d\mu = \int_A Q_\xi(\mathbb{1}) d\mu + \lambda_\xi \int_A \mu_\xi(\mathbb{1}) d\mu.$$

$\ll C n^{-\alpha}, \alpha < 1$        $\xrightarrow{n \rightarrow \infty} 1$

$$\mathbb{E}(e^{i\xi \frac{S_n F}{\sqrt{n}}}) \approx \lambda_\xi \int_A \mu_\xi(\mathbb{1}) d\mu + o(1) + o(1)$$

Approximate formula, good in the limit  $n \rightarrow +\infty$ .  
We have replaced an operator with a scalar!

All is left is to estimate  $\lambda'_0$  and  $\lambda''_0$  (we already know that  $\lambda_\xi$  is smooth).

$$\mathcal{L}_\xi R_\xi = \lambda_\xi R_\xi$$

$$\mathcal{L}'_0 R_0 + \mathcal{L}_0 R'_0 = \lambda'_0 R_0 + \lambda_0 R'_0$$

$$\mathcal{L}(iF) + \mathcal{L}(R'_0) = \lambda'_0 + R'_0$$

$$\text{with } \mathcal{L}'_0 = \frac{d(\mathcal{L}(e^{i\xi F}))}{d\xi} \Big|_{\xi=0} = \mathcal{L}(iF)$$

Integrating:

$$\int_A \mathcal{L}(iF) d\mu + \int_A \mathcal{L}(R'_0) d\mu = \lambda'_0 + \int_A R'_0 d\mu. \quad \text{so } \lambda'_0 = 0.$$

$$R'_0 = i(I - \mathcal{L})^{-1} \mathcal{L}(F)$$

$I - \mathcal{L}: \text{Lip}_0 \rightarrow \text{Lip}_0$  is invertible, and  $R'_0 = \text{Lip}_0$  since  $\mu(h_\xi) = 1 \forall \xi$ .

Differentiate twice, and integrate.

$$\mathcal{L}''_0(\mathbb{1}) + \mathcal{L}'_0 R'_0 + R''_0 = \lambda''_0 + \mathcal{L}'_0 R'_0 + R''_0$$

$$\int_A \mathcal{L}(-F^2) + \mathcal{L}(\mathcal{L}(iF) \cdot i(I - \mathcal{L})^{-1} \mathcal{L}(F)) d\mu = \lambda''_0$$

$$\lambda''_0 = -\sigma^2(F), \quad \text{with } \sigma^2(F) = \int_A F^2 d\mu + \mathcal{L}(\mathcal{L}(iF) \cdot i(I - \mathcal{L})^{-1} \mathcal{L}(F)) d\mu = \int_A F^2 d\mu + \mathcal{L} \sum_{k=1}^{+\infty} \int_A (F \cdot \mathcal{L}^k(F)) d\mu$$

Hence  $\lambda_{\xi/\sqrt{n}} = 1 - \frac{\sigma^2(F) \xi^2}{2n} + o(n^{-3/2})$ , where  $\mathbb{E}(e^{i\xi \frac{S_n F}{\sqrt{n}}}) \xrightarrow{n \rightarrow \infty} e^{-\frac{\sigma^2(F) \xi^2}{2}}$ .

Theorem (Central Limit Theorem)

If  $\sigma^2(F) > 0$  (i.e. if  $F$  is not a coboundary), then  $\frac{S_n F}{\sqrt{n}}$  converges to  $\mathcal{N}(0, \sigma^2(F))$  in distribution.

### Gibbs-Markov maps

All the computations above work as well with Gibbs-Markov maps. One improvement is possible: we can choose

$F$  such that

$$\|F\|_{\text{Lip}(a)} := \sup_{x \neq y} \frac{|F(x) - F(y)|}{d(x, y)} < +\infty, \quad F \in L^2(A, \mu), \quad \int_A F d\mu = 0.$$

Then  $\xi \rightarrow e^{i\xi F}$  is not continuous on  $\mathcal{L}(\text{Lip}(a))$ , but  $\xi \rightarrow \mathcal{L}_\xi$  is, thanks to the regularizing effect

of  $\mathcal{L}$ . Since  $a$  can be contable,  $F$  may be unbounded. For instance, taking for

## 5- The local central limit theorem

Let  $(A, \mu, T)$  be a Gibbs-Markov map,  $F: A \rightarrow \mathbb{Z}^d$  constant on elements of  $\alpha$ ,  $F \in \mathcal{L}^2$ ,  $\int F d\mu = 0$ .

