Chapter 3

Invariants : Chern classes and cohomology

In this chapter, we will focus on some invariants of toric sheaves, namely characteristic classes and cohomology groups. In both cases, we will define those invariants through their properties. Our goal is then to give simple descriptions in terms of Σ-families. This is achieved in Theorem 3.2.5 for Chern classes, and in Corollary 3.3.6 and Theorem 3.3.8 for cohomology. We then give applications of those results, by computing these invariants for some classical examples, such as ideal and tangent sheaves.

3.1 Resolutions

One of the main tools to compute the aforementioned invariants are *resolutions*. In this first section, we will describe Klyachko's resolution for locally free toric sheaves over smooth toric varieties, and discuss Perling's resolution for toric reflexive sheaves over normal varieties.

Klyachko's resolution

Let $\mathcal{E}$ be a coherent sheaf on a variety X. A *locally-free resolution* for $\mathcal{E}$ is an exact sequence

$$\ldots \to \mathcal{F}_i \to \ldots \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{E} \to 0 \tag{3.1}$$

of coherent sheaves such that $\mathcal{F}_i$ is locally free for all $i \geq 0$. If there is $r \in \mathbb{N}^*$ such that $\mathcal{F}_i = 0$ for all $i \geq r$, we say that the resolution is *finite*, and if all the $\mathcal{F}_i$ split, we call this sequence a *free resolution*. Finally, if $\mathcal{E}$ is a toric sheaf and (3.1) is in the category of toric sheaves, we will use the terminology *toric resolution*. Given a coherent sheaf on a smooth projective variety, one can always find a finite locally-free resolution, cf [25, Corollary II.5.18 and Exercise III.6.9]. In the smooth and complete toric case, Klyachko produced an *explicit* finite, free, toric resolution for any locally free toric sheaf, out if its family of filtrations [39]. The construction goes as follows. Let $\mathcal{E}$ be a locally free toric sheaf with family of filtrations $(E^\rho(\bullet))$ over a smooth and complete toric variety X. Over each affine subset $U_\sigma \subset X$, with $\sigma \in \Sigma$, the sheaf $\mathcal{E}$ is free by Proposition 1.3.25. Hence, by Proposition 2.2.14, we can find a decomposition

$$E = \bigoplus_{i=1}^{r} E_{m_i} \tag{3.2}$$

with weights $(m_i)_{1\leq i\leq r} \in M^r$ such that for each $\rho \in \sigma(1)$,

$$E^\rho(\langle m, u_\rho\rangle) = \bigoplus_{m_i \leq_\rho m} E_{m_i}.$$

We which to extend this splitting to the whole of X. To do so, following Perling's presentation [58, Section 6.1], we introduce the function

$$\begin{array}{rccc} \nu : & \Sigma(1) & \to & \mathbb{Z} \\ & \rho & \mapsto & \min\{i \in \mathbb{Z} \mid E^\rho(i) = E\}. \end{array} \tag{3.3}$$

We then define a family of filtrations $(F^\rho_\sigma(\bullet))$ by setting for each $\rho \in \Sigma(1)$:

$$\forall j \in \mathbb{Z},\ F^\rho_\sigma(j) = \begin{cases} E^\rho(j) & \text{if} \quad \rho \in \sigma(1) \\ \{0\} & \text{if} \quad \rho \notin \sigma(1) \text{ and } j < \nu(\rho) \\ E & \text{if} \quad \rho \notin \sigma(1) \text{ and } j \geq \nu(\rho), \end{cases} \tag{3.4}$$

with associated reflexive T-sheaf $\mathcal{F}_\sigma$, cf Theorem 1.3.20. The sheaves $(\mathcal{F}_\sigma)_{\sigma\in\Sigma}$ satisfy the following properties.

Lemma 3.1.1. *Assuming that X is smooth and $\mathcal{E}$ locally free, the sheaves $(\mathcal{F}_\sigma)_{\sigma\in\Sigma}$ defined by (3.4) satisfy the following :*

(i) *For each $\sigma \in \Sigma$, $\mathcal{F}_\sigma$ is a splittable locally free T-sheaf,*

(ii) *For each $\sigma \in \Sigma$, there is an equivariant injection $\phi_\sigma : \mathcal{F}_\sigma \to \mathcal{E}$ that restricts to an isomorphism over U_σ,*

(iii) *For each pair $\tau \leq \sigma$ of cones in Σ, there is an equivariant injection $\phi_{\tau\sigma}$: $\mathcal{F}_\tau \to \mathcal{F}_\sigma$ that restricts to an isomorphism over U_τ.*

Proof. For (i), fix $\sigma \in \Sigma$. We have a decomposition

$$E = \bigoplus_{i=1}^r E_{m_i}$$

for weights $[m_i] \in M/(\sigma^\perp \cap M)$ such that for each $\rho \in \sigma(1)$,

$$F^\rho(\langle m, u_\rho\rangle) = E^\rho(\langle m, u_\rho\rangle) = \bigoplus_{m_i \leq_\rho m} E_{m_i}.$$

Then, by (3.4), each $F^\rho(j)$ is a direct sum of the E_{m_i}'s, and Klyachko's compatibility criterion is satisfied, as well as Proposition 2.2.14(iii). Hence, $\mathcal{F}_\sigma$ is a splittable locally-free T-sheaf.

For (ii), note that for each $\sigma \in \Sigma$, the identity $1_E : E \to E$ induces an inclusion $F^\rho_\sigma(j) \subset E^\rho(j)$ for each $\rho \in \Sigma(1)$ and $j \in \mathbb{Z}$, which is an isomorphism when $\rho \in \sigma(1)$. The result follows by Theorem 1.3.20. The same argument shows (iii). ∎

For $\sigma \in \Sigma$, the knowledge of the weights $(m_i)_{1\le i\le r} \in M^r$ of the splitting (3.2) enables to describe explicitly the decomposition of $\mathcal{F}_\sigma$ into a direct sum of line bundles. Indeed, we find in Exercise 3.1.7 :

$$\mathcal{F}_\sigma = \Big(\bigoplus_{i=1}^{r} \mathcal{O}_X\big(-\sum_{\rho\in\sigma(1)} \langle m_i, u_\rho\rangle\, D_\rho\big)\Big) \otimes_{\mathcal{O}_X} \mathcal{O}_X\big(-\sum_{\rho\notin\sigma(1)} \nu(\rho)\, D_\rho\big). \tag{3.5}$$

The families of filtrations of the sheaves $(\mathcal{F}_\sigma)_{\sigma\in\Sigma}$ contain all the information about $(E^\rho(\bullet))$. We would like to produce a toric free resolution of $\mathcal{E}$ out of the $(\mathcal{F}_\sigma)_{\sigma\in\Sigma}$. For this, we need first a surjective morphism from a split locally free T-sheaf to $\mathcal{E}$. The most natural choice would be to consider the sum of all the inclusions $\phi_\sigma : \mathcal{F}_\sigma \to \mathcal{E}$, for $\sigma \in \Sigma$. We could actually restrict the sum over maximal dimensional cones. However, to extend this first map to a complex, we will consider instead a signed sum, defined as follows. We fix an ordering of the rays $\Sigma(1) = \{\rho_0, \ldots, \rho_p\}$. For each maximal dimensional cone $\sigma \in \Sigma(n)$, written $\sigma = \rho_{i_1} + \ldots + \rho_{i_n}$ with $i_1 < \ldots < i_n$, set $j_0 < \ldots < j_{p-n}$ to be the elements in $\{0, \ldots, p\} \setminus \{i_1, \ldots, i_n\}$. We then define ε_σ to be $+1$ if $(\rho_{i_1}, \ldots, \rho_{i_n}, \rho_{j_0}, \ldots, \rho_{j_{p-n}})$ is obtained from $(\rho_0, \ldots, \rho_p)$ by an even permutation, and -1 otherwise. We then have the following :

Lemma 3.1.2. *Assume that X is smooth and $\mathcal{E}$ is locally free, and consider the sheaves $(\mathcal{F}_\sigma)_{\sigma\in\Sigma}$ defined by (3.4). Then the map*

$$\sum_{\sigma\in\Sigma(n)} \varepsilon_\sigma\, \phi_\sigma : \bigoplus_{\sigma\in\Sigma(n)} \mathcal{F}_\sigma \to \mathcal{E} \tag{3.6}$$

is a surjective morphism of toric sheaves.

Proof. The result is straightforward on each U_σ, for $\sigma \in \Sigma(n)$, by Lemma 3.1.1(ii). As by the orbit-cone correspondence $(U_\sigma)_{\sigma\in\Sigma(n)}$ is an affine cover of X, the result follows. ∎

For each $0 \le k \le n$, we introduce a split locally free T-sheaf :

$$\mathcal{F}_k = \bigoplus_{\sigma\in\Sigma(k)} \mathcal{F}_\sigma. \tag{3.7}$$

We set $\phi_n : \mathcal{F}_n \to \mathcal{E}$ to be the surjective morphism defined in Lemma 3.1.2. We then define a morphism for each $0 \le k \le n-1$:

$$\phi_k : \mathcal{F}_k \to \mathcal{F}_{k+1}$$

in the following way. The ordering on $\Sigma(1)$ induces in turn an ordering on each set of rays $\sigma(1)$, with $\sigma \in \Sigma$, and an orientation for each basis[1] of $\sigma \cap N$ consisting in

[1] By basis here we mean a minimal set of generators for the semigroup

primitive generators (u_i)'s of rays (ρ_i)'s of σ. For each pair of cones $\tau \leq \sigma$ of Σ with $\dim(\tau) = k \geq 1$, write $\tau = \rho_{i_1} + \ldots + \rho_{i_k}$ with $i_1 < \ldots < i_k$, and $\sigma(1) \setminus \tau(1) = (\rho_{j_1}, \ldots, \rho_{j_l})$ with $j_1 < \ldots < j_l$. Then we set $\varepsilon_{\tau\sigma} = \pm 1$, with the rule that $\varepsilon_{\tau\sigma} = +1$ if $(u_{i_1}, \ldots, u_{i_k}, u_{j_1}, \ldots, u_{j_l})$ is a positively oriented basis of $N \cap \sigma$ and $\varepsilon_{\tau\sigma} = -1$ if not. We then define ϕ_k on $\mathcal{F}_\tau$, for $\tau \in \Sigma(k)$, by setting

$$(\phi_k)_{|\mathcal{F}_\tau} = \sum_{\substack{\tau \leq \sigma, \\ \sigma \in \Sigma(k+1)}} \varepsilon_{\tau\sigma}\, \phi_{\tau\sigma} \tag{3.8}$$

if $k \geq 1$, and

$$\phi_0 = \sum_{\rho \in \Sigma(1)} \phi_{0\rho}.$$

We then have :

Theorem 3.1.3. *Assume that X is smooth and complete, and let $\mathcal{E}$ be a locally free toric sheaf on X. Let $(\mathcal{F}_k)_{0 \leq k \leq n}$ be the split sheaves defined by (3.4) and (3.7), and the maps $(\phi_k)_{0 \leq k \leq n}$ be defined by (3.6) and (3.8). Then*

$$0 \to \mathcal{F}_0 \xrightarrow{\phi_0} \mathcal{F}_1 \xrightarrow{\phi_1} \mathcal{F}_2 \to \ldots \xrightarrow{\phi_{n-1}} \mathcal{F}_n \xrightarrow{\phi_n} \mathcal{E} \to 0 \tag{3.9}$$

defines a finite, free, toric resolution of $\mathcal{E}$.

Proof. We first show that (3.9) defines a complex, that is, for each k,

$$\phi_{k+1} \circ \phi_k = 0.$$

It is enough to check this on each summand $\mathcal{F}_\tau$, with $\tau \in \Sigma(k)$. We first assume that $k + 1 \leq n - 1$. By definition, on $\mathcal{F}_\tau$,

$$\phi_{k+1} \circ \phi_k = \sum_{\tau \leq \sigma \leq \sigma'} \varepsilon_{\sigma\sigma'} \varepsilon_{\tau\sigma}\, \phi_{\sigma\sigma'} \circ \phi_{\tau\sigma}, \tag{3.10}$$

where in the sum $\sigma \in \Sigma(k+1)$ and $\sigma' \in \Sigma(k+2)$. It is straightforward to verify that $\phi_{\sigma\sigma'} \circ \phi_{\tau\sigma} = \phi_{\tau\sigma'}$. Then, the contribution of $\phi_{\tau\sigma'}$ in (3.10) is given by

$$\varepsilon_{\tau\sigma_1}\varepsilon_{\sigma_1\sigma'} + \varepsilon_{\tau\sigma_2}\varepsilon_{\sigma_2\sigma'} = 0$$

where σ_1 and σ_2 are the two $(k+1)$-dimensional cones of Σ containing τ and contained in σ' (we use here smoothness of Σ, that implies that each k-dimensional cone is spanned by its k rays). A similar argument shows that $\phi_n \circ \phi_{n-1} = 0$. Note that this requires completeness and smoothness of Σ, which implies that each cone $\tau \in \Sigma(n-1)$ is contained in exactly two n-dimensional cones of Σ.

By Lemma 3.1.1, what is left is showing exactness of (3.9). Clearly, ϕ_0 is injective. Surjectivity of ϕ_n was shown in Lemma 3.1.2. Denote, for each k, the family of filtrations of $\mathcal{F}_k$ by $(F_k^\rho(\bullet))$, and by

$$\phi_k : (F_k^\rho(\bullet)) \to (F_{k+1}^\rho(\bullet))$$

the associated morphisms of filtrations. We will prove that for each $\rho \in \Sigma(1)$ and each $j \in \mathbb{Z}$, we have exactness of the complex :

$$0 \to F_0^\rho(j) \xrightarrow{\phi_0} F_1^\rho(j) \xrightarrow{\phi_1} \ldots \to F_n^\rho(j) \xrightarrow{\phi_n} E^\rho(j) \to 0. \tag{3.11}$$

Note that

$$F_k^\rho(j) = \bigoplus_{\tau\in\Sigma(k)} F_\tau^\rho(j).$$

If $j \geq \nu(\rho)$, for each $\sigma \in \Sigma$, $F_\sigma^\rho(j) = E$. Then, the complex (3.11) is obtained by tensoring

$$0 \to \mathbb{Z} \xrightarrow{\delta_0} \bigoplus_{\rho'\in\Sigma(1)} \mathbb{Z} \xrightarrow{\delta_1} \bigoplus_{\tau\in\Sigma(2)} \mathbb{Z} \xrightarrow{\delta_2} \ldots \xrightarrow{\delta_{n-1}} \bigoplus_{\sigma\in\Sigma(n)} \mathbb{Z} \xrightarrow{\delta_n} \mathbb{Z} \to 0 \tag{3.12}$$

by E over $\mathbb{Z}$, where the boundary operators are given by

$$(\delta_k(x_\tau)_{\tau\in\Sigma(k)})_\sigma = \sum_{\substack{\tau\leq\sigma\\ \tau\in\Sigma(k)}} \varepsilon_{\tau\sigma} x_\tau.$$

One recognizes the augmented simplicial cohomology complex for the simplicial decomposition of a sphere $S^{n-1} \subset N_\mathbb{R}$ given by the cells $(\sigma \cap S^{n-1})_{\sigma\in\Sigma}$, for a fixed euclidean structure on $N_\mathbb{R}$. Hence, (3.12) is exact, and so is (3.11) when $j \geq \nu(\rho)$. When $j < \nu(\rho)$, we this time have

$$F_k^\rho(j) = \bigoplus_{\substack{\tau\in\Sigma(k)\\ \rho\in\tau(1)}} E^\rho(j).$$

Then, in that case, the complex (3.11) is obtained by tensoring the complex

$$0 \to \bigoplus_{\rho} \mathbb{Z} \xrightarrow{\delta_1} \bigoplus_{\substack{\tau\in\Sigma(2)\\ \rho\in\tau(1)}} \mathbb{Z} \xrightarrow{\delta_2} \ldots \xrightarrow{\delta_{n-1}} \bigoplus_{\substack{\sigma\in\Sigma(n)\\ \rho\in\sigma(n)}} \mathbb{Z} \xrightarrow{\delta_n} \mathbb{Z} \to 0 \tag{3.13}$$

by $E^\rho(j)$ over $\mathbb{Z}$. Again, this is exact, being the augmented cohomology complex of the simplicial complex $\{\sigma \cap S^{n-1} \mid \rho \in \sigma(1),\ \sigma \in \Sigma\}$. Hence, (3.11) is exact. Then, for any $\sigma \in \Sigma$, and any $m \in M$, one has

$$\Gamma(U_\sigma, \mathcal{F}_k)_m = \bigoplus_{\tau\in\Sigma(k)} \Big(\bigcap_{\rho\in\sigma(1)} F_\tau^\rho(\langle m, u_\rho\rangle)\Big).$$

Together with exactness of (3.11), and Theorem 1.2.21, this implies that (3.9) is exact, cf Exercise 3.1.8. ∎

Perling's resolution

On normal varieties, for a given coherent sheaf, a finite locally free resolution may not exist. However, for reflexive toric sheaves over normal toric varieties, Perling obtained the following [58, Chapter 6].

