Chapter 2

Examples of toric sheaves

We focus now on examples and basic constructions of toric sheaves. The first section is devoted to classical examples related to the geometry of the variety. Along the way, we obtain a simple proof of Zariski–Lipman conjecture, Theorem 2.1.10, for toric varieties. In the following section, we describe algebraic operations on T-sheaves in terms of Σ-families, and derive Kaneyama and Klyachko's splitting result for the projective space in Corollary 2.2.18. In the last section, we discuss pullbacks and pushforwards for toric sheaves.

2.1 First examples

In this section, we study the most common examples of torsion-free sheaves on normal varieties, in the T-equivariant context. We start with rank one equivariant reflexive sheaves, corresponding to torus invariant Weil divisors, and provide their families of filtrations. In the irreducible case, we generalise this to ideal sheaves of T-invariant subvarieties. Finally, in higher rank, we show that the tangent sheaf is naturally endowed with an equivariant structure, that we describe in details. This leads to the proof of the Zariski–Lipman conjecture in the toric case.

Rank one reflexive sheaves

We refer the reader to [16, Chapter 4, Sections 0 and 1] for the background material on Weil divisors on normal toric varieties. Let X be a normal toric variety. Denote by $\mathbb{C}(X)$ its field of rational functions. We set $\mathrm{Cl}(X)$ to be its *class group*, that is the group of Weil divisors – formal linear combinations of prime divisors – modulo principal divisors – divisors of the form $\mathrm{div}(f)$ for some $f \in \mathbb{C}(X)^*$. Set $\mathrm{Div}_T(X)$ to be the set of T-invariant Weil divisors on X. An important result is the following [16, Theorem 4.1.3]:

Proposition 2.1.1. *There is an exact sequence*

$$M \to \mathrm{Div}_T(X) \to \mathrm{Cl}(X) \to 0 \tag{2.1}$$

where the first map is $m \mapsto \mathrm{div}(\chi^m)$ and the second sends a divisor to its class $D \mapsto [D]$. If X has no torus factor, the sequence is also exact on the left.

A toric variety is said to have *no torus factor* when $\{u_\rho \mid \rho \in \Sigma(1)\}$ spans $N_{\mathbb{R}}$, where recall $u_\rho \in N$ is the primitive generator of the ray ρ such that

$$\rho = \mathbb{R} \cdot u_\rho.$$

We will also need the following facts [16, Propositions 4.1.1 and 4.1.2]:

Proposition 2.1.2. *Let $\rho \in \Sigma(1)$ with associated prime divisor $D_\rho = \overline{O(\rho)}$. Then the associated valuation*

$$\nu_\rho := \nu_{D_\rho} : \mathbb{C}(X)^* \to \mathbb{Z}$$

is given on χ^m, $m \in M$, by

$$\nu_\rho(\chi^m) = \langle m, u_\rho \rangle.$$

Then, for any $m \in M$, the principal divisor $\operatorname{div}(\chi^m)$ *is given by*

$$\operatorname{div}(\chi^m) = \sum_{\rho' \in \Sigma(1)} \langle m, u_{\rho'} \rangle \, D_{\rho'}.$$

To any Weil divisor D, one can associate a rank one reflexive sheaf $\mathcal{O}_X(D)$ defined on open sets $U \subset X$ by

$$\Gamma(U, \mathcal{O}_X(D)) := \{f \in \mathbb{C}(X)^* \mid \big(\operatorname{div}(f) + D\big)_{|U} \geq 0\} \cup \{0\}.$$

Conversely, any rank one reflexive sheaf on X is isomorphic to $\mathcal{O}_X(D)$ for some Weil divisor D, an $\mathcal{O}_X(D) \simeq \mathcal{O}_X(D')$ if and only if $[D] = [D'] \in \operatorname{Cl}(X)$ (see e.g. [16, Chapter 8.0]). Then we have the following:

Proposition 2.1.3. *Let $\mathcal{L}$ be a rank one reflexive sheaf on X. Then there are integers $(a_\rho)_{\rho \in \Sigma(1)}$ such that*

$$\mathcal{L} \simeq \mathcal{O}_X\big(\sum_{\rho \in \Sigma(1)} a_\rho \, D_\rho\big).$$

In particular, $\mathcal{L}$ is isomorphic to a T-equivariant rank 1 reflexive sheaf.

Proof. The first statement follows from Proposition 2.1.1. It is then enough to show that $\mathcal{L} := \mathcal{O}_X\big(\sum_{\rho \in \Sigma(1)} a_\rho \, D_\rho\big)$ admits a T-equivariant structure. Let $\sigma \in \Sigma$. By definition,

$$\begin{aligned} \Gamma(U_\sigma, \mathcal{L}) &= \{f \in \mathbb{C}(X)^* \mid \big(\operatorname{div}(f) + \sum_\rho a_\rho D_\rho\big)_{|U_\sigma} \geq 0\} \cup \{0\} \\ &= \{f \in \mathbb{C}(X)^* \mid \operatorname{div}(f)_{|U_\sigma} + \sum_{\rho \in \sigma(1)} a_\rho D_\rho \geq 0\} \cup \{0\}. \end{aligned}$$

Recall then that the fraction field of X is

$$\mathbb{C}(X) \simeq \mathbb{C}(T) = \mathbb{C}(\chi^m \mid m \in M),$$

and is isomorphic to formal Laurent series in n-variables. As $\sum_\rho a_\rho D_\rho$ is T-invariant, for each $t \in T$, one then has an isomorphism defined on generators of $\mathbb{C}(X)$ by

$$\begin{array}{rccc} \Phi_{\mathcal{L},t}: & \Gamma(U_\sigma, t^*\mathcal{L}) & \to & \Gamma(U_\sigma, \mathcal{L}) \\ & \chi^m & \mapsto & \chi^m(t)\chi^m. \end{array}$$

The $(\Phi_{\mathcal{L},t})_{t\in T}$'s satisfy the cocycle condition (1.6) and induce a T-equivariant structure on $\mathcal{L}$. ∎

From now on, when referring to a rank one reflexive T-sheaf, we will assume that the T-equivariant structure is given by $\Phi_{\mathcal{L}}$, as in the proof of Proposition 2.1.3. Such sheaves are described by families of filtrations, thanks to Theorem 1.3.20.

Proposition 2.1.4. *Let $\mathcal{L} = \mathcal{O}_X\big(\sum_{\rho\in\Sigma(1)} a_\rho\, D_\rho\big)$ be a rank one T-equivariant reflexive sheaf. Denote by $(L^\rho(\bullet))_{\rho\in\Sigma(1)}$ the associated family of filtrations. Then for any $\rho \in \Sigma(1)$,*

$$L^\rho(i) = \begin{cases} \{0\} & \text{if} \quad i < -a_\rho \\ \mathbb{C} & \text{if} \quad i \geq -a_\rho. \end{cases}$$

Of course, there is a slight abuse of terminology in the above proposition, as 'the' family of filtrations is only defined up to an isomorphism.

Proof. Let $\rho \in \Sigma(1)$. From the proof of Proposition 2.1.3,

$$\Gamma(U_\rho, \mathcal{L}) = \{f \in \mathbb{C}(X)^* \mid \mathrm{div}(f)_{|U_\rho} + a_\rho D_\rho \geq 0\} \cup \{0\}.$$

Let $m \in M$. Then, $\mathrm{div}(\chi^m)$ is T-invariant. By the orbit-cone correspondence, $\mathrm{div}(\chi^m)_{|U_\rho}$ is supported on D_ρ. By Proposition 2.1.2, $\nu_\rho(\chi^m) = \langle m, u_\rho\rangle$. Hence, $\mathrm{div}(\chi^m)_{|U_\rho} + a_\rho D_\rho \geq 0$ if and only if $\langle m, u_\rho\rangle + a_\rho \geq 0$. We deduce that the degree-m component of $\Gamma(U_\rho, \mathcal{L})$ is given by

$$\Gamma(U_\rho, \mathcal{L})_m = \begin{cases} \{0\} & \text{if} \quad \langle m, u_\rho\rangle < -a_\rho \\ \mathbb{C}\cdot\chi^m & \text{if} \quad \langle m, u_\rho\rangle \geq -a_\rho. \end{cases}$$

The result follows, as by definition

$$L^\rho(\langle m, u_\rho\rangle) \simeq \Gamma(U_\rho, \mathcal{L})_m.$$

∎

Ideal sheaves

Fix $\tau \in \Sigma(k)$, and consider the T-invariant subvariety $V(\tau) = \overline{O(\tau)} \subset X$. We define

$$\mathcal{I}_\tau \subset \mathcal{O}_X$$

to be the ideal sheaf of $V(\tau)$. As $V(\tau)$ is T-invariant, the T-equivariant structure Φ_X on $\mathcal{O}_X$ restricts to $\mathcal{I}_\tau$ and induces a T-equivariant structure on $\mathcal{I}_\tau$, cf Example 1.2.3. We then consider $\mathcal{I}_\tau$ as a T-equivariant sheaf on X, with the T-equivariant structure $(\Phi_X)_{|\mathcal{I}_\tau}$. As $\mathcal{I}_\tau$ is a subsheaf of $\mathcal{O}_X$, it is torsion-free. Hence, by Theorem 1.3.11, it is described by a family of multifiltrations.

Proposition 2.1.5. *Assume that X is smooth. Let (I_m^σ) be the family of multifiltrations associated to $\mathcal{I}_\tau$. Then, for any $\sigma \in \Sigma$, one has either $\tau \nleq \sigma$, and*

$$I_m^\sigma = \begin{cases} \{0\} & \text{if} \quad 0 \nleq_\sigma m \\ \mathbb{C} & \text{if} \quad 0 \leq_\sigma m, \end{cases}$$

or $\tau \leq \sigma$ and then

$$I_m^\sigma = \begin{cases} \{0\} & \text{if} \quad 0 \nleq_\sigma m, \\ \{0\} & \text{if} \quad 0 \leq_\sigma m,\ m \in \tau^\perp \cap M, \\ \mathbb{C} & \text{if} \quad 0 \leq_\sigma m,\ m \notin \tau^\perp \cap M. \end{cases}$$

Note that if $\tau \leq \sigma$, then the condition that a class $[m] = 0$ in $M/(\tau^\perp \cap M)$ makes sense. Hence, the family of multifiltrations in Proposition 2.1.5 is well defined.

Proof. Fix $\sigma \in \Sigma$. If $\tau \nleq \sigma$, then by the orbit-cone correspondence, $V(\tau) \cap U_\sigma = \varnothing$ and $(\mathcal{I}_\tau)_{|U_\sigma} = (\mathcal{O}_X)_{|U_\sigma}$. In that case, the result follows from Examples 1.2.3 and 1.2.16. Assume now that $\tau \leq \sigma$. By smoothness of X, we can find a $\mathbb{Z}$-basis $(e_i)_{1 \leq i \leq n}$ of N such that

$$\tau = \mathbb{R}_+ \cdot e_1 + \ldots + \mathbb{R}_+ \cdot e_k \subset N_\mathbb{R}$$

and

$$\sigma = \mathbb{R}_+ \cdot e_1 + \ldots + \mathbb{R}_+ \cdot e_l = \tau + \mathbb{R}_+ \cdot e_{k+1} + \ldots + \mathbb{R}_+ \cdot e_l.$$

Consider the dual basis $(e_i^*)_{1 \leq i \leq n}$ for M. Then, in U_σ, we have

$$V(\tau) \cap U_\sigma = \bigcap_{i=1}^{k} \{\chi^{e_i^*} = 0\}.$$

Let $m = m_1 e_1^* + \ldots + m_n e_n^* \in M$. We deduce that $\chi^m \in \Gamma(U_\sigma, \mathcal{I}_\tau)$ if and only if for all $0 \leq i \leq l$, $m_i \geq 0$ and there is $1 \leq i \leq k$ such that $m_i \geq 1$. This is equivalent to $0 \leq_\sigma m$ and $m \notin \tau^\perp \cap M$. The result follows by the construction of the family of multifiltrations associated to $\mathcal{I}_\tau$. ∎

Example 2.1.6. We consider the singular quadric surface given by

$$X = \operatorname{Spec}(\mathbb{C}[u, v, w]/\langle uv - w^2 \rangle)$$

as in Example 1.1.5.(2). The fan Σ of X consists in one 2-dimensional cone $\sigma = \mathbb{R}_+ \cdot (1,0) + \mathbb{R}_+ \cdot (1,2)$, two rays $\rho_1 = \mathbb{R}_+ \cdot (1,0)$ and $\rho_2 = \mathbb{R}_+ \cdot (1,2)$, and the 0-dimensional cone $\{0\}$. Let $\tau := \rho_1$ and $V = V(\tau) \subset X$. We describe the family of multifiltrations (I_m^τ) in Figure 2.1. For each cone $\upsilon \in \Sigma$, we represent the cone $\upsilon^\vee$ by the shaded zone. To each weight $m \in \mathbb{Z}^2$ is attached the vector space I_m^υ, and bigger dots correspond precisely to weights m such that $I_m^\upsilon = \mathbb{C}$.

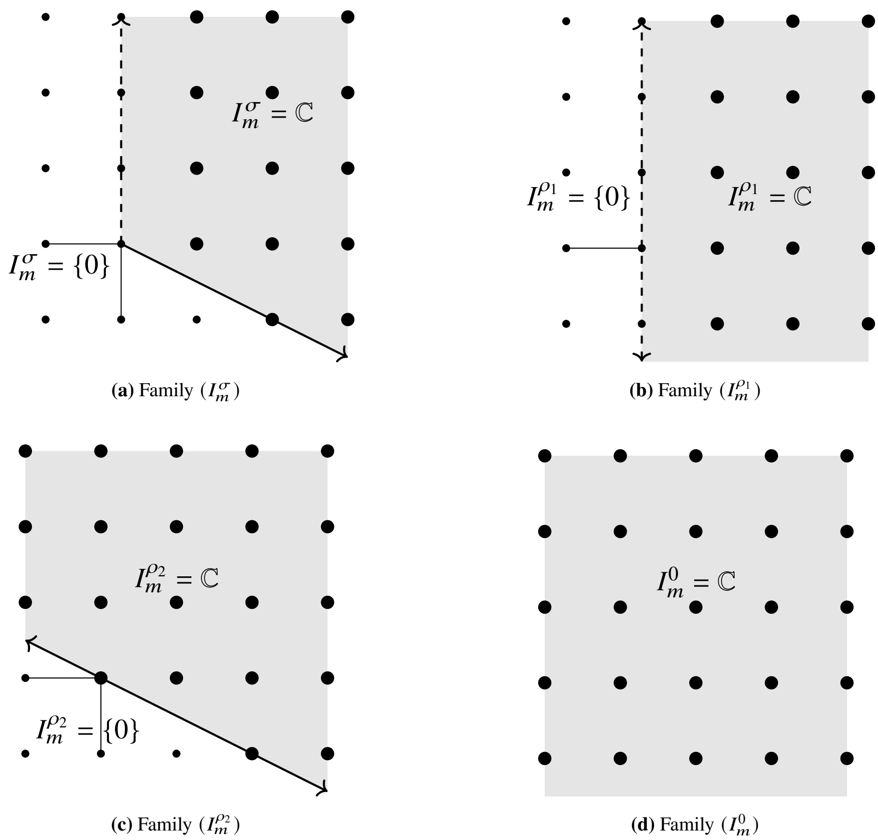

Figure 2.1. The υ-families (I_m^υ).