What can be said about  $\mathbb{P}(S_n F = 0)$ ?

Let  $\Sigma^2(F) := \langle \nu, \Sigma^2(F) \nu \rangle := \sigma^2 \langle \nu, F \rangle$ .

Definition

Let  $H := \{(\xi, \lambda) : \xi \in \mathbb{R}/2\pi\mathbb{Z}, |\lambda| = 1, \lambda \in S_p(\mathcal{L}_\xi^m)\}$ . It is a subgroup of  $\mathbb{R}/2\pi\mathbb{Z} \times S^1$ .

$(\tilde{A}, \tilde{\mu}, \tilde{T})$  is ergodic if  $1 \in S_p(\mathcal{L})$  is simple and  $\lambda \neq 1$  whenever  $(\xi, \lambda) \in H$  and  $\xi \neq 0$ .

$(\tilde{A}, \tilde{\mu}, \tilde{T})$  is aperiodic if  $1 \in S_p(\mathcal{L})$  is simple and  $H = \{(0, 1)\}$ .

Theorem (Local central limit theorem)

Assume that  $(\tilde{A}, \tilde{\mu}, \tilde{T})$  is aperiodic.

Then  $\Sigma^2(F) > 0$  and  $\mathbb{P}(S_n F = 0) \underset{n \rightarrow \infty}{\sim} \frac{1}{\Sigma \det(\Sigma(F)) n}$ .

Proof

$$\mathbb{P}(S_n F = 0) = \mathbb{E}(\mathbb{1}_0(S_n F)) = \frac{1}{(2\pi)^d} \int_A \int_{\mathbb{R}^d} e^{i \langle \xi, S_n F \rangle} d\xi d\mu = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_A \mathcal{L}_\xi^m(\mathbb{1}) d\mu d\xi.$$

Let  $B(0, \varepsilon)$  be a small neighborhood of 0 in  $\mathbb{R}^d$ .

On  $B(0, \varepsilon)^c$ ,  $\|\mathcal{L}_\xi^m(\mathbb{1})\|_{L^1} \leq C n^m$ . (aperiodicity!)

On  $B(0, \varepsilon)$ ,  $\mathcal{L}_\xi^m(\mathbb{1}) = \lambda_\xi^m \mu_\xi(\mathbb{1}) + O(n^m)$   
↑ uniform in  $\varepsilon$ .

$$\text{Hence } \mathbb{P}(S_n F = 0) = \frac{1}{(2\pi)^d} \int_{B(0, \varepsilon)} \lambda_\xi^m \mu_\xi(\mathbb{1}) d\xi + O(n^m).$$

The end of the proof is the same as for random walks.

Formule exacte, faisant apparaître l'opérateur de transfert.

Approximate formula, useful in an ergodic regime.

## 6- References

For the spectral properties of the transfer operator:

Positive transfer operators and decay of correlations, V. Baladi, World Scientific, 2000

Central limit theorem: Nagaev-Guivarch's method.

## III- Probabilistic potential theory for dynamical systems

### 1- Setting

Let  $(A, \mu, T)$  be a Gibbs-Markov map,  $F: A \rightarrow \mathbb{Z}^d$  constant on elements of  $\alpha$ .

Assume that  $(\tilde{A}, \tilde{\mu}, \tilde{T})$  is ergodic and recurrent.

