Chapter 1

Equivariant sheaves on toric varieties and Σ-families

In this chapter, after a quick review on toric varities, we introduce *toric sheaves* and their Σ-families, following Perling [58, 59]. We then characterise Σ-families of coherent, torsion free, and reflexive toric sheaves. Finally, we conclude with Klyachko's characterisation of locally free toric sheaves in terms of families of filtrations, Theorem 1.3.27.

1.1 Toric varieties and fans

In this section we recall the fan description of toric varieties and introduce the notations that will be used throughout the text. We assume that the reader has already met toric varieties, but if not, the examples and exercises may help to catch up with the basic notions. We refer to the classical textbooks, e.g. [16, 21, 54], for a more detailed treatment and proofs. In particular, our notations and presentation closely follow [16].

Definition

We denote by T a (multiplicative) torus, that is an affine variety isomorphic to $(\mathbb{C}^*)^n$ for some $n \in \mathbb{N}^*$, such that the group multiplication and inverse maps are regular, i.e. morphisms in the category of algebraic varieties.

Definition 1.1.1. A *toric variety* is an algebraic variety X endowed with a regular torus action

$$\alpha : T \times X \to X$$

such that X contains a dense and open torus orbit isomorphic to T.

The fact that the open and dense T-orbit is a copy of T implies that the group action is faithfull (or effective). We will also use the following notation for the action of $t \in T$ on $x \in X$:

$$t \cdot x := \alpha(t, x).$$

Examples 1.1.2. Here is a list of classical examples of toric varieties :

(1) The torus T itself, endowed with the multiplication map,

(2) The affine space $\mathbb{C}^n$, with action

$$\begin{array}{cccc} \alpha : & (\mathbb{C}^*)^n \times \mathbb{C}^n & \to & \mathbb{C}^n \\ & \big((t_1, \dots, t_n),\ (x_1, \dots, x_n)\big) & \mapsto & (t_1 x_1, \dots, t_n x_n), \end{array}$$

(3) The projective space $\mathbb{P}^n$, with action

$$\begin{array}{cccc} \alpha : & (\mathbb{C}^*)^n \times \mathbb{P}^n & \to & \mathbb{C}^n \\ & \big((t_1,\ldots,t_n),\ [x_0,x_1,\ldots,x_n]\big) & \mapsto & [x_0, t_1x_1,\ldots,t_nx_n], \end{array}$$

(4) The cuspidal curve $V(x^3-y^2)\subset\mathbb{C}^2$, with $\mathbb{C}^*$-action

$$t\cdot(x,y)=(t^2x,t^3y),$$

(5) Whitney's umbrella $V(x^2-y^2z)\subset\mathbb{C}^3$, with $(\mathbb{C}^*)^2$-action

$$(t_1,t_2)\cdot(x,y,z)=(t_1t_2x,t_1y,t_2^2z)$$

(6) The conifold $V(xy-zw)\subset\mathbb{C}^4$, with $(\mathbb{C}^*)^3$-action

$$(t_1,t_2,t_3)\cdot(x,y,z,w)=(t_1x,t_2y,t_3z,t_3^{-1}t_2t_1w).$$

(7) The rational normal cone $V(I)\subset\mathbb{C}^{n+1}$, where the ideal I is generated by

$$I=\langle x_ix_{j+1}-x_{i+1}x_j \mid 0\le i,j\le n-1\rangle\subset\mathbb{C}[x_0,x_1,\ldots,x_n]$$

and where the $(\mathbb{C}^*)^2$-action is given by

$$(t_1,t_2)\cdot(x_0,\ldots,x_n)=(t_1^nx_0,t_1^{n-1}t_2x_1,\ldots,t_1t_2^{n-1}x_{n-1},t_2^nx_n).$$

One may notice in the last examples that the defining ideals are generated by binomials of the form

$$\prod_{i=1}^n x_i^{a_i}-\prod_{i=1}^n x_i^{b_i}$$

where $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_n)$ are in $\mathbb{N}^n$. This property is actually satisfied by any affine toric variety, and the converse is true, cf [16, Proposition 1.1.11].

We let N to be the set of one-parameter subgroups of T. This is a lattice - an abelian group isomorphic to $\mathbb{Z}^n$ - and we denote by $M=\mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$ its dual, also called the character lattice of T. We will set $\langle\cdot,\cdot\rangle$ the duality pairing defined on $M\times N$. More explicitly, if $T=(\mathbb{C}^*)^n$, then $N=\mathbb{Z}^n$, and each $u\in\mathbb{Z}^n$ gives rise to the one-parameter subgroup λ_u defined by

$$\begin{array}{cccc} \lambda_u : & \mathbb{C}^* & \to & (\mathbb{C}^*)^n \\ & t & \mapsto & (t^{u_1},\ldots,t^{u_n}). \end{array}$$

An element $m\in M=\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n,\mathbb{Z})\simeq\mathbb{Z}^n$ gives in turn a character

$$\begin{array}{cccc} \chi^m : & (\mathbb{C}^*)^n & \to & \mathbb{C}^* \\ & (t_1,\ldots,t_n) & \mapsto & \prod_{i=1}^n t_i^{m_i}. \end{array}$$

The duality pairing then reads

$$\langle m, u \rangle := m_1 u_1 + \ldots + m_n u_n.$$

In general, any one parameter subgroup $\lambda_u : \mathbb{C}^* \to T$ can be composed by any character function $\chi^m : T \to \mathbb{C}^*$, providing a regular group homomorphism

$$\begin{array}{rccc} \chi^m \circ \lambda_u : & \mathbb{C}^* & \to & \mathbb{C}^* \\ & t & \mapsto & t^{\langle m,u \rangle}. \end{array}$$

The affine case

We focus now on the affine case, and will use the notation $\mathbb{C}[Y]$ to denote the algebra of regular functions of an affine variety Y. If X is an affine toric variety with embedded torus $\iota : T \to X$, consider the associated morphism of algebras of regular functions :

$$\iota^* : \mathbb{C}[X] \to \mathbb{C}[T]. \tag{1.1}$$

This is an injection, for $\iota(T)$ is dense in X. Recall that, up to an isomorphism, $\mathbb{C}[T]$ is the ring of Laurent polynomials in n variables. Then, $\mathbb{C}[T]$ splits as a direct sum :

$$\mathbb{C}[T] = \bigoplus_{m \in M} \mathbb{C} \cdot \chi^m,$$

where $\chi^m : T \to \mathbb{C}^*$ denotes the character function of $m \in M$. Each weight $m \in M$ corresponds to a one-dimensional irreducible representation for the regular representation of T on $\mathbb{C}[T]$ defined, for $t \in T$ and $f \in \mathbb{C}[T]$, by

$$t \cdot f = f \circ \alpha(t^{-1}, \cdot). \tag{1.2}$$

With this definition the character function χ^m is homogeneous of degree $-m$:

$$\forall t \in T\,,\; t \cdot \chi^m = \chi^{-m}(t) \cdot \chi^m.$$

Similarly, one defines a representation of T on $\mathbb{C}[X]$ induced by the regular torus action α via Formula (1.2), and obtains a weight space decomposition - by the *complete reducibility theorem* -

$$\mathbb{C}[X] = \bigoplus_{m \in S} \mathbb{C}[X]_m \tag{1.3}$$

where

$$\mathbb{C}[X]_m := \left\{ f \in \mathbb{C}[X] \mid \forall t \in T,\; t \cdot f = \chi^{-m}(t) f \right\}. \tag{1.4}$$