Tangent and cotangent sheaves

Recall that the sheaf of Kähler differentials on X (see e.g. [25, Chapter II.8]), also called *cotangent sheaf*, denoted Ω_X^1, is defined as the sheaf of $\mathcal{O}_X$-modules generated on affine charts $U \subset X$ by elements of the form df, for $f \in \Gamma(U, \mathcal{O}_X)$, subject to the relations :

(1) For all $(f, g) \in \Gamma(U, \mathcal{O}_X)^2$, and $(a, b) \in \mathbb{C}^2$, $d(af + bg) = adf + bdg$,

(2) For all $(f, g) \in \Gamma(U, \mathcal{O}_X)^2$, $d(fg) = fdg + gdf$.

The sheaf Ω^1_X is a coherent sheaf that carries interesting information about the variety X. In particular, Ω^1_X is locally free if and only if X is smooth, cf [25, Theorem II.8.15]. On each T-invariant affine chart U_σ, $\sigma \in \Sigma$, the $\mathbb{C}$-algebra homomorphism

$$\alpha^* : \mathbb{C}[S_\sigma] \to \mathbb{C}[T] \otimes \mathbb{C}[S_\sigma]$$

induces a homomorphism of $\mathbb{C}[T] \otimes \mathbb{C}[S_\sigma]$-modules :

$$d\alpha : \quad \Gamma(T \times U_\sigma, \alpha^*\Omega^1_X) \quad \to \quad \Gamma(T \times U_\sigma, \Omega^1_{T\times X}).$$

We then obtain an equivariant structure Φ on the sheaf of Kähler differentials by composing :

$$\Phi : \alpha^*\Omega^1_X \xrightarrow{d\alpha} \Omega^1_{T\times X} \xrightarrow{\simeq} \pi_1^*\Omega^1_T \oplus \pi_2^*\Omega^1_X \xrightarrow{\mathrm{pr}_2} \pi_2^*\Omega^1_X \tag{2.2}$$

where pr_2 is the projection on the second factor, cf Exercise 2.1.12. With this equivariant structure, we obtain the following family of filtrations.

Proposition 2.1.7. *Assume that X is smooth. Let $(\Lambda^\rho(\bullet))_{\rho\in\Sigma(1)}$ be the family of filtrations associated to Ω^1_X. Then for any $\rho \in \Sigma(1)$,*

$$\Lambda^\rho(i) = \begin{cases} \{0\} & \text{if} \quad i \leq -1 \\ \rho^\perp & \text{if} \quad i = 0 \\ M_\mathbb{C} & \text{if} \quad i \geq 1. \end{cases}$$

In the above proposition, we extended the duality pairing $\mathbb{C}$-linearly to $M_\mathbb{C} \times N_\mathbb{C}$, where $N_\mathbb{C} = N \otimes_\mathbb{Z} \mathbb{C}$ and $M_\mathbb{C} = M \otimes_\mathbb{Z} \mathbb{C}$. Note that the description of $\Gamma(U_\sigma, \Omega^1_X)$ in the following proof provides the weights $(m_1, \ldots, m_n) = (e_1^*, \ldots, e_n^*)$ of Klyachko's compatibility condition, cf Theorem 1.3.27.

Proof. As X is smooth, Ω^1_X is locally-free. It is then characterised by a family of multifiltrations, subject to Klyachko's compatibility conditions. Let $\sigma \in \Sigma(n)$. By smoothness, we may assume that σ is generated as a cone by a basis $(e_1, \ldots, e_n)$ of N. Let $(e_1^*, \ldots, e_n^*)$ denote the dual basis of M. Then, $\Gamma(U_\sigma, \Omega^1_X)$ is the $\mathbb{C}[S_\sigma]$-module generated by the elements

$$dx_i := d\chi^{e_i^*}, \ 1 \leq i \leq n.$$

By definition of the T-action on sections, cf Equation (1.11), we see that for $t \in T$,

$$t \cdot dx_i = \chi^{-e_i^*}(t)\, dx_i.$$

Hence, we obtain an isomorphism of M-graded $\mathbb{C}[S_\sigma]$-modules :

$$\Gamma(U_\sigma, \Omega^1_X) \simeq \bigoplus_{i=1}^n \mathbb{C}[S_\sigma] \cdot dx_i \simeq \bigoplus_{i=1}^n \mathbb{C}[S_\sigma][e_i^*].$$

We then have for each $\rho \in \sigma(1)$, and each $[m] \in M/(\rho^{\perp} \cap M)$:

$$\Gamma(U_\rho, \Omega^1_X)_m \simeq \bigoplus_{e_i^* \leq_\rho m} \mathbb{C} \cdot \chi^{m-e_i^*}\, dx_i.$$

The result follows by the definition of the filtration $(\Lambda^\rho(i))_{i\in\mathbb{Z}}$. ∎

We next introduce the *tangent sheaf* :

$$\mathcal{T}_X := (\Omega^1_X)^\vee = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X).$$

As $\mathcal{T}_X$ is the dual of the coherent sheaf Ω^1_X, it is reflexive, even when X is not smooth. It is also naturally T-equivariant, cf Exercise 1.3.35. Hence, it is described by a family of filtrations as well.

Proposition 2.1.8. *Let $(TX^\rho(\bullet))_{\rho\in\Sigma(1)}$ be the family of filtrations associated to $\mathcal{T}_X$. Then for any $\rho \in \Sigma(1)$,*

$$TX^\rho(i) = \begin{cases} \{0\} & \text{if} \quad i \leq -2 \\ \mathbb{C}\cdot u_\rho & \text{if} \quad i = -1 \\ N_{\mathbb{C}} & \text{if} \quad i \geq 0. \end{cases}$$

Recall that u_ρ is the primitive generator of the ray ρ.

Proof. We set $e_1 = u_\rho$, that we complete into a $\mathbb{Z}$-basis $(e_1, \ldots, e_n)$ of N, with dual basis $(e_1^*, \ldots, e_n^*)$. On U_ρ, by the orbit-cone correspondence and by Lemma 1.3.3, Ω^1_X and $\mathcal{T}_X$ are locally-free. From the proof of Proposition 2.1.7, $\Gamma(U_\rho, \Omega^1_X)$ is generated as a $\mathbb{C}[S_\rho]$-module by the elements $dx_i := d\chi^{e_i^*}$, $1 \leq i \leq n$. Let

$$\frac{\partial}{\partial x_i} : \Gamma(U_\rho, \Omega^1_X) \to \mathbb{C}[S_\sigma]$$

be homomorphisms of $\mathbb{C}[S_\rho]$-modules be defined by

$$\frac{\partial}{\partial x_i}(dx_j) = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j. \end{cases}$$

Then, $\Gamma(U_\rho, \mathcal{T}_X)$ is spanned as a $\mathbb{C}[S_\rho]$-module by the elements $\dfrac{\partial}{\partial x_i}$, $1 \leq i \leq n$. By definition of the dual equivariant structure, each $\dfrac{\partial}{\partial x_i}$ has degree $-e_i^*$ in the weight space decomposition of $\Gamma(U_\rho, \mathcal{T}_X)$. As

$$\rho^\perp \cap M = \mathbb{Z}\cdot e_2^* \oplus \ldots \oplus \mathbb{Z}\cdot e_n^*$$

and

$$M/(\rho^\perp \cap M) \simeq \mathbb{Z}\cdot e_1^*,$$

we easily deduce the family of filtrations for $\mathcal{T}_X$ on U_ρ. ∎

Example 2.1.9. Let $X = U_\sigma$ be an affine toric surface associated to a cone $\sigma = \rho_1 + \rho_2$ with two generating rays ρ_1 and ρ_2, and denote by (TX_m^σ) the family of multifiltrations for its tangent sheaf. As this family is reflexive, we have for each $m = (m_1, m_2) \in M \simeq \mathbb{Z}^2$:

$$TX_m^\sigma = \begin{cases} \{0\} & \text{if} \quad m_1 \leq -1\ ,\ m_2 \leq -1, \\ \mathbb{C} \cdot u_{\rho_1} & \text{if} \quad m_1 = -1\ ,\ m_2 \geq 0, \\ \mathbb{C} \cdot u_{\rho_2} & \text{if} \quad m_2 = -1\ ,\ m_1 \geq 0, \\ N_\mathbb{C} & \text{if} \quad m_1 \geq 0\ ,\ m_2 \geq 0. \end{cases}$$

We describe this family in Figure 2.2, where to each weight m is associated TX_m^σ, the small dots corresponding to the lines $TX_m^{\rho_i}$, and the big ones to the space $N_\mathbb{C}$.

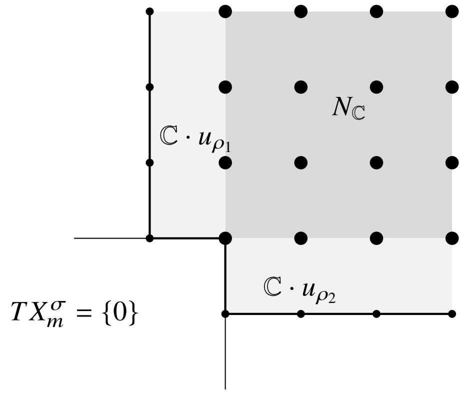

Figure 2.2. Family (TX_m^σ) for the tangent sheaf.

Zariski–Lipman conjecture

We know that a variety is smooth if and only if its sheaf of Kähler differentials is locally free. In fact, it is conjectured that smoothness should also be equivalent to locally freeness of the tangent sheaf. This is known as the Zariski–Lipman conjecture. In the toric case, one can use the simple description of the tangent sheaf to give a proof of this conjecture.

Theorem 2.1.10. *Let X be a normal toric variety. Then X is smooth if and only if its tangent sheaf $\mathcal{T}_X$ is locally free.*

The proof that follows was given in [62], although the result was already known for toric varieties by other arguments. We refer the interested reader to [62] and reference therein.

Proof. If X is smooth, then Ω_X^1 is locally free and so is $\mathcal{T}_X$. For the converse, assume that $\mathcal{T}_X$ is locally free. Smoothness is a local condition, hence we restrict to $U_\sigma \subset X$, for some $\sigma \in \Sigma$. We may without loss assume that σ is of dimension n, and that U_σ has no torus factor, so that $\{u_\rho \mid \rho \in \sigma(1)\}$ spans $N_{\mathbb{R}}$. Then, U_σ is smooth if and only if σ is smooth [16, Theorem 1.3.12], which, by definition, is equivalent to the fact that $\{u_\rho \mid \rho \in \sigma(1)\}$ is a $\mathbb{Z}$-basis for N.

As $\{u_\rho,\ \rho \in \sigma(1)\}$ spans $N_{\mathbb{R}}$, $\sigma^\perp = \{0\}$. Then, by locally freeness, the family of filtrations from Proposition 2.1.8 satisfies Klyachko's compatibility criterion, Theorem 1.3.27. Thus we can find a decomposition

$$N_{\mathbb{C}} = \bigoplus_{j=1}^{n} E_{m_j}$$

with, for $\rho \in \sigma(1)$, and $i \in \{-2, -1, 0\}$,

$$TX^\rho(i) = \bigoplus_{\langle m_j, u_\rho \rangle \leq i} E_{m_j}. \tag{2.3}$$

First, from

$$TX^\rho(-2) = \{0\},$$

we deduce that for all $1 \leq j \leq n$, and all $\rho \in \sigma(1)$, $\langle m_j, u_\rho \rangle \geq -1$. Secondly, taking $i = -1$ in (2.3),

$$\mathbb{C} \cdot u_\rho = \bigoplus_{\langle m_j, u_\rho \rangle \leq -1} E_{m_j}.$$

Thus, for each $\rho \in \sigma(1)$ we can find j_0 and $m_\rho := m_{j_0}$ with $\langle m_\rho, u_\rho \rangle = -1$ and such that $\mathbb{C} \cdot u_\rho = E_{m_\rho}$. Note that σ contains no line by strong convexity, so if $\rho \neq \rho'$, then $u_\rho \notin \mathbb{Z} \cdot u_{\rho'}$ and $\mathbb{C} \cdot u_\rho \neq \mathbb{C} \cdot u_{\rho'}$. As $\{u_\rho \mid \rho \in \sigma(1)\}$ spans $N_{\mathbb{C}}$ over $\mathbb{C}$,

$$N_{\mathbb{C}} = \sum_{\rho \in \sigma(1)} \mathbb{C} \cdot u_\rho = \bigoplus_{\rho \in \sigma(1)} E_{m_\rho} \subset N_{\mathbb{C}}.$$

Then,

$$N_{\mathbb{C}} = \bigoplus_{\rho \in \sigma(1)} E_{m_\rho}$$

and $\sigma(1)$ contains exactly n elements. Let $\rho, \rho' \in \sigma(1)$ be two distinct rays. Necessarily,

$$\mathbb{C} \cdot u_\rho \cap \mathbb{C} \cdot u_{\rho'} = \{0\}$$

and by (2.3) with $i = -, 1$, m_ρ must satisfy $\langle m_\rho, u_{\rho'} \rangle \geq 0$.

Last, taking now $i = 0$ in (2.3), we deduce from $TX^\rho(0) = N_{\mathbb{C}}$ that $\langle m_\rho, u_{\rho'} \rangle = 0$. To conclude, for all $\rho, \rho' \in \sigma(1)$,

$$\langle m_\rho, u_{\rho'} \rangle = \begin{cases} 1 & \text{if} \quad \rho = \rho' \\ 0 & \text{if} \quad \rho \neq \rho'. \end{cases}$$

Hence, each element $u \in N$ can be uniquely written

$$u = \sum_{\rho \in \sigma(1)} \langle m_\rho, u \rangle \, u_\rho$$

with $\langle m_\rho, u \rangle \in \mathbb{Z}$ for each $\rho \in \sigma(1)$. Thus, $\{u_\rho \mid \rho \in \sigma(1)\}$ is a basis of N, which concludes the proof. ∎

2.1.1 Exercises for 2.1

Exercise 2.1.11 (Structure sheaf of invariant subvarieties). Let X be a smooth toric variety with fan Σ and let $\tau \in \Sigma$. Set $Y := V(\tau)$, and let $\iota : Y \to X$ denote the inclusion morphism. Then there is a short exact sequence :

$$0 \to \mathcal{I}_\tau \to \mathcal{O}_X \to \iota_* \mathcal{O}_Y \to 0, \tag{2.4}$$

where recall $\mathcal{I}_\tau$ denotes the ideal sheaf of Y.

(1) Show that $\iota_* \mathcal{O}_Y$ inherits a T-equivariant structure from (2.4).

(2) If (E_m^σ) (resp. (I_m^σ)) stands for the family of multifiltrations for $\mathcal{O}_X$ (resp. for $\mathcal{I}_\tau$), show that for any $\sigma \in \Sigma$, the σ-family (V^σ, ψ^σ) for $(\iota_* \mathcal{O}_Y)_{|U_\sigma}$ is given, for $m \in M$, by

$$V_m^\sigma \simeq E_m^\sigma / I_m^\sigma$$

and, for $m \leq_\sigma m'$,

$$\psi_{m,m'}^\sigma = \begin{cases} 1_{\mathbb{C}} & \text{if} \quad V_m^\sigma = V_{m'}^\sigma = \mathbb{C}, \\ 0 & \text{otherwise.} \end{cases}$$

(3) Give a precise description of the spaces $(V_m^\sigma)_{m \in M}$ when $\tau \leq \sigma$.