Let  $I \subset \mathbb{Z}^d$  finite. The first hitting time is

$$\varphi(x, i) := \inf\{n \geq 1: \tilde{T}^n(x, i) \in [I]\},$$

$$[I] \approx A \times I.$$

The induced transformation is:

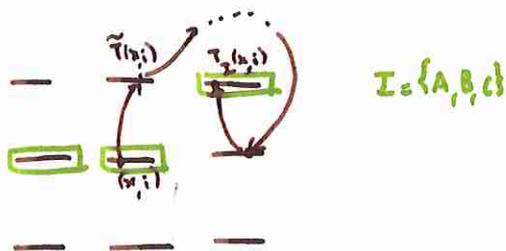
$$T_I(x, i) := \tilde{T}^{\varphi(x, i)}(x, i).$$

Then  $([I], \mu_I, T_I)$  is measure-preserving with  $\mu_I = \frac{1}{|I|} \tilde{\mu}|_I$ .

### Question

What can be said about the transition probabilities

$$P_{I, ij} := \mu(\{x; T_I(x, i) \in [j]\})?$$



For  $I$  fixed, things are difficult. Spectral methods give ~~asymptotic~~ more easily asymptotic results. How do we translate this question in an asymptotic setting?

### Families of shapes

We consider a family  $(I_\varepsilon)_{\varepsilon > 0}$  of subsets of  $\mathbb{Z}^d$ , indexed by a common finite set  $I$ .

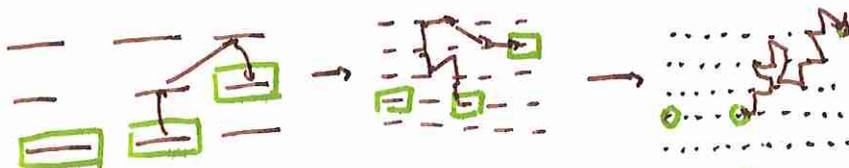
We assume that

$$\lim_{\varepsilon \rightarrow 0} \min_{i \neq j \in I_\varepsilon} |i - j| = +\infty.$$

For instance:

\*  $I \subset \mathbb{R}^d$  finite

\*  $I_\varepsilon = \{\text{nearest neighbours of } \varepsilon^{-1}i : i \in I\}$ .



Let  $T_\varepsilon = T_{I_\varepsilon}$ ,  $P_\varepsilon = P_{I_\varepsilon}$  etc. What can we say about  $P_\varepsilon$  when  $\varepsilon$  vanishes?

### Remark

Each  $P_\varepsilon$  is stochastic and preserves the counting measure.

## 2- Koopman operator and transfer operator

Can we use probabilistic potential theory, but with which kernel?

### The Koopman operator

Let  $f, g \in L^\infty(\tilde{A}, \tilde{\mu})$ ,  $\text{Supp}(f) \subset [I_\epsilon]$ .

If  $(I - K_\epsilon)g = f$ , then  $(I - K_\epsilon)g = f$ .

$$g - g \circ \tilde{T} = f \quad g - g \circ T_\epsilon = f$$

This is actually easy to prove. However, most Lipschitz functions are not coboundaries (Linsic: each periodic orbit is an obstruction, and there are infinitely many such orbits), so - in the context of Lipschitz functions - we cannot find a suitable  $g$ .

Note however that

$$P_\epsilon = \pi_{\tilde{x}}^* K_\epsilon \pi_x^*, \quad \text{with } \pi_x: \begin{cases} \text{Lip}([I], \epsilon) \rightarrow \mathbb{C}^{\mathbb{I}} \\ f \mapsto (\int f(x, i) d\rho)_{i \in \mathbb{I}} \end{cases}$$

$$\pi_x^*: \begin{cases} \mathbb{C}^{\mathbb{I}} \rightarrow \text{Lip}([I], \epsilon) \\ (f_i)_{i \in \mathbb{I}} \mapsto (f(x, i) = f_i) \end{cases}$$

So  $P_\epsilon$  can be seen has an averaged version of  $K_\epsilon$ .

### The transfer operator

Let  $f, g \in L^\infty(\tilde{A}, \tilde{\mu})$ ,  $\text{Supp}(f) \subset [I_\epsilon]$ .