Theorem 3.1.4. *Let $\mathcal{E}$ be a reflexive toric sheaf over a normal toric variety X. Then, $\mathcal{E}$ admits a toric resolution*

$$0 \to \mathcal{F}_d \to \mathcal{F}_{d-1} \to \ldots \to \mathcal{F}_0 \to \mathcal{E} \to 0$$

such that each $\mathcal{F}_i$ is a direct sum of reflexive rank 1 toric sheaves.

This resolution of toric reflexive sheaves is very useful, as rank 1 reflexive sheaves are easy to describe, cf Proposition 2.1.4, and the resolution is explicitly computed from the families of filtrations of $\mathcal{E}$, cf [58, Sections 6.2 and 6.4]. As a straightforward corollary, one obtains :

Corollary 3.1.5. *Let $\mathcal{E}$ be a reflexive toric sheaf over a smooth toric variety X. Then $\mathcal{E}$ admits a finite, free and toric resolution.*

Again, we emphasize that this resolution can be explicitly computed from the family of filtrations of the sheaf $\mathcal{E}$.

Remark 3.1.6. In contrast with Klyachko's resolution, Perling's resolution doesn't require X to be complete. Moreover, while each term in Klyachko's resolution has a very simple description, Perling's resolution is often more efficient for computations, as its terms are of smaller rank than the ones of Klyachko's.

We won't give a proof of Theorem 3.1.4, as it relies on representation theory for partially ordered sets, cf [58, Section 5], which goes beyond the scope of this book. However, in Exercise 3.1.9, the interested reader will find a simple proof in the rank 2 case, and for the tangent sheaf in Exercise 3.1.10.

3.1.1 Exercises for 3.1

Exercise 3.1.7 (Klyachko's free sheaves). Let $\mathcal{E}$ be a locally free T-sheaf over a smooth toric variety X, with family of filtrations $(E^\rho(\bullet))$. Let $\sigma \in \Sigma$ and define $\mathcal{F}_\sigma$ as in (3.4). Let $(m_i)_{1\le i\le r} \in M^r$ be weights as in (3.2).

(1) Show that on U_σ,

$$\mathcal{F}_\sigma \simeq \bigoplus_{i=1}^{r} \mathcal{O}_X\big(-\sum_{\rho\in\sigma(1)} \langle m_i, u_\rho\rangle\, D_\rho\big).$$

(2) Show that

$$\mathcal{F}_\sigma = \bigoplus_{i=1}^{r} \mathcal{O}_X\big(- \sum_{\rho\in\sigma(1)} \langle m_i, u_\rho\rangle\, D_\rho - \sum_{\rho\notin\sigma(1)} \nu(\rho)\, D_\rho\big)$$

and conclude the proof of Formula (3.5).

Hint : use Propositions 2.1.4 and 2.2.10.

Exercise 3.1.8 (Proof of Theorem 3.1.3). We use the notations of the proof of Theorem 3.1.3.

(1) Give an explicit description of the spaces $\Gamma(U_\sigma, \mathcal{F}_k)_m$ in terms of the family of filtrations for $\mathcal{E}$.

(2) Deduce the exactness of

$$0 \to \Gamma(U_\sigma, \mathcal{F}_0)_m \xrightarrow{\phi_0} \Gamma(U_\sigma, \mathcal{F}_1)_m \xrightarrow{\phi_1} \dots \to \Gamma(U_\sigma, \mathcal{F}_n)_m \xrightarrow{\phi_n} \Gamma(U_\sigma, \mathcal{E})_m \to 0.$$

(3) Conclude the proof of Theorem 3.1.3.

Exercise 3.1.9 (Perling's resolution). Let $\mathcal{E}$ be a rank 2 reflexive sheaf over a normal toric variety X. Denote by $(E^\rho(\bullet))$ the family of filtrations for $\mathcal{E}$, given by :

$$\forall\, \rho \in \Sigma(1),\ \forall\, j \in \mathbb{Z},\ E^\rho(j) = \begin{cases} \{0\} & \text{if} \quad j < a_\rho \\ L_\rho & \text{if} \quad a_\rho \le j < b_\rho \\ \mathbb{C}^2 & \text{if} \quad b_\rho \le j \end{cases}$$

for some lines $L_\rho \subset \mathbb{C}^2$ and integers $a_\rho \le b_\rho$.

(1) Assume that X is smooth and that there is at most two different lines in $\{L_\rho \mid \rho \in \Sigma(1)\}$. Show in that case that $\mathcal{E}$ is locally free.

We assume now that there are at least 3 distinct lines amongst the L_ρ's, with corresponding $a_\rho < b_\rho$.

(2) Assume that $n \geq 3$. Show that $\mathcal{E}$ is not locally free.

We will construct Perling's resolution for $\mathcal{E}$, cf Theorem 3.1.4. Consider the $(n+1)$-dimensional complex vector space

$$F := \bigoplus_{\rho\in\Sigma(1)} L_\rho$$

and the rank $(n+1)$ toric reflexive sheaf $\mathcal{F}$ whose family of filtrations is given by

$$F^\rho(i) = \begin{cases} \{0\} & \text{if} \quad i < a_\rho \\ L_\rho & \text{if} \quad a_\rho \le i < b_\rho \\ F & \text{if} \quad b_\rho \le i. \end{cases}$$

(3) Show that the sum of the inclusions $L_\rho \to \mathbb{C}^2$ induces a surjective T-equivariant morphism $\phi : \mathcal{F} \to \mathcal{E}$.

(4) Prove that $\mathcal{F}$ is a direct sum of rank 1 reflexive sheaves, namely

$$\mathcal{F} \simeq \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}_X(-a_\rho D_\rho - \sum_{\rho'\in\Sigma(1)\setminus\{\rho\}} b_{\rho'} D_{\rho'}).$$

(5) Prove that the kernel of ϕ is a direct sum of rank 1 reflexive sheaves, namely

$$\ker\phi \simeq \big(\mathcal{O}_X(-\sum_{\rho\in\Sigma(1)} b_\rho D_\rho)\big)^{\oplus(n-1)}.$$

(6) The sequence $0 \to \ker\phi \to \mathcal{F} \to \mathcal{E} \to 0$ is Perling's resolution for $\mathcal{E}$. Compare it to Klyachko's resolution.

Exercise 3.1.10 (The Euler sequence). Let X be a smooth toric variety. We will construct the (dual) *Euler sequence* :

$$0 \to \mathcal{O}_X^{\oplus p} \to \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}_X(D_\rho) \to \mathcal{T}_X \to 0,$$

where p is the Picard rank of X. Define a toric reflexive sheaf $\mathcal{F}$ of rank $|\Sigma(1)|$ on X by setting its family of filtrations to be :

$$F^\rho(i) = \begin{cases} \{0\} & \text{if} \quad i < -1 \\ \mathbb{C}\cdot u_\rho & \text{if} \quad i = -1 \\ \bigoplus_{\rho\in\Sigma(1)} \mathbb{C}\cdot u_\rho & \text{if} \quad i \geq 0. \end{cases}$$

(1) Prove that the sum of the inclusions $\mathbb{C}\cdot u_\rho \to N_\mathbb{C}$ induces a surjective morphism

$$\phi : \mathcal{F} \to \mathcal{T}_X.$$

(2) Show that

$$\mathcal{F} \simeq \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}_X(D_\rho).$$

(3) Prove that

$$\mathcal{K} := \ker\phi \simeq \mathcal{O}_X^{\oplus p}$$

for some $p \in \mathbb{N}$.

(4) Show that

$$0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{T}_X \to 0$$

is a free, toric resolution of $\mathcal{T}_X$ (this is actually Perling's resolution).

(5) Compare it to Klyachko's resolution.

Hints : for (1), use Proposition 2.1.8. For (2), use Propositions 2.1.4. For (3), use Klyachko's compatibility criterion.

Exercise 3.1.11 (Koszul resolution). Let X be a smooth toric variety, and consider $\iota : V \longrightarrow X$ the inclusion of some orbit closure $V = V(\tau)$ for $\tau \in \Sigma$. We will construct the *Koszul resolution* of $\iota_* \mathcal{O}_V$ by direct sums of toric line bundles. For this, we introduce the toric vector bundle of rank $r = \operatorname{codim}(V)$:

$$\mathcal{F} = \bigoplus_{\rho \in \tau(1)} \mathcal{O}_X(D_\rho)$$

and a section $s = (s_\rho)_{\rho \in \tau(1)} \in H^0(X, \mathcal{F})$, whose component s_ρ is a defining section for $D_\rho = \{s_\rho = 0\}$.

(1) Show that $V(\tau) = \{s = 0\}$.

(2) Prove that convolution by s induces the Koszul complex

$$0 \to \Lambda^r \mathcal{F}^\vee \to \ldots \to \Lambda^2 \mathcal{F}^\vee \to \mathcal{F}^\vee \to \mathcal{O}_X \to \iota_* \mathcal{O}_V \to 0.$$

(3) Show that this complex is a toric free resolution of $\iota_* \mathcal{O}_V$.

3.2 Characteristic classes

On smooth varieties, one can associate *characteristic classes* to any coherent sheaf. Those are cohomology classes, and provide simple topological invariants in the locally free case. In this section, we provide formulae and methods to compute them in the toric setting.

Definition

On a smooth projective variety X, one can attach to any coherent sheaf $\mathcal{E}$ a set of cohomology classes $(c_i(\mathcal{E}))_{0\leq i\leq n}$, called *Chern classes*. Denoting by

$$c(\mathcal{E}) := c_0(\mathcal{E}) + c_1(\mathcal{E}) + \ldots + c_n(\mathcal{E})$$

the *total Chern class*, they satisfy the following properties (cf [16, Chapter 13.1] or [55, Chapter I.1] and reference therein):

(1) $c_0(\mathcal{E}) = 1$ and for $0 \leq i \leq n$, $c_i(\mathcal{E}) \in H^{2i}(X, \mathbb{Z})$,

(2) Normalisation : $c(\mathcal{O}_X) = 1$,

(3) Multiplicativity : for any exact sequence of coherent sheaves

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0,$$

one has

$$c(\mathcal{F}) = c(\mathcal{E}) \cdot c(\mathcal{G}),$$

(4) Naturality : if $\mathcal{E}$ is locally free and $f : Y \to X$ is a morphism, then

$$c(f^*\mathcal{E}) = f^*c(\mathcal{E}).$$

The Chern classes can actually be defined by the previous properties for locally free sheaves, and then extended to coherent sheaves by using multiplicativity and the existence of finite locally free resolutions. They are trivial for trivial locally free sheaves by (2) and (3), and in general give a measure of 'non triviality'. In addition to the previous properties, they also satisfy the following.

Proposition 3.2.1. *The Chern classes of a coherent sheaf $\mathcal{E}$ of rank r over a smooth projective variety X satisfy :*

(i) *If $\mathcal{E}$ is locally free, $c_i(\mathcal{E}) = 0$ for $i > r$,*

(ii) *If $\mathcal{E}$ is locally free, $c_i(\mathcal{E}^\vee) = (-1)^i c_i(\mathcal{E})$ for all i,*

(iii) *If $\mathcal{E}$ is torsion free, $c_1(\Lambda^r\mathcal{E}) = c_1(\mathcal{E})$,*

(iv) *If $\mathcal{E}$ is reflexive and $\mathcal{L}$ is invertible, then for $i \leq r$,*

$$c_i(\mathcal{E} \otimes \mathcal{L}) = \sum_{k=0}^{i} \binom{r-i+k}{k} c_{i-k}(\mathcal{E}) \cdot c_1(\mathcal{L})^k,$$

(v) *If D is a Cartier divisor on X, then*

$$c_1(\mathcal{O}_X(D)) = [D] \in H^2(X, \mathbb{Z}).$$

Examples

The first Chern class of a locally free toric sheaf can be easily computed from the properties in Proposition 3.2.1.

Proposition 3.2.2. *Let $\mathcal{E}$ be a locally free toric sheaf on a smooth toric variety X, with associated family of filtrations $(E^\rho(\bullet))$. Then*

$$c_1(\mathcal{E}) = -\sum_{\rho \in \Sigma(1)} \Big(\sum_{i \in \mathbb{Z}} i \big(\dim E^\rho(i) - \dim E^\rho(i-1) \big) \Big) [D_\rho] \tag{3.14}$$

Proof. From Proposition 3.2.1.(iii), $c_1(\mathcal{E}) = c_1(\Lambda^r \mathcal{E})$, where r is the rank of $\mathcal{E}$. As $\Lambda^r \mathcal{E}$ is a rank one locally free sheaf, it is equivalent to $\mathcal{O}_X(D)$ for some invariant Cartier divisor D on X. Then, $c_1(\mathcal{E}) = [D]$ by Proposition 3.2.1.(v). The result follows from the description of the family of filtrations for $\Lambda^r \mathcal{E}$ in Proposition 2.2.10 and from Proposition 2.1.4. ∎

Formula (3.14) is actually valid for any coherent toric sheaf on projective and smooth toric varieties, as we will see later in this section. We will now use Perling's resolution to compute efficiently Chern classes of reflexive toric sheaves on examples. The first is the tangent sheaf.