Exercise 2.1.12 (Kähler differentials). The module of Kähler differentials of a given $\mathbb{C}$-algebra R, denoted Ω_R, is the R-module generated by the symbols df, modulo the relations

(i) For any $(f, g) \in R^2$, and $(a, b) \in \mathbb{C}^2$, $d(af + bg) = adf + bdg$,

(ii) For any $(f, g) \in R^2$, $d(fg) = fdg + gdf$.

The following shows that the sheaf of Kähler differentials is well defined, and carries a natural T-equivariant structure.

(1) Let R_f be the localisation of R at a non-nilpotent element f. Show that $\Omega_{R_f} \simeq \Omega_R \otimes R_f$.

(2) Let $\phi : R \to S$ be a $\mathbb{C}$-algebra homomorphism. Show that this naturally induces a morphism of S-modules :

$$d\phi : \Omega_R \otimes_R S \to \Omega_S.$$

(3) Show that the morphism Φ defined in (2.2) is well defined, and an isomorphism.

(4) Show that Φ satisfies the cocycle condition of an equivariant structure.

Exercise 2.1.13 (Logarithmic differentials). Let X be a smooth toric variety with fan Σ, and consider a subset $\Delta \subset \Sigma(1)$. Set

$$D := \sum_{\rho \in \Delta} D_\rho \subset X.$$

We define the sheaf of *logarithmic forms* on X along D, denoted $\Omega^1_X(\log D)$, as follows. For an affine chart $U_\sigma \subset X$, we set

$$\Gamma(U_\sigma, \Omega^1_X(\log D)) = \Big(\bigoplus_{\rho \in \sigma(1) \cap \Delta} \mathbb{C}[S_\sigma] \cdot \frac{d\chi^{m_\rho}}{\chi^{m_\rho}} \Big) \oplus \Big(\bigoplus_{\rho \in \sigma(1) \setminus \Delta} \mathbb{C}[S_\sigma] \cdot d\chi^{m_\rho} \Big)$$

where $(m_\rho)_{\rho \in \sigma(1)}$ stands for the basis of M dual to $(u_\rho)_{\rho \in \sigma(1)}$.

(1) Show that $\Omega^1_X(\log D)$ is a well defined locally free sheaf.

(2) Show that $\Omega^1_X(\log D)$ is T-equivariant.

(3) Give the family of filtrations for $\Omega^1_X(\log D)$.

2.2 Algebraic operations

Algebraic operations for sheaves of $\mathcal{O}_X$-modules preserve T-equivariance, and thus admit a transcription into the language of Σ-families. We give in this section formulae for those operations, focusing on the simpler families of filtrations. In relation to direct sums, we provide a splitting criterion in the locally free case that is due to Klyachko. This leads to a proof, in the toric case, of a strong version of Hartshorne's conjecture on locally free sheaves over the projective space.

Subfamilies

We start this section with the description of 'subobjects' in the various categories that were introduced in Chapter 1.

Definition 2.2.1. Let $(\mathcal{E}, \Phi)$ be a T-equivariant sheaf. A subsheaf $\mathcal{F} \subset \mathcal{E}$ is a *T-equivariant subsheaf* if Φ restricts to an isomorphism $\alpha^*\mathcal{F} \xrightarrow{\simeq} \pi_2^*\mathcal{F}$.

If $\mathcal{F} \subset \mathcal{E}$ is a T-equivariant subsheaf of $(\mathcal{E}, \Phi)$, we will denote by $\Phi_{|\mathcal{F}}$ the restricted isomorphism $\alpha^*\mathcal{F} \xrightarrow{\simeq} \pi_2^*\mathcal{F}$. Note that the cocycle condition (1.5) for $\Phi_{|\mathcal{F}}$ is automatically satisfied. As we are working over Noetherian schemes, a T-equivariant subsheaf of a coherent (resp. torsion-free) T-equivariant sheaf is also coherent (resp. torsion-free). Hence, it corresponds to a *subfamily*, that we now define.

Definition 2.2.2. Let (E^σ, χ^σ) be a σ-family. A *sub-σ-family* is the data F^σ of vector subspaces $F^\sigma_m \subset E^\sigma_m$ for each $m \in M$ such that for $m \leq_\sigma m'$,

$$\chi^\sigma_{m,m'}(F^\sigma_m) \subset F^\sigma_{m'}.$$

We will denote by χ^σ the restricted morphisms so that (F^σ, χ^σ) is itself a σ-family, and denote by $F^\sigma \subset E^\sigma$ the inclusion of a sub-σ-family.

The equivalence between the category of σ-families and the category $\mathrm{qCoh}^T(X)$ implies that sub-σ-families correspond to T-equivariant subsheaves over an affine chart U_σ.

Definition 2.2.3. Let (E^Σ, η^Σ) be a Σ-family. A *sub-Σ-family*, or simply *subfamily*, is the data F^Σ of sub-σ-families $F^\sigma \subset E^\sigma$ for each $\sigma \in \Sigma$ such that for $\tau \leq \sigma$, the isomorphism $\eta_{\tau\sigma}$ restricts to an isomorphism $\iota^*_{\tau\sigma}F^\sigma \xrightarrow{\simeq} F^\tau$. We will denote by (F^Σ, η^Σ) the Σ-family obtained by restriction of the isomorphisms $(\eta_{\tau\sigma})_{\tau\leq\sigma}$, and by $F^\Sigma \subset E^\Sigma$ the fact that (F^Σ, η^Σ) is a subfamily.

As for σ-families, sub-Σ-families correspond to equivariant subsheaves in the category $\mathrm{Coh}^T(X)$.

Definition 2.2.4. Let (E^σ_m) be a family of multifiltrations. A *subfamily* of multifiltrations, or simply *subfamily*, is the data of a vector subspace $F \subset E$ and of a family of multifiltrations (F^σ_m) for F such that for each $\sigma \in \Sigma$, and each $m \in M$, $F^\sigma_m \subset E^\sigma_m$.

One defines similarly subfamilies for families of filtrations. Subfamilies of families of multifiltrations (resp. of filtrations) correspond to torsion-free T-subsheaves (resp. reflexive T-subsheaves). Be careful though that in general a torsion-free subsheaf of a reflexive sheaf is not reflexive, cf [25, Section 2]. Finally, we can define *equivariant subbundles*.

Definition 2.2.5. Let $(\underline{\mathcal{E}}, \underline{\Phi})$ be a T-vector bundle. A T*-subbundle* is a subbundle $\underline{\mathcal{F}} \subset \underline{\mathcal{E}}$ such that the T-structure $\underline{\Phi}$ restricts to a T-structure on $\underline{\mathcal{F}}$.

Equivariant subbundles correspond to subfamilies of filtrations $(F^\rho(\bullet))_{\rho\in\Sigma(1)} \subset (E^\rho(\bullet))_{\rho\in\Sigma(1)}$ such that Klyachko's compatibility criterion for E induces the compatibility criterion for F, cf Theorem 1.3.27. The simplest examples of T-subbundles, or more generally T-subsheaves, are direct summands in direct sums, that we define next.

Algebraic operations for T-sheaves

The content of the next proposition is that algebraic operations preserve equivariance.

Proposition 2.2.6. *Let $(\mathcal{E}, \Phi)$ and $(\mathcal{F}, \Psi)$ be two T-equivariant sheaves. Then $\mathcal{E} \oplus \mathcal{F}$, $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ and $\mathcal{E}^\vee$ admit natural T-equivariant structures.*

Proof. For the direct sum, using the facts that

$$\alpha^*(\mathcal{E} \oplus \mathcal{F}) \simeq \alpha^*\mathcal{E} \oplus \alpha^*\mathcal{F}$$

and

$$\pi_2^*(\mathcal{E} \oplus \mathcal{F}) \simeq \pi_2^*\mathcal{E} \oplus \pi_2^*\mathcal{F}$$

we can define (with a little abuse of notations) an isomorphism

$$\Phi \oplus \Psi : \alpha^*(\mathcal{E} \oplus \mathcal{F}) \to \pi_2^*(\mathcal{E} \oplus \mathcal{F}).$$

As Φ and Ψ satisfy (1.5), it is straightforward to check that the cocycle condition is satisfied for $\Phi \oplus \Psi$.

Similarly, using that

$$\alpha^*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = \alpha^*\mathcal{E} \otimes_{\mathcal{O}_{T\times X}} \alpha^*\mathcal{F}$$

and

$$\pi_2^*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = \pi_2^*\mathcal{E} \otimes_{\mathcal{O}_{T\times X}} \pi_2^*\mathcal{F},$$

we can define

$$\Phi \otimes \Psi : \alpha^*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) \to \pi_2^*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$$

by tensoring Φ and Ψ. While the tensor product doesn't preserve injectivity in general, it preserves isomorphisms. Indeed, there exists inverses Φ^{-1} and Ψ^{-1}, and by the functorial properties of the tensor product, $\Phi^{-1} \otimes \Psi^{-1}$ is an inverse for $\Phi \otimes \Psi$. As the tensor product commutes with pullbacks, the cocycle condition is satisfied for $\Phi \otimes \Psi$, providing a T-structure on $\mathcal{E} \otimes \mathcal{F}$.

Finally, the case of the dual sheaf was left in Exercise 1.3.35 and can be delt with similar general arguments on algebraic operations on sheaves of $\mathcal{O}_X$-modules. ∎

Let $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$-modules and $p \in \mathbb{N}^*$. Recall that the p-th tensor power of $\mathcal{E}$, denoted $T^p\mathcal{E}$, is given by

$$T^p\mathcal{E} := (\underbrace{\mathcal{E} \otimes \ldots \otimes \mathcal{E}}_{p-\text{times}}).$$

Then, we can define the p-th symmetric power $\mathrm{Sym}^p\mathcal{E}$ (resp. p-th exterior power $\Lambda^p\mathcal{E}$) of $\mathcal{E}$ to be the sheaf of $\mathcal{O}_X$-modules obtained as the sheafification of the presheaf

$$U \mapsto \mathrm{Sym}^p_{\mathcal{O}_X(U)}(\mathcal{E}(U))$$

(resp. of the presheaf

$$U \mapsto \Lambda^p_{\mathcal{O}_X(U)}(\mathcal{E}(U)).)$$

Proposition 2.2.7. *Let $\mathcal{E}$ be a T-equivariant sheaf. Then $T^p\mathcal{E}$, $\mathrm{Sym}^p\mathcal{E}$ and $\Lambda^p\mathcal{E}$ admit natural T-equivariant structures.*

Proof. For $T^p\mathcal{E}$, the result follows from Proposition 2.2.6. Recall that $\mathrm{Sym}^p\mathcal{E}$ can alternatively be defined as the quotient of $T^p\mathcal{E}$ by the subsheaf $\mathcal{S}^p$ generated by sections of the form

$$s_1 \otimes \ldots \otimes s_p - s_{\phi(1)} \otimes \ldots \otimes s_{\phi(p)},$$

where $\phi \in \mathfrak{S}_p$ runs over all permutations of the indices. The sheaf $\mathcal{S}^p$ is T-equivariant for the equivariant structure $\Phi \otimes \ldots \otimes \Phi$ on $T^n\mathcal{E}$, as can be seen for example by checking that $\Phi_t \otimes \ldots \otimes \Phi_t$ induces an isomorphism from $t^*\mathcal{S}^p$ to $\mathcal{S}^p$. As $\mathrm{qCoh}^T(X)$ is abelian, we deduce that $\mathrm{Sym}^p\mathcal{E}$ inherits a T-equivariant structure. A similar argument works for $\Lambda^p\mathcal{E}$. ∎

Corollary 2.2.8. *The canonical sheaf*

$$\mathcal{K}_X := (\Lambda^n\Omega^1_X)^{\vee\vee}$$

of X admits a T-equivariant structure induced by the one on Ω^1_X.

Algebraic operations on families of filtrations

The simplest operation on T-sheaves is given by direct sums, and translates as follows in terms of Σ-families.

Definition 2.2.9. The *direct sum* of two σ-families (E^σ, χ^σ) and (F^σ, ψ^σ) is the σ-family $(G^\sigma, \omega^\sigma)$, denoted $E^\sigma \oplus F^\sigma$, defined by

$$\forall m \in M\,,\ G^\sigma_m = E^\sigma_m \oplus F^\sigma_m$$

and for $m \leq_\sigma m'$,

$$\omega^\sigma_{m,m'} := \chi^\sigma_{m,m'} \oplus \psi^\sigma_{m,m'}.$$

Similarly, the direct sum of two Σ-families (E^Σ, η^Σ) and $(F^\Sigma, \theta^\Sigma)$ is the Σ-family $(G^\Sigma, \kappa^\Sigma)$, denoted $E^\Sigma \oplus F^\Sigma$, defined by

$$\forall \sigma \in \Sigma\,,\ G^\sigma = E^\sigma \oplus F^\sigma$$

and for $\tau \leq \sigma$,

$$\kappa_{\tau\sigma} := \eta_{\tau\sigma} \oplus \theta_{\tau\sigma}.$$

Note that in the above definition we used that for $\tau \leq \sigma$, $\iota^*_{\tau\sigma}(E^\sigma \oplus F^\sigma)$ is naturally isomorphic to $\iota^*_{\tau\sigma}(E^\sigma) \oplus \iota^*_{\tau\sigma}(F^\sigma)$. Clearly, direct sums of Σ-families correspond to direct sums of T-equivariant sheaves. In the same way, the definition for direct sums of families of multifiltrations (resp. filtrations) is straightforward, and correspond to direct sums of torsion-free (resp. reflexive) T-sheaves. When two families of filtrations satisfy Klyachko's compatibility condition, so does their direct sum - this corresponds to direct sums of T-vector bundles.

The tensor product operation is more subtle, as it may introduce torsion in general. Tensor products of coherent sheaves (resp. locally free sheaves) are coherent (resp. locally free), but torsion-freeness or reflexivity may be lost through tensorisation. In the following, we will restrict ourselves to families of filtrations, for their simplicity. We refer the reader to Exercise 2.2.22 for the tensor product of a family of multifiltrations by a family of filtrations. To ease notations, we will denote by $(E^\rho(\bullet))$ a family of filtrations $(E^\rho(\bullet))_{\rho\in\Sigma(1)}$, omitting the reference to the fan Σ when the situation is clear enough.

Proposition 2.2.10. *Let $\mathcal{E}$ and $\mathcal{F}$ be two reflexive T-sheaves with families of filtrations $(E^\rho(\bullet))$ and $(F^\rho(\bullet))$ respectively, with $r = \operatorname{rank}(\mathcal{E})$. Then :*

(i) *The family of filtrations $((E \oplus F)^\rho(\bullet))$ for $\mathcal{E} \oplus \mathcal{F}$ is given by*

$$\forall \rho \in \Sigma(1)\,,\ \forall k \in \mathbb{Z},\ (E \oplus F)^\rho(k) = E^\rho(k) \oplus F^\rho(k).$$

(ii) *The family of filtrations $((E \otimes F)^\rho(\bullet))$ for $(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})^{\vee\vee}$ is given by*

$$\forall \rho \in \Sigma(1)\,,\ \forall k \in \mathbb{Z},\ (E \otimes F)^\rho(k) = \sum_{i+j=k} E^\rho(i) \otimes F^\rho(j).$$

(iii) *The family of filtrations* $((E^*)^\rho(\bullet))$ *for* $\mathcal{E}^\vee$ *is given by*

$$\forall \rho \in \Sigma(1)\,,\ \forall k \in \mathbb{Z}\,,\ (E^*)^\rho(k) = (E/E^\rho(-k-1))^*.$$

(iv) *The family of filtrations* $(\det(E)^\rho(\bullet))$ *for* $(\Lambda^r\mathcal{E})^{\vee\vee}$ *is given by*

$$\forall \rho \in \Sigma(1)\,,\ \forall k \in \mathbb{Z}\,,\ \det(E)^\rho(k) = \begin{cases} \{0\} & \text{if} \quad k < k_0 \\ \Lambda^r E & \text{if} \quad k_0 \leq k \end{cases}$$

where

$$k_0 = \sum_{i\in\mathbb{Z}} i\big(\dim(E^\rho(i)) - \dim(E^\rho(i-1))\big).$$

The T-equivariant structures that we consider in Proposition 2.2.10 are the ones described in the previous section.