If  $(I - \tilde{L}_\epsilon)g = f$ , then  $(I - L_\epsilon)g = f$ .

In addition,  $P_\epsilon = \pi_x^* L_\epsilon \pi_{\tilde{x}}^*$  is, up to transposition, an averaged version of  $L_\epsilon$ .

### Roadmap

$f$  Lipschitz, supported on  $[I_\epsilon]$

$g := (I - \tilde{L})^{-1} f$  Can be computed by Fourier transform

$$g = (I - L_\epsilon)^{-1} f$$

Invert  $(I - L_\epsilon)^{-1}$  How?

### 3- Statement of the main theorem

We have:

\* a discretization of  $\mathbb{Z}^d$ :  $P_\epsilon^t$ .

\* a discretization of  $(I - \mathcal{L}_\epsilon)^{-1}$ :  $S_\epsilon$ , defined by

$$S_\epsilon = \prod_x (I - \mathcal{L}_\epsilon)^{-1} \pi^*$$

$$\langle R, S_\epsilon p \rangle = |\Sigma| \int_{[\Sigma]} \pi^*(R) \cdot (I - \mathcal{L}_\epsilon)^{-1} \pi^*(p) d\mu_{\Sigma}^{-1}$$

acting on  $C_0^{\mathbb{Z}^d}$ .

#### Theorem

Assume that  $(\tilde{A}, \tilde{\mu}, \tilde{T})$  is ergodic and recurrent. There is equivalence between

\*  $P_\epsilon^t = I - \epsilon R + o(\epsilon)$ , with  $R$  irreducible

\*  $S_\epsilon = \epsilon^{-1} S + o(\epsilon^{-1})$ , with  $S$  irreducible

not the same notion of irreducibility!

If this is the case,  $S = (R^t)^{-1}$ ,  $R^t$  acting on  $C_0^{\mathbb{Z}^d}$ .

#### Remark

$R$  is a bi-R-matrix (row sum = sum columns = 1,  $> 0$  on the diagonal,  $< 0$  elsewhere). Irreducible if  $\exists$  negative paths.

#### Example

Posner-Frobenius:  $\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$ ,  $S_\epsilon(R) \geq 0$  ( $> 0$  on  $C_0^{\mathbb{Z}^d}$ )

$\mathbb{Z}^d$ -extension,  $F \in \mathbb{L}^2(A, \mu)$ ,  $I_\epsilon = \epsilon^{-1} I + o(\epsilon^{-1})$  with  $I \in \mathbb{R}^{\mathbb{Z}^d}$ .

$$\langle R, S_\epsilon p \rangle = |\Sigma| \int_{\Sigma} \pi^*(R) (I - \mathcal{L}_\epsilon)^{-1} \pi^*(p) d\xi.$$

We find  $S_\epsilon \sim \frac{|h(\xi)|}{\pi \det(\Sigma)} Id$

Main contribution comes from  $\lambda(\xi)$ , in  $\frac{1}{|\Sigma| |\lambda(\xi)|}$ .

$$P_{\epsilon, ij} \sim \frac{\pi \det(\Sigma)}{|\Sigma| |h(\xi)|} \text{ if } i \neq j, \quad P_{\epsilon, ii} = 1 - \frac{|\Sigma| - 1}{|\Sigma|} \cdot \frac{\pi \det(\Sigma)}{|h(\xi)|} + o(|h(\xi)|^{-1}).$$

### 4- Sketch of the proof

Fast-slow systems: see Roberts's talk.

~~There are a few possible approaches. For instance: see  $\mathcal{L}_\epsilon$  as a perturbation of  $\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$  acting on  $\mathbb{R}^2$  (see works of Keller, Liverani and Dolgopyat, Wright on metastable states). For my talk, I will take a probabilistic point of view.~~

I shall focus on the direct implication; the converse direction rely on it.

If the ~~systems~~ sites are far apart:

$$\boxed{=}_{\epsilon_0}$$

$$\boxed{=}_{\epsilon_1}$$



Prob:  $1 - O(\epsilon)$ .