Proposition 3.2.3. *Let X be a smooth toric variety. Then*

$$c(\mathcal{T}_X) = \prod_{\rho \in \Sigma(1)} \big(1 + [D_\rho]\big) \in H^\bullet(X, \mathbb{Z}).$$

Proof. From Exercise 3.1.10, there is an exact sequence

$$0 \to \mathcal{O}_X^{\oplus p} \to \bigoplus_{\rho \in \Sigma(1)} \mathcal{O}_X(D_\rho) \to \mathcal{T}_X \to 0.$$

By the defining properties of Chern classes,

$$c(\mathcal{O}_X^{\oplus p}) = c(\mathcal{O}_X)^p = 1$$

and then

$$\begin{aligned} c(\mathcal{T}_X) &= c\Big(\bigoplus_{\rho \in \Sigma(1)} \mathcal{O}_X(D_\rho)\Big) \\ &= \prod_{\rho \in \Sigma(1)} c(\mathcal{O}_X(D_\rho)). \end{aligned}$$

By Proposition 3.2.1,

$$c(\mathcal{O}_X(D_\rho)) = 1 + [D_\rho],$$

and the result follows. ∎

A second example is given with rank 2 reflexive sheaves. Let $\mathcal{E}$ be a rank 2 reflexive T-sheaf on X with family of filtrations $(E^\rho(\bullet))$. We may assume that there are lines $L_\rho \subset \mathbb{C}^2$ and integers $a_\rho \le b_\rho$ such that for each $\rho \in \Sigma(1)$ and $j \in \mathbb{Z}$:

$$E^\rho(j) = \begin{cases} \{0\} & \text{if} \quad j < a_\rho \\ L_\rho & \text{if} \quad a_\rho \le j < b_\rho \\ \mathbb{C}^2 & \text{if} \quad b_\rho \le j. \end{cases}$$

Up to tensoring by $\mathcal{O}_X(\sum_{\rho\in\Sigma(1)} b_\rho D_\rho)$, by Propositions 2.2.10 and 2.1.4, we may assume that $b_\rho = 0$ for each $\rho \in \Sigma(1)$. In that case,

$$E^\rho(j) = \begin{cases} \{0\} & \text{if} \quad j < a_\rho \\ L_\rho & \text{if} \quad a_\rho \le j < 0 \\ \mathbb{C}^2 & \text{if} \quad 0 \le j \end{cases}$$

for some $(a_\rho)_{\rho\in\Sigma(1)} \in \mathbb{Z}_{\le 0}^{|\Sigma(1)|}$. We will say that $\mathcal{E}$ is *normalised*, and refer to the integers $(a_\rho)_{\rho\in\Sigma(1)}$ and lines $(L_\rho)_{\rho\in\Sigma(1)}$ as the *parameters* of $(E^\rho(\bullet))$.

Proposition 3.2.4. *Let $\mathcal{E}$ be a normalised rank 2 toric reflexive sheaf on X, with parameters $(a_\rho)_{\rho\in\Sigma(1)}$. Then*

$$c(\mathcal{E}) = \prod_{\rho\in\Sigma(1)} \big(1 - a_\rho[D_\rho]\big) \in H^\bullet(X, \mathbb{Z}).$$

In the non normalised case, one may recover the Chern classes of $\mathcal{E}$ after untwisting by using Proposition 3.2.1.

Proof. By Exercise 3.1.9, there is a resolution of $\mathcal{E}$ given by

$$0 \to \mathcal{O}_X^{\oplus(n-1)} \to \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}_X(-a_\rho D_\rho) \to \mathcal{E} \to 0.$$

Then, the proof follows the same arguments as the one of Proposition 3.2.3. ∎

Klyachko's formula

In general, the exact sequences given by Perling's resolution have more than 3 terms, and finding them may become tedious. Using Klyachko's resolution for locally free toric sheaves instead, we will find a closed formula for the total Chern class in terms of the Σ-family. We will see that this formula actually holds for any coherent toric sheaf. To state it, let us first introduce some notations. Let (E^σ_m) be a finite Σ-family corresponding to a coherent sheaf $\mathcal{E}$ over a smooth projective toric variety X. Fix a cone $\sigma \in \Sigma(d)$, that we may write

$$\sigma = \mathbb{R}_+ e_1 + \ldots + \mathbb{R}_+ e_d \subset N_\mathbb{R}$$

for $(e_1, \ldots, e_d)$ part of a $\mathbb{Z}$-basis of N. We will follow Kool's notation [42]. We rewrite the σ-family $(E^\sigma_m)_{m\in M}$ as $(E^\sigma(\lambda_1, \ldots, \lambda_d))_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{Z}^d}$, where each $m \in M/(\sigma^\perp \cap M)$ is uniquely written

$$m = \lambda_1 e_1^* + \ldots + \lambda_d e_d^*,$$

by mean of the isomorphism

$$M/(\sigma^\perp \cap M) \simeq \mathbb{Z} \cdot e_1^* \oplus \ldots \oplus \mathbb{Z} \cdot e_d^*.$$

For each $1 \le i \le d$, we introduce the $\mathbb{Z}$-linear operator Δ_i on the free abelian group generated by the vector spaces $\{E^\sigma(\lambda_1, \ldots, \lambda_d)\}_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{Z}^d}$ defined in the following way. For $(\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d$:

$$\Delta_i\left(E^\sigma(\lambda_1, \ldots, \lambda_d)\right) = E^\sigma(\lambda_1, \ldots, \lambda_d) - E^\sigma(\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_d).$$

We then set for any $(\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d$:

$$[E^\sigma](\lambda_1, \ldots, \lambda_d) = \Delta_1(\ldots(\Delta_d\left(E^\sigma(\lambda_1, \ldots, \lambda_d)\right))\ldots).$$

We then extend the definition of the dimension in a $\mathbb{Z}$-linear way on the free abelian group generated by the multifiltration, and we can consider the integer

$$\dim\left([E^\sigma](m)\right) := \dim\left([E^\sigma](\lambda_1, \ldots, \lambda_d)\right) \in \mathbb{Z},$$

where we used the identification $m = \lambda_1 e_1^* + \ldots + \lambda_d e_d^*$. For example, for a ray $\rho \in \Sigma(1)$, and $m = \lambda_1 e_1^*$,

$$\dim([E^\rho](m)) = \dim(E^\rho(\lambda_1)) - \dim(E^\rho(\lambda_1 - 1)),$$

while for $\sigma \in \Sigma(2)$, and $m = \lambda_1 e_1^* + \lambda_2 e_2^*$,

$$\begin{aligned}\dim([E^\sigma](m)) = &\dim(E^\sigma(\lambda_1, \lambda_2)) \\ &- \dim(E^\sigma(\lambda_1 - 1, \lambda_2)) - \dim(E^\sigma(\lambda_1, \lambda_2 - 1)) \\ &+ \dim(E^\sigma(\lambda_1 - 1, \lambda_2 - 1)).\end{aligned}$$

In general, the formula is given by

$$\dim([E^\sigma](m)) = \sum_{\mu\in\{0,1\}^d} (-1)^{(\sum_{i=0}^d \mu_i)} \dim\left(E^\sigma(\lambda_1 - \mu_1, \ldots, \lambda_d - \mu_d)\right) \qquad (3.15)$$

We then obtain Klyachko's formula[2], cf [39, Theorem 3.2.1].

Theorem 3.2.5. *Let $\mathcal{E}$ be a coherent toric sheaf over a smooth projective toric variety, with Σ-family (E^σ_m). Then :*

$$c(\mathcal{E}) = \prod_{\sigma\in\Sigma}\prod_{m\in M/(\sigma^\perp\cap M)}\left(1-\sum_{\rho\in\sigma(1)}\langle m,u_\rho\rangle\,[D_\rho]\right)^{(-1)^{\operatorname{codim}(\sigma)}\dim([E^\sigma](m))} \tag{3.16}$$

The proof was only stated for locally free sheaves in [39], and its extension to coherent sheaves follows an argument by Kool [42, Proposition 3.16]. Note that finiteness of (E^σ_m) ensures that the product on the right hand side of (3.16) is actually finite, cf Exercise 3.2.10.

Proof. Assume first that $\mathcal{E}$ is locally free. Using Klyachko's resolution, Theorem 3.1.3:

$$0\to\mathcal{F}_0\to\bigoplus_{\rho\in\Sigma(1)}\mathcal{F}_\rho\to\ldots\to\bigoplus_{\sigma\in\Sigma(n)}\mathcal{F}_\sigma\to\mathcal{E}\to 0$$

we find

$$\begin{aligned} c(\mathcal{E}) &= \prod_{i=0}^{n} c(\mathcal{F}_i)^{(-1)^{n-i}}\\ &= \prod_{i=0}^{n}\prod_{\sigma\in\Sigma(i)} c(\mathcal{F}_\sigma)^{(-1)^{n-i}}\\ &= \prod_{\sigma\in\Sigma} c(\mathcal{F}_\sigma)^{(-1)^{\operatorname{codim}\sigma}}. \end{aligned}$$

By Equation (3.5), we obtain

$$c(\mathcal{F}_\sigma) = \prod_{i=1}^{r}\Big(1-\sum_{\rho\in\sigma(1)}\langle m_i^\sigma,u_\rho\rangle\,[D_\rho]-\sum_{\rho\notin\sigma(1)}\nu(\rho)\,[D_\rho]\Big),$$

where the weights $(m_i^\sigma)_{1\le i\le r}$ are given by (3.2) and define Klyachko's compatibility condition on U_σ, while $\nu(\rho)$ is defined by (3.3). Then, to prove Formula (3.16) we first show that :

$$\prod_{i=1}^{r}\Big(1-\sum_{\rho\in\sigma(1)}\langle m_i^\sigma,u_\rho\rangle\,[D_\rho]\Big)=\prod_{m\in M/(\sigma^\perp\cap M)}\Big(1-\sum_{\rho\in\sigma(1)}\langle m,u_\rho\rangle\,[D_\rho]\Big)^{\dim([E^\sigma](m))}. \tag{3.17}$$

[2] we use increasing filtrations rather than decreasing filtrations as in [39], which accounts for the difference in signs in our formulae

If $\mathcal{E}$ is of rank $r = 1$, as

$$E^{\sigma}_{m} = \bigcap_{\rho \in \sigma(1)} E^{\rho}(\langle m, u_{\rho} \rangle),$$

we deduce that $\dim([E^{\sigma}](m)) = 0$ unless $m = m^{\sigma}_{1}$, in which case $\dim([E^{\sigma}](m)) = 1$. Hence, (3.17) follows in that case. A similar argument gives (3.17) for higher ranks, using that each space E^{σ}_{m} is a direct sum of some of the $E^{\sigma}_{m^{\sigma}_{i}}$'s, together with Klyachko's compatibility condition. If we had $\nu(\rho) = 0$ for each $\rho \in \Sigma(1)$, this would conclude the proof in the locally free case. But we may achieve this condition, up to tensoring $\mathcal{E}$ by $\mathcal{O}_X(\sum_{\rho \in \Sigma(1)} \nu(\rho) D_{\rho})$. This affects the family of multifiltrations of $\mathcal{E}$ by sending

$$E^{\sigma}_{m} \mapsto E^{\sigma}_{m - m_{\sigma}},$$

where $[m_{\sigma}] \in M/(\sigma^{\perp} \cap M)$ is defined by $\langle m_{\sigma}, u_{\rho} \rangle = \nu(\rho)$, cf Exercise 2.2.22. Twisting back to $\mathcal{E}$, Formula (3.16) remains valid, by using the change of variables $m \mapsto m + m_{\sigma}$ for each $\sigma \in \Sigma$. This achieves the proof in the locally free case.

In the general case, we can use the existence of a finite toric resolution of $\mathcal{E}$ by locally free toric sheaves :

$$0 \to \mathcal{E}_q \to \mathcal{E}_{q-1} \to \ldots \to \mathcal{E}_0 \to \mathcal{E} \to 0.$$

To obtain such toric resolutions, one first uses Serre's global generation theorem [25, Theorem II.5.17], which asserts that there is $k \in \mathbb{N}$ such that the natural map

$$\phi : H^0(X, \mathcal{E} \otimes \mathcal{O}_X(k)) \otimes \mathcal{O}_X \to \mathcal{E} \otimes \mathcal{O}_X(k)$$

is surjective, for $\mathcal{O}_X(1)$ some ample line bundle (see Chapter 4). Choosing an equivariant structure on $\mathcal{E} \otimes \mathcal{O}_X(k)$ (which is possible by Proposition 2.1.3), we may assume that ϕ is equivariant, by choosing a basis of equivariant sections in $H^0(X, \mathcal{E} \otimes \mathcal{O}_X(k))$. Then, we obtain a T-equivariant surjective morphism

$$\mathcal{E}_0 \to \mathcal{E}$$

by tensoring ϕ by $\mathcal{O}_X(-k)$. Iterating the process, we end up with the desired toric resolution. Formula (3.16) then applies to each $\mathcal{E}_k$ in the resolution. But for each $\sigma \in \Sigma$ and each $m \in M$, the resolution yields an exact sequence

$$0 \to (E^{\sigma}_{q})_m \to (E^{\sigma}_{q-1})_m \to \ldots \to (E^{\sigma}_{0})_m \to E^{\sigma}_{m} \to 0,$$

where $((E^{\sigma}_{k})_m)$ is the family of multifiltrations for $\mathcal{E}_k$. Hence

$$\dim E^{\sigma}_{m} = \sum_{i=0}^{q} (-1)^i \dim((E^{\sigma}_{i})_m).$$

The result then follows by multiplicativity of the total Chern class. ∎

As an application, we extend Formula (3.14) to coherent sheaves on projective smooth toric varieties.

Corollary 3.2.6. *Let $\mathcal{E}$ be a coherent toric sheaf on a smooth projective toric variety X, with associated Σ-family (E_m^σ). Then*

$$c_1(\mathcal{E}) = -\sum_{\rho\in\Sigma(1)} \Big(\sum_{i\in\mathbb{Z}} i\big(\dim E^\rho(i) - \dim E^\rho(i-1)\big)\Big)[D_\rho] \tag{3.18}$$

Proof. We follow here [42, Corollary 3.18]. The first Chern class of $\mathcal{E}$ is the contribution of degree 2 in Formula (3.16). Hence

$$\begin{aligned} c_1(\mathcal{E}) &= -\sum_{\sigma\in\Sigma}\sum_{m\in M/(\sigma^\perp\cap M)} (-1)^{\operatorname{codim}(\sigma)} \dim([E^\sigma](m)) \sum_{\rho\in\sigma(1)} \langle m, u_\rho\rangle [D_\rho] \\ &= -\sum_{\rho\in\Sigma(1)}\sum_{\rho\in\sigma(1)} (-1)^{\operatorname{codim}(\sigma)} \Big(\sum_{m\in M/(\sigma^\perp\cap M)} \dim([E^\sigma](m))\,\langle m, u_\rho\rangle\Big)[D_\rho] \end{aligned}$$

Because of the terms $\langle m, u_\rho\rangle$, using Formula (3.15), the contributions from the terms $\dim E_m^\sigma$ cancel out whenever $\dim\sigma \geq 2$, and we end with

$$c_1(\mathcal{E}) = -\sum_{\rho\in\Sigma(1)} \Big(\sum_{\rho\in\sigma(1)} (-1)^{\operatorname{codim}\sigma}\Big)\Big(\sum_{i\in\mathbb{Z}} i\big(\dim E^\rho(i) - \dim E^\rho(i-1)\big)\Big)[D_\rho].$$

The result then follows from $\sum_{\rho\in\sigma(1)} (-1)^{\operatorname{codim}\sigma} = 1$, cf Exercise 3.2.12. ∎

Chern classes of structure sheaves

With Klyachko's formula (3.16), one can in principle compute the Chern classes of any coherent toric sheaf. We will now derive a simpler formula for structure sheaves of invariant subvarieties.