Proof. The proof for (i) is straightforward from the definition of direct sums, as for any $\sigma \in \Sigma$ and any $m \in M$,

$$\Gamma(U_\sigma, \mathcal{E} \oplus \mathcal{F})_m = \Gamma(U_\sigma, \mathcal{E})_m \oplus \Gamma(U_\sigma, \mathcal{F})_m.$$

To prove (ii), we may restrict ourselves to affine sets $U_\rho \subset X$, where $\rho \in \Sigma(1)$ is a ray. By Lemma 1.3.3, the restrictions of $\mathcal{E}$ and $\mathcal{F}$ to U_ρ are locally free, and thus

$$(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})^{\vee\vee}_{|U_\rho} \simeq \mathcal{E}_{|U\rho} \otimes_{\mathcal{O}_X(U_\rho)} \mathcal{F}_{|U_\rho}.$$

Then, we have

$$\Gamma(U_\rho, \mathcal{E} \otimes \mathcal{F}) = \Gamma(U_\rho, \mathcal{E}) \otimes_{\mathbb{C}[S_\rho]} \Gamma(U_\rho, \mathcal{F}).$$

By definition of the equivariant structure on the tensor product, sections $e \in \Gamma(U_\rho, \mathcal{E})_m$ and $f \in \Gamma(U_\rho, \mathcal{F})_{m'}$ give sections

$$e \otimes f \in \Gamma(U_\rho, \mathcal{E} \otimes \mathcal{F})_{m+m'}.$$

The result then follows by definition of the associated families of filtrations.

For (iii), we restrict again to U_ρ, where $\mathcal{E}$ is locally free. Then, by construction of the dual equivariant structure, the contraction map

$$\begin{array}{ccc} \Gamma(U_\rho, \mathcal{E}) \otimes_{\mathbb{C}[S_\rho]} \Gamma(U_\rho, \mathcal{E}^\vee) & \to & \mathbb{C}[S_\rho] \\ e \otimes f & \mapsto & f(e) \end{array}$$

is a degree zero morphism of $\mathbb{C}[S_\rho]$-modules. By Proposition 1.3.25, $\Gamma(U_\rho, \mathcal{E})$ is a free module with generators $(e_i)_{1\leq i\leq r} \in \Gamma(U_\rho, \mathcal{E})^r$ of weights $(m_i)_{1\leq i\leq r} \in M^r$. Then, if $(e_i^*)_{1\leq i\leq r}$ denotes the dual basis, we have for all (i, j) :

$$e_j^*(e_i) = \delta_{ij}$$

where δ_{ij} stands for the Kronecker delta function. By equivariance of contraction, we deduce that $e_i^* \in \Gamma(U_\rho, \mathcal{E}^\vee)_{-m_i}$. Hence,

$$\Gamma(U_\rho, \mathcal{E}) = \bigoplus_{i=1}^{r} \mathbb{C}[S_\sigma] \cdot e_i \simeq \bigoplus_{i=1}^{r} \mathbb{C}[S_\sigma][m_i].$$

while

$$\Gamma(U_\rho, \mathcal{E}^\vee) = \bigoplus_{i=1}^{r} \mathbb{C}[S_\sigma] \cdot e_i^* \simeq \bigoplus_{i=1}^{r} \mathbb{C}[S_\sigma][-m_i].$$

By construction of the associated families of filtrations, for $j \in \mathbb{Z}$,

$$E^\rho(j) = \bigoplus_{\langle m_i, u_\rho \rangle \leq j} \mathbb{C} \cdot e_i$$

and

$$E^\rho(j) = \bigoplus_{-\langle m_i, u_\rho \rangle \leq j} \mathbb{C} \cdot e_i^*.$$

We deduce that

$$E^\rho(-(j+1)) \simeq \left(E/E^\rho(j)\right)^*$$

and the result follows.

Finally, to prove (iv), we use a similar argument and restrict to U_ρ. We find that

$$\det(E)^\rho(k) = \sum_{i_1,\dots,i_r \mid \sum_j i_j = k} E^\rho(i_1) \wedge \ldots \wedge E^\rho(i_r).$$

As $\Lambda^r E$ is one dimensional, and $k \mapsto \det(E)^\rho(k)$ is a filtration for $\Lambda^r E$, we then have

$$\det(E)^\rho(k) = \begin{cases} 0 & \text{if} \quad k < k_0 \\ \Lambda^r E & \text{if} \quad k \geq k_0 \end{cases}$$

where k_0 is the smallest integer such that there is a partition $i_1, \dots, i_r$ of k with $E^\rho(i_1) \wedge \ldots \wedge E^\rho(i_r) \neq \{0\}$. From the fact that $(E^\rho(i))_{i \in \mathbb{Z}}$ forms a filtration of vector spaces, we deduce that k_0 must be the sum of the integers i such that the dimension of $E^\rho(i)$ changes, counted with multiplicity. Hence

$$k_0 = \sum_{i \in \mathbb{Z}} i\big(\dim(E^\rho(i)) - \dim(E^\rho(i-1))\big),$$

which ends the proof. ∎

We deduce from last proposition two useful corollaries.

Corollary 2.2.11. *Let $\mathcal{E}$ be a reflexive T-sheaf with family of filtrations $(E^{\rho}(\bullet))$. Let $\mathcal{L} = \mathcal{O}_X(D)$ be the rank one reflexive T-sheaf associated to the torus invariant divisor $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$. Then the family of filtrations $((E \otimes L)^{\rho}(\bullet))$ for the twisted sheaf $(\mathcal{E} \otimes \mathcal{O}_X(D))^{\vee\vee}$ is given by*

$$\forall \rho \in \Sigma(1)\,,\ \forall k \in \mathbb{Z}\,,\ (E \otimes L)^{\rho}(k) = E^{\rho}(k + a_\rho)$$

Proof. The family of filtrations for $\mathcal{L}$ is given in Proposition 2.1.4. The result then follows from Proposition 2.2.10(ii). ∎

We describe here the family of filtrations for the canonical sheaf.

Corollary 2.2.12. *The family of filtrations $(K_X^{\rho}(\bullet))$ for the canonical sheaf $\mathcal{K}_X$ is given by*

$$K_X^{\rho}(k) = \begin{cases} \{0\} & \text{if} \quad k < 1 \\ \Lambda^n M_{\mathbb{C}} & \text{if} \quad 1 \le k. \end{cases}$$

In particular, $\mathcal{K}_X \simeq \mathcal{O}_X(-\sum_{\rho\in\Sigma(1)} D_\rho)$ and $-\sum_{\rho\in\Sigma(1)} D_\rho$ is a torus invariant canonical divisor on X.

Proof. The description of the family of filtrations for $\mathcal{K}_X$ follows from the one for Ω_X^1 in Proposition 2.1.7, together with Corollary 2.2.8 and Proposition 2.2.10(iv). The last statement is then a consequence of the description of families of filtrations for rank 1 sheaves, Proposition 2.1.4. ∎

Splitting criterion

In this section, we consider a locally free T-equivariant sheaf $\mathcal{E}$ of rank r over the normal toric variety X. Denote by $(E^{\rho}(\bullet))_{\rho\in\Sigma(1)}$ the associated family of filtrations of the r-dimensional vector space E. Recall that it is subject to Klyachko's compatibility criterion. For each $\sigma \in \Sigma$, we denote by $(m_1^{\sigma}, \ldots, m_r^{\sigma}) \in M^r$ the weights that satisfy, for $\rho \in \sigma(1)$,

$$E^{\rho}(j) = \bigoplus_{\langle m_i^{\sigma}, u_\rho\rangle \le j} E_{m_i^{\sigma}}\,, \tag{2.5}$$

where

$$E = \bigoplus_{i=1}^{r} E_{m_i^{\sigma}}$$

is the splitting as in Theorem 1.3.27. Recall that for a maximal dimensional cone $\sigma \in \Sigma(n)$, those weights are unique up to permutations, cf Remark 1.3.28.

Definition 2.2.13. A T-equivariant reflexive sheaf $\mathcal{E}$ is *splittable* if there exist rank one reflexive T-sheaves $(\mathcal{L}_i)_{1\le i\le r}$ such that

$$\mathcal{E} \simeq \bigoplus_{i=1}^{r} \mathcal{L}_i$$

where the isomorphism is in the category of T-sheaves.

We may as well say that a splittable sheaf *splits*, or *is split*. The following discussion on splittable locally free sheaves goes back to Klyachko's original work [39, Section 6].

Proposition 2.2.14. *Assume that X is smooth, and let $\mathcal{E}$ be a locally free T-sheaf. The following are equivalent :*

(i) *$\mathcal{E}$ is splittable,*

(ii) *There exists a decomposition $E = \bigoplus_{i=1}^r E_i$ such that for each $\sigma \in \Sigma$, up to permutation, we have $E_{m_i^\sigma} = E_i$,*

(iii) *There exists a decomposition $E = \bigoplus_{i=1}^r E_i$ such that for each $\rho \in \Sigma(1)$, and each $j \in \mathbb{Z}$, $E^\rho(j)$ is a direct sum of some of the E_i's.*

We can rephrase the last point in Proposition 2.2.14 by saying that there exists a basis of E such that for each $\rho \in \Sigma(1)$, and each $j \in \mathbb{Z}$, $E^\rho(j)$ is a coordinate subspace of E in this basis. This admits a characterisation in terms of a *distributive lattice*, but we won't use it in this book. We refer the interested reader to Klyachko's original paper [39].

Proof. We first prove $(i) \Rightarrow (ii)$. Assume that $\mathcal{E} \simeq \bigoplus_i \mathcal{L}_i$. Denote by $(L_i^\rho(\bullet))_{\rho\in\Sigma(1)}$ the family of filtrations for each $\mathcal{L}_i$. Set

$$E = \bigoplus_{i=1}^r L_i$$

where L_i is the direct limit of $(L_i^\rho(j))_{j\in\mathbb{Z}}$. By (2.5), for $\sigma \in \Sigma(n)$, $\rho \in \sigma(1)$ and each $j \in \mathbb{Z}$,

$$\bigoplus_{i=1}^r L_i^\rho(j) = \bigoplus_{\langle m_i^\sigma, u_\rho\rangle \leq j} E_{m_i^\sigma}.$$

As each $L_i^\rho(j)$ is either $\{0\}$ or L_i, this implies that the spaces $(E_{m_i^\sigma})_{1\leq i\leq r}$ can be chosen to be equal to the (L_i)'s.

The implication $(ii) \Rightarrow (iii)$ follows directly from (2.5). We now prove that $(iii) \Rightarrow (ii)$. It is enough to consider the case $\sigma \in \Sigma(n)$, the general case following from similar arguments. By Klyachko's compatibility condition, there is a decomposition

$$E = \bigoplus_{i=1}^r E_{m_i^\sigma}$$

for some weights $m_i^\sigma \in M$ and spaces $E_{m_i^\sigma} \subset E$ such that for any $\rho \in \sigma(1)$, and all $j \in \mathbb{Z}$,

$$E^\rho(j) = \bigoplus_{\langle m_i^\sigma, u_\rho\rangle \leq j} E_{m_i^\sigma} = \bigoplus_{i\in I_j^\rho} E_i$$

where $I_j^\rho \subset \{1, \dots, r\}$ and $j \mapsto I_j^\rho$ is increasing. This implies that there is a partition

$$\{1, \dots, r\} = \bigcup_k S_k$$

such that for each k,

$$\bigoplus_{i \in S_k} E_{m_i^\sigma} = \bigoplus_{i \in S_k} E_i$$

and such that all the quotients $E^\rho(j)/E^\rho(j-1)$, $\rho \in \sigma(1)$, $j \in \mathbb{Z}$, are isomorphic to one of the spaces $\bigoplus_{i \in S_k} E_i$, for some k. Then, we may chose the spaces $E^\sigma_{m_i}$ to be equal to the E_i's, up to permutations (note that the spaces $E_{m_i^\sigma}$ may not be uniquely determined if some weights amongst the m_i^σ's are equal, cf Remark 1.3.28).

We conclude by showing $(ii) \Rightarrow (i)$. We then define for each $1 \le i \le r$ a family of filtrations $(L_i^\rho(\bullet))_{\rho \in \Sigma(1)}$ for a one dimensional $\mathbb{C}$-vector space L_i as follows. We set $L_i = E_i$ and for $\rho \in \Sigma(1)$, we pick $\sigma \in \Sigma(n)$ such that $\rho \in \sigma(1)$. We then set for $j \in \mathbb{Z}$:

$$L_i^\rho(j) = \begin{cases} \{0\} & \text{if} \quad j < \langle m_i^\sigma, u_\rho \rangle, \\ L_i & \text{if} \quad j \ge \langle m_i^\sigma, u_\rho \rangle. \end{cases}$$

Each coordinate $\langle m_i^\sigma, u_\rho \rangle$ of each weight m_i^σ in the basis dual to $(u_\rho)_{\rho \in \sigma(1)}$ is determined by being the least integer j such that $E_{m_i^\sigma} = E_i$ appears in the decomposition of $E^\rho(j)$ coming from (2.5). Hence, $L_i^\rho(j)$ is well defined and doesn't depend on the choice of σ containing ρ. By construction and from (2.5) again, we also have

$$\forall \rho \in \Sigma(1), \; \forall j \in \mathbb{Z}, \; \bigoplus_{i=1}^r L_i^\rho(j) = E^\rho(j).$$

Setting $\mathcal{L}_i$ to be the rank one locally free sheaf associated to $(L_i^\rho(\bullet))_{\rho \in \Sigma(1)}$, we obtain the splitting

$$\mathcal{E} = \bigoplus_{i=1}^r \mathcal{L}_i.$$

■

Remark 2.2.15. In the general case, when X is not smooth, the implications $(1) \Rightarrow (2) \Rightarrow (3)$ in Proposition 2.2.14 are still valid, with the same proof.

Proposition 2.2.14 has many interesting applications, once it is coupled with the following technical lemma :

Lemma 2.2.16. *Let $(E_i)_{i \in I}$ be a countable family of vector subspaces of some r-dimensional vector space E. Assume that for all $J \subset I$ with $|J| = r + 1$, there is a basis $\mathcal{B}_J$ of E such that any E_j, for $j \in J$, is a coordinate subspace of E in the basis $\mathcal{B}_J$. Then there exists a basis $\mathcal{B}$ of E such that for any $i \in I$, E_i is a coordinate subspace of E in the basis $\mathcal{B}$.*

We leave the proof in Exercise 2.2.23. We then have the following :

Corollary 2.2.17. *Assume that X is smooth. Then the following conditions are equivalent :*

(i) *All locally free T-sheaves of rank r are splittable on X,*

(ii) *Any set of $(r+1)$ elements in $\Sigma(1)$ spans a cone in Σ.*

In particular, this shows that in the smooth case, if $|\Sigma(1)| \geq n+1$ (i.e. X is not affine), for $n \leq r \leq |\Sigma(1)|-1$, there exists non-splittable locally free T-sheaves of rank r on X.