Then the probability of traveling between two sites is low. In addition, if we go back to the site we started from,

$$\inf \{n \geq 1 : \tilde{T}^n(x, i) \in [\Sigma_\epsilon]\} = \inf \{n \geq 1 : \tilde{T}^n(x, i) \in [i]\} = \inf \{n \geq 0 : \tilde{T}^n(x, 0) \in [0]\}, \text{ so we have just applied } \boxed{=}_{\epsilon_0}.$$

Hence: \* most of the time, we apply  $\boxed{=}_{\epsilon_0}$ .

\* rarely (with probability  $O(\epsilon)$ ), we transit from one site to another.

~~Assumption~~

Fact

$\mathcal{L}_{\varepsilon 0}$  is quasi-compact with essential spectral radius at most  $\lambda^{-1}$  and ergodic.

Assumption (not at all trivial to remove; interestingly, the argument I used differs for  $\mathbb{Z}$  and  $\mathbb{Z}^d$ -extensions).

$\mathcal{L}_{\varepsilon 0}$  is mixing.

There are a few possible approaches. Following Keller, Liverani and Dolgopyat, Wright (works on metastable states), one could see  $\mathcal{L}_\varepsilon$  as a perturbation of  $\begin{pmatrix} \mathcal{L}_0 & 0 \\ 0 & \mathcal{R}_0 \end{pmatrix}$ , and work from there. I had actually at first a more probabilistic approach, a bit

less efficient (although, in the end, the result is the same), but which I find quite fun.

~~Assumption~~

Idea

We have a system with two time scales:

\* on a time scale  $\mathcal{O}(1)$ :  $\mathcal{L}_{\varepsilon 0}$  mixes exponentially fast on each site.

\* on a time scale  $\mathcal{O}(\varepsilon^{-1})$ : transitions between sites become noticeable.

Invariant cones ( $\mathcal{O}(1)$  time scale)

On a short time scale:

\*  $\mathcal{L}_{\varepsilon 0}$  contracts on each site:  $\mathcal{L}_{\varepsilon 0}^n(f(\cdot, i))$  is exponentially close to  $\int_{\mathbb{I}} f d\nu$  (multiplicative effect).

\* transitions between sites add an additive effect of order  $\varepsilon$ .

When the two are taken into account, if we start from  $\pi^* f$  (constant on each site), then we will stay  $\varepsilon$ -close

to functions which are constant on each site. Let:

$$C_K(\varepsilon) := \left\{ f: \mathbb{I} \rightarrow \mathbb{C}, f(\cdot, i) \in \text{Lip}(\lambda_\varepsilon) \forall i, f \geq 0, \|(I - \pi^* \pi_*) f\|_{\text{Lip}(\lambda_\varepsilon)} \leq K \varepsilon \|f\|_{\ell^1} \right\}.$$

Lemma (cone contraction)

Let  $\lambda_1 > 0$ . For all large enough  $K, n$  and small enough  $\varepsilon$ ,

$$\mathcal{L}^n(C_K(\varepsilon)) \subset C_{\lambda_1 K}(\varepsilon).$$

Hence, starting from  $C_K(\varepsilon)$ , we stay  $\varepsilon$ -close to constants at all times.



Decay of correlations ( $\mathcal{O}(\varepsilon^{-1})$  time scale)

• If  $f \in C_K(\varepsilon)$ , then

$$\pi_* \mathcal{L}_\varepsilon f = (I - \varepsilon R + \sigma(\varepsilon)) \pi_* f.$$

→ error  $\mathcal{P}_\varepsilon - (I - \varepsilon R)$ :  $\sigma(\varepsilon)$

→ error from the fact that  $f$  is not in  $\pi^* \mathbb{C}^{\mathbb{I}}$ :  $\sigma(\varepsilon^2)$ .