Proposition 3.2.7. *Let X be a smooth projective toric variety, and let $\iota : V \to X$ be the inclusion of the invariant subvariety $V = V(\tau)$, for some $\tau \in \Sigma(d)$. Then*

$$c(\iota_*\mathcal{O}_V) = \prod_{I\subset\tau(1)} \Big(1 - \sum_{\rho\in I}[D_\rho]\Big)^{(-1)^{|I|}}.$$

Proof. Recall that $\iota_*\mathcal{O}_V$ has a Σ-family (E_m^σ) given by

$$E_m^\sigma = \begin{cases} \{0\} & \text{if} \quad \tau \not\leq \sigma, \\ \{0\} & \text{if} \quad \tau \leq \sigma \text{ and } m \notin \tau^\perp\cap\sigma^\vee, \\ \mathbb{C} & \text{if} \quad \tau\leq\sigma \text{ and } m \in \tau^\perp\cap\sigma^\vee, \end{cases}$$

cf Proposition 2.1.5 and Exercise 2.1.11. We then use Klyachko's formula (3.16) and find

$$c(\iota_*\mathcal{O}_V) = \prod_{\tau \leq \sigma} \prod_{m \in M/(\sigma^\perp \cap M)} \left(1 - \sum_{\rho \in \sigma(1)} \langle m, u_\rho \rangle \, [D_\rho]\right)^{(-1)^{\operatorname{codim}(\sigma)} \dim([E^\sigma](m))}.$$

We will use the identification

$$M/(\tau \cap M) \simeq \mathbb{Z} \cdot e_1^* \oplus \ldots \oplus \mathbb{Z} \cdot e_d^*$$

where $(e_1, \ldots, e_d)$ are primitive generators of the rays of τ and $(e_i^*)_{1 \leq i \leq d}$ form a dual basis of $M/(\tau^\perp \cap M)$. Now, for τ, the only non zero space amongst the E_m^τ's is E_0^τ. Its contribution to the product

$$P(\tau) := \prod_{m \in M/(\tau^\perp \cap M)} \left(1 - \sum_{\rho \in \tau(1)} \langle m, u_\rho \rangle \, [D_\rho]\right)^{(-1)^{\operatorname{codim}(\tau)} \dim([E^\tau](m))}$$

comes when $\dim([E^\tau](m)) = \pm 1$, that is, by definition of $[E^\sigma](m)$, when $m = \mu$, for some $\mu = \mu_1 e_1^* + \ldots + \mu_d e_d^*$, with coordinates $\mu_i \in \{0, 1\}$. We then find

$$P(\tau) = \prod_{\mu = \mu_1 e_1^* + \ldots + \mu_d e_d^*} \left(1 - \sum_{\rho \in \tau(1)} \langle \mu, u_\rho \rangle \, [D_\rho]\right)^{(-1)^{\operatorname{codim}(\tau)} (-1)^{|\mu|}}$$

where the sum is over elements μ with coordinates $\mu_i \in \{0, 1\}$, and $|\mu| = \mu_1 + \ldots + \mu_d$. This gives

$$P(\tau) = \prod_{I \subset \tau(1)} \big(1 - \sum_{\rho \in I} [D_\rho]\big)^{(-1)^{\operatorname{codim}(\tau)} (-1)^{|I|}}.$$

We then compute the contribution

$$P(\sigma) = \prod_{m \in M/(\sigma^\perp \cap M)} \left(1 - \sum_{\rho \in \sigma(1)} \langle m, u_\rho \rangle \, [D_\rho]\right)^{(-1)^{\operatorname{codim}(\sigma)} \dim([E^\sigma](m))}$$

for $\tau \leq \sigma$. We use again an identification

$$M/(\sigma^\perp \cap M) \simeq \mathbb{Z} \cdot e_1^* \oplus \ldots \mathbb{Z} \cdot e_k^*$$

where $(e_i)_{1 \leq i \leq k}$ is the set of primitive ray generators for σ such that $(e_i)_{1 \leq i \leq d}$ is the set of generators for τ. Given the σ-family (E_m^σ), E_m^σ is non zero only when $m \in \tau^\perp \cap \sigma^\vee$. Then, to have $\dim([E^\sigma](m)) \neq 0$, there must be $\mu = \mu_1 e_1^* + \ldots + \mu_k e_k^*$, $\mu_i \in \{0, 1\}$, such that $m \in \mu + \tau^\perp \cap \sigma^\vee$. Working in the basis $(e_i^*)_{1 \leq i \leq k}$, we see that for fixed m, the set

$$(\tau^\perp \cap \sigma^\vee) \cap \{m - \mu \mid \mu_i \in \{0, 1\}\}$$

is either empty, or has cardinality 2^p, with $p = 0$ if and only if m writes itself $m = \nu_1 e_1^* + \ldots + \nu_d e_d^*$, $\nu_i \in \{0, 1\}$. When $p \geq 1$, we have $\dim([E^\sigma](m)) = 0$. We conclude that

$$P(\sigma) = \prod_{\mu=\mu_1 e_1^* + \ldots + \mu_d e_d^*} \left(1 - \sum_{\rho \in \sigma(1)} \langle \mu, u_\rho \rangle [D_\rho]\right)^{(-1)^{\operatorname{codim}(\sigma)}(-1)^{|\mu|}}$$

where again the $\mu_i' s$ in the sum belong to $\{0, 1\}$. This gives

$$P(\sigma) = \prod_{I \subset \tau(1)} (1 - \sum_{\rho \in I} [D_\rho])^{(-1)^{\operatorname{codim}(\sigma)}(-1)^{|I|}}.$$

The end of the proof then follows from

$$c(\iota_* \mathcal{O}_V) = \prod_{\tau \leq \sigma} P(\sigma)$$

together with

$$\sum_{\tau \leq \sigma} (-1)^{\operatorname{codim} \sigma} = 1,$$

the latter following from combinatorial arguments, cf Exercise 3.2.12. ■

The formula of Proposition 3.2.7 can also be obtained by using the Koszul resolution, cf Exercise 3.1.11.

3.2.1 Exercises for 3.2

In all the exercises of this section, X is assumed to be smooth and projective.

Exercise 3.2.8 (Chern classes of log tangent sheaves). Let $\Delta \subset \Sigma(1)$, and set

$$D := \sum_{\rho \in \Delta} D_\rho \subset X.$$

Consider the *logarithmic tangent sheaf* :

$$\mathcal{T}_X(-\log D) = \Omega_X^1(\log D)^\vee,$$

cf Exercise 2.1.13.

(1) Compute the family of filtrations for $\mathcal{T}_X(-\log D)$.

(2) Deduce, as for $\mathcal{T}_X$, a finite free resolution for $\mathcal{T}_X(-\log D)$.

(3) Conclude that

$$c(\mathcal{T}_X(-\log D)) = \prod_{\rho \in \Sigma(1) \setminus \Delta} (1 + [D_\rho]).$$

Exercise 3.2.9 (Third Chern class and locally freeness). Let $\mathcal{E}$ be a normalised rank 2 toric reflexive sheaf on the complex projective space $\mathbb{P}^n$. Let $(E^\rho(\bullet))$ be its family of filtrations, with parameters $(a_\rho)_{\rho\in\Sigma(1)} \in \mathbb{Z}_{\leq 0}^{|\Sigma(1)|}$ and $(L_\rho)_{\rho\in\Sigma(1)}$, L_ρ being a line in $\mathbb{C}^2$. We use the isomorphism

$$H^\bullet(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}[H]/\langle H^{n+1}\rangle,$$

and view the Chern classes as integers via $c_i \in \mathbb{Z} \mapsto c_i \cdot H^i \in H^{2i}(\mathbb{P}^n, \mathbb{Z})$.

(1) Show that $c_i(\mathcal{E}) \geq 0$ for $0 \leq i \leq n$.

(2) Prove that $c_3(\mathcal{E}) = 0$ if and only if $\mathcal{E}$ is locally free.

(3) Extend this last result in the non normalised case.

Hint : for (2), show that the vanishing of c_3 implies that at most two lines in $(L_\rho)_{\rho\in\Sigma(1)}$ actually appears in $(E^\rho(\bullet))$. Then use Proposition 2.2.14.

Exercise 3.2.10 (Finiteness of Klyachko's formula). Show that (3.16) is a finite product, using finiteness of the Σ-family.

Exercise 3.2.11 (The Chern character). The *Chern character* $\mathrm{ch}(\cdot) \in H^\bullet(X, \mathbb{Q})$ for coherent sheaves on X is a characteristic class that satisfies the following defining properties :

(1) For an invertible sheaf $\mathcal{L}$,

$$\mathrm{ch}(\mathcal{L}) = \exp(c_1(\mathcal{L})),$$

(2) If $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ is a short exact sequence of coherent sheaves,

$$\mathrm{ch}(\mathcal{F}) = \mathrm{ch}(\mathcal{E}) + \mathrm{ch}(\mathcal{G}).$$

Prove that if $\mathcal{E}$ is a toric coherent sheaf with finite Σ-family (E_m^σ), then

$$\mathrm{ch}(\mathcal{E}) = \sum_{\sigma\in\Sigma}\sum_{m\in M/(\sigma^\perp\cap M)} (-1)^{\operatorname{codim}\sigma} \dim\big([E^\sigma](m)\big) \exp\Big(-\sum_{\rho\in\sigma(1)} \langle m, u_\rho\rangle\, [D_\rho]\Big).$$

Hint : follow the proof of Theorem 3.2.5.

Exercise 3.2.12 (A combinatorial formula). Let $\sigma_0 \in \Sigma$ be a cone in a smooth and complete fan Σ. Show that

$$\sum_{\sigma_0\leq\sigma} (-1)^{\operatorname{codim}\sigma} = 1,$$

where the sum is over cones $\sigma \in \Sigma$ containing σ_0. Hint : use the simplicial complex obtained by cutting cones $\sigma_0 \leq \sigma$ with a sphere in $N_\mathbb{R}$, as in the proof of Theorem 3.1.3.

3.3 Cohomology

Given a sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ on X, one can define *cohomology groups* $H^\bullet(X,\mathcal{E})$. For specific choices of sheaves associated to X, such as the sheaves of differential forms or the tangent sheaf, cohomology groups carry important geometric information about the variety itself. In this section, we will describe some of the specific aspects of cohomology for toric sheaves. The key result, Theorem 3.3.8, shows that it can be computed using *Klyachko–Čech complex*. We then derive some examples, leading to vanishing results for sheaves of p-forms. Finally, we explain how Serre duality for toric sheaves corresponds to Alexander duality for (co)homology of locally contractible compact subsets of the sphere.

Definition

We refer to Appendix A or [16, Section 9.0] for a brief account on cohomology, and to [25, Chapter III] for a more detailed treatment. We will focus here on the main properties of those groups, and methods to compute them. The cohomology groups satisfy the following :

(1) For each $i \in \mathbb{N}$, the assignment

$$\begin{array}{ccc} \mathrm{Mod}(X) & \to & \mathfrak{Ab} \\ \mathcal{E} & \mapsto & H^i(X,\mathcal{E}) \end{array}$$

is an additive functor from the category of sheaves of $\mathcal{O}_X$-modules to the category of abelian groups,

(2) For any sheaf of $\mathcal{O}_X$-modules,

$$H^0(X,\mathcal{E}) = \Gamma(X,\mathcal{E}),$$

(3) Given a short exact sequence of sheaves of $\mathcal{O}_X$-modules :

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0,$$

one has a *long exact sequence in cohomology* :

$$\begin{array}{cccccccc} 0 \to & H^0(X,\mathcal{E}) & \to & H^0(X,\mathcal{F}) & \to & H^0(X,\mathcal{G}) & \xrightarrow{\partial_0} & \\ & H^1(X,\mathcal{E}) & \to & H^1(X,\mathcal{F}) & \to & H^1(X,\mathcal{G}) & \xrightarrow{\partial_1} & \ldots \\ & H^i(X,\mathcal{E}) & \to & H^i(X,\mathcal{F}) & \to & H^i(X,\mathcal{G}) & \xrightarrow{\partial_i} & \ldots \end{array}$$

where the morphisms $\partial_i : H^i(X,\mathcal{G}) \to H^{i+1}(X,\mathcal{E})$ are called *connecting homomorphisms*.

Item (1) above says that cohomology groups are *abelian*, and that for any morphism $\phi : \mathcal{E} \to \mathcal{F}$ of sheaves of $\mathcal{O}_X$-modules, one has associated morphisms in cohomology $h^i(\phi) : H^i(X, \mathcal{E}) \to H^i(X, \mathcal{F})$, in such a way that

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) & \to & \mathrm{Hom}(H^i(X, \mathcal{E}), H^i(X, \mathcal{F})) \\ \phi & \mapsto & h^i(\phi) \end{array}$$

is a morphism of abelian groups. The properties $(1) - (2) - (3)$ are almost[3] enough to characterize cohomology groups, and their existence is ensured by the fact that $\mathrm{Mod}(X)$ *has enough injectives*, cf Appendix A. From those properties, one says that the cohomology groups are the *derived functors* of the functor of global sections Γ. In addition to those defining properties, cohomology groups also satisfy the following fundamental vanishing results.