Proof. Assume (ii), and let $\mathcal{E}$ be a rank r locally free T-sheaf on X. Denote by $(E^\rho(\bullet))_{\rho\in\Sigma(1)}$ its family of filtrations. For any choice $(\rho_i)_{1\leq i\leq r} \in \Sigma(1)^{r+1}$, we can consider the restriction of $\mathcal{E}$ to the affine set U_σ, for

$$\sigma = \rho_1 + \ldots + \rho_{r+1} \in \Sigma.$$

By Proposition 1.3.25, $\mathcal{E}_{|U_\sigma}$ is splittable. By Proposition 2.2.14, it follows that there exists a basis of E such that for any $\rho \in \sigma(1)$ and any $j \in \mathbb{Z}$, $E^\rho(j)$ is a coordinate subspace of E in this basis. We can then apply Lemma 2.2.16 to the spaces $(E^\rho(j))_{\rho\in\Sigma(1),j\in\mathbb{Z}}$, and find a basis of E such that all the spaces $E^\rho(j)$, $\rho \in \Sigma(1)$, $j \in \mathbb{Z}$, are coordinate subspaces in this basis. By Proposition 2.2.14 again, we conclude that $\mathcal{E}$ is splittable.

To prove $(i) \Rightarrow (ii)$, we show the contraposition. Assume that there is a set $\Delta \subset \Sigma(1)$ of $(r+1)$ rays that do not span a cone in Σ. We then define a family of filtrations by setting, for $\rho \notin \Delta$, and $j \in \mathbb{Z}$,

$$E^\rho(j) = \begin{cases} \{0\} & \text{if} \quad j < 0, \\ \mathbb{C}^r & \text{if} \quad j \geq 0, \end{cases}$$

and for $\rho \in \Delta$, and $j \in \mathbb{Z}$,

$$E^\rho(j) = \begin{cases} \{0\} & \text{if} \quad j < -1, \\ L^\rho & \text{if} \quad j = -1, \\ \mathbb{C}^r & \text{if} \quad j \geq 0, \end{cases}$$

where the $(L^\rho)_{\rho\in\Delta}$ are lines in general position in $\mathbb{C}^r$. As the rays of Δ don't span any cone in Σ, for any $\sigma \in \Sigma$, $|\sigma(1) \cap \Delta| \leq r$. Hence, for any $\sigma \in \Sigma$, we may find a splitting

$$\mathbb{C}^r = \Big(\bigoplus_{\rho\in\Delta\cap\sigma(1)} L^\rho\Big) \oplus F_\sigma$$

for some vector space $F_\sigma \subset \mathbb{C}^r$. Using this, together with the fact that σ is smooth, one can easily check that Klyachko's compatibility criterion holds for $(E^\rho(\bullet))_{\rho\in\Sigma(1)}$, which then defines a locally free T-sheaf of rank r on X. However, as E is r-dimensional and the $(r+1)$ lines $L_\rho \subset E$ are in general position, Proposition 2.2.14 (iii) is not satisfied, and $\mathcal{E}$ is not splittable. ∎

Corollary 2.2.18. *Let X be a n-dimensional smooth toric variety.*

(i) *All rank n locally free T-sheaves on X split if and only if X is affine,*

(ii) *All rank $(n-1)$ locally free T-sheaves on X split if and only if X is affine, or the projective space.*

Proof. As X is smooth, X is affine if and only if $|\Sigma(1)| = n$. By Corollary 2.2.17, (i) follows. If X is affine or the projective space, each set of n rays in the fan spans a cone, which implies by Corollary 2.2.17 the splitting of locally free T-sheaves of rank $n-1$. To show the converse, note that the only smooth n-dimensional toric variety with $|\Sigma(1)| = n+1$ is the projective space, up to isomorphism. When $|\Sigma(1)| \geq n+2$, one can show that there exists a set of n rays in Σ that do not span a cone, cf Exercise 2.2.24. The result follows. ∎

The fact that for $r < n$, all rank r locally free T-sheaves on $\mathbb{CP}^n$ are splittable is a result due independently to Klyachko [39] and Kaneyama [38]. We refer to the Notes 2.3.1 for more historical background on the interesting splitting problem for low rank vector bundles on the projective space.

2.2.1 Exercises for 2.2

Exercise 2.2.19 (Symmetric and exterior powers). Let $\mathcal{E}$ be a reflexive T-sheaf with family of filtrations $(E^\rho(\bullet))$. Describe, for $p \in \mathbb{N}^*$, the families of filtrations for $(\mathrm{Sym}^p \mathcal{E})^{\vee\vee}$ and $(\Lambda^p \mathcal{E})^{\vee\vee}$ in terms of $(E^\rho(\bullet))$. Hint : you may follow the proof of Proposition 2.2.10.

Exercise 2.2.20 (Gorenstein varieties). A normal variety is called *Gorenstein* if its canonical divisor is Cartier or, equivalently, if its canonical sheaf is locally free. Prove that a normal toric variety is Gorenstein if and only if for any $\sigma \in \Sigma$, there exists $m_\sigma \in M$ such that for all $\rho \in \sigma(1)$, $\langle m_\sigma, u_\rho \rangle = 1$. Hint : use Corollary 2.2.12 together with Klyachko's compatibility condition.

Exercise 2.2.21 (The Krull–Schmidt property). Let $\mathfrak{C}$ be an abelian category[1]. A *direct sum decomposition* for an object $A \in \mathfrak{C}$, written

$$A = A_1 \oplus \ldots \oplus A_p,$$

is the data of objects $(A_i)_{1\leq i \leq p} \in \mathfrak{C}^p$ together with a monomorphism $\iota_i : A_i \longrightarrow A$ and an epimorphism $\pi_i : A \longrightarrow A_i$ for each $1 \leq i \leq p$ such that :

(i) For all $1 \leq i \leq p$, $\pi_i \circ \iota_i = 1_{A_i}$,

[1]The Krull-Schmidt property only requires having an additive category to be defined in general.

(ii) For all $i \neq j$, $\pi_j \circ \iota_i = 0$,

(iii) $\sum_{i=1}^{p} \iota_i \circ \pi_i = 1_A$.

A direct sum decomposition as above is *trivial* if there exists j such that ι_j (or π_j) is an isomorphism (in that case, for $k \neq j$, $\pi_k = 0$ and $\iota_k = 0$). An object $A \in \mathfrak{C}$ is *indecomposable* if all its direct sum decompositions are trivial.

We say that $\mathfrak{C}$ satisfies the *Krull-Schmidt property* if for any $A \in \mathfrak{C}$, and any two direct sum decompositions for A :

$$A = B_1 \oplus \ldots \oplus B_p = C_1 \oplus \ldots \oplus C_q$$

with $B_i, C_j \in \mathfrak{C}$ indecomposable, we have $p = q$ and there exists a permutation $\phi \in \mathfrak{S}_p$ such that $C_i = B_{\phi(i)}$. By Atiyah's result [3, Theorem 1 and Corollary of Lemma 3], a $\mathbb{C}$-linear abelian category[2] satisfies the Krull–Schmidt property as soon as for any couple $(A, B) \in \mathfrak{C}^2$, the set $\mathrm{Hom}(A, B)$ is a finite dimensional $\mathbb{C}$-vector space. We will use this to show that for a normal toric variety X, $\mathrm{Coh}^T(X)$ and $\mathrm{Ref}^T(X)$ satisfy the Krull-Schmidt property.

(1) Let $\sigma \in \Sigma$. Show that the category of M-graded $\mathbb{C}[S_\sigma]$-modules with their degree 0 morphisms is a $\mathbb{C}$-linear abelian category.

(2) Show that the category of M-graded finitely generated $\mathbb{C}[S_\sigma]$-modules with degree 0 morphisms satisfies the Krull–Schmidt property.

(3) Deduce that $\mathrm{Coh}^T(U_\sigma)$ satisfies the Krull–Schmidt property.

(4) Conclude that $\mathrm{Coh}^T(X)$ satisfies the Krull–Schmidt property.

Exercise 2.2.22 (Tensoring multifiltrations by filtrations). Let

$$\mathcal{L} = \mathcal{O}_X\Big(\sum_{\rho \in \Sigma(1)} a_\rho D_\rho\Big)$$

be a rank one locally free sheaf over a smooth toric variety X, endowed with its natural T-equivariant structure, cf Proposition 2.1.3.

(1) Show that the family of multifiltrations of $\mathcal{L}$ is given by

$$L_m^\sigma = \begin{cases} \mathbb{C} & \text{if} \quad \forall \rho \in \sigma(1)\,,\ \langle m, u_\rho \rangle \geq -a_\rho, \\ \{0\} & \text{otherwise} \end{cases}$$

(2) Show that for each $\sigma \in \Sigma$, there exists a unique $[m_\sigma] \in M/(\sigma^\perp \cap M)$ such that $\langle m_\sigma, u_\rho \rangle = -a_\rho$ for all $\rho \in \sigma(1)$.

[2] An abelian category is $\mathbb{C}$-linear if $\mathrm{Hom}(A, B)$ is a $\mathbb{C}$-vector space for all pair (A, B) of objects, and if the composition is $\mathbb{C}$-bilinear

(3) Let (E_m^σ) be the family of multifiltrations of a torsion-free T-sheaf $\mathcal{E}$. Show that the family of multifiltrations $(E \otimes L)_m^\sigma$ for $\mathcal{E} \otimes \mathcal{L}$ is given by

$$(E \otimes L)_m^\sigma = E_{m-m_\sigma}^\sigma.$$

Exercise 2.2.23 (Proof of Lemma 2.2.16). Let E be a $\mathbb{C}$-vector space of dimension r. Consider $(E_i)_{i\in I}$ a countable family of vector subspaces. Assume that for all $J \subset I$ with $|J| = r+1$, there is a basis $\mathcal{B}_J$ of E such that for all $j \in J$, E_j is a coordinate subspace of E in the basis $\mathcal{B}_J$. Our goal is to show the existence of a basis $\mathcal{B}$ of E such that for any $i \in I$, E_i is a coordinate subspace of E in this basis. We proceed by induction on r.

(1) Prove the result when $r = 1$ and $r = 2$.

(2) Let $j_0 \in J$, with $E_{j_0} \subsetneq E$. Show that there exists a basis of E_{j_0} such that for any $j \in J$, $E_j \cap E_{j_0}$ is a coordinate subspace of E_{j_0} in this basis.

(3) Assume that for all $J \subset I$ with $|J| \leq r$, $\sum_{j\in J} E_j \subsetneq E$.

  (a) Show that there exists $E' \subsetneq E$ such that for all $j \in J$, $E_j \subset E'$.

  (b) Using the induction hypothesis, conclude in that case.

(4) Assume that there exists $J_0 \subset I$, with $|J_0| \leq r$ such that $\sum_{j\in J_0} E_j = E$.

  (a) Show that for any $i \in I$, $E_i = \sum_{j\in J_0} E_i \cap E_j$.

  (b) Conclude by using an induction on $p = |J_0|$.

Exercise 2.2.24 (Proof of Corollary 2.2.18). Let Σ be a smooth fan in $N_\mathbb{R}$ with $\mathrm{rank}(N) = n$. Assume that $\mathrm{Span}_\mathbb{R}\{u_\rho \,|\rho \in \Sigma(1)\} = N_\mathbb{R}$ and $|\Sigma(1)| \geq n+2$. The goal of the exercise is to show that there is a set of n rays in $\Sigma(1)$ that do not span a cone in Σ. We may assume that there is a cone $\sigma \in \Sigma(n)$, and by smoothness, that $(u_\rho)_{\rho\in\sigma(1)}$ is the canonical basis $(e_1, \dots, e_n)$ of $\mathbb{Z}^n \simeq N$.

(1) Denote by $v_1 = (x_1, \dots, x_n)$ and $v_2 = (y_1, \dots, y_n)$ the primitive generators of two rays $\rho_1, \rho_2 \in \Sigma(1) \setminus \sigma(1)$, written in the basis $(e_i)_{1\leq i\leq n}$. Show that we can restrict to the case when $x_n = \pm 1$ and $y_1 = \pm 1$. Hint : use the facts that $(v_1, e_1, \dots, e_{n-1})$ should generate a cone in $\Sigma(n)$, and similarly for $(v_2, e_2, \dots, e_n)$.

(2) Show then that $\pm 1 + x_1 y_n = 0$.

(3) Conclude that either $(v_1, e_2, \dots, e_n)$ or $(v_1, e_1, \dots, e_{n-1})$ cannot span a smooth n-dimensional cone in Σ.

Exercise 2.2.25 (Extension of locally-free T-sheaves). Let $\mathcal{E}$ be a locally free T-sheaf of rank r over an open T-invariant subset $U \subset X$. Set $Z = X \setminus U$, and assume that $\mathrm{codim}(Z) > r+1$.

(1) Using the orbit-cone correspondence, show that if $\sigma \in \Sigma$ satisfies $\dim(\sigma) \leq r+1$, then $U_\sigma \subset U$.

(2) Deduce that $\mathcal{E}$ extends as a locally free T-sheaf over X. Hint : you may use Proposition 2.2.14 and Lemma 2.2.16.

2.3 Pullbacks and pushforwards

In this section, we consider pullbacks and pushforwards of T-sheaves along toric morphisms. Restricting ourselves to the locally free or reflexive cases, in some examples, we describe how to compute the family of filtrations of the reflexive hull of a pulled back (resp. pushed forward) T-sheaf from the initial one. We start by recalling the definition of a toric morphism and by providing some classical examples.

Toric morphisms

Let X and Y be two normal toric varieties, with fans Σ_X and Σ_Y respectively. We will let a subscript X or Y to stand for the 'toric package' associated to each of these toric varieties, e.g. denote by N_X the lattice of X, T_Y the torus of Y, etc.

Definition 2.3.1. A *toric morphism* from X to Y is a morphism of varieties $\phi : X \to Y$ whose restriction to T_X induces a group homomorphism

$$\phi_{|T_X} : T_X \to T_Y.$$

Let $\phi : X \to Y$ be a toric morphism. One can show that ϕ is equivariant with respect to the torus actions, cf Exercise 2.3.18, meaning that the following diagram is commutative :

$$\begin{tikzcd} T_X \times X \arrow[r, "\alpha_X"] \arrow[d, "\phi\times\phi"] & X \arrow[d, "\phi"] \\ T_Y \times Y \arrow[r, "\alpha_Y"] & Y. \end{tikzcd} \tag{2.6}$$

In the above diagram, and in the sequel, we will sometimes denote by ϕ the restriction $\phi_{|T_X}$ to the tori. The restriction of ϕ to the tori induces a morphism of lattices of one parameter subgroups :

$$\underline{\phi} : N_X \to N_Y$$

and dualy a morphism between the character lattices :

$$\underline{\phi}^\vee : M_Y \to M_X.$$

We denote with a subscript $\mathbb{K}$ the $\mathbb{K}$-linear extensions of those morphisms, for $\mathbb{K} \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. Note that by equivariance, for any $\sigma_X \in \Sigma_X$, we have the existence of a cone $\sigma_Y \in \Sigma_Y$ such that

$$\underline{\phi}_{\mathbb{R}}(\sigma_X) \subset \sigma_Y.$$

This property is called *compatibility* of $\underline{\phi}$ with the fans Σ_X and Σ_Y. Note that

$$\phi_{|U_{\sigma_X}} : U_{\sigma_X} \to U_{\sigma_Y}$$

with associated morphism of algebras denoted

$$\phi^* : \mathbb{C}[S_{\sigma_Y}] \to \mathbb{C}[S_{\sigma_X}].$$

Conversely, any morphism of lattices $\underline{\psi} : N_X \to N_Y$ that is compatible with the fans Σ_X and Σ_Y naturally induces a toric morphism $\psi : X \to Y$, cf [16, Theorem 3.3.4]. The first examples of toric morphisms that we have met are the inclusions of affine charts $U_\tau \subset U_\sigma$ when $\tau \leq \sigma$. We will describe now other classical examples.