This defines an *M-grading* on $\mathbb{C}[X]$, meaning that

$$\mathbb{C}[X]_m \cdot \mathbb{C}[X]_{m'} \subset \mathbb{C}[X]_{m+m'}.$$

Note that our definition (1.4) for $\mathbb{C}[X]_m$ uses the *opposite grading*, so that $\chi^m \in \mathbb{C}[T]_m$ has degree $+m$, rather than $-m$, for our choice of M-grading.

Examples 1.1.3. We give some examples of M-gradings.

(1) For the affine space $X = \mathbb{C}^n$,

$$\mathbb{C}[X] = \mathbb{C}[x_1, \dots, x_n],$$

which splits as

$$\mathbb{C}[X] = \bigoplus_{m \in \mathbb{N}^n} \mathbb{C} \cdot x_1^{m_1} \dots x_n^{m_n}.$$

With our convention, we simply have

$$\mathbb{C}[X]_m = \mathbb{C} \cdot x_1^{m_1} \dots x_n^{m_n}.$$

(2) If $X = V(x^3 - y^2)$, the cuspidal curve, then

$$\begin{aligned}\mathbb{C}[X] =& \mathbb{C}[x, y]/\langle x^3 - y^2 \rangle \\ \simeq& \mathbb{C}[t^2, t^3]\end{aligned}$$

and

$$\mathbb{C}[t^2, t^3] = \bigoplus_{i \in \mathbb{N} \setminus \{1\}} \mathbb{C} \cdot t^i \subset \mathbb{C}[t^{\pm 1}].$$

(3) For the umbrella $X = V(x^2 - y^2 z) \subset \mathbb{C}^3$, we have

$$\begin{aligned}\mathbb{C}[X] =& \mathbb{C}[x, y, z]/\langle x^2 - y^2 z \rangle \\ \simeq& \mathbb{C}[t_1 t_2, t_1, t_2^2],\end{aligned}$$

and

$$\mathbb{C}[t_1 t_2, t_1, t_2^2] = \bigoplus_{m \in S} \mathbb{C} \cdot t_1^{m_1} t_2^{m_2} \subset \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$$

where

$$S = \mathbb{N} \cdot (1, 1) + \mathbb{N} \cdot (1, 0) + \mathbb{N} \cdot (0, 2) \subset \mathbb{Z}^2.$$

(4) Finally, if we consider the conifold $X = V(xy - zw) \subset \mathbb{C}^4$, then

$$\begin{aligned}\mathbb{C}[X] =& \mathbb{C}[x, y, z, w]/\langle xy - zw \rangle \\ \simeq& \mathbb{C}[t_1, t_2, t_3, t_1 t_2 t_3^{-1}]\end{aligned}$$

and

$$\mathbb{C}[t_1, t_2, t_3, t_1 t_2 t_3^{-1}] = \bigoplus_{m \in S} \mathbb{C} \cdot t_1^{m_1} t_2^{m_2} t_3^{m_3} \subset \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]$$

where

$$S = \mathbb{N} \cdot (1, 0, 0) + \mathbb{N} \cdot (0, 1, 0) + \mathbb{N} \cdot (0, 0, 1) + \mathbb{N} \cdot (1, 1, -1) \subset \mathbb{Z}^3.$$

The splitting (1.3) is the same as the one induced by the inclusion ι^* in Equation (1.1), and $S \subset M$ is an *affine semigroup*, that is a finitely generated sub-semigroup of M. The algebra $\mathbb{C}[X]$ is then the *semigroup algebra* of S. Any affine toric variety can be obtained as the spectrum of the semigroup algebra of some affine semigroup. This is where convex geometry, with cones, steps onto the stage.

Cones

We set $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ on which we extend linearly the duality pairing (still denoted $\langle \cdot, \cdot \rangle$). Consider the cone $\sigma_S \subset M_{\mathbb{R}}$ generated by S :

$$\sigma_S := \left\{ \sum_{i \in I} \lambda_i m_i \mid \lambda_i \in \mathbb{R}_+,\ m_i \in S \right\}.$$

It turns out that this cone is a *rational polyhedral cone* of dimension n. This mean that there exists a finite set of generators $(m_1, \ldots, m_d) \in M$ such that

$$\sigma_S = \mathbb{R}_+ \cdot m_1 + \ldots + \mathbb{R}_+ \cdot m_d.$$

Examples 1.1.4. Here are the cones for the examples 1.1.3.

(1) The affine semigroup S for the affine plane $\mathbb{C}^2$ with standard torus action is $S = \mathbb{N}^2$. The cone σ_S for the affine space $\mathbb{C}^2$ is drawn in Figure 1.1. The shaded zone corresponds to the cone, the black dots are the elements of S, the generators are the dots labelled by the m_i's.

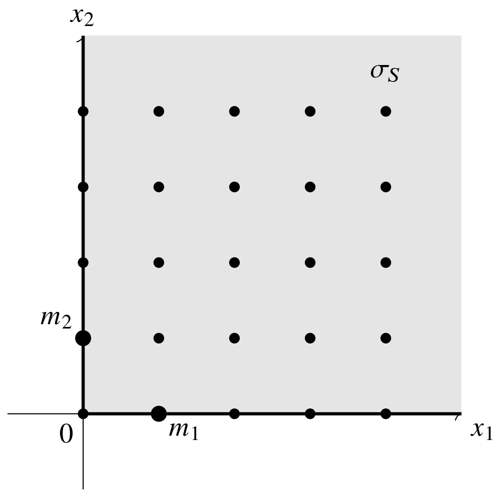

Figure 1.1. The cone $\sigma_S = \mathbb{R}_+ \cdot (1, 0) + \mathbb{R}_+ \cdot (0, 1)$ of $\mathbb{C}^2$.

(2) For the cuspidal curve, $S = \mathbb{N} \setminus \{1\}$, and the cone $\sigma_S = \mathbb{R}_+$ is generated by $m = 2 \in \mathbb{N}$. In the Figure 1.2, the white dots are the integral points in σ_S not contained in S.

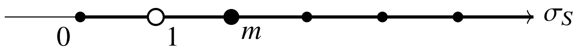

Figure 1.2. The cone $\sigma_S = \mathbb{R}_+ \cdot (2)$ of the cuspidal curve.