Theorem 3.3.1 (Grothendieck). *Let X be an n-dimensional variety and let $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$-modules on X. Then for $i > n$, $H^i(X, \mathcal{E}) = 0$.*

Hence, we only consider cohomology groups $H^i(X, \mathcal{E})$ for $0 \leq i \leq \dim(X)$. This result was proved by Grothendieck in greater generality, for Noetherian topological spaces and sheaves of abelian groups. We refer to [25, Theorem III.2.7] for a proof and reference to the original result. For affine varieties, we have :

Theorem 3.3.2 (Serre vanishing). *Let $\mathcal{E}$ be a quasicoherent sheaf of $\mathcal{O}_X$-modules over an affine variety U. Then $H^i(U, \mathcal{E}) = 0$ for all $i > 0$.*

We refer to [25, Theorem III.3.7] for a proof. Another very useful result is *Serre duality*, cf [25, Section III.7], that we will use in the following form :

Theorem 3.3.3 (Serre duality). *Let $\mathcal{E}$ be a locally free sheaf over a smooth and complete variety X of dimension n. Then for any $0 \leq i \leq n$,*

$$H^i(X, \mathcal{E})^* \simeq H^{n-i}(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{K}_X).$$

In practice, to compute cohomology groups, one uses *Čech cohomology*. Let $\mathcal{U} = \{U_i \mid 1 \leq i \leq l\}$ be an affine open cover of X, that is affine subsets $U_i \subset X$ such that

$$\bigcup_{i=1}^{l} U_i = X.$$

[3] One also needs compatibility of the connecting homomorphisms ∂_i with morphisms between short exact sequences of sheaves, as well as a universality property, to characterize the functors $H^i(X, \cdot)$

For a given sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ on X, the group of pth Čech cochains for $\mathcal{E}$ is defined by :

$$\check{C}^p(\mathcal{U}, \mathcal{E}) := \bigoplus_{1 \le i_0 < i_1 < \ldots < i_p \le l} \Gamma(U_{i_0} \cap \ldots \cap U_{i_p}, \mathcal{E}).$$

The Čech cochains come with *differentials* :

$$d^p : \check{C}^p(\mathcal{U}, \mathcal{E}) \to \check{C}^{p+1}(\mathcal{U}, \mathcal{E}).$$

Those are morphisms defined on an element $\alpha \in \check{C}^p(\mathcal{U}, \mathcal{E})$ by

$$d^p \alpha = \sum_{1 \le i_0 < i_1 < \ldots < i_{p+1} \le l} (d^p \alpha)_{i_0 \ldots i_{p+1}}$$

with $(d^p \alpha)_{i_0 \ldots i_{p+1}} \in \Gamma(U_{i_0} \cap \ldots \cap U_{i_{p+1}}, \mathcal{E})$ given by

$$(d^p \alpha)_{i_0 \ldots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \, (\alpha_{i_0 \ldots \hat{i}_k \ldots i_{p+1}})_{|U_{i_0} \cap \ldots \cap U_{i_{p+1}}},$$

where the subscript $\hat{i}_k$ means that i_k is omitted, and where $(\alpha_{i_0 \ldots \hat{i}_k \ldots i_{p+1}})_{|U_{i_0} \cap \ldots \cap U_{i_{p+1}}}$ is the image of $\alpha_{i_0 \ldots \hat{i}_k \ldots i_{p+1}}$ through the restriction map. One can check that $(\check{C}^\bullet(\mathcal{U}, \mathcal{E}), d^\bullet)$ forms a complex, and then define the *Čech cohomology groups* :

$$\check{H}^i(\mathcal{U}, \mathcal{E}) := \frac{\ker \, d^i}{\operatorname{Im} d^{i-1}}$$

where d^{-1} is the trivial map $0 \to \check{C}^0(\mathcal{U}, \mathcal{E})$. A far reaching corollary of Serre vanishing is the following, cf [25, Theorem III.4.5].

Theorem 3.3.4. *Let $\mathcal{E}$ be a quasicoherent sheaf over a variety X. Then for any affine open cover $\mathcal{U}$ of X, there are natural isomorphisms for each i :*

$$\check{H}^i(\mathcal{U}, \mathcal{E}) \simeq H^i(X, \mathcal{E}).$$

Cohomology of Σ-families

Consider now a normal and complete toric variety X with fan Σ. We have an affine open cover

$$\mathcal{U}_\Sigma := \{U_\sigma \mid \sigma \in \Sigma(n)\}.$$

The fact that the union of the U_σ's, where σ runs through maximal dimensional cones, covers X comes from completeness of X and the orbit-cone correspondence. Fix an ordering

$$\{\sigma_1, \ldots, \sigma_l\} = \Sigma(n)$$

of the maximal dimensional facets of Σ. For each $I \subset \{1, \ldots, l\}$, we set

$$\sigma_I := \bigcap_{i \in I} \sigma_i$$

and we have by Definition 1.1.7 :

$$U_{\sigma_I} = \bigcap_{i \in I} U_{\sigma_i}.$$

Let $E^\bullet := (E^\Sigma, \eta^\Sigma)$ be a Σ-family associated to a quasicoherent T-sheaf $\mathcal{E}$ over X. We have an M-grading, for each $I \subset \{1, \ldots, l\}$,

$$\Gamma(U_{\sigma_I}, \mathcal{E}) \simeq \bigoplus_{m \in M} E_m^{\sigma_I}.$$

We then set, for $p \geq 0$, and $m \in M$,

$$C^p(\mathcal{U}_\Sigma, E^\bullet)_m := \bigoplus_{|I|=p+1} E_m^{\sigma_I},$$

where the set I runs through subsets of $\{1, \ldots, l\}$ with cardinality $p+1$. Recall that for any $J \subset I \subset \{1, \ldots, l\}$, there is an inclusion

$$\iota_{\sigma_I \sigma_J} : U_{\sigma_I} \to U_{\sigma_J}$$

with associated restriction morphism

$$\iota^*_{\sigma_I \sigma_J} : E_m^{\sigma_J} \to \iota^*_{\sigma_I \sigma_J} E_m^{\sigma_J}$$

and an isomorphism

$$\eta_{\sigma_I \sigma_J} : \iota^*_{\sigma_I \sigma_J} E_m^{\sigma_J} \to E_m^{\sigma_I}.$$

We then introduce the differentials :

$$d_m^p : C^p(\mathcal{U}_\Sigma, E^\bullet)_m \to C^{p+1}(\mathcal{U}_\Sigma, E^\bullet)_m$$

defined on an element $s = (s_J) \in \bigoplus_{|J|=p+1} E_m^{\sigma_J}$ by

$$d_m^p s = \sum_{\substack{I=(i_0,\ldots,i_{p+1}) \\ i_0<i_1<\ldots<i_{p+1}}} \sum_{\substack{J \subset I,\, |J|=p+1 \\ J=(i_0,\ldots,\hat{i}_k,\ldots,i_{p+1})}} (-1)^k \eta_{\sigma_I \sigma_J} \circ \iota^*_{\sigma_I \sigma_J}(s_J).$$

We leave to the reader the proof of the following in Exercise 3.3.23 :

Lemma 3.3.5. *For each m, $(C^\bullet(\mathcal{U}_\Sigma, E^\bullet)_m, d_m^\bullet)$ forms a differential complex. Moreover, the Čech complex of $(\mathcal{U}_\Sigma, \mathcal{E})$ is isomorphic to the direct sum over M :*

$$(\check{C}^\bullet(\mathcal{U}_\Sigma, \mathcal{E}), d^\bullet) \simeq \bigoplus_{m \in M} (C^\bullet(\mathcal{U}_\Sigma, E^\bullet)_m, d_m^\bullet).$$

We then define the cohomology groups associated to $(C^\bullet(\mathcal{U}_\Sigma, E^\bullet)_m, d_m^\bullet)$, for each $m \in M$, and $i \geq 0$:

$$H^i(\mathcal{U}_\Sigma, E^\bullet)_m := \frac{\ker d_m^i}{\operatorname{Im} d_m^{i-1}},$$

where again d_m^{-1} is the trivial map.

Corollary 3.3.6. *The cohomology groups $H^i(X, \mathcal{E})$ are M-graded. More precisely,*

$$H^i(X, \mathcal{E}) \simeq \bigoplus_{m \in M} H^i(\mathcal{U}_\Sigma, E^\bullet)_m.$$

Proof. As $\mathcal{E}$ is quasicoherent, we can apply Theorem 3.3.4 with the affine cover $\mathcal{U}$. Then, the result follows from Lemma 3.3.5. ∎

In the definition of $C^\bullet(\mathcal{U}_\Sigma, E^\bullet)_m$, different choices of $I, I' \subset \{1, \ldots, l\}$, even with $|I| = |I'|$, may lead to an equality of the associated opens $U_{\sigma_I} = U_{\sigma_{I'}} = U_\sigma$, and to some 'overcount' of the associated spaces E_m^σ. Hence, computing the cohomology of the complex $(C^\bullet(\mathcal{U}_\Sigma, E^\bullet)_m, d_m^\bullet)$ may not be the most efficient way to recover the cohomology of $\mathcal{E}$. We can consider a simpler, and somehow more natural, differential complex, that will fix this issue. As in [11, 12], we fix an orientation on each cone $\sigma \in \Sigma$ and signs $\varepsilon_{\tau\sigma}$ for each pair $\tau \preceq \sigma$, $\dim \sigma = \dim \tau + 1$, with $\varepsilon_{\tau\sigma} = +1$ when the orientation of σ induces the orientation of τ by restriction, and $\varepsilon_{\tau\sigma} = -1$ otherwise. We then introduce complexes $(C^\bullet(\Sigma, E^\bullet)_m, \partial_m^\bullet)$ for each $m \in M$ as follows. For each $0 \leq i \leq n$, set the cochains to be

$$C^i(\Sigma, E^\bullet)_m := \bigoplus_{\sigma \in \Sigma(n-i)} E_m^\sigma.$$

The differentials are given, for $0 \leq i \leq n-1$, by

$$\begin{array}{rccc} \partial_m^i : & C^i(\Sigma, E^\bullet)_m & \to & C^{i+1}(\Sigma, E^\bullet)_m \\ & (s_\sigma)_{\sigma \in \Sigma(n-i)} & \mapsto & \displaystyle\sum_{\tau \in \Sigma(n-i-1)} \sum_{\substack{\tau \preceq \sigma \\ \sigma \in \Sigma(n-i)}} \varepsilon_{\tau\sigma} \, \eta_{\tau\sigma} \circ \iota_{\tau\sigma}^* s_\sigma \end{array}$$

and by

$$\partial_m^n : C^n(\Sigma, E^\bullet)_m \to 0,$$

and

$$\partial_m^{-1} : 0 \to C^0(\Sigma, E^\bullet)_m.$$

We leave to the reader the verification that

$$\partial_m^{i+1} \circ \partial_m^i = 0,$$

that follows from Definition 1.2.20.(2) and the same argument that was used to show that (3.9) is a complex (see also [16, Proof of Lemma 12.3.3]). Following [1], we introduce the following terminology.

Definition 3.3.7. The complex $(C^\bullet(\Sigma, E^\bullet)_m, \partial^\bullet_m)$ is called the m-th Klyachko–Čech complex.

We can introduce the associated cohomology groups

$$H^i(\Sigma, E^\bullet)_m := \frac{\ker \partial^i_m}{\operatorname{Im} \partial^{i-1}_m}.$$

The main result of this section is the following.

Theorem 3.3.8. *Assume that $\mathcal{E}$ is a coherent toric sheaf over a normal and projective toric variety X. Let $E^\bullet = (E^\Sigma, \eta^\Sigma)$ be the associated Σ-family. Then,*

$$H^i(X, \mathcal{E}) \simeq \bigoplus_{m \in M} H^i(\Sigma, E^\bullet)_m.$$

Before giving a proof of this theorem, we will present some corollaries. The first one gives a formula for the graded version of the *Euler characteristic*. The latter is defined by :

$$\chi(\mathcal{E}) = \sum_{i=0}^{n} (-1)^i \dim(H^i(X, \mathcal{E})),$$

for any coherent sheaf X on a complete variety X. When $\mathcal{E}$ is a coherent toric sheaf, one can consider for each $m \in M$, the weight-m Euler characteristic :

$$\chi(\mathcal{E})_m := \sum_{i=0}^{n} (-1)^i \dim \left(H^i(X, \mathcal{E})_m\right).$$

Corollary 3.3.9. *Assume that $\mathcal{E}$ is a coherent toric sheaf over a normal and projective toric variety X. Let $E^\bullet = (E^\Sigma, \eta^\Sigma)$ stand for the associated Σ-family. Then for each $m \in M$,*

$$\chi(\mathcal{E})_m = \sum_{\sigma \in \Sigma} (-1)^{\operatorname{codim}(\sigma)} \dim(E^\sigma_m).$$

When $\mathcal{E}$ is torsion-free, the complex $(C^\bullet(\Sigma, E^\bullet)_m, \partial^\bullet_m)$ takes a simpler form. In this case, $E^\bullet = (E^\sigma_m)$ simply is a family of multifiltrations for a finite dimensional vector space E, and for each m, the complex $(C^\bullet(\Sigma, E^\bullet)_m, \partial^\bullet_m)$ reads :

$$0 \to \bigoplus_{\sigma \in \Sigma(n)} E^\sigma_m \xrightarrow{\partial^0_m} \ldots \to \bigoplus_{\sigma \in \Sigma(2)} E^\sigma_m \xrightarrow{\partial^{n-2}_m} \bigoplus_{\rho \in \Sigma(1)} E^\rho_m \xrightarrow{\partial^{n-1}_m} E \to 0 \tag{3.19}$$

with for $(s_\sigma) \in C^i(\Sigma, E^\bullet)_m$, and $\tau \in \Sigma(n-i-1)$:

$$(\partial^i_m(s_\sigma))_\tau = \sum_{\substack{\tau \leq \sigma \\ \sigma \in \Sigma(n-i)}} \varepsilon_{\tau\sigma}\, s_\sigma.$$

Another corollary of Theorem 3.3.8 then is :

Corollary 3.3.10. *Let $\mathcal{E}$ be a torsion-free toric sheaf on a normal and projective toric variety X, with associated Σ-family $E^\bullet := (E_m^\sigma)_{m\in M}$. Then for each $m \in M$,*

$$H^0(X,\mathcal{E})_m = \bigcap_{\sigma\in\Sigma(n)} E_m^\sigma$$

and

$$H^n(X,\mathcal{E})_m = E/\sum_{\rho\in\Sigma(1)} E_m^\rho.$$

Proof of Theorem 3.3.8

This section is devoted to the proof of Theorem 3.3.8. Our method will rely on techniques from homological algebra (e.g. acyclic resolutions or short exact sequences) that are common in computing cohomology theories. We will use a sheaf version of the complexes $(C^\bullet(\Sigma, E^\bullet)_m, \partial_m)$, for a given Σ-family $E^\bullet = (E^\Sigma, \eta^\Sigma)$ associated to a quasicoherent toric sheaf $\mathcal{E}$. We define, for each $0 \le i \le n$:

$$\mathcal{C}^i(\Sigma,\mathcal{E}) := \bigoplus_{\sigma\in\Sigma(n-i)} (\iota_\sigma)_*(\mathcal{E}_{|U_\sigma}),$$

where $\iota_\sigma : U_\sigma \to X$ is the inclusion map. For each cone $\sigma \in \Sigma$, $\mathcal{E}_{|U_\sigma}$ clearly is a toric sheaf on U_σ. From Proposition 2.3.14, $\mathcal{C}^i(\Sigma,\mathcal{E})$ is naturally a quasicoherent toric sheaf. More precisely, for each $\tau \in \Sigma$, and $m \in M$, we have for each $\sigma \in \Sigma$:

$$\begin{array}{rcl} \Gamma\big(U_\tau, (\iota_\sigma)_*(\mathcal{E}_{|U_\sigma})\big)_m & = & \Gamma(U_\tau \cap U_\sigma, \mathcal{E}_{|U_\sigma})_m \\ & = & \Gamma(U_\tau \cap U_\sigma,\, \mathcal{E})_m \\ & = & E_m^{\tau\cap\sigma}, \end{array}$$

and then

$$\Gamma(U_\tau, \mathcal{C}^i(\Sigma,\mathcal{E}))_m = \bigoplus_{\sigma\in\Sigma(n-i)} E_m^{\tau\cap\sigma}. \tag{3.20}$$

Hence we can define morphisms for $0 \le i \le n-1$ by setting

$$\begin{array}{cccc} \partial^i : & \mathcal{C}^i(\Sigma,\mathcal{E}) & \to & \mathcal{C}^{i+1}(\Sigma,\mathcal{E}) \\ & (s_\sigma)_{\sigma\in\Sigma(n-i)} & \mapsto & \displaystyle\sum_{\tau\in\Sigma(n-i-1)} \sum_{\substack{\tau\le\sigma \\ \sigma\in\Sigma(n-i)}} \varepsilon_{\tau\sigma}\, \eta_{\tau\sigma} \circ \iota^*_{\tau\sigma} s_\sigma. \end{array}$$