Example 2.3.2. First, for any toric variety X, and any $p \in \mathbb{N}^*$, one can define the morphism of lattices

$$\begin{array}{rccc} \underline{\phi}_p : & N & \to & N \\ & u & \mapsto & p \cdot u. \end{array}$$

This corresponds to a toric morphism $\phi_p : X \to X$ with restriction on $T \simeq (\mathbb{C}^*)^n$ given by

$$\phi_p(t_1, \dots, t_n) = (t_1^p, \dots, t_n^p).$$

This morphism is then a finite cover of X onto itself. In this example,

$$N' := \underline{\phi}_p(N) = p \cdot N \subset N$$

and N' has finite index in N. We considered ϕ_p has a non-trivial morphism from X to itself. We could have considered instead the toric variety Z associated to (N', Σ), and the inclusion

$$\underline{\psi} : N' \to N.$$

This is compatible with Σ and yields a toric morphism $\psi : Z \to X$. In fact, this construction works for any sublattice $N' \subset N$ of finite index. If we let

$$G := N/N',$$

there is an induced action of G on Z, and ψ is the quotient:

$$\psi : Z \to Z/G \simeq X,$$

cf Exercise 2.3.20.

Example 2.3.3. Let

$$\underline{\phi} : N_X \to N_Y$$

be a surjective morphism of lattices compatible with Σ_X and Σ_Y. We define

$$N_0 := \ker \underline{\phi}$$

so that we have an exact sequence

$$0 \to N_0 \to N_X \to N_Y \to 0.$$

Set

$$\Sigma_0 := \{\sigma \in \Sigma_X \mid \sigma \subset \ker \underline{\phi}_{\mathbb{R}}\}.$$

This is a fan of cones in $(N_0)_{\mathbb{R}}$ and we can then consider the toric variety X_0 associated to (N_0, Σ_0). Following [16, Definition 3.3.18], we say that Σ_X is *split* by Σ_0 and Σ_Y if there exists a subfan $\hat{\Sigma}_Y \subset \Sigma_X$ such that :

(1) The map

$$\begin{array}{ccc} \hat{\Sigma}_Y & \to & \Sigma_Y \\ \sigma & \mapsto & \underline{\phi}_{\mathbb{R}}(\sigma) \end{array}$$

is well defined and bijective, with

$$\forall \sigma \in \hat{\Sigma}_Y\,,\ \underline{\phi}_{\mathbb{R}}(\sigma \cap N_X) = \underline{\phi}_{\mathbb{R}}(\sigma) \cap N_Y.$$

(2) The map

$$\begin{array}{ccc} \Sigma_0 \times \hat{\Sigma}_Y & \to & \Sigma \\ (\sigma_0, \sigma_Y) & \mapsto & \sigma_0 + \sigma_Y \end{array}$$

is well defined and a bijection.

From [16, Theorem 3.3.19], in this situation, the toric morphism $\phi : X \to Y$ is a *locally trivial fiber bundle* with fiber X_0.

Example 2.3.4. We say that the fan Σ_X is a *refinement* of Σ_Y if every cone of Σ_X is included in a cone of Σ_Y and $|\Sigma_X| = |\Sigma_Y|$. In this situation, $N_X = N_Y$, and the identity $\underline{\phi} := 1 : N_X \to N_Y$ induces a toric morphism $\phi : X \to Y$. Refinements include important classes of toric morphisms, such as blow-ups [16, Definition 3.3.17] and resolution of singularities [16, Theorem 11.1.9]. We describe here the special case of the blow-up along a torus fixed point. We then assume that $\sigma \in \Sigma$ is a smooth cone, so that $(u_\rho)_{\rho \in \sigma(1)}$ forms a basis of N. We consider

$$u_\sigma := \sum_{\rho \in \sigma(1)} u_\rho.$$

The *star subdivision* of Σ along σ is defined to be the fan $\Sigma^*(\sigma)$ of cones in $N_{\mathbb{R}}$ given by

$$\Sigma^*(\sigma) = \big(\Sigma \setminus \{\sigma\}\big) \cup \Sigma'(\sigma)$$

where $\Sigma'(\sigma)$ is the set of cones spanned by subsets of $\{u_\rho \mid \rho \in \sigma(1)\} \cup \{u_\sigma\}$ that do not contain $(u_\rho)_{\rho \in \sigma(1)}$. By construction, the identity from N to itself is compatible with $\Sigma^*(\sigma)$ and Σ, and induces a toric morphism

$$\phi : X^* \to X$$

where X^* is the toric variety associated to $(N, \Sigma^*(\sigma))$. One can show that X^* is the blow-up of X along the torus fixed-point $\gamma_\sigma \in U_\sigma$, and ϕ is the blow-down map [16, Proposition 3.3.15]. In Figure 2.3, we present the fan presentation of the blow-up of $\mathbb{C}^2$ at the torus fixed point 0. The fan Σ is the cone $\sigma = \mathbb{R}_+ \cdot e_1 + \mathbb{R}_+ \cdot e_2$ together with its faces.

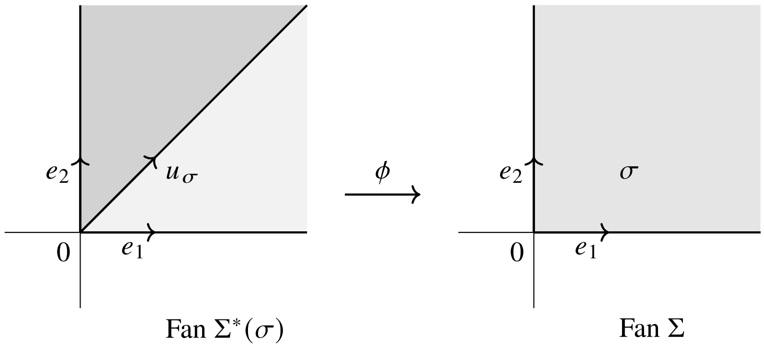

Figure 2.3. The blow-up of 0 in $\mathbb{C}^2$.

Remark 2.3.5. Given a cone $\sigma \in \Sigma$, the orbit closure $V(\sigma) = \overline{O(\sigma)}$ is a torus-invariant subvariety of X. One should be careful though that the inclusion $V(\sigma) \subset X$ is not a toric morphism. Indeed, the lattice of $V(\sigma)$ appears to be a quotient $N(\sigma)$ of the lattice of X, and the fan of $V(\sigma)$ is given by projections of cones $\sigma \leq \tau$ through the quotient map $N_\mathbb{R} \to N(\sigma)_\mathbb{R}$, cf Exercise 1.1.16. One may always find a splitting of

$$1 \to T_\sigma \to T \to T/T_\sigma \to 1$$

where T_σ is the stabiliser of $O(\sigma)$. Given such a splitting $s : T/T_\sigma \to T$, denote by $T(\sigma)$ its image. It is isomorphic to the torus of $V(\sigma)$, and induces an equivariant morphism

$$(\iota, s) : (V(\sigma), T(\sigma)) \to (X, T).$$

However, $\iota : V(\sigma) \to X$ does not restrict to s in general, and this pair is not a toric morphism.

Pullbacks

Let $\phi : X \to Y$ be a toric morphism between two normal toric varieties. Recall from [25, Chapter II.1] that for a sheaf of $\mathcal{O}_Y$-modules $\mathcal{E}$ on Y, the inverse image sheaf $\phi^{-1}\mathcal{E}$ is defined to be the sheafification of the presheaf

$$U \mapsto \varinjlim \Gamma(\tilde{U}, \mathcal{E})$$

where the limit is taken over open charts $\tilde{U}$ containing $\phi(U)$. Then, the pulled back sheaf $\phi^*\mathcal{E}$ is by definition

$$\phi^*\mathcal{E} := \phi^{-1}\mathcal{E} \otimes_{\phi^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Recall the following from [25, Proposition II.5.8]:

Proposition 2.3.6. *We have the following properties for pullbacks :*

(i) *If $\mathcal{E} \in \mathrm{qCoh}(Y)$, then $\phi^*\mathcal{E} \in \mathrm{qCoh}(X)$,*

(ii) *If $\mathcal{E} \in \mathrm{Coh}(Y)$, then $\phi^*\mathcal{E} \in \mathrm{Coh}(X)$,*

(iii) *If $\mathcal{E}$ is locally free on Y, then $\phi^* \in \mathcal{E}$ is locally free on X.*

Remark 2.3.7. Be careful though that the pullback of a torsion-free (resp. reflexive) sheaf $\mathcal{E}$ is not necessarily torsion-free (resp. reflexive). To remedy this, one considers the *torsion-free pullback* (resp. *reflexive pullback*), that is the quotient of $\phi^*\mathcal{E}$ by its torsion (resp. the reflexive hull of $\phi^*\mathcal{E}$).

Pullbacks preserve torus equivariance :

Proposition 2.3.8. *Let $\phi : X \to Y$ be a toric morphism. Then :*

(i) *If $\mathcal{E} \in \mathrm{qCoh}^{T_Y}(Y)$, then $\phi^*\mathcal{E} \in \mathrm{qCoh}^{T_X}(X)$,*

(ii) *If $\mathcal{E} \in \mathrm{Coh}^{T_Y}(Y)$, then $\phi^*\mathcal{E} \in \mathrm{Coh}^{T_X}(X)$,*

(ii) *If $\mathcal{E}$ is a locally free T_Y-sheaf, then $\phi^*\mathcal{E}$ is a locally free T_X-sheaf.*

Proof. By Proposition 2.3.6, it is enough to show that pullback along ϕ sends T_Y-equivariant sheaves of $\mathcal{O}_Y$-modules to T_X-equivariant sheaves of $\mathcal{O}_X$-modules. Let Φ be a T_Y-equivariant structure on a sheaf of $\mathcal{O}_Y$-modules $\mathcal{E}$ on Y. Consider the pullback of Φ along

$$\phi \times \phi : T_X \times X \to T_Y \times Y.$$

By the functorial properties of pullbacks, we deduce that

$$(\phi \times \phi)^*\Phi : (\phi \times \phi)^*\alpha_Y^*\mathcal{E} \to (\phi \times \phi)^*\pi_{2,Y}^*\mathcal{E}$$

is an isomorphism. By equivariance and commutativity of the square (2.6), we have

$$(\phi \times \phi)^*\alpha_Y^*\mathcal{E} \simeq \alpha_X^*\phi^*\mathcal{E}$$

and similarly

$$(\phi \times \phi)^*\pi_{2,Y}^*\mathcal{E} \simeq \pi_{2,X}^*\phi^*\mathcal{E}.$$

Hence $(\phi\times\phi)^*\Phi$ induces an isomorphism between $\alpha_X^*\phi^*\mathcal{E}$ and $\pi_{2,X}^*\phi^*\mathcal{E}$. The cocycle condition (1.5) for $(\phi\times\phi)^*\Phi$ follows from commutativity of the squares :

$$\begin{array}{ccc}
T_X\times T_X\times X & \xrightarrow{\beta_X} & T_X\times X \\
\downarrow{\scriptstyle \phi\times\phi\times\phi} & & \downarrow{\scriptstyle \phi\times\phi} \\
T_Y\times T_Y\times Y & \xrightarrow{\beta_Y} & T_Y\times Y,
\end{array}$$

where (β_X,β_Y) stands for one of the couples $(\mu_X\times 1_X,\mu_Y\times 1_Y)$, $(\pi_{23,X},\pi_{23,Y})$ or $(1_{T_X}\times\alpha_X, 1_{T_Y}\times\alpha_Y)$. Indeed, by commutativity of those squares, we have :

$$\begin{array}{rcl}
(\mu_X\times 1_X)^*(\phi\times\phi)^*\Phi & = & (\phi\times\phi\times\phi)^*(\mu_Y\times 1_Y)^*\Phi \\
& = & (\phi\times\phi\times\phi)^*\left(\pi_{23,Y}^*\Phi\circ(1_{T_Y}\times\alpha_Y)^*\Phi\right) \\
& \simeq & (\phi\times\phi\times\phi)^*\left(\pi_{23,Y}^*\Phi\right)\circ(\phi\times\phi\times\phi)^*\left((1_{T_Y}\times\alpha_Y)^*\Phi\right) \\
& = & \pi_{23,X}^*(\phi\times\phi)^*\Phi\circ(1_X\times\alpha_X)^*(\phi\times\phi)^*\Phi,
\end{array}$$

where in the second equality we used the cocycle condition for Φ. This concludes the proof. ∎

In Exercise 2.3.19, you will find an alternative proof that the pullback of a T-vector bundle along a toric morphism still admits a T-structure. As it seems slightly tedious to describe in full generality the Σ-families of pulled back T-sheaves, we will consider the simpler locally free case. We will need the following lemma, whose proof is left in Exercise 2.3.18.

Lemma 2.3.9. *Let $\phi : X\to Y$ be a toric morphism. Then for any $\sigma'\in\Sigma_X$, there exists a unique minimal cone $\sigma\in\Sigma_Y$ containing $\underline{\phi}_{\mathbb{R}}(\sigma')$. Moreover, $\phi(\gamma_{\sigma'})=\gamma_\sigma$ and $\phi(O(\sigma'))\subset O(\sigma)$.*

Recall that the distinguished point $\gamma_\sigma\in U_\sigma$ is defined in Exercise 1.1.15, and lives in the torus orbit $O(\sigma)$. In the following proposition, the equivariant structure on the pulled back sheaf $\phi^*\mathcal{E}$ is the one naturally induced by pulling back the equivariant structure on $\mathcal{E}$, as in the proof of Proposition 2.3.8.

Proposition 2.3.10. *Let $\phi : X\to Y$ be a toric morphism. Let $\mathcal{E}$ be a locally free T_Y-sheaf on Y with family of filtrations $(E^\rho(\bullet))_{\rho\in\Sigma_Y(1)}$. Denote by $((\phi^*E)^\rho(\bullet))_{\rho\in\Sigma_X(1)}$ the family of filtrations of $\phi^*\mathcal{E}$. Let $\rho'\in\Sigma_X(1)$ and let $\sigma\in\Sigma_Y$ be the minimal cone such that $\underline{\phi}_{\mathbb{R}}(\rho')\subset\sigma$. Then we have :*

(i) *If $\sigma=\{0\}$, then*

$$\forall j\in\mathbb{Z},\ (\phi^*E)^{\rho'}(j)=\begin{cases}\{0\} & \text{if } j<0\\ E & \text{if } j\geq 0,\end{cases}$$

(ii) *If* $\sigma = \rho \in \Sigma_Y(1)$ *with* $\underline{\phi}(u_{\rho'}) = a_\rho\, u_\rho$ *for some* $a_\rho \in \mathbb{N}^*$, *then*

$$\forall j \in \mathbb{Z},\ (\phi^* E)^{\rho'}(j) = E^\rho\Big(\Big\lfloor \frac{j}{a_\rho} \Big\rfloor\Big),$$

(iii) *If* $\sigma = \sum_{i=1}^r \rho_i$ *with* $(\rho_i)_{1\le i\le r} \in (\Sigma(1))^r$ *and* $\underline{\phi}(u_{\rho'}) = \sum_{i=1}^r a_i\, u_{\rho_i}$ *for some* $(a_i)_{1\le i\le r} \in (\mathbb{N}^*)^r$, *then*

$$\forall j \in \mathbb{Z},\ (\phi^* E)^{\rho'}(j) = \sum_{j_1+\ldots+j_r=j} \bigcap_{i=1}^r E^{\rho_i}\Big(\Big\lfloor \frac{j_i}{a_i} \Big\rfloor\Big).$$

Of course, point (ii) in Proposition 2.3.10 is a special case of point (iii). We stated it separately as it arises in many interesting applications.