(3) For Whitney's umbrella,

$$S = \mathbb{N} \cdot (1,1) + \mathbb{N} \cdot (1,0) + \mathbb{N} \cdot (0,2) \subset \mathbb{Z}^2.$$

We deduce that the cone σ_S is the same as for the affine plane. However, the set of integral points in σ_S is different, cf Figure 1.3. Again, we denote by bigger black dots the generators $(m_i)_{1 \leq i \leq 3}$ of S, and by white dots the elements of $\sigma_S \cap M \setminus S$.

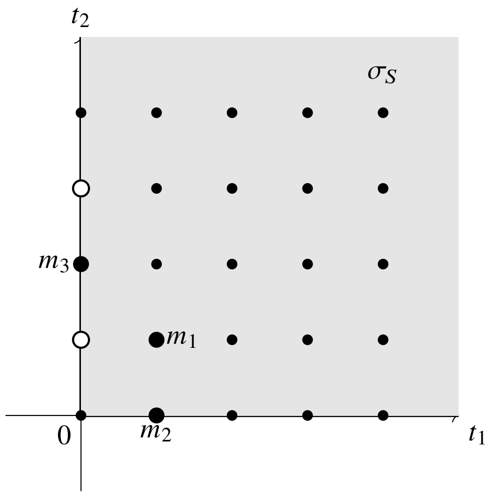

Figure 1.3. The cone of Whitney's umbrella.

(4) Finally, we consider the conifold $V(xy - zw)$. The affine semigroup S has four generators $(m_i)_{1 \leq i \leq 4}$, the first three being the elements of the canonical basis of $M \simeq \mathbb{Z}^3$, and the last one $m_4 = (1, 1, -1)$. The cone σ_S is depicted in Figure 1.4. This time, we have $\sigma_S \cap M = S$, and we only represent the generators of S with black dots.

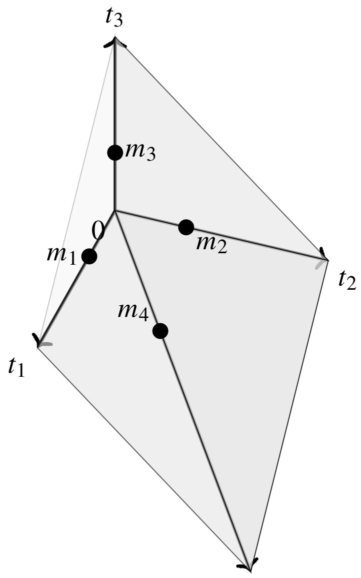

Figure 1.4. The cone of the conifold.

As illustrated in Examples 1.1.4, some care must be taken here as in general the inclusion $S \subset M \cap \sigma_S$ may be strict. The equality corresponds to the case when S is *saturated* in M, that is $km \in S$ for $(k, m) \in \mathbb{N}^* \times M$ implies $m \in S$. In that situation, we may write

$$X = \mathrm{Spec}(\mathbb{C}[S]) = \mathrm{Spec}(\mathbb{C}[\sigma_S \cap M])$$

where we use the notation $\mathbb{C}[S]$ for the semigroup algebra of S. One can show that this saturation property of the semigroup S is equivalent to X being *normal*, cf Exercise 1.1.13. For example, the affine space and the conifold in Examples 1.1.4 are normal, while the cuspidal curve and Whitney's umbrella are not. In the other direction, if $\sigma \subset M_{\mathbb{R}}$ is a rational polyhedral cone, one can show that $\sigma \cap M$ is an affine semigroup - this is Gordan's Lemma, Exercise 1.1.12. The conclusion is that any normal toric affine variety corresponds to a rational polyhedral cone of maximal dimension $\sigma \subset M_{\mathbb{R}}$ via the map

$$\sigma \subset M_{\mathbb{R}} \longrightarrow \mathrm{Spec}(\mathbb{C}[\sigma \cap M]).$$

This is not yet fully satisfactory, as we would like the faces of a given cone to correspond to affine subsets of the associated affine variety. To remedy this, we dualize. For $\sigma \subset N_{\mathbb{R}}$ a rational polyhedral cone, introduce its *dual cone* $\sigma^{\vee} \subset M_{\mathbb{R}}$:

$$\sigma^{\vee} := \left\{ m \in M_{\mathbb{R}} \mid \forall u \in \sigma\,,\ \langle m, u \rangle \geq 0 \right\}.$$

This is again a rational and polyhedral cone. One defines the dual of a cone in $M_{\mathbb{R}}$ in a similar way and one can check that for any such cones $(\sigma^\vee)^\vee = \sigma$. Then, maximal dimensional cones are dual to *strictly convex* cones, that is cones not containing any line. To ease terminology, from now on and until the end of the book,

the term *cone* will now refer to a *rational polyhedral cone*.

Then, to any strictly convex cone $\sigma \subset N_{\mathbb{R}}$, one can assign a normal, toric, affine variety

$$U_\sigma := \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M]).$$

We will then use the shorthand notation

$$S_\sigma := \sigma^\vee \cap M$$

so that $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$.

Examples 1.1.5. We introduce two more examples of normal affine toric varieties.

(1) If we consider the cone $\rho = \mathbb{R}_+ \cdot (1,0)$ in $N_{\mathbb{R}} = \mathbb{R}^2$, we obtain as dual cone

$$\rho^\vee = \mathbb{R}_+ \cdot (1,0) + \mathbb{R} \cdot (0,1) \subset M_{\mathbb{R}} \simeq \mathbb{R}^2.$$

Then,

$$S_\rho = \rho^\vee \cap M = \mathbb{N} \cdot (1,0) + \mathbb{N} \cdot (0,1) + \mathbb{N} \cdot (0,-1),$$

and

$$\mathbb{C}[S_\rho] = \mathbb{C}[x_1, x_2, x_2^{-1}]$$

so that

$$U_\rho = \mathrm{Spec}(\mathbb{C}[S_\rho]) = \mathbb{C} \times \mathbb{C}^*.$$

(2) Consider the two dimensional cone $\sigma \subset N_{\mathbb{R}} = \mathbb{R}^2$ generated by $(1,0)$ and $(1,2)$. The dual cone is

$$\sigma^\vee = \mathbb{R}_+ \cdot (0,1) + \mathbb{R}_+ \cdot (2,-1) \subset M_{\mathbb{R}} \simeq \mathbb{R}^2.$$

This is depicted in Figures 1.5 and 1.6. We obtain

$$S_\sigma = \mathbb{N} \cdot (0,1) + \mathbb{N} \cdot (2,-1) + \mathbb{N} \cdot (1,0).$$

Note that while $\sigma^\vee$ has two generators $m_1 = (0,1)$ and $m_2 = (2,-1)$ as a cone, the semigroup $S_\sigma = M \cap \sigma^\vee$ requires an additional third generator $m_3 = (1,0)$ to be spaned over $\mathbb{N}$. We then obtain

$$\mathbb{C}[S_\sigma] = \mathbb{C}[y, x^2y^{-1}, x] \simeq \mathbb{C}[u,v,w]/\langle uv - w^2 \rangle.$$

Then, the affine toric variety $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ is a quadric surface with an isolated singularity at zero.