Moreover, we have an injective morphism :

$$\iota : \mathcal{E} \to \mathcal{C}^0(\Sigma,\mathcal{E})$$

given by the direct sum over $\Sigma(n)$ of the restriction maps ι_σ. We leave again to the reader checking that the following is a complex :

$$0 \to \mathcal{E} \xrightarrow{\iota} \mathcal{C}^0(\Sigma,\mathcal{E}) \xrightarrow{\partial^0} \mathcal{C}^1(\Sigma,\mathcal{E}) \xrightarrow{\partial^1} \dots \xrightarrow{\partial^{n-1}} \mathcal{C}^n(\Sigma,\mathcal{E}) \to 0, \tag{3.21}$$

that is $\partial^0 \circ \iota = 0$ and for each $1 \leq i \leq n-1$, $\partial^i \circ \partial^{i-1} = 0$. To prove Theorem 3.3.8, we will show that (3.21) is an *acyclic resolution* for $\mathcal{E}$, assuming finiteness of the Σ-family $E^\bullet$. This means that for each $0 \leq i \leq n$, the sheaf $\mathcal{C}^i(\Sigma, \mathcal{E})$ has vanishing cohomology groups except in degree zero, and that (3.21) is exact. We first show acyclicity :

Lemma 3.3.11. *Let $\mathcal{E}$ be a quasicoherent toric sheaf on X. For each $0 \leq i \leq n$ and each $j \geq 1$, one has*

$$H^j(X, \mathcal{C}^i(\Sigma, \mathcal{E})) = 0.$$

Proof. Being additive, the cohomology functors send direct sums to direct sums. It is then enough to show that for each σ, $(\iota_\sigma)_*(\mathcal{E}_{|U_\sigma})$ has vanishing cohomology in degree greater or equal to one. As

$$\mathcal{U}_\sigma := \{U_\sigma \cap U_\tau = U_{\sigma\cap\tau} \mid \tau \in \Sigma(n)\}$$

is an affine cover of U_σ, by the Čech description of cohomology, one sees that

$$\check{H}^j(\mathcal{U}_\Sigma, (\iota_\sigma)_*(\mathcal{E}_{|U_\sigma})) = \check{H}^j(\mathcal{U}_\sigma, \iota_\sigma^*\mathcal{E}).$$

Together with Theorem 3.3.4, we deduce that

$$H^j(X, (\iota_\sigma)_*(\mathcal{E}_{|U_\sigma}) \simeq H^j(U_\sigma, \iota_\sigma^*\mathcal{E}).$$

The result then follows from Serre vanishing Theorem 3.3.2. ∎

To prove exactness of the complex (3.21), we will reduce ourselves to the case of rank one reflexive sheaves, by mean of Perling's resolution. This method relies on the following three lemmas.

Lemma 3.3.12. *Let $\mathcal{E}$ be a quasicoherent toric sheaf on X. Then $\iota : \mathcal{E} \to \mathcal{C}^0(\Sigma, \mathcal{E})$ is injective and* $\operatorname{Im}\iota = \ker \partial^0$.

Proof. This follows from the fact that $\mathcal{E}$ is a sheaf. First, the U_σ, $\sigma \in \Sigma(n)$, cover X. Hence, if the restriction of a section $s \in \Gamma(U_\tau, \mathcal{E})$ to each $U_\tau \cap U_\sigma$, $\sigma \in \Sigma(n)$, vanishes, then $s = 0$. For the second point, $(s_\sigma)_{\sigma\in\Sigma(n)}$ being in the kernel of ∂^0 precisely means that the sections s_{σ_1} and s_{σ_2} agree on the overlaps $\sigma_1 \cap \sigma_2$, for each pair $(\sigma_1, \sigma_2) \in \Sigma(n)^2$. We deduce that they glue over X and come by restriction from a section of $\mathcal{E}$. ∎

In what follows, we let $(\mathcal{C}^\bullet(\Sigma, \mathcal{E}), \partial^\bullet)$ to stand for the complex (3.21).

Lemma 3.3.13. *Any exact sequence*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

of quasicoherent toric sheaves induces an exact sequence of complexes :

$$0 \to (\mathcal{C}^\bullet(\Sigma, \mathcal{E}), \partial^\bullet) \to (\mathcal{C}^\bullet(\Sigma, \mathcal{F}), \partial^\bullet) \to (\mathcal{C}^\bullet(\Sigma, \mathcal{G}), \partial^\bullet) \to 0.$$

By *exact sequence of complexes*, we mean that for each $0 \leq i \leq n$, there is an exact sequence

$$0 \to \mathcal{C}^i(\Sigma, \mathcal{E}) \to \mathcal{C}^i(\Sigma, \mathcal{F}) \to \mathcal{C}^i(\Sigma, \mathcal{G}) \to 0, \tag{3.22}$$

with commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{C}^i(\Sigma, \mathcal{E}) & \longrightarrow & \mathcal{C}^i(\Sigma, \mathcal{F}) & \longrightarrow & \mathcal{C}^i(\Sigma, \mathcal{G}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial^i} & & \downarrow{\scriptstyle \partial^i} & & \downarrow{\scriptstyle \partial^i} & & \\ 0 & \longrightarrow & \mathcal{C}^{i+1}(\Sigma, \mathcal{E}) & \longrightarrow & \mathcal{C}^{i+1}(\Sigma, \mathcal{F}) & \longrightarrow & \mathcal{C}^{i+1}(\Sigma, \mathcal{G}) & \longrightarrow & 0. \end{array} \tag{3.23}$$

Proof. Denote by (E_m^σ), (F_m^σ) and (G_m^σ) the σ families of $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$, respectively. For each $\sigma \in \Sigma$, $\tau \in \Sigma$ and $m \in M$, restriction to $U_{\tau\cap\sigma}$ yields exactness of

$$0 \to E_m^{\tau\cap\sigma} \to F_m^{\tau\cap\sigma} \to G_m^{\tau\cap\sigma} \to 0.$$

Then, by the description of the sections of $\mathcal{C}^i(\Sigma, \mathcal{E})$, cf (3.20), we have induced exact sequences

$$0 \to \Gamma(U_\tau, \mathcal{C}^i(\Sigma, \mathcal{E}))_m \to \Gamma(U_\tau, \mathcal{C}^i(\Sigma, \mathcal{F}))_m \to \Gamma(U_\tau, \mathcal{C}^i(\Sigma, \mathcal{G}))_m \to 0.$$

By Exercise 3.3.20, this proves the existence of the exact sequences (3.22). Commutativity of (3.23) follows from the fact that morphisms of sheaves commute with restriction maps, by definition. ∎

Finally, we have :

Lemma 3.3.14. *Consider an exact sequence of quasicoherent toric sheaves on X :*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0.$$

Then if two of the complexes $(\mathcal{C}^\bullet(\Sigma, \mathcal{E}), \partial^\bullet)$, $(\mathcal{C}^\bullet(\Sigma, \mathcal{F}), \partial^\bullet)$ *or* $(\mathcal{C}^\bullet(\Sigma, \mathcal{G}), \partial^\bullet)$ *are exact, so is the third one.*

In the proof that follows, we will denote by

$$\mathcal{H}^i(\Sigma, \mathcal{E}) = \frac{\ker \partial^i}{\operatorname{Im} \partial^{i-1}}$$

the cohomology *sheaves* of the complex $(\mathcal{C}^\bullet(\Sigma, \mathcal{E}), \partial^\bullet)$.

Proof. From Lemma 3.3.13, we have a short exact sequence of complexes

$$0 \to (\mathcal{C}^\bullet(\Sigma, \mathcal{E}), \partial^\bullet) \to (\mathcal{C}^\bullet(\Sigma, \mathcal{F}), \partial^\bullet) \to (\mathcal{C}^\bullet(\Sigma, \mathcal{G}), \partial^\bullet) \to 0.$$

This induces in turn a long exact sequence in cohomology, cf Appendix B :

$$\begin{array}{llllllll} 0 \to & \mathcal{H}^0(\Sigma,\mathcal{E}) & \to & \mathcal{H}^0(\Sigma,\mathcal{F}) & \to & \mathcal{H}^0(\Sigma,\mathcal{G}) & \to & \\ & \mathcal{H}^1(\Sigma,\mathcal{E}) & \to & \mathcal{H}^1(\Sigma,\mathcal{F}) & \to & \mathcal{H}^1(\Sigma,\mathcal{G}) & \to & \ldots \\ & \mathcal{H}^i(\Sigma,\mathcal{E}) & \to & \mathcal{H}^i(\Sigma,\mathcal{F}) & \to & \mathcal{H}^i(\Sigma,\mathcal{G}) & \to & \ldots \end{array}$$

By Lemma 3.3.12, the sheaves $\mathcal{H}^0(\Sigma,\mathcal{E})$, $\mathcal{H}^0(\Sigma,\mathcal{F})$ and $\mathcal{H}^0(\Sigma,\mathcal{G})$ all vanish. Exactness of a complex $(\mathcal{C}^\bullet(\Sigma,\mathcal{E}),\partial^\bullet)$ is the same as vanishing of all sheaves $\mathcal{H}^i(\Sigma,\mathcal{E})$, $i \geq 0$. Hence, the result follows. ∎

Together with Lemma 3.3.11, the following proposition is the key to prove Theorem 3.3.8.

Proposition 3.3.15. *Assume that X is normal and projective. Let $\mathcal{E}$ be a coherent toric sheaf on X. Then the complex* (3.21) *is exact.*

Proof. As in the proof of Theorem 3.2.5, $\mathcal{E}$ admits a finite locally free resolution by toric sheaves (the proof of this fact doesn't use smoothness). By a direct iteration of Lemma 3.3.14, we may then assume that $\mathcal{E}$ is locally free, hence reflexive. Then, $\mathcal{E}$ admits a resolution by direct sums of rank one reflexive sheaves, cf Theorem 3.1.4. Hence, using Lemma 3.3.14 again, it is actually enough to deal with the case when $\mathcal{E} = \mathcal{L}$ is a rank one reflexive toric sheaf on X. By Proposition 2.1.3, there exists integers $(a_\rho)_{\rho\in\Sigma(1)} \in \mathbb{Z}^{|\Sigma(1)|}$ such that

$$\mathcal{L} \simeq \mathcal{O}_X\Big(\sum_{\rho\in\Sigma(1)} a_\rho\, D_\rho\Big).$$

We set (L_m^σ) to be a family of multifiltrations of $\mathcal{L}$. By proposition 2.1.4, and Proposition 1.3.16, for any $\sigma \in \Sigma$, and any weight $m \in M$, we may assume that

$$L_m^\sigma = \begin{cases} \{0\} & \text{if} \quad \exists\,\rho \in \sigma(1)\,,\ \langle m, u_\rho\rangle < -a_\rho \\ \mathbb{C} & \text{if} \quad \forall\,\rho \in \sigma(1)\,,\ \langle m, u_\rho\rangle \geq -a_\rho. \end{cases}$$

We wish to prove that $(\mathcal{C}^\bullet(\Sigma,\mathcal{L}),\partial^\bullet)$ is an exact sequence of quasicoherent toric sheaves. By Theorem 1.2.21, it is enough to prove exactness at the level of Σ-families. This amounts to showing exactness, for each $\tau \in \Sigma$, and $m \in M$, of the complex $(\Gamma(U_\tau,\mathcal{C}^\bullet(\Sigma,\mathcal{L}))_m,\partial^\bullet)$ (see Exercise 3.3.20). By (3.20), this complex is

$$0 \to L_m^\tau \to \bigoplus_{\sigma\in\Sigma(n)} L_m^{\tau\cap\sigma} \xrightarrow{\partial_m^0} \ldots \xrightarrow{\partial_m^{n-2}} \bigoplus_{\rho\in\Sigma(1)} L_m^{\tau\cap\rho} \xrightarrow{\partial_m^{n-1}} \mathbb{C} \to 0 \qquad (3.24)$$

with

$$(\partial_m^i(s_\sigma))_{\sigma'} = \sum_{\substack{\sigma'\leq\sigma \\ \sigma\in\Sigma(n-i)}} \varepsilon_{\sigma'\sigma}\, s_\sigma.$$

From the description of (L_m^σ), the contribution of $L_m^{\tau\cap\sigma}$ to this latter complex will be non-zero precisely when for all $\rho \in \tau(1) \cap \sigma(1)$, $\langle m, u_\rho\rangle \geq -a_\rho$. Let then $\Sigma_{\hat{\tau}_m} \subset \Sigma$ be the subset of cones $\sigma \in \Sigma$ such that for all $\rho \in \sigma(1)$, either $\rho \notin \tau(1)$, or $\rho \in \tau(1)$ and $\langle m, u_\rho\rangle \geq -a_\rho$. Equivalently, $\Sigma_{\hat{\tau}_m}$ is the largest subfan of Σ that do not contain the set of rays

$$\tau_m := \{\rho \in \tau(1) \mid \langle m, u_\rho\rangle < -a_\rho\} \subset \tau(1).$$

Then, we reduced ourselves to showing exactness of the complex

$$0 \to L_m^\tau \to \bigoplus_{\sigma\in\Sigma_{\hat{\tau}_m}(n)} \mathbb{C}_\sigma \xrightarrow{\partial_m^0} \ldots \to \bigoplus_{\rho\in\Sigma_{\hat{\tau}_m}(1)} \mathbb{C}_\rho \xrightarrow{\partial_m^{n-1}} \mathbb{C} \to 0, \tag{3.25}$$

where each $\mathbb{C}_\sigma$ is a copy of $\mathbb{C}$. We recognize the augmented homology complex for the CW-complex given by $\Sigma_{\hat{\tau}_m} \cap S^{n-1}$, where $S^{n-1} \subset N_\mathbb{R}$ is the sphere given by a fixed euclidean structure on $N_\mathbb{R}$ (see Appendix B). We further note that either $\tau_m = \emptyset$, and then $|\Sigma_{\hat{\tau}_m}| \cap S^{n-1} = S^{n-1}$, or $\tau_m \neq 0$, and then $|\Sigma_{\hat{\tau}_m}| \cap S^{n-1}$ is contractible. Indeed, for each cone $\sigma \in \Sigma$ such that $\sigma(1) \cap \tau_m \neq \emptyset$, the cell $\sigma \cap S^{n-1}$ admits a deformation retract to a point in $\tau_m \cap S^{n-1}$, and $\tau \cap S^{n-1}$ is contractible. In both cases, this proves exactness of (3.25), which concludes the proof. ∎

We can now conclude the proof of Theorem 3.3.8.