Proof. Assume that $\sigma = 0$. Then by Lemma 2.3.9, $\phi(O(\rho')) \subset T_Y$. Using the orbit-cone correspondence, this implies that $\phi(U_{\rho'}) \subset T_Y$. Hence, by definition,

$$\Gamma(U_{\rho'}, \phi^*\mathcal{E}) = \Gamma(T_Y, \mathcal{E}) \otimes_{\phi^*} \mathbb{C}[S_{\rho'}].$$

As $\Gamma(T_Y, \mathcal{E}) \simeq E \otimes \mathbb{C}[M_Y]$, this yields

$$\Gamma(U_{\rho'}, \phi^*\mathcal{E}) \simeq E \otimes \mathbb{C}[S_{\rho'}].$$

By definition,

$$(\phi^* E)^{\rho'}(\langle m', u_{\rho'}\rangle) = \Gamma(U_{\rho'}, \phi^*\mathcal{E})_{m'}$$

for any $m' \in M_X$. The proof of (i) follows.

Assume now that $\sigma = \rho$ is a ray in Σ_Y. As $\underline{\phi}(u_{\rho'}) \in N_Y$, there exists $a_\rho \in \mathbb{N}^*$ such that $\underline{\phi}(u_{\rho'}) = a_\rho\, u_\rho$. This time, using Lemma 2.3.9, we have $\phi(U_{\rho'}) \subset U_\rho$ but $\phi(U_{\rho'}) \subsetneq T_Y$ as $\phi(\gamma_{\rho'}) = \gamma_\rho$ and $\gamma_\rho \notin T_Y$. Hence

$$\Gamma(U_{\rho'}, \phi^*\mathcal{E}) = \Gamma(U_\rho, \mathcal{E}) \otimes_{\phi^*} \mathbb{C}[S_{\rho'}].$$

By definition, for $m' \in M_X$, we have

$$(\phi^* E)^{\rho'}(\langle m', u_{\rho'}\rangle) = \Gamma(U_{\rho'}, \phi^*\mathcal{E})_{m'}.$$

Given a section $e_m \in \Gamma(U_\rho, \mathcal{E})_m$ and a section $s_p \in \mathbb{C}[S_{\rho'}]_p$, the torus T_X acts on $s_p \otimes e_m$ with weight $\underline{\phi}^\vee(m) + p$. As for any $m \in M_Y$ and any $p \in S_{\rho'}$,

$$\begin{aligned}\langle \underline{\phi}^\vee(m) + p, u_{\rho'}\rangle &= \langle m, \underline{\phi}(u_{\rho'})\rangle + \langle p, u_\rho\rangle\\ &= a_\rho\langle m, u_\rho\rangle + \langle p, u_\rho\rangle,\end{aligned}$$

we deduce that

$$(\phi^* E)^{\rho'}(j) = E^\rho\Big(\Big\lfloor \frac{j}{a_\rho} \Big\rfloor\Big).$$

The last case follows with a similar argument. By Lemma 2.3.9 and the orbit-cone correspondence, we have that U_σ is the minimal T_Y-invariant affine chart containing $\phi(U_{\rho'})$, and thus for any $m' \in M_X$:

$$(\phi^* E)^{\rho'}(\langle m', u_{\rho'}\rangle) = \Gamma(U_{\rho'}, \phi^*\mathcal{E})_{m'} = \big(\Gamma(U_\sigma, \mathcal{E}) \otimes_{\phi^*} \mathbb{C}[S_{\rho'}]\big)_{m'}.$$

Again, the T_X-action has weight $\underline{\phi}^\vee(m)$ on sections of $\Gamma(U_\sigma, \mathcal{E})_m$. We then compute

$$\begin{aligned} \langle \underline{\phi}^\vee(m), u_{\rho'}\rangle =& \langle m, \underline{\phi}(u_{\rho'})\rangle \\ =& \sum_{i=1}^r a_i \, \langle m, u_{\rho_i}\rangle. \end{aligned}$$

As $((\phi^* E)^\rho(\bullet))$ is reflexive, we have for any $m \in M_Y$:

$$\Gamma(U_\sigma, \mathcal{E})_m = \bigcap_{i=1}^r E^{\rho_i}(\langle m, u_{\rho_i}\rangle),$$

and the result follows. ∎

Proposition 2.3.10 enables to compute the families of filtrations for pullbacks of locally free T_Y-sheaves for all the toric morphisms $\phi : X \longrightarrow Y$ from Examples 2.3.2, 2.3.3 and 2.3.4. Indeed, Proposition 2.3.10.(ii) covers all the cases of Example 2.3.2. For the locally trivial fibrations as in Examples 2.3.3, any ray $\rho' \in \Sigma_X$ is either sent to $\{0\}$ or to a ray of Σ_Y. Finally, the pullback along a blow-up, Example 2.3.4, can be obtained with Proposition 2.3.10.(iii).

Example 2.3.11. Let $Y = \mathbb{C}^2$ and consider the blow-up along the origin :

$$\phi : X \longrightarrow Y$$

with $X = \mathrm{Bl}_0(\mathbb{C}^2)$. The pullback $\phi^* TY$ of the tangent sheaf is then a T_X-equivariant locally free sheaf, with family of filtrations $(\phi^* TY)^\rho(\bullet)$. The fan of X contains three rays, namely $\rho_1 = \mathbb{R}_+ \cdot (1,0)$ and $\rho_2 = \mathbb{R}_+ \cdot (0,1)$ coming from the cone of $\mathbb{C}^2$, and the additional $\rho = \mathbb{R}_+ \cdot (1,1)$ obtained in the blow-up process, cf Example 2.3.4. From Propositions 2.1.8 and 2.3.10, we obtain

$$(\phi^* TY)^{\rho_i}(j) = \left\{ \begin{array}{ccc} \{0\} & \text{if} & j \leq -2 \\ \mathbb{C} \cdot u_{\rho_i} & \text{if} & j = -1 \\ N_\mathbb{C} & \text{if} & j \geq 0. \end{array} \right.$$

for $1 \leq i \leq 2$, and for ρ we have :

$$(\phi^* TY)^{\rho}(j) = \left\{ \begin{array}{ccc} \{0\} & \text{if} & j \leq -2 \\ N_\mathbb{C} & \text{if} & j \geq -1, \end{array} \right.$$

which follows from

$$\big(TY^{\rho_1}(-1) \cap TY^{\rho_2}(0)\big) + \big(TY^{\rho_1}(0) \cap TY^{\rho_2}(-1)\big) = \mathbb{C} \cdot u_{\rho_1} + \mathbb{C} \cdot u_{\rho_2} = N_\mathbb{C}.$$

Pushforwards

We now consider pushforwards. If $\mathcal{E}$ is a sheaf of $\mathcal{O}_X$-modules on X, its pushforward along ϕ is the sheaf of $\mathcal{O}_Y$-modules $\phi_*\mathcal{E}$ defined by

$$U \mapsto \Gamma(U, \phi_*\mathcal{E}) = \Gamma(\phi^{-1}(U), \mathcal{E}).$$

Note that its $\mathcal{O}_Y$-module structure is naturally induced by

$$\phi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\phi^{-1}(U)).$$

From [25, Proposition II.5.8], we have this time :

Proposition 2.3.12. *We have the following properties for pushforwards :*

(i) *If* $\mathcal{E} \in \mathrm{qCoh}(X)$*, then* $\phi_*\mathcal{E} \in \mathrm{qCoh}(Y)$*,*

(ii) *If* ϕ *is proper and* $\mathcal{E} \in \mathrm{Coh}(X)$*, then* $\phi_*\mathcal{E} \in \mathrm{Coh}(Y)$*.*

Recall that in our setting, $\phi : X \to Y$ is a *proper morphism* if it is *universally closed*, meaning that for all varieties Z and morphisms $\psi : Z \to Y$, the projection π_Z defined by

$$\begin{tikzcd} X \times_Y Z \arrow[r, "\pi_X"] \arrow[d, "\pi_Z"] & X \arrow[d, "\phi"] \\ Z \arrow[r, "\psi"] & Y, \end{tikzcd}$$

is a closed map for the Zariski topology[3]. From [16, Theorem 3.4.11] we extract the following:

Theorem 2.3.13. *Let* $\phi : X \to Y$ *be a toric morphism. Then the following are equivalent :*

(i) ϕ *is a proper morphism,*

(ii) ϕ *is proper for the analytic topology,*

(iii) $\underline{\phi}_{\mathbb{R}}^{-1}(|\Sigma_Y|) = |\Sigma_X|$.

In contrast with pullbacks, pushforwards of torus equivariant sheaves may not admit natural torus equivariant structures. To explain this, let's consider the affine situation. Let $\phi : U_{\sigma'} \to U_\sigma$ be a toric morphism between affine toric varieties with cones $\sigma' \subset N'_{\mathbb{R}}$ and $\sigma \subset N_{\mathbb{R}}$ respectively. A sheaf $\mathcal{E} \in \mathrm{qCoh}^{T'}(U_{\sigma'})$ corresponds to an M'-graded $\mathbb{C}[S_{\sigma'}]$-algebra :

$$\Gamma(U_{\sigma'}, \mathcal{E}) = \bigoplus_{m' \in M'} \Gamma(U_{\sigma'}, \mathcal{E})_{m'},$$

[3] We only consider morphisms between reduced irreducible varieties over $\mathbb{C}$, hence do not require 'separated' and 'finite type' in our definition of proper morphisms, those properties being automatically satisfied.

while the quasicoherent sheaf $\phi_*\mathcal{E}$ is defined by

$$\Gamma(U_\sigma, \phi_*\mathcal{E}) = \Gamma(U_{\sigma'}, \mathcal{E})$$

with $\mathbb{C}[S_\sigma]$-module structure induced by $\phi^* : \mathbb{C}[S_\sigma] \to \mathbb{C}[S_{\sigma'}]$. We wish to produce an M-grading on $\Gamma(U_\sigma, \phi_*\mathcal{E})$ out of its M'-grading. The map $\underline{\phi}^\vee : M \to M'$ naturally suggests to define for each $m \in M$:

$$\Gamma(U_\sigma, \phi_*\mathcal{E})_m := \Gamma(U_{\sigma'}, \mathcal{E})_{\underline{\phi}^\vee(m)}.$$

Unfortunately, this would imply that

$$\Gamma(U_\sigma, \phi_*\mathcal{E}) = \bigoplus_{m\in M} \Gamma(U_\sigma, \phi_*\mathcal{E})_m = \bigoplus_{m\in M} \Gamma(U_{\sigma'}, \mathcal{E})_{\underline{\phi}^\vee(m)}$$

which may be different from $\Gamma(U_{\sigma'}, \mathcal{E})$, unless $\underline{\phi}^\vee$ is an isomorphism. We will then require further properties for a toric morphism $\underline{\phi} : X \to Y$ in order to define a pushforward between $\mathrm{qCoh}^{T_X}(X)$ and $\mathrm{qCoh}^{T_Y}(Y)$. The most natural one, motivated by the affine situation, is that $\underline{\phi}$ is an isomorphism, and hence $T_X \simeq T_Y$.

Proposition 2.3.14. *Let $\phi : X \to Y$ be a toric morphism such that $\underline{\phi} : N_X \to N_Y$ is an isomorphism, and let $T = T_X \simeq T_Y$. Then :*

(i) *If $\mathcal{E} \in \mathrm{qCoh}^T(X)$ then $\phi_*\mathcal{E} \in \mathrm{qCoh}^T(Y)$,*

(ii) *If in addition ϕ is proper and $\mathcal{E} \in \mathrm{Coh}^T(X)$, then $\phi_*\mathcal{E} \in \mathrm{Coh}^T(Y)$.*

Proof. Without loss, we will assume that $T_Y = T$. By Proposition 2.3.12, it is enough to show that the pushforward of a T-equivariant quasicoherent sheaf along ϕ admits a T-equivariant structure. Let $(\mathcal{E}, \Phi) \in \mathrm{qCoh}^T(X)$. We consider the pushforward :

$$(\phi \times \phi)_*\Phi : (\phi \times \phi)_*\alpha_X^*\mathcal{E} \to (\phi \times \phi)_*\pi_{2,X}^*\mathcal{E}.$$

The functorial properties of pushforwards imply that $(\phi \times \phi)_*$ is an isomorphism. Note that $\pi_{2,Y} : T \times Y \to Y$ is flat. This implies flatness of the action $\alpha_Y : T \times Y \to Y$, as α_Y is the composition of $\pi_{2,Y}$ with the isomorphism

$$\begin{array}{ccc} T \times Y & \to & T \times Y \\ (t, y) & \mapsto & (t, t \cdot y). \end{array}$$

Then, we may apply flat base change to the Cartesian squares

$$\begin{tikzcd} T \times X \arrow[r, "\alpha_X"] \arrow[d, "\phi\times\phi"] & X \arrow[d, "\phi"] \\ T \times Y \arrow[r, "\alpha_Y"] & Y, \end{tikzcd}$$

and

$$\begin{array}{ccc} T \times X & \xrightarrow{\pi_{2,X}} & X \\ \Big\downarrow{\scriptstyle \phi\times\phi} & & \Big\downarrow{\scriptstyle \phi} \\ T \times Y & \xrightarrow{\pi_{2,Y}} & Y, \end{array}$$

to obtain an isomorphism (that we still denote by $(\phi \times \phi)_*\Phi$) :

$$(\phi \times \phi)_*\Phi : \alpha_Y^* \phi_* \mathcal{E} \to \pi_{2,Y}^* \phi_* \mathcal{E}$$

(see e.g. [25, Proposition III.9.3]). As the product $\mu : T \times T \to T$ is flat, we can argue as in the proof of Proposition 2.3.8, using flat base change again, to show that $(\phi \times \phi)_*\Phi$ satisfies the cocycle condition and defines an equivariant structure on $\phi_*\mathcal{E}$. ∎

While the hypothesis of Proposition 2.3.14 may look very restrictive, they still cover the wide range of toric morphisms given by refinements, as in Example 2.3.4. As an application, we have the following.

Proposition 2.3.15. *Let Σ_X be a refinement of a fan Σ_Y and consider $\phi : X \to Y$ the associated morphism of toric varieties. Let $\mathcal{E}$ be a reflexive T-sheaf on X with family of filtrations $(E^\rho(\bullet))_{\rho\in\Sigma_X(1)}$. Then the associated family of filtrations for $(\phi_*\mathcal{E})^{\vee\vee}$, denoted $((\phi_*E)^\rho(\bullet))_{\rho\in\Sigma_Y(1)}$, is given, for $\rho \in \Sigma_Y(1)$, by*

$$\forall j \in \mathbb{Z},\ (\phi_*E)^\rho(j) = E^\rho(j).$$

In particular, $(\phi_*\Omega^1_X)^{\vee\vee} = \Omega^1_Y$, $(\phi_*\mathcal{T}_X)^{\vee\vee} = \mathcal{T}_Y$ and $(\phi_*\mathcal{K}_X)^{\vee\vee} = \mathcal{K}_Y$.