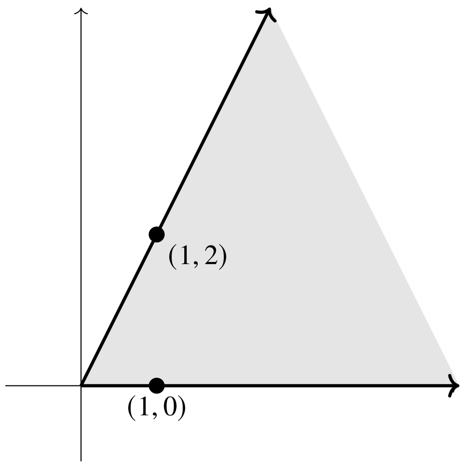

Figure 1.5. The cone σ.

The fan description

We proceed now to the construction of more general normal toric varieties. This can be done by gluing together a collection of affine toric varieties with the same torus T in a T-equivariant way. The resulting collection of cones will patch to form a *fan* of cones in $N_{\mathbb{R}}$. For $m \in M_{\mathbb{R}}$, denote the $\langle \cdot, \cdot \rangle$-orthogonal of m by

$$m^{\perp} := \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle = 0\}.$$

Let $\sigma \subset N_{\mathbb{R}}$ be a (strictly convex) cone. A *face* of σ is a cone $\tau \subset \sigma$ of the form

$$\tau = m^{\perp} \cap \sigma$$

for some $m \in \sigma^{\vee} \cap M$. We will denote $\tau \leq \sigma$ the fact that τ is a face of σ, and $\tau < \sigma$ if in addition $\tau \neq \sigma$. A *facet* is a codimension one face, and a *ray* is a one dimensional face. By construction, if $\tau \leq \sigma$, we have a T-equivariant inclusion

$$U_{\tau} \hookrightarrow U_{\sigma}.$$

Indeed, one can show that if $\tau = m^{\perp} \cap \sigma$ for some $m \in \sigma^{\vee} \cap M$, then

$$\tau^{\vee} \cap M = \sigma^{\vee} \cap M + \mathbb{N} \cdot (-m),$$

and

$$\mathbb{C}[\tau^{\vee} \cap M] = \mathbb{C}[\sigma^{\vee} \cap M]_{\chi^m}$$

is the localisation of the algebra of U_{σ} by the ideal (χ^m). As the closed set defined by χ^m is T-invariant, we deduce that the regular T-action on U_{σ} restricts to the one on U_{τ}.

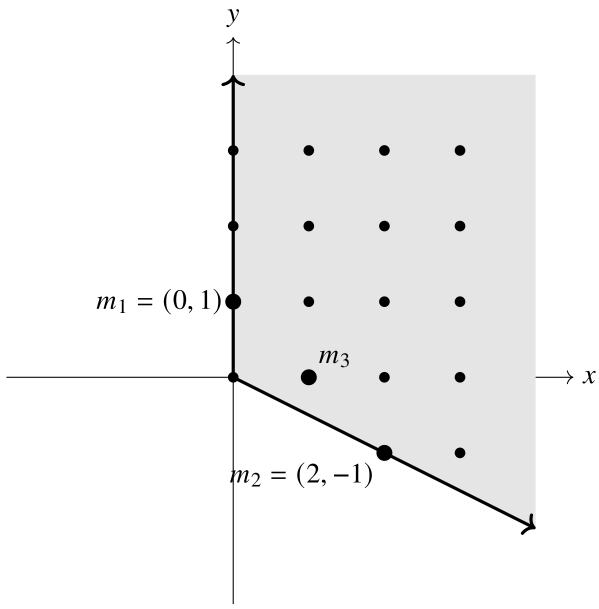

Figure 1.6. The dual cone $\sigma^\vee$ with integral points.

Example 1.1.6. As an example, consider the cone

$$\sigma = \mathbb{R}_+ \cdot (1, 0) + \mathbb{R}_+ \cdot (0, 1) \subset \mathbb{R}^2.$$

Its dual cone is

$$\sigma^\vee = \mathbb{R}_+ \cdot (1, 0) + \mathbb{R}_+ \cdot (0, 1),$$

and σ corresponds to the affine variety

$$U_\sigma = \operatorname{Spec}(\mathbb{C}[x_1, x_2]) = \mathbb{C}^2.$$

We have a facet $\rho = \mathbb{R}_+ \cdot (1, 0)$, given by $\rho = m^\perp \cap \sigma$, for $m = (0, 1) \in M \simeq \mathbb{Z}^2$. We see that

$$\rho^\vee \cap M = \sigma^\vee \cap M + \mathbb{N} \cdot (0, -1)$$

which corresponds to the localisation

$$\mathbb{C}[x_1, x_2, x_2^{-1}] = \mathbb{C}[x_1, x_2]_{(x_2)}.$$

The inclusion $\rho \subset \sigma$ yields the inclusion

$$U_\rho = \mathbb{C} \times \mathbb{C}^* \subset \mathbb{C}^2 = U_\sigma.$$

This leads to the definition of a fan.

Definition 1.1.7. A fan Σ in N is a finite set of strictly convex cones in $N_{\mathbb{R}}$ satisfying the following :

(1) If $\sigma \in \Sigma$ and $\tau \leq \sigma$, then $\tau \in \Sigma$,

(2) If $(\sigma_1, \sigma_2) \in \Sigma^2$, then for $i \in \{1, 2\}$, one has $\sigma_1 \cap \sigma_2 \leq \sigma_i$.

Given a fan Σ in N, one can form a normal toric variety X_Σ by gluing together the collection $\{U_\sigma \mid \sigma \in \Sigma\}$ along the common intersections $U_{\sigma_1 \cap \sigma_2} \subset U_{\sigma_i}$, $i \in \{1, 2\}$, and all normal toric varieties arise this way.

Example 1.1.8. Figure 1.7 represents the fan of the projective plane, with three 2-dimensional cones, three rays, and the zero cone. We leave as an exercise to the reader checking that this fan indeed corresponds to the 2-dimensional projective space, cf Exercise 1.1.17.

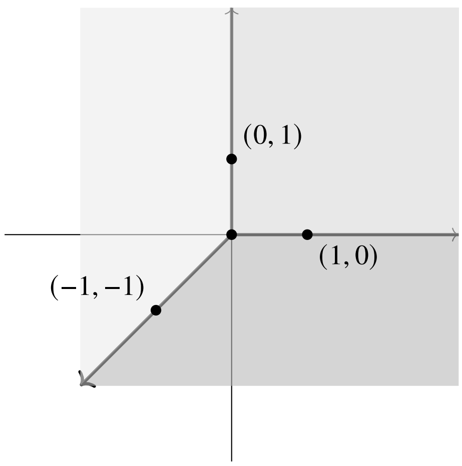

Figure 1.7. The fan of the projective plane.