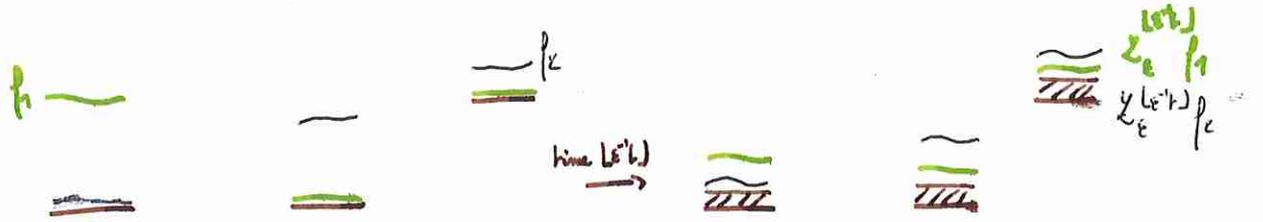
Recursively,

$$\pi_* \mathcal{L}_\varepsilon^n f = (I - \varepsilon R + \sigma_n(\varepsilon)) \dots (I - \varepsilon R + \sigma_1(\varepsilon)) \pi_* f$$

We can take  $n \approx \varepsilon^{-1} t$  (time scale of  $\mathcal{O}(\varepsilon^{-1})$ ). We get (controlling the error):

$$\pi_* \mathcal{L}_\varepsilon^{\lfloor \varepsilon^{-1} t \rfloor} f \xrightarrow{\varepsilon \rightarrow 0} e^{-tK} \pi_* f \quad \forall t > 0$$

• We can get a Nakhor coupling argument going:



Couple the shaded part, start again with the difference.

Setting  $\|P\|_{K,\varepsilon} = \inf \{ \max \{ \|P_1\|_{L^1}, \|P_2\|_{L^1} \}, P_1, P_2 \in C_{K,\varepsilon}(\mathbb{R}), P = P_1 - P_2 \}$ , we get, for all  $P \in (L^1(\mathbb{R}))_0$ :

$$\| \Sigma^{L\varepsilon^{-1}t} P \|_{K,\varepsilon} \leq C e^{-\rho t} \|P\|_{K,\varepsilon},$$

where  $\rho$  can be chosen as close as one wishes of the  $\min \{ \operatorname{Re}(\lambda) : \lambda \in \Sigma_P(\mathbb{R}), \lambda \neq 0 \}$ .



Payoff

Let  $P, f \in C_0^\infty$ . We have

$$\langle h, S_\varepsilon P \rangle = \langle P, (I - \Sigma)^{-1} \pi^* P \rangle$$

$$= \langle \pi^* h, (I - \Sigma_\varepsilon)^{-1} \pi^* P \rangle$$

$$\text{But } (I - \Sigma_\varepsilon)^{-1} \pi^* P = \sum_{n=0}^{+\infty} \Sigma_\varepsilon^n \pi^* P = e^{-\frac{t}{\varepsilon} \Sigma_\varepsilon} \pi^* P$$

n. ||\_{K,\varepsilon} bounded

$$= \varepsilon^{-1} \int_0^{+\infty} \Sigma_\varepsilon^{L\varepsilon^{-1}t} \pi^* P dt$$

$$\rightarrow \int_0^{+\infty} e^{-\frac{t}{\varepsilon} \Sigma_\varepsilon} \pi^* P dt = \int_0^{+\infty} e^{-tR} P dt = R_0^{-1} P.$$

$$\text{Hence } \langle h, S_\varepsilon P \rangle \approx e^{-1} \langle h, R_0^{-1} P \rangle \square$$

Converse

Trivial if  $|I|=2$  (up to renormalization of  $\varepsilon$ , we can always assume that  $P_\varepsilon = I - \varepsilon \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ ).

Otherwise, assuming that  $S_\varepsilon = \varepsilon^{-1} S + o(\varepsilon^{-1})$  with  $S$  irreducible, we use the  $|I|=2$  case as a bootstrap to ensure that we can get  $P_\varepsilon = I - \varepsilon R + o(\varepsilon)$  with  $R$  irreducible, then conclude with the direct direction.