Proof. As Σ_X refines Σ_Y, any ray $\rho \in \Sigma_Y(1)$ belongs to $\Sigma_X(1)$. Moreover, by definition of $\underline{\phi}$, we have that $\phi_{|U_\rho}$ is the identity from $U_\rho \subset X$ to $U_\rho \subset Y$. On $U_\rho \subset Y$, $(\phi_*\mathcal{E})^{\vee\vee}$ is locally free and hence is equal to $\phi_*\mathcal{E}$, cf Lemma 1.3.3. Then, the result follows directly by definition of pushforwards. The final statement comes from the fact that Ω^1_Y, $\mathcal{T}_Y$ and $\mathcal{K}_Y$ are characterised by their families of filtrations, cf Propositions 2.1.7 and 2.1.8, and Corollary 2.2.12. ∎

We may extend Proposition 2.3.14 in two different ways, both requiring $\underline{\phi} : N_X \to N_Y$ to be surjective. First, assume that we have an exact sequence of lattices

$$0 \to N_0 \to N_X \xrightarrow{\underline{\phi}} N_Y \to 0.$$

Denote by $T_0 \subset T_X$ the subtorus $T_0 = N_0 \otimes_{\mathbb{Z}} \mathbb{C}^*$. Consider Σ_X and Σ_Y two fans in N_X and N_Y respectively, compatible with $\underline{\phi}$, and let $\phi : X \to Y$ be the associated morphism. For any $\mathcal{E} \in \mathrm{qCoh}^{T_X}(X)$, we define the T_0-*invariant pushforward* of $\mathcal{E}$ along ϕ to be the sheaf $\phi_*^{T_0}\mathcal{E}$ on Y given on affines $U_\sigma \subset Y$, $\sigma \in \Sigma_Y$, by

$$\Gamma(U_\sigma, \phi_*^{T_0}\mathcal{E}) = \Gamma(\phi^{-1}(U_\sigma), \mathcal{E})^{T_0},$$

where the supscript T_0 stands for T_0-invariant elements. This is a well defined quasicoherent sheaf of $\mathcal{O}_Y$-modules. Indeed, as ϕ is equivariant, $\phi^{-1}(U_\sigma)$ is an open T_X-invariant subset of X. The T_X-equivariant structure then provides a regular representation of T_X on $\Gamma(\phi^{-1}(U_\sigma), \mathcal{E})$ in the usual way, hence an M_X-grading :

$$\Gamma(\phi^{-1}(U_\sigma), \mathcal{E}) = \bigoplus_{m \in M_X} \Gamma(\phi^{-1}(U_\sigma), \mathcal{E})_m.$$

Note that

$$M_Y = \mathrm{Hom}_{\mathbb{Z}}(N_Y, \mathbb{Z}) \simeq N_0^\perp \subset M_X$$

and that

$$\Gamma(\phi^{-1}(U_\sigma), \mathcal{E})^{T_0} = \bigoplus_{m \in N_0^\perp} \Gamma(\phi^{-1}(U_\sigma), \mathcal{E})_m.$$

For each $\sigma' \in \Sigma_X$ such that $U_{\sigma'} \subset \phi^{-1}(U_\sigma)$, the morphism $\phi^* : \mathbb{C}[S_\sigma] \to \mathbb{C}[S_{\sigma'}]$ factorizes through $\mathbb{C}[S_{\sigma'} \cap N_0^\perp]$. We deduce that $\Gamma(\phi^{-1}(U_\sigma), \mathcal{E})^{T_0}$ carries a $\mathbb{C}[S_\sigma]$-module structure. Moreover, it is $M_Y \simeq N_0^\perp$-graded, and thus $\phi_*^{T_0}(\mathcal{E})_{|U_\sigma}$ admits a T_Y-equivariant structure. We leave in Exercise 2.3.22 the proof of the following :

Proposition 2.3.16. *Let $\phi : X \to Y$ be a toric morphism such that $\underline{\phi} : N_X \to N_Y$ is surjective. Denote by $T_0 \subset T_X$ the subtorus generated by* $\ker \underline{\phi}$*. Then :*

(i) *If $\mathcal{E} \in \mathrm{qCoh}^{T_X}(X)$, then $\phi_*^{T_0}\mathcal{E} \in \mathrm{qCoh}^{T_Y}(Y)$,*

(ii) *If in addition ϕ is proper and $\mathcal{E} \in \mathrm{Coh}^{T_X}(X)$, then $\phi_*^{T_0}\mathcal{E} \in \mathrm{Coh}^{T_Y}(Y)$.*

In fact, surjectivity of $\underline{\phi}$ is enough to ensure that pushforwards maps $\mathrm{qCoh}^{T_X}(X)$ to $\mathrm{qCoh}^{T_Y}(Y)$, although in a non canonical way.

Proposition 2.3.17. *Let $\phi : X \to Y$ be a toric morphism such that $\underline{\phi} : N_X \to N_Y$ is surjective. Then :*

(i) *If $\mathcal{E} \in \mathrm{qCoh}^{T_X}(X)$, then $\phi_*\mathcal{E} \in \mathrm{qCoh}^{T_Y}(Y)$,*

(ii) *If in addition ϕ is proper and $\mathcal{E} \in \mathrm{Coh}^{T_X}(X)$, then $\phi_*\mathcal{E} \in \mathrm{Coh}^{T_Y}(Y)$.*

Proof. Using Proposition 2.3.12, it is enough to prove point (i). Let $(\mathcal{E}, \Phi) \in \mathrm{qCoh}^{T_X}(X)$. Using surjectivity of $\underline{\phi} : N_X \to N_Y$, we may find a splitting $\psi : N_Y \to N_X$ such that $\underline{\phi} \circ \psi = 1_{N_Y}$. This induces an injection $T_Y \to T_X$, and we will then consider T_Y as a subtorus of T_X. Pulling back Φ along the injection

$$T_Y \times X \to T_X \times X,$$

we obtain a T_Y-equivariant structure Ψ on $\mathcal{E}$, that is an isomorphism

$$\Psi : (\alpha_X)^*_{|T_Y \times X}\mathcal{E} \to (\pi_{2,X})^*_{|T_Y \times X}\mathcal{E}$$

satisfying the similar cocycle condition (1.5), restricting all maps to $T_Y \subset T_X$. Then, we can pushforward Ψ along $1_{T_Y} \times \phi : T_Y \times X \to T_Y \times Y$ to obtain an equivariant structure on $\phi_* \mathcal{E}$, the proof following the one of Proposition 2.3.14. ∎

Note that the equivariant structure obtained on $\phi_* \mathcal{E}$ in Proposition 2.3.17 depends on the choice of the splitting ψ, as seen in Exercise 2.3.23.

2.3.1 Exercises for 2.3

Exercise 2.3.18 (Basic properties of toric morphisms). Let $\phi : X \to Y$ be a toric morphism.

(1) Show that ϕ is equivariant.

(2) Let $\sigma \in \Sigma_X$. Show that there is a unique minimal cone $\sigma' \in \Sigma_Y$ such that $\underline{\phi}_{\mathbb{R}}(\sigma) \subset \sigma'$.

(3) Prove that $\phi(\gamma_\sigma) = \gamma_{\sigma'}$, where γ_σ is the distinguished point defined in Exercise 1.1.15.

(4) Prove that $\phi(O(\sigma)) \subset O(\sigma')$ and $\phi(V(\sigma)) \subset V(\sigma')$.

Hint for (1) : reduce to the affine case and use density of the torus.

Exercise 2.3.19 (Pullbacks of T-vector bundles). Let $\underline{\mathcal{E}} \to Y$ be a vector bundle of rank r over a normal toric variety Y. Denote by $(g_{\sigma\sigma'} : U_{\sigma\cap\sigma'} \to \mathrm{Gl}_r(\mathbb{C}))_{\sigma,\sigma' \in \Sigma_Y^2}$ the transition maps of $\underline{\mathcal{E}}$.

(1) Show that a T_Y-equivariant structure on $\underline{\mathcal{E}}$ is equivalent to the data of representations $(\varrho_\sigma : T_Y \to \mathrm{Gl}_r(\mathbb{C}))_{\sigma \in \Sigma_Y}$ such that for all $x \in U_{\sigma\cap\sigma'}$ and all $t \in T_Y$:

$$g_{\sigma\sigma'}(\alpha_Y(t,x)) \cdot \varrho_{\sigma'}(t) = \varrho_\sigma(t) \cdot g_{\sigma\sigma'}(x).$$

(2) Let $\phi : X \to Y$ be a toric morphism. For any $\sigma' \in \Sigma_X$, let $\sigma \in \Sigma_Y$ be minimal such that $\underline{\phi}_{\mathbb{R}}(\sigma') \subset \sigma$, and define $\phi^* \varrho_{\sigma'} := \varrho_\sigma \circ \phi$. Show that the data $(\phi^* \varrho_\sigma : T_X \to \mathrm{Gl}_r(\mathbb{C}))_{\sigma \in \Sigma_X}$ gives a T_X-equivariant structure on $\phi^* \underline{\mathcal{E}}$.

Exercise 2.3.20 (Finite quotients). Consider $N_X \subset N_Y$ a sublattice of finite index and set $G = N_Y / N_X$. Let Σ be a fan in N_Y, and consider X and Y the toric varieties associated to (N_X, Σ) and (N_Y, Σ), respectively. The inclusion $\underline{\phi} : N_X \to N_Y$ induces a toric morphism $\phi : X \to Y$.

(1) Show that $N_X \to N_Y$ induces an exact sequence

$$1 \to G \to T_X \to T_Y \to 1,$$

and thus a G-action on X.

(2) Let $\sigma \in \Sigma$. Show that the image of

$$\phi^* : \mathbb{C}[\sigma^\vee \cap M_Y] \to \mathbb{C}[\sigma^\vee \cap M_X]$$

is given by the ring of invariants $\mathbb{C}[\sigma^\vee \cap M_X]^G$ under the regular representation of G on $\mathbb{C}[\sigma^\vee \cap M_X]$.

(3) Deduce that $Y = X/G$ and that $\phi : X \to Y$ is the quotient morphism.

Assume that $Y = U_\sigma$ is affine and let $\mathcal{E} \in \mathrm{qCoh}^{T_Y}(Y)$ with σ-family (E^σ, χ^σ).

(4) Compute the σ-family of $\phi^*\mathcal{E}$ in terms of (E^σ, χ^σ).

Exercise 2.3.21 (Locally trivial fibrations). Let $\phi : X \to Y$ be a locally trivial fibration as in Example 2.3.3. Let $\mathcal{E} \in \mathrm{qCoh}^{T_Y}(Y)$. Describe the σ-families of $\phi^*\mathcal{E}$ in terms of the σ-families of $\mathcal{E}$.

Exercise 2.3.22 (Invariant pushforward). Give the proof of Proposition 2.3.16. Hint : for point (ii), note that $\phi_*^{T_0}\mathcal{E}$ is a subsheaf of $\phi_*\mathcal{E}$.

Exercise 2.3.23 (Equivariant structures on pushforwards). Consider a toric morphism between two affine normal toric varieties $\phi : U_{\sigma'} \to U_\sigma$. Assume that $\underline{\phi} : N' \to N$ is surjective. For $\mathcal{E} \in \mathrm{qCoh}^{T_X}(U_{\sigma'})$, describe the possible M-graded structures on $\Gamma(U_\sigma, \phi_*\mathcal{E})$, cf Proposition 2.3.17.

Exercise 2.3.24 (Descent of toric sheaves). Let $\phi : X \to Y$ be a toric morphism such that $\underline{\phi} : N_X \to N_Y$ is surjective. Denote by $T_0 \subset T_X$ the subtorus generated by $\ker \underline{\phi}$. Assume that $\phi_*^{T_0}\mathcal{O}_X \simeq \mathcal{O}_Y$. We say that a sheaf $\mathcal{E} \in \mathrm{qCoh}^{T_X}(X)$ *descends* to Y if there exists $\mathcal{F} \in \mathrm{qCoh}^{T_Y}(Y)$ such that $\mathcal{E} = \phi^*\mathcal{F}$.

(1) Show that for any $\mathcal{F} \in \mathrm{qCoh}^{T_Y}(Y)$, $\phi_*^{T_0}\phi^*\mathcal{F} \simeq \mathcal{F}$.

(2) Deduce that $\mathcal{E} \in \mathrm{qCoh}^{T_X}(X)$ descends to Y if and only if $\mathcal{E} \simeq \phi^*\phi_*^{T_0}\mathcal{E}$.

Notes for Chapter 2

Locally free sheaves on the projective space

The splitting problem for locally free sheaves over the projective space goes back to the 70's. In [26], Hartshorne conjectured that for $n \geq 7$, there is no unsplittable locally free sheaf of rank 2 on the complex projective space. This is related to another conjecture [24] on low codimensional subvarieties of $\mathbb{P}^n$. It is well known that rank 1 vector bundles correspond to divisors. In codimension 2, the Hartshorne–Serre correspondence gives a criterion for the existence of a rank 2 vector bundle with a section whose zero locus correspond to a given subvariety [55, Chapter 1.5]. This relates unsplittable vector bundles to non-complete intersections. The latter being very hard to construct (see e.g. [30]), and behaving topologically as complete intersections, this lead to the aforementioned conjectures. We refer the interested readers to [61] for a survey on the origin of these problems.

Then, the splitting of T-equivariant vector bundles on the projective space, Corollary 2.2.18, was proved independently by Kaneyama [38] and Klyachko [39], and can be seen as a strong version of Hartshorne's conjecture for toric vector bundles. More recently, Ilten and Suess improved this result to vector bundles equivariant under a torus action of lower dimension, cf [33]. In the other direction, Cotignoli and Sterian have shown that an extension of Hartshorne's conjecture to toric Fano manifolds fails in general [15]. However, the original conjecture remains open.

Pure equivariant sheaves

Recall that a coherent sheaf $\mathcal{E} \neq 0$ on X is said to be *pure* if for any coherent subsheaf $0 \subsetneq \mathcal{F} \subset \mathcal{E}$, on has $\dim(\mathrm{Supp}(\mathcal{F})) = \dim(\mathrm{Supp}(\mathcal{E}))$, where $\mathrm{Supp}(\mathcal{F})$ is the support of $\mathcal{F}$, that is the closure of the set of points $x \in X$ where $\mathcal{F}_x \neq 0$. Pure sheaves generalise torsion-free sheaves, that correspond to pure sheaves with support of maximal dimension. In [42], Kool shows that for a coherent T-sheaf $\mathcal{E}$, being pure is equivalent to the equivariant version of purity : for any T-equivariant non-zero coherent subsheaf $\mathcal{F} \subset \mathcal{E}$, $\dim(\mathrm{Supp}(\mathcal{F})) = \dim(\mathrm{Supp}(\mathcal{E}))$ [42, Proposition 2.5]. Then, he describes the Σ-families of pure T-equivariant sheaves over smooth toric varieties, cf [42, Section 2]. Those results generalise the much simpler case of the pure sheaf $\iota_* \mathcal{O}_Y$ studied in Exercise 2.1.11, where $\iota : Y \to X$ denote the inclusion of an irreducible T-invariant subvariety.