Geometric properties of normal toric varieties can be recovered from the associated fans. Completeness and smoothness of toric varieties correspond to the following notions for fans. We say that a fan Σ in N is *complete* if its support, denoted $|\Sigma|$, that is the union of all its cones, is equal to the whole space $N_{\mathbb{R}}$. We will say that a fan is *smooth* if all its cones are smooth. A cone $\sigma \subset N_{\mathbb{R}}$ is smooth if its primitive ray generators $(u_\rho)_{\rho \in \sigma(1)}$ form a subset of a $\mathbb{Z}$-basis of N (we denote by $\sigma(1)$ the set of rays of σ, and $u_\rho \in N$ the primitive generator of ρ). Complete fans yields complete varieties (see e.g. [16, Section 3.4]), while smooth fans correspond to smooth toric varieties [16, Theorem 1.3.12]. Finally, a *simplicial* toric variety is a toric variety whose fan consists in simplicial cones, that are cones whose primitive ray generators form a subset of a $\mathbb{Q}$-basis of $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$. For examples, the affine space is smooth, the projective space is smooth and complete, and the quadric surface from Example 1.1.5.(2) is simplicial.

A very nice feature in the fan description of a normal toric variety X_Σ is that the torus orbits of X_Σ are in bijection with the cones in Σ. This will be referred to as the *orbit-cone correspondence*. A convenient way to define this bijection is to assign to each $\sigma \in \Sigma$ the unique orbit of minimal dimension $O(\sigma)$ in U_σ. The detailed construction is given in Exercise 1.1.15. What we will use in the following sections is listed in the following proposition, cf [16, Theorem 3.2.6].

Proposition 1.1.9. *Let Σ be a fan in N. Then*

(i) *For each $\sigma \in \Sigma$,* $\dim(\sigma) = \mathrm{codim}(O(\sigma)) = \dim(X_\Sigma) - \dim(O(\sigma))$*,*

(ii) *For each $\sigma \in \Sigma$,*

$$U_\sigma = \bigcup_{\tau \leq \sigma} O(\tau),$$

(iii) *For each $\tau \in \Sigma$,*

$$\overline{O(\tau)} = \bigcup_{\tau \leq \sigma} O(\sigma).$$

For $0 \leq k \leq \dim(X)$, we set $\Sigma(k)$ to be the set of k-dimensional cones in Σ. For a given $\sigma \in \Sigma(k)$, the closure of $O(\sigma)$ in X_Σ will be denoted $V(\sigma)$. It is a $(n-k)$-dimensional toric variety, see Exercise 1.1.16. In particular, one sees that rays of Σ are in bijection with irreducible torus invariant divisors. For a given ray $\rho \in \Sigma(1)$, we will use the notation $D_\rho = V(\rho)$.

Example 1.1.10. In Figure 1.7, the three rays correspond to the three divisors of the projective plane given by $x_i = 0$ in homogeneous coordinates $[x_0, x_1, x_2]$, the cone $\{0\}$ corresponds to the torus $T \subset \mathbb{P}^2$, and the three 2-dimensional cones correspond to the three fixed points under the torus action.

Morphisms of toric varieties are characterized as follows. Let N' be another lattice, with dual M', and Σ' be a fan in N'. Denote by $X' = X_{\Sigma'}$ the associated toric variety. Then, a morphism of lattices $\underline{\phi} : N \to N'$ such that its $\mathbb{R}$-linear extension $\underline{\phi}_\mathbb{R} : N_\mathbb{R} \to N'_\mathbb{R}$ maps cones of Σ into cones of Σ' - $\underline{\phi}$ is said to be *compatible with the fans* Σ and Σ' - induces an equivariant morphism $\phi : X \to X'$. Locally, if $\underline{\phi}_\mathbb{R}(\sigma) \subset \tau$ for a pair $(\sigma, \tau) \in \Sigma \times \Sigma'$,

$$\phi : U_\sigma \to U_\tau$$

is the morphism of varieties corresponding to the $\mathbb{C}$-algebra morphism

$$\phi^* : \mathbb{C}[\tau^\vee \cap M'] \to \mathbb{C}[\sigma^\vee \cap M]$$

induced by the dual map $\underline{\phi}^\vee : M' \to M$. Note that this equivariant morphism induces by restriction a morphism of groups $\phi : T \to T'$, where $T' = N' \otimes_\mathbb{Z} \mathbb{C}^*$ is the torus of X'. Conversely, one can show that any *toric morphism* $\phi : X \to X'$ - a morphism of varieties that restricts to a group homomorphism between tori - is equivariant for the torus actions, and induces a morphism of lattices $\underline{\phi} : N \to N'$ compatible with the fans Σ and Σ' in the above sense. Examples will be given in Chapter 2, Section 2.3, where morphisms will be studied in more details in relation to toric sheaves.

To sum up, we have defined a functor from the category of fans with their compatible morphisms to the category of normal toric varieties with toric morphisms; one can check that this functor induces an equivalence between those two categories, Exercise 1.1.19. Later on, we will need more results about toric varieties. Rather than listing all of them in this first section, we will move on to the main topic of this book, and come back to background material when necessary.

1.1.1 Exercises for 1.1

In all the exercises, Σ denotes a fan in a lattice N with dual lattice M.

Exercise 1.1.11 (Torus action and points). Let $S \subset M$ be an affine semigroup. Show that there is a correspondence between :

(1) closed points in $\mathrm{Spec}(\mathbb{C}[S])$,

(2) $\mathbb{C}$-algebra morphisms $\mathbb{C}[S] \to \mathbb{C}$,

(3) semigroup morphisms $S \to \mathbb{C}$ (with multiplicative action on $\mathbb{C} = \mathbb{C}^* \cup \{0\}$).

Show that the torus action

$$\alpha : T \times \mathrm{Spec}(\mathbb{C}[S]) \to \mathrm{Spec}(\mathbb{C}[S])$$

corresponds to the $\mathbb{C}$-algebra morphism defined on generators by

$$\begin{array}{cccc} \alpha^* : & \mathbb{C}[S] & \to & \mathbb{C}[M] \otimes \mathbb{C}[S] \\ & \chi^m & \mapsto & \chi^m \otimes \chi^m . \end{array}$$

Exercise 1.1.12 (Gordan's Lemma). Let $\sigma \in N_{\mathbb{R}}$ be a (rational polyhedral) cone. Show that $\sigma^\vee \cap M$ is a finitely generated semigroup. Hint : consider a finite set of generators $\{m_1, \dots, m_d\}$ for the polyhedral cone $\sigma^\vee$, and the set

$$\Big\{ \sum_{i=1}^{d} \lambda_i m_i \mid \forall\, i\,,\, 0 \leq \lambda_i < 1 \Big\} \cap M .$$

Exercise 1.1.13 (Normalisation). Let $S \subset M$ be an affine semigroup. Set $\sigma = (\sigma_S)^\vee \subset N_{\mathbb{R}}$, where $\sigma_S \subset M_{\mathbb{R}}$ is the cone spanned by elements of S. Assume that $\dim(\sigma_S) = \mathrm{rank}(M)$. Show that the morphism

$$U_\sigma \to \mathrm{Spec}(\mathbb{C}[S])$$

induced by the inclusion $\mathbb{C}[S] \subset \sigma^\vee \cap M$ is the normalisation map.