Proof of Theorem 3.3.8. Let $\mathcal{E}$ be a coherent toric sheaf over X, with Σ-family $E^\bullet$. By Proposition 3.3.15, $(\mathcal{C}^\bullet(\Sigma,\mathcal{E}), \partial^\bullet)$ is exact. By Lemma 3.3.11, for each $0 \leq i \leq n$, $\mathcal{C}^i(\Sigma,\mathcal{E})$ have vanishing cohomology in positive degrees. Then, it is a standard fact that the global sections of an acyclic resolution computes cohomology for a given sheaf. We recall the argument for completeness. First, note that for each $0 \leq i \leq n$,

$$\Gamma(X, \mathcal{C}^i(\Sigma,\mathcal{E}))_m = C^i(\Sigma, E^\bullet)_m,$$

and that the restriction of the differential ∂^i agrees with the differential ∂_m^i of the complex $C^\bullet(\Sigma, E^\bullet)_m$. Then, we proceed by induction on the cohomological degree. First, consider the quotient obtained from the injection $\mathcal{E} \to \mathcal{C}^0(\Sigma,\mathcal{E})$:

$$0 \to \mathcal{E} \to \mathcal{C}^0(\Sigma,\mathcal{E}) \xrightarrow{\partial^0} \mathcal{Q}_0 \to 0.$$

This yields a long exact sequence in cohomology :

$$\begin{array}{ccccccc} 0 \to & H^0(X,\mathcal{E}) & \to & H^0(X,\mathcal{C}^0(\Sigma,\mathcal{E})) & \xrightarrow{\partial^0} & H^0(X,\mathcal{Q}_0) & \to \\ & H^1(X,\mathcal{E}) & \to & H^1(X,\mathcal{C}^0(\Sigma,\mathcal{E})) & \to & H^1(X,\mathcal{Q}_0) & \to \quad \ldots \end{array}$$

As the morphisms $H^i(X,\mathcal{E}) \to H^i(X,\mathcal{C}^0(\Sigma,\mathcal{E}))$ and $H^i(X,\mathcal{C}^0(\Sigma,\mathcal{E})) \to H^i(X,\mathcal{Q}_0)$ are homogeneous of degree zero for the M-grading (this follows from the Čech description for example), so are the boundary maps in the long exact sequence. We then have long exact sequences in cohomology, for each $m \in M$:

$$\begin{array}{lllllllll} 0 \to & H^0(X,\mathcal{E})_m & \to & C^0(\Sigma, E^\bullet)_m & \xrightarrow{\partial_m^0} & H^0(X,\mathcal{Q}_0)_m & \to & \\ & H^1(X,\mathcal{E})_m & \to & 0 & \to & H^1(X,\mathcal{Q}_0)_m & \to & \\ & H^2(X,\mathcal{E})_m & \to & 0 & \to & H^2(X,\mathcal{Q}_0)_m & \to & \dots \end{array}$$

We deduce that

$$H^0(X,\mathcal{E})_m \simeq \ker \partial_m^0 = H^0(\Sigma, E^\bullet)_m,$$

as well as

$$H^1(X,\mathcal{E})_m \simeq \frac{H^0(X,\mathcal{Q}_0)_m}{\operatorname{Im} \partial_m^0}$$

and that for $i \geq 2$,

$$H^i(X,\mathcal{E})_m \simeq H^{i-1}(X,\mathcal{Q}_0)_m.$$

We then consider the short exact sequence

$$0 \to \mathcal{Q}_0 \to \mathcal{C}^1(\Sigma,\mathcal{E}) \xrightarrow{\partial^1} \mathcal{Q}_1 \to 0.$$

As $\mathcal{C}^1(\Sigma,\mathcal{E})$ has no cohomology in higher degree, the associated long exact sequence gives

$$H^0(X,\mathcal{Q}_0)_m \simeq \ker \partial_m^1,$$

which proves that

$$H^1(X,\mathcal{E})_m \simeq H^1(\Sigma, E^\bullet)_m,$$

and the result follows by iterating this process. ∎

Remark 3.3.16. Klyachko's original proof of Theorem 3.3.8 used Serre duality and a result of Leray on degeneration of specific spectral sequences, cf [39, Theorem 4.1.1 and Remark 4.1.2]. It was then only valid for locally free toric sheaves. We extend here this result to the coherent case, over normal and projective toric varieties. Using a little more algebraic topology, one can extend the result of Theorem 3.3.8, and its corollaries, to quasicoherent toric sheaves over normal and complete toric varieties (hence dropping the projectivity assumption). This requires the use of sheaves over fans (see Note 3.3.1), and Godement's resolution, cf [31, Section 3]. We also mention another approach based on methods from algebra and the *Canonical Čech complex* of [48, Section 6, Example 6.6], together with Cox's presentation of toric sheaves as in Note 1.3.1. We refer the interested reader to [60, Proposition 1.3.14].

Examples : rank one reflexive sheaves

Let X be a normal and projective toric variety, and

$$\mathcal{L} = \mathcal{O}_X\big(\sum_{\rho\in\Sigma(1)} a_\rho\, D_\rho\big)$$

be a rank one reflexive toric sheaf, for some integers $(a_\rho)_{\rho\in\Sigma(1)} \in \mathbb{Z}^{|\Sigma(1)|}$, and consider its family of multifiltrations $L^\bullet = (L^\sigma_m)$. We have

$$L^\sigma_m = \begin{cases} \{0\} & \text{if} \quad \exists\, \rho \in \sigma(1)\,,\ \langle m, u_\rho\rangle < -a_\rho \\ \mathbb{C} & \text{if} \quad \forall\, \rho \in \sigma(1)\,,\ \langle m, u_\rho\rangle \geq -a_\rho. \end{cases}$$

From Theorem 3.3.8, for each $m \in M$, the cohomology groups $H^\bullet(X,\mathcal{L})_m$ can be computed from Klyachko–Čech complex $(C^\bullet(\Sigma, L^\bullet)_m, \partial^\bullet_m)$:

$$0 \to \bigoplus_{\sigma\in\Sigma(n)} L^\sigma_m \xrightarrow{\partial^0_m} \ldots \to \bigoplus_{\sigma\in\Sigma(2)} L^\sigma_m \xrightarrow{\partial^{n-2}_m} \bigoplus_{\rho\in\Sigma(1)} L^\rho_m \xrightarrow{\partial^{n-1}_m} \mathbb{C} \to 0.$$

The space L^σ_m is non-zero if and only if for all $\rho \in \sigma(1)$, $\langle m, u_\rho\rangle \geq -a_\rho$. Set

$$\Sigma_{\mathcal{L},m} := \{\sigma \in \Sigma \mid \forall\, \rho \in \sigma(1),\ \langle m, u_\rho\rangle \geq -a_\rho\} \subset \Sigma. \tag{3.26}$$

It is straightforward to check that $\Sigma_{\mathcal{L},m}$ is a fan. Then, the cohomology groups $H^\bullet(X,\mathcal{L})_m$ are computed by the following complex :

$$0 \to \bigoplus_{\sigma\in\Sigma_{\mathcal{L},m}(n)} \mathbb{C}_\sigma \xrightarrow{\partial^0_m} \ldots \to \bigoplus_{\sigma\in\Sigma_{\mathcal{L},m}(2)} \mathbb{C}_\sigma \xrightarrow{\partial^{n-2}_m} \bigoplus_{\rho\in\Sigma_{\mathcal{L},m}(1)} \mathbb{C}_\rho \xrightarrow{\partial^{n-1}_m} \mathbb{C} \to 0.$$

This is precisely the augmented homology complex for the CW-complex given by $\Sigma_{\mathcal{L},m} \cap S^{n-1}$, where $S^{n-1} \subset N_\mathbb{R}$ is the sphere given by a fixed euclidean structure on $N_\mathbb{R}$ (cf Appendix B). Hence we obtained :

Proposition 3.3.17. *Let $\mathcal{L} = \mathcal{O}_X(\sum_{\rho\in\Sigma(1)} a_\rho\, D_\rho)$ be a rank one reflexive sheaf on a normal and projective toric variety X. Then for each $m \in M$ and each $0 \leq i \leq n-1$, one has*

$$H^i(X,\mathcal{L})_m \simeq \tilde{H}_{n-1-i}(|\Sigma_{\mathcal{L},m}| \cap S^{n-1}),$$

where $\tilde{H}^\bullet$ stands for the reduced homology groups, and $\Sigma_{\mathcal{L},m}$ is defined in (3.26).

Note that $H^n(X,\mathcal{L})$ can easily be computed from Corollary 3.3.10, which concludes the description of the cohomology groups of rank one reflexive sheaves. As a particular case, we recover a vanishing result due to Demazure :

Corollary 3.3.18. *Let X be a normal and projective toric variety. Then for each $i \geq 1$,*

$$H^i(X,\mathcal{O}_X) = 0.$$

Proof. If $\mathcal{L} = \mathcal{O}_X$, we see that the fan $\Sigma_{\mathcal{L},m}$, for some $m \in M$, is given by the largest subfan of Σ contained in the halfspace $m^\vee \subset N_\mathbb{R}$. Hence, by completeness of Σ, $|\Sigma_{\mathcal{L},m}| \cap S^{n-1}$ is contractible, and we find $H^i(X, \mathcal{O}_X)_m = 0$ for all $m \in M$ and $1 \le i \le n-1$ by Proposition 3.3.17. The vanishing of $H^n(X, \mathcal{O}_X)$ follows easily from Corollary 3.3.10 and completeness of the fan. ∎

Examples : sheaves of differential forms

The sheaf of degree p differential forms is defined by

$$\Omega_X^p := (\Lambda^p(\Omega_X^1))^{\vee\vee}.$$

In the toric case, the cohomology of those sheaves is rather restricted.

Theorem 3.3.19. *Let X be a smooth projective toric variety. Then, for any $p \neq q$,*

$$H^p(X, \Omega_X^q) = 0.$$

This result actually holds over simplicial and complete toric varieties, but the proof requires a more general version of Serre duality, cf [16, Chapter 9, Section 3]. This vanishing result is due to Danilov [18, Section 12] when $p < q$, and the proof given here follows Ishida's ideas [35], and the presentation from [16, Theorem 9.3.2] (see also [54, Section 3.1] and reference therein).

Proof. Introduce *Ishida's complex* :

$$0 \to \Omega_X^q \to \mathcal{K}^0(\Sigma, q) \xrightarrow{\phi^0} \mathcal{K}^1(\Sigma, q) \to \dots \xrightarrow{\phi^{q-1}} \mathcal{K}^q(\Sigma, q) \to 0,$$

where

$$\mathcal{K}^j(\Sigma, q) := \bigoplus_{\sigma \in \Sigma(j)} \Lambda^{q-j}(\sigma^\perp \cap M) \otimes_\mathbb{Z} \mathcal{O}_{V(\sigma)},$$

the map

$$\Omega_X^q \to \mathcal{K}^0(\Sigma, q) = \Lambda^q M \otimes_\mathbb{Z} \mathcal{O}_X$$

is the q-th exterior power of the injection $\Omega_X^1 \to M \otimes_\mathbb{Z} \mathcal{O}_X$, cf Proposition 2.1.7, and the maps

$$\phi^j : \mathcal{K}^j(\Sigma, q) \to \mathcal{K}^{j+1}(\Sigma, q)$$

are defined by sending an element $\alpha_\tau \otimes s_\tau$ in $(\tau^\perp \cap M) \otimes_\mathbb{Z} \mathcal{O}_{V(\tau)}$ to

$$\phi^j(\alpha_\tau \otimes s_\tau) = \sum_{\substack{\tau < \sigma \\ \sigma = \tau + \rho}} (\iota_{u_\rho} \alpha_\tau) \otimes (s_\tau)_{|V(\sigma)},$$

where ι_{u_ρ} denotes the inner product of the form α_τ with the primitive generator u_ρ of the ray ρ. Each of the terms in the Ishida complex is a coherent toric sheaf (we denoted

$\mathcal{O}_V$ for $\iota_*\mathcal{O}_V$, where $\iota : V \longrightarrow X$ is the inclusion map). We leave in Exercise 3.3.26 the proof that this complex is exact. By definition of the pushforward, for any invariant subvariety $V \subset X$, we have

$$H^\bullet(X, \iota_*\mathcal{O}_V) \simeq H^\bullet(V, \mathcal{O}_V).$$

Together with Corollary 3.3.18, we deduce that for any $j \geq 0$ and any $k \geq 1$,

$$H^k(X, \mathcal{K}^j(\Sigma, q)) = 0.$$

An inductive argument as in the proof of Theorem 3.3.8 shows then that $H^p(X, \Omega_X^q) = 0$ for $p > q$. For the case when $p < q$, we use Serre duality, Theorem 3.3.3, applied to the sheaf Ω_X^q. This gives

$$H^p(X, \Omega_X^q)^* \simeq H^{n-p}(X, (\Omega_X^q)^\vee \otimes \mathcal{K}_X).$$

But $\mathcal{K}_X = \Omega_X^n$, and

$$(\Omega_X^q)^\vee \otimes \mathcal{K}_X \simeq \mathcal{H}om(\Omega_X^q, \Omega_X^n) \simeq \Omega_X^{n-q}$$

so

$$H^p(X, \Omega_X^q)^* \simeq H^{n-p}(X, \Omega_X^{n-q}).$$

As $n - p < n - q$, the result follows from the previous case. ∎

The computations for the cohomology groups $H^p(X, \Omega_X^p)$, and their twisted versions, can be found in [45]. In Exercise 3.3.27, another example of computations is given, namely the cohomology for tangent sheaves of toric surfaces.

Alexander and Serre dualities

We finish this section by explaining how Alexander duality, cf Appendix B, implies Serre duality for toric line bundles, assuming X to be smooth. Let

$$\mathcal{L} = \mathcal{O}_X\Big(\sum_{\rho \in \Sigma(1)} a_\rho D_\rho\Big)$$

be a toric line bundle, for some integers $(a_\rho)_{\rho \in \Sigma(1)} \in \mathbb{Z}^{|\Sigma(1)|}$, with family of multi-filtrations $L^\bullet = (L_m^\sigma)$ satisfying

$$L_m^\sigma = \begin{cases} \{0\} & \text{if} \quad \exists\, \rho \in \sigma(1)\,,\ \langle m, u_\rho \rangle < -a_\rho \\ \mathbb{C} & \text{if} \quad \forall\, \rho \in \sigma(1)\,,\ \langle m, u_\rho \rangle \geq -a_\rho. \end{cases}$$

Denote by $\check{\mathcal{L}} := \mathcal{L}^\vee \otimes \mathcal{K}_X$, and set $\check{L}^\bullet = (\check{L}_m^\sigma)$ be the associated family of multifiltrations. From Exercise 3.3.28, one has

$$\check{L}_{-m}^\sigma = \begin{cases} \{0\} & \text{if} \quad \exists\, \rho \in \sigma(1)\,,\ \langle m, u_\rho \rangle \geq -a_\rho \\ \mathbb{C} & \text{if} \quad \forall\, \rho \in \sigma(1)\,,\ \langle m, u_\rho \rangle < -a_\rho. \end{cases}$$

We then introduce the subfans of Σ :

$$\Sigma_{\mathcal{L},m} = \{\sigma \in \Sigma \mid \forall \rho \in \sigma(1),\ \langle m, u_\rho\rangle \geq -a_\rho\}$$

and

$$\Sigma_{\check{\mathcal{L}},-m} = \{\sigma \in \Sigma \mid \forall \rho \in \sigma(1),\ \langle m, u_\rho\rangle < -a_\rho\}.$$

From the discussion in the proof of Proposition 3.3.17, we know that the cohomology of $(C^\bullet(\Sigma, \check{L}^\bullet)_{-m}, \partial^\bullet_{-m})$ computes the reduced homology of $\check{K} := S^{n-1} \cap |\Sigma_{\check{\mathcal{L}},-m}|$. On the other hand, it is straightforward to check that the cohomology of the dual complex of $(C^\bullet(\Sigma, \mathcal{L})_m, \partial^\bullet_m)$ computes the reduced cohomology of $K := S^{n-1} \cap |\Sigma_{\mathcal{L},m}|$. As the rays of Σ all belong to $\Sigma_{\check{\mathcal{L}},-m}(1) \cup \Sigma_{\mathcal{L},m}(1)$, the space K is a deformation retract of $S^{n-1} \setminus \check{K}$. Hence, the reduced (co)homology of K is isomorphic to the one of $S^{n-1} \setminus \check{K}$. By Alexander duality, we deduce that the i-th cohomology for the dual complex of $(C^\bullet(\Sigma, \mathcal{L})_m, \partial^\bullet_m)$ is isomorphic to the $(n-i)$-th cohomology for $(C^\bullet(\Sigma, \check{L}^\bullet)_{-m}, \partial^\bullet_{-m})$, recovering Serre duality in degrees $1 \leq i \leq n-1$. In general, Serre duality may be viewed as a version of Alexander duality with local systems of coefficients, cf Exercise 3.3.28.

3.3.1 Exercises for 3.3

Exercise 3.3.20 (Exact sequences of Σ-families). Let

$$0 \to \mathcal{E} \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{G} \to 0 \tag{3.27}$$

be a complex of quasicoherent toric sheaves on X, with associated complex of Σ-families:

$$0 \longrightarrow (E^\Sigma, \eta^\Sigma) \xrightarrow{\Phi^\Sigma} (F^\Sigma, \mu^\Sigma) \xrightarrow{\Psi^\Sigma} (G^\Sigma, \nu^\Sigma) \longrightarrow 0.$$

Show that (3.27) is exact[4] if and only if for each $\sigma \in \Sigma$, the associated complex of σ-families

$$0 \longrightarrow E^\sigma_m \xrightarrow{\phi^\sigma_m} F^\sigma_m \xrightarrow{\psi^\sigma_m} G^\sigma_m \longrightarrow 0$$

is exact for each $m \in M$. Hint : use the long exact sequence in cohomology and Serre vanishing on each $U_\sigma \subset X$.

Exercise 3.3.21 (Ext sheaves). Given a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on X, the sheaves $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \cdot)$ are defined as the right derived functors of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \cdot)$. In practice, one may compute them by using a locally free resolution [25, Proposition 6.5]. Given

$$\ldots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0$$

[4] Exactness means ϕ is injective, $\operatorname{Im}\phi = \ker\psi$ and ψ is surjective.

a locally free resolution of $\mathcal{F}$, then for any sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$,

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{E}) = h^i(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}_\bullet,\mathcal{E})).$$

Show that if X is a smooth toric variety and $\mathcal{F}$ is a coherent toric sheaf, then for any quasicoherent toric sheaf $\mathcal{E}$, $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{E})$ is a quasicoherent toric sheaf. Hint : use the existence of a locally free toric resolution, as in the proof of Theorem 3.2.5.

Exercise 3.3.22 (Higher direct images). Given a toric morphism between normal toric varieties $\phi : X \to Y$, for any quasicoherent sheaf $\mathcal{E}$ on X, one obtains *higher direct images* $R^i\phi_*\mathcal{E}$, $i \geq 0$. Those are quasicoherent sheaves on Y defined on affines $U_\sigma \subset Y$ by

$$\Gamma(U_\sigma, R^i\phi_*\mathcal{E}) = H^i(\phi^{-1}(U_\sigma),\mathcal{E}).$$

Assume that $\mathcal{E}$ is a quasicoherent toric sheaf and that $\underline{\phi} : N_X \to N_Y$ is surjective. Show then that $R^i\phi_*\mathcal{E}$ admits a T_Y-equivariant structure. Hint : reduce to the case when Y is affine and follow the arguments of Proposition 2.3.17.

Exercise 3.3.23 (Differential complex of Σ-families). Let $E^\bullet = (E^\Sigma, \eta^\Sigma)$ be a Σ-family over a complete fan Σ. Denote by $\mathcal{E}$ the associated T-sheaf.

(1) Show that for each $m \in M$, $(C^\bullet(\Sigma, E^\bullet)_m, d^\bullet_m)$ forms a differential complex.

(2) Show that the Čech differential preserves the M-grading on sections of $\mathcal{E}$ over toric affine open sets.

(3) Conclude the proof of Lemma 3.3.5.

Exercise 3.3.24 (Complexes on $\mathbb{P}^n$). Let $E^\bullet$ be a Σ-family on the projective space. Show that the complexes $(C^\bullet(\mathcal{U}_\Sigma, E^\bullet)_m, d^\bullet_m)$ and $(C^\bullet(\Sigma, E^\bullet)_m, \partial^\bullet_m)$ coincide for all $m \in M$.

Exercise 3.3.25 (Hirzebruch–Riemann–Roch). Let X be a smooth and projective variety and $\mathcal{E}$ be a coherent sheaf on X. The Hirzebruch–Riemann–Roch theorem relates the Euler characteristic of $\mathcal{E}$ to the Chern character $\mathrm{ch}(\mathcal{E})$ of $\mathcal{E}$, as defined in Exercise 3.2.11. The formula is given by

$$\chi(\mathcal{E}) = \int_X \mathrm{ch}(\mathcal{E})\,\mathrm{Td}(X), \tag{3.28}$$

where

$$\int_X : H^{2n}(X,\mathbb{Q}) \to H_0(X,\mathbb{Q}) \simeq \mathbb{Q}$$

is the Poincaré dual map and $\mathrm{Td}(X) \in H^\bullet(X,\mathbb{Q})$ is a characteristic class, the *Todd class* (see e.g. [16, Chapter 13, Section 1]). Show that to prove (3.28) for coherent toric sheaves, it is enough to prove it for (toric) line bundles. Hint : show the existence of a finite and free equivariant resolution.

We refer the interested reader to [16, Chapter 13], and reference therein, for a proof of an equivariant version of Hirzebruch–Riemann–Roch theorem for toric line bundles, from which the classical case follows.

Exercise 3.3.26 (Ishida's complex). Consider Ishida's complex, as defined in the proof of Theorem 3.3.19.

(1) Show that this is indeed a complex of coherent toric sheaves.

(2) We will show exactness if Ishida's complex. Let $\sigma_0 \in \Sigma$ and $m \in M \cap \sigma_0^\vee$.

  (a) Show that

$$\Gamma(U_{\sigma_0}, \mathcal{K}^j(\Sigma, q))_m = \bigoplus_{\substack{\sigma\in\Sigma(j)\\ \sigma\leq\sigma_0,\ \sigma\subset m^\perp}} \Lambda^{q-j}\big(\sigma^\perp \cap M_{\mathbb{C}}\big).$$

  (b) Let $\mathcal{B} = (e_i^*)_{1\leq i\leq n}$ be a basis of $M_{\mathbb{C}}$ such that $(e_1^*, \dots, e_r^*)$ is a basis for $\sigma_0^\perp \cap M_{\mathbb{C}}$. For each cone $\sigma \leq \sigma_0$, give a basis of $\Lambda^{q-j}\big(\sigma^\perp \cap M_{\mathbb{C}}\big)$ in terms of $\mathcal{B}$.

  (c) Deduce exactness of $(\Gamma(U_{\sigma_0}, \mathcal{K}^\bullet(\Sigma, q))_m, \phi_m^\bullet)$.

  (d) Conclude.

Exercise 3.3.27 (Deformation theory of toric surfaces). Cohomology of the tangent sheaf carries important information about the deformation theory of the variety. In this exercise, we will compute the cohomology of the tangent sheaf of smooth toric surfaces, recovering results from [34]. Let X be a smooth toric surface. We may order the rays of $\Sigma(1)$ counter clockwise, and denote them accordingly by $\{\rho_1, \rho_2, \dots, \rho_d\}$.

(1) Using Corollary 3.3.10, show that $H^2(X, \mathcal{T}_X) = 0$.

(2) Using Theorem 3.3.8, show that for any $m \in M$,

$$\dim(H^1(X, \mathcal{T}_X)_m) = \sharp\{\rho_i \in \Sigma(1) \mid \langle m, u_{\rho_i}\rangle = -1,\ \langle m, u_{\rho_{i\pm1}}\rangle < 0\},$$

where by convention $\rho_0 = \rho_d$ and $\rho_{d+1} = \rho_1$.

(3) Recover those results using the Euler sequence, Exercise 3.1.10, together with Proposition 3.3.17.

Exercise 3.3.28 (Serre duality). Let $\mathcal{E}$ be a rank r locally free toric sheaf over a smooth and projective toric variety X. Denote by $E^\bullet = (E_m^\sigma)$ its family of multifiltrations. For each $\sigma \in \Sigma$, let $(m_i^\sigma)_{1\leq i\leq r} \in M/(\sigma^\perp \cap M)$ and

$$E = \bigoplus_{i=1}^{r} E_{m_i^\sigma}$$

be the eigenspace decomposition from Klyachko's compatibility, Theorem 1.3.27, such that for each $\rho \in \sigma(1)$, and each $m \in M$,

$$E^\rho(\langle m, u_\rho \rangle) = \bigoplus_{m_i^\sigma \leq_\rho m} E_{m_i^\sigma}.$$

We fix $0 \leq k \leq n$ and $m \in M$.

(1) (a) Let $\sigma \in \Sigma(n-k)$. Show that

$$(E_m^\sigma)^* \simeq \bigoplus_{m_i^\sigma \leq_\sigma m} E^*_{m_i^\sigma}.$$

(b) Deduce that

$$C^k(\Sigma, E^\bullet)^*_m \simeq \bigoplus_{\sigma \in \Sigma(n-k)} \bigoplus_{m_i^\sigma \leq_\sigma m} E^*_{m_i^\sigma}.$$

(c) Describe the dual complex $(C^\bullet(\Sigma, E^\bullet)_m, \partial_m^\bullet)^*$ of $(C^\bullet(\Sigma, E^\bullet)_m, \partial_m^\bullet)$ in terms of the spaces $E^*_{m_i^\sigma}$.

(2) We denote by $\check{E}^\bullet = (\check{E}_m^\sigma)$ the family of multifiltrations for the sheaf $\mathcal{E}^\vee \otimes \mathcal{K}_X$. Let $\sigma \in \Sigma(k)$. We set $\check{m}_i^\sigma \in M/(\sigma^\perp \cap M)$ defined by $\langle \check{m}_i^\sigma, u_\rho \rangle = -\langle m_i^\sigma, u_\rho \rangle + 1$ for all $\rho \in \sigma(1)$, and $E_{\check{m}_i^\sigma} := E_{m_i^\sigma}$.

(a) Show that

$$\check{E}^\sigma_{-m} \simeq \bigoplus_{m \leq_\sigma -\check{m}_i^\sigma} E^*_{\check{m}_i^\sigma}.$$

(b) Describe the complex $(C^\bullet(\Sigma, \check{E}^\bullet)_{-m}, \partial^\bullet_{-m})$ in terms of the $E^*_{\check{m}_i^\sigma}$'s.

(3) Interpret Serre duality, Theorem 3.3.3, for the complexes $(C^\bullet(\Sigma, E^\bullet)_m, \partial_m^\bullet)^*$ and $(C^\bullet(\Sigma, \check{E}^\bullet)_{-m}, \partial^\bullet_{-m})$.

Exercise 3.3.29 (Simplicity of $\mathcal{T}_{\mathbb{P}^n}$). We will show that $\mathcal{T}_{\mathbb{P}^n}$ is *simple*, that is

$$H^0(\mathbb{P}^n, \mathrm{End}(\mathcal{T}_{\mathbb{P}^n})) = \mathbb{C} \cdot \mathrm{Id}_{\mathcal{T}_{\mathbb{P}^n}}.$$

This shows in particular that $\mathcal{T}_{\mathbb{P}^n}$ is *indecomposable*.

(1) Show that the family of filtrations $(E^\rho(\bullet))$ for $\mathrm{End}(\mathcal{T}_{\mathbb{P}^n})$ is

$$\forall j \in \mathbb{Z},\ E^\rho(j) = \begin{cases} \{0\} & \text{if} \quad j < -1 \\ \rho^\perp \otimes \mathbb{C} \cdot u_\rho & \text{if} \quad j = -1 \\ \rho^\perp \otimes N_\mathbb{C} + M_\mathbb{C} \otimes \mathbb{C} \cdot u_\rho & \text{if} \quad j = 0 \\ M_\mathbb{C} \otimes N_\mathbb{C} & \text{if} \quad j \geq 1. \end{cases}$$

(2) Deduce that $H^0(\mathbb{P}^n, \mathrm{End}(\mathcal{T}_{\mathbb{P}^n}))_m = 0$ if $m \neq 0$.

(3) Show that $H^0(\mathbb{P}^n, \mathrm{End}(\mathcal{T}_{\mathbb{P}^n}))_0 = \mathbb{C} \cdot \mathrm{Id}_{\mathcal{T}_{\mathbb{P}^n}}$ and conclude.

Notes for Chapter 3

On the resolution property

In Section 3.1, we have seen that toric vector bundles on smooth toric varieties admit finite resolutions by direct sums of line bundles. A variety is said to have the *resolution property* if any of its coherent sheaves admits a (non necessarily finite) resolution by locally free sheaves. A question raised in [63] is whether any variety (or specific stacks or schemes) has the resolution property. The answer is known to be positive for quasiprojective varieties, for $\mathbb{Q}$-factorial varieties, and for normal surfaces (see the introduction in [57] and reference therein). In the first two cases, such resolutions are actually free. In contrast with the results presented in Section 3.1, Payne constructed complete toric threefolds with no nontrivial toric vector bundles of rank less than or equal to three [57]. It is not known whether Payne's examples have the resolution property. This highlights the subtlety of this question, even in the toric context.

Equivariant cohomology and classes

We have restricted ourselves to the study of classical Chern classes and cohomology groups for toric sheaves. Being equivariant, toric sheaves actually admit equivariant versions of those invariants. Those carry more informations, and can be used to recover the classical ones. We refer the interested reader to [16, Chapter 13] or [8, Part III], and reference therein.

Cohomology and sheaves on fans

A fan Σ naturally carries a topological structure where the open sets are given by subfans. Given a Σ-family (E_m^σ), we obtain for each weight $m \in M$ a sheaf on Σ whose stalk at $\sigma \in \Sigma$ is E_m^σ, and with restriction maps induced by the morphisms $(\eta_{\tau\sigma} \circ \iota_{\tau\sigma}^*)_{\tau \leq \sigma}$. Using the theory of sheaves on fans and Godement's resolution, the results from [31, Section 3] show that Theorem 3.3.8 extends to normal complete toric varieties. The use of sheaves on fans goes back to [12] and [11]. It has applications in equivariant K-theory [4], and in combinatorial cohomology for fans [5].

More vanishing

We limited ourselves to the vanishing results Corollary 3.3.18 and Theorem 3.3.19. One should be aware that there exists a wide range of vanishing theorems for cohomology of toric sheaves. Most of them involve twisting a toric sheaf by a line bundle satisfying a positivity condition, as described in the following chapter. We refer to the comprehensive treatment in [16, Sections 9.2 and 9.3] for more vanishing results and references.