Exercise 1.1.14 (Cones and faces). Let $\sigma \subset N_{\mathbb{R}}$ be a cone. Show that the intersection of two faces of σ is again a face of σ. Show that if $\tau \leq \sigma$ and $\eta \leq \tau$, then $\eta \leq \sigma$.

Exercise 1.1.15 (Orbit-cone correspondence). Let $\sigma \in \Sigma$ and set $S_\sigma = \sigma^\vee \cap M$.

(1) Show that

$$\begin{array}{rccc} \gamma_\sigma : & S_\sigma & \to & \mathbb{C} \\ & m & \mapsto & \begin{cases} 1 & \text{if } m \in \sigma^\perp \\ 0 & \text{otherwise} \end{cases} \end{array}$$

defines a semigroup morphism. We will also denote by $\gamma_\sigma \in U_\sigma$ the associated closed point.

(2) Show that if $\tau \leq \sigma$, $\gamma_\tau \in U_\sigma$.

(3) Set $O(\sigma) = T \cdot \gamma_\sigma$. Show that

$$\begin{aligned} O(\sigma) &= \{\gamma : S_\sigma \to \mathbb{C} \mid \gamma(m) \neq 0 \Leftrightarrow m \in \sigma^\perp\} \\ &\simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*) \simeq T_{N(\sigma)} \end{aligned}$$

where $T_{N(\sigma)} := N(\sigma) \otimes_{\mathbb{Z}} \mathbb{C}^*$ for $N(\sigma) = N/N_\sigma$ and $N_\sigma \subset N$ is the lattice generated by $\sigma \cap N$.

Exercise 1.1.16 (Orbit closures as toric varieties). Use the orbit-cone correspondence to show that for each $\sigma \in \Sigma$, the orbit closure $V(\sigma)$ is a toric variety, with fan

$$\mathrm{Star}(\sigma) := \left\{\overline{\tau} \in N(\sigma)_{\mathbb{R}} \mid \sigma \leq \tau, \tau \in \Sigma\right\},$$

where $\overline{\tau}$ denotes the image of τ under the projection $N_{\mathbb{R}} \to N(\sigma)_{\mathbb{R}}$.

Exercise 1.1.17 (The projective space). Let $(e_1, \ldots, e_n)$ be the canonical basis of $\mathbb{Z}^n$, and set $e_0 = -e_1 - e_2 - \ldots - e_n$. For each finite set $I \subset \{0, \ldots, n\}$ with $|I| \leq n$, set

$$\sigma_I := \sum_{i \in I} \mathbb{R}_+ \cdot e_i \subset \mathbb{R}^n.$$

Let

$$\Sigma := \left\{\sigma_I \mid I \subset \{0, \ldots, n\},\ |I| \leq n\right\}.$$

(1) Show that Σ defines a fan in $\mathbb{Z}^n$.

(2) Show that the associated toric variety is isomorphic to the projective space.

(3) Let $\sigma \in \Sigma$. Describe the associated orbit closure $V(\sigma) \subset \mathbb{P}^n$.

Exercise 1.1.18 (Hirzebruch surfaces). Let $p \in \mathbb{N}$. In $\mathbb{Z}^2$, with canonical basis (e_1, e_2), consider the complete fan whose rays are spanned by the primitive elements

$$\{e_1, e_2, -e_2, -e_1 + pe_2\}$$

as depicted in Figure 1.8. The associated toric variety is called the *p-th Hirzebruch surface*, denoted $\mathbb{F}_p$. Show that there exists a toric morphism from $\mathbb{F}_p$ to $\mathbb{P}^1$.

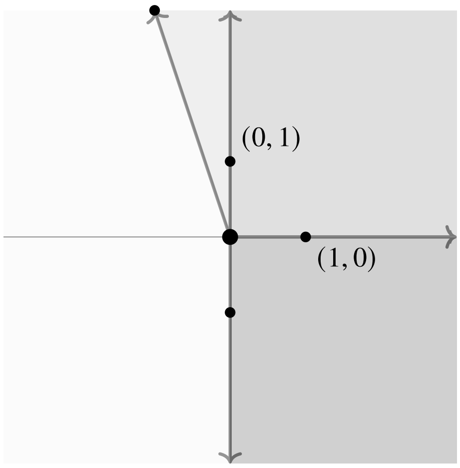

Figure 1.8. The fan of $\mathbb{F}_p$.

Exercise 1.1.19 (The category of fans). Check that the functor $(\Sigma, N) \longrightarrow X_\Sigma$ described in Section 1.1 induces an equivalence of categories.

1.2 Equivariant sheaves and Σ-families

We now introduce equivariant sheaves over toric varieties and provide their linear algebraic description following Perling [58, 59]. In the sequel, Σ will stand for a fan with associated lattice N and toric variety X. We will then keep the notations of the previous section, e.g. T for the torus, M for the character lattice, etc

Equivariant sheaves

Denote by π_i the projection onto the i-th factor of $T \times X$, $\mu : T \times T \longrightarrow T$ the multiplication and recall that $\alpha : T \times X \longrightarrow X$ stands for the regular torus action. We will also use $\pi_{23} : T \times T \times X \longrightarrow T \times X$ the projection on the last two factors, and for a given variety Y, $1_Y : Y \longrightarrow Y$ stands for the identity morphism.

Definition 1.2.1. Let $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$-modules on X. A *T-equivariant structure* on $\mathcal{E}$ is an isomorphism

$$\Phi : \alpha^*\mathcal{E} \longrightarrow \pi_2^*\mathcal{E}$$

satisfying the following cocycle condition :

$$(\mu \times 1_X)^*\Phi = \pi_{23}^*\Phi \circ (1_T \times \alpha)^*\Phi. \tag{1.5}$$

A *torus equivariant sheaf* on X is a pair $(\mathcal{E}, \Phi)$ where $\mathcal{E}$ is a sheaf of $\mathcal{O}_X$-modules and Φ a T-equivariant structure on $\mathcal{E}$.

We will also refer to torus equivariant sheaves as *toric sheaves*, *T-equivariant sheaves*, *T-sheaves*, or even *equivariant sheaves*. In most situations, the isomorphism Φ will not be specified, and we will simply denote by $\mathcal{E}$ a T-equivariant sheaf. For each $t \in T$, denote by α_t the isomorphism $\alpha(t, \cdot) : X \longrightarrow X$ induced by the action of t. Let $(\mathcal{E}, \Phi)$ be a T-equivariant sheaf. We set for all $t \in T$:

$$t^*\mathcal{E} := \alpha_t^*\mathcal{E}.$$

This is a sheaf of $\mathcal{O}_X$-modules. Pulling back Φ along the inclusion

$$\iota_t : \{t\} \times X \to T \times X$$

induces an isomorphism between $t^*\mathcal{E}$ and $\mathcal{E}$:

$$\Phi_t : t^*\mathcal{E} \simeq \iota_t^*\alpha^*\mathcal{E} \longrightarrow \iota_t^*\pi_2^*\mathcal{E} \simeq \mathcal{E}.$$

The cocycle condition (1.5) gives for all $(s, t) \in T^2$:

$$\Phi_{st} = \Phi_t \circ (t^*\Phi_s), \tag{1.6}$$

where $t^*\Phi_s = \alpha_t^*\Phi_s$ and where we use $(st)^*\mathcal{E} \simeq t^*(s^*\mathcal{E})$. This is obtained by pulling back (1.5) along the inclusion

$$\iota_{(s,t)} : \{(s,t)\} \times X \to T^2 \times X.$$

Remark 1.2.2. One can show that the data of isomorphisms $(\Phi_t)_{t\in T}$ satisfying (1.6) is enough to recover an equivariant structure, provided that the associated representations on the space of sections of $\mathcal{E}$ are *regular*, cf Definition 1.2.8 and Exercise 1.2.28 below.

Example 1.2.3. The structure sheaf $\mathcal{O}_X$ admis a T-equivariant structure defined as follows. On each T-invariant set U_σ, we have

$$\Gamma(T \times U_\sigma, \alpha^*\mathcal{O}_X) = (\mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[S_\sigma]) \otimes_{\alpha^*} \mathbb{C}[S_\sigma]$$

while

$$\Gamma(T \times U_\sigma, \pi_2^*\mathcal{O}_X) = (\mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[S_\sigma]) \otimes_{\pi_2^*} \mathbb{C}[S_\sigma] \simeq \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[S_\sigma]$$

where in the first equality we see $\mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[S_\sigma]$ as a $\mathbb{C}[S_\sigma]$-module through the action

$$\begin{array}{cccc} \alpha^* : & \mathbb{C}[S_\sigma] & \to & \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[S_\sigma] \\ & \chi^q & \mapsto & \chi^q \otimes \chi^q \end{array}$$

while in the second through the projection

$$\begin{array}{cccc} \pi_2^* : & \mathbb{C}[S_\sigma] & \to & \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[S_\sigma] \\ & \chi^q & \mapsto & 1 \otimes \chi^q. \end{array}$$

We can define

$$\begin{array}{cccc} \Phi_X : & \Gamma(T \times U_\sigma, \alpha^*\mathcal{O}_X) & \to & \Gamma(T \times U_\sigma, \pi_2^*\mathcal{O}_X) \\ & (\chi^m \otimes \chi^p) \otimes_{\alpha^*} \chi^q & \mapsto & \chi^{m+q} \otimes \chi^{p+q}, \end{array} \tag{1.7}$$

where $(\chi^m, \chi^p, \chi^q) \in \mathbb{C}[M] \times \mathbb{C}[S_\sigma] \times \mathbb{C}[S_\sigma]$ are generators. This is an isomorphism of $\mathbb{C}[M] \otimes \mathbb{C}[S_\sigma]$-modules. For each $t \in T$, one then has

$$\begin{array}{cccc} \Phi_{X,t} : & \Gamma(U_\sigma, t^*\mathcal{O}_X) & \to & \Gamma(U_\sigma, \mathcal{O}_X) \\ & \chi^q & \mapsto & \chi^q(t)\chi^q \end{array}$$

from which (1.6) easily follows. Hence Φ_X defines an equivariant structure on $\mathcal{O}_X$.

Given a sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$, a T-equivariant structure on $\mathcal{E}$ may not exist. When it does, we may obtain other T-equivariant structures as follows. Let Φ be a T-equivariant structure on $\mathcal{E}$. Given a character $\chi^m \in \operatorname{Hom}_{\mathbb{Z}}(T, \mathbb{C}^*)$, one can *shift* Φ by setting for all $t \in T$:

$$(\chi^m \cdot \Phi)_t := \chi^m(t)\Phi_t.$$

The isomorphisms $(\chi^m \cdot \Phi)_{t\in T}$ obviously satisfy the cocycle condition (1.6) and define a new T-equivariant structure on $\mathcal{E}$. For *simple* equivariant sheaves, all T-equivariant structures are obtained in this way.

Definition 1.2.4. A sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ is *simple* if

$$\mathrm{Hom}(\mathcal{E},\mathcal{E}) = \mathbb{C}\cdot \mathrm{Id}_{\mathcal{E}}.$$

Simplicity is a necessary condition for stability, and stable sheaves arise as building blocks for coherent sheaves on varieties, cf Chapter 5.

Lemma 1.2.5. *Let $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$-modules. Assume that $\mathcal{E}$ is simple. If Φ is a T-equivariant structure on $\mathcal{E}$, then any other T-equivariant structure on $\mathcal{E}$ is of the form $\chi^m\cdot\Phi$, for some $m\in M$.*

Proof. Let Ψ be an equivariant structure on $\mathcal{E}$. For each $t\in T$, we may consider the isomorphism $\Phi_t\circ\Psi_t^{-1}$ of $\mathcal{E}$. As $\mathcal{E}$ is simple, there is $\chi(t)\in\mathbb{C}^*$ such that $\Phi_t\circ\Psi_t^{-1} = \chi(t)\cdot\mathrm{Id}_{\mathcal{E}}$. The cocycle condition (1.6) then implies that χ is a morphism. ∎

Example 1.2.6. When X is complete, $\mathcal{O}_X$ is a simple sheaf. Hence from Example 1.2.3, and Lemma 1.2.5, all equivariant structures on $\mathcal{O}_X$ are of the form $\chi^m\cdot\Phi_X$, for $m\in M$ and where Φ_X is defined in (1.7).

We can form a category of T-equivariant sheaves.

Definition 1.2.7. A T-equivariant morphism (or simply T-morphism) between two T-equivariant sheaves $(\mathcal{E},\Phi)$ and $(\mathcal{F},\Psi)$ is a morphism $\theta:\mathcal{E}\to\mathcal{F}$ of sheaves of $\mathcal{O}_X$-modules such that

$$\pi_2^*\theta\circ\Phi = \Psi\circ\alpha^*\theta. \tag{1.8}$$

We will denote by $\mathrm{Hom}^T(\mathcal{E},\mathcal{F})$ the $\mathbb{C}$-vector space of T-equivariant morphisms.

Equivalently, a morphism of sheaves of $\mathcal{O}_X$-modules between two T-equivariant sheaves $(\mathcal{E},\Phi)$ and $(\mathcal{F},\Psi)$ is T-equivariant if for all $t\in T$,

$$\theta\circ\Phi_t = \Psi_t\circ t^*\theta. \tag{1.9}$$

The class of T-equivariant sheaves together with T-equivariant morphisms form a category, that we will denote by $\mathrm{Mod}^T(X)$. By restricting the class of objects to quasicoherent (resp. coherent) sheaves of $\mathcal{O}_X$-modules, one obtains the abelian categories of T-equivariant quasicoherent sheaves (resp. of T-equivariant coherent sheaves), denoted $\mathrm{qCoh}^T(X)$ (resp. $\mathrm{Coh}^T(X)$), see Exercise 1.2.26.

M-graded modules

We restrict our attention to the case when X is affine, $X = U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ for some strictly convex cone $\sigma\subset N_{\mathbb{R}}$. Our goal in this section is to provide an algebraic characterisation of T-equivariant sheaves on U_σ. Let $(\mathcal{E},\Phi)$ be a quasicoherent toric sheaf on U_σ. Through Grothendieck's $\sim$-functor, the sheaf $\mathcal{E}$ is characterised by the

$\mathbb{C}[S_\sigma]$-module $\Gamma(U_\sigma, \mathcal{E})$. As for regular functions, the T-action on U_σ induces a T-representation on $\Gamma(U_\sigma, \mathcal{E})$ defined, for $t \in T$ and $f \in \Gamma(U_\sigma, \mathcal{E})$, by

$$t \cdot f = \Phi_{t^{-1}}(\alpha^*_{t^{-1}} f) \tag{1.10}$$

where $\alpha^*_{t^{-1}} f \in \Gamma(U_\sigma, \alpha^*_{t^{-1}}\mathcal{E})$. The fact that this defines a representation follows from the cocycle condition (1.6).

Definition 1.2.8. A *regular representation* of T on a $\mathbb{C}$-vector space V is a linear action of T on V such that for any finite dimensional vector subspace $W \subset V$:

(1) The vector space $T \cdot W := \mathrm{Span}_{\mathbb{C}}\{t \cdot v \mid t \in T,\ v \in W\}$ is finite dimensional.

(2) The representation $T \to \mathrm{GL}(T \cdot W)$ is a morphism of algebraic groups.

The above representation of T on $\Gamma(U_\sigma, \mathcal{E})$ is regular. This can be seen as follows. Consider the composition

$$\Gamma(U_\sigma, \mathcal{E}) \xrightarrow{\alpha^*} \Gamma(T \times U_\sigma, \alpha^*\mathcal{E}) \xrightarrow{\Phi} \Gamma(T \times U_\sigma, \pi_2^*\mathcal{E}) \xrightarrow{\simeq} \mathbb{C}[M] \otimes_{\mathbb{C}} \Gamma(U_\sigma, \mathcal{E}).$$

This gives a morphism

$$\Phi^* : \Gamma(U_\sigma, \mathcal{E}) \to \mathbb{C}[M] \otimes_{\mathbb{C}} \Gamma(U_\sigma, \mathcal{E})$$

called the *dual action*. For each t, the inclusion $\{t^{-1}\} \times U_\sigma \to T \times U_\sigma$ yields

$$\iota^*_{t^{-1}} : \mathbb{C}[M] \otimes_{\mathbb{C}[S_\sigma]} \Gamma(U_\sigma, \mathcal{E}) \to \Gamma(U_\sigma, \mathcal{E}).$$

Then by construction, the representation of T on $\Gamma(U_\sigma, \mathcal{E})$ is also given by

$$t \cdot f = \iota^*_{t^{-1}} \circ \Phi^*(f). \tag{1.11}$$

This easily implies that it is regular, cf Exercise 1.2.27. In particular, $\Gamma(U_\sigma, \mathcal{E})$ admits a *weight space decomposition*, as for $\mathbb{C}[S_\sigma]$:

$$\Gamma(U_\sigma, \mathcal{E}) = \bigoplus_{m \in M} \Gamma(U_\sigma, \mathcal{E})_m$$

where

$$\Gamma(U_\sigma, \mathcal{E})_m = \left\{ f \in \Gamma(U_\sigma, \mathcal{E}) \mid \forall\, t \in T,\ t \cdot f = \chi^{-m}(t) f \right\}.$$

In conclusion, $\Gamma(U_\sigma, \mathcal{E})$ is an M-graded $\mathbb{C}[S_\sigma]$-module, meaning that the product of weitghed sections respects the grading :

$$\mathbb{C}[S_\sigma]_m \times \Gamma(U_\sigma, \mathcal{E})_{m'} \to \Gamma(U_\sigma, \mathcal{E})_{m+m'}.$$

Example 1.2.9. The M-graded module structure on $\Gamma(U_\sigma, \mathcal{O}_X) = \mathbb{C}[S_\sigma]$ associated to the equivariant structure Φ_X of Example 1.2.3 simply corresponds to the weight space decomposition in Equation 1.3:

$$\mathbb{C}[S_\sigma] = \bigoplus_{m \in S_\sigma} \mathbb{C} \cdot \chi^m.$$

For $m_0 \in M$, the shifted equivariant structure $\chi^{m_0} \cdot \Phi_X$ yields the same algebra, but with shifted grading $m \mapsto m + m_0$, i.e. such that the regular function χ^m has weight $m + m_0$. We will denote this shifted M-graded algebra by $\mathbb{C}[S_\sigma][m_0]$. Explicitly,

$$\mathbb{C}[S_\sigma][m_0]_m = \mathbb{C}[S_\sigma]_{m-m_0} = \mathbb{C} \cdot \chi^{m-m_0}$$

when $m - m_0 \in \sigma^\vee$, and is trivial otherwise.

Based on Kaneyama's original result [37, Proposition 3.4], Perling noticed that $\mathcal{E} \mapsto \Gamma(U_\sigma, \mathcal{E})$ actually induces an equivalence of categories [58, Proposition 2.31]:

Proposition 1.2.10. *The global section functor $\Gamma(U_\sigma, \cdot)$ induces an equivalence of categories between $\mathrm{qCoh}^T(U_\sigma)$ and M-graded $\mathbb{C}[S_\sigma]$-modules with degree zero morphisms.*

Recall that a *degree-m morphism*, for $m \in M$, between two M-graded $\mathbb{C}[S_\sigma]$-modules is a morphism of $\mathbb{C}[S_\sigma]$-modules $\phi : A \to B$ that sends the weight spaces $A_{m'}$ to $B_{m'+m}$.

Proof. The discussion above shows that any $\mathcal{E} \in \mathrm{Ob}(\mathrm{qCoh}^T(U_\sigma))$ yields an M-graded $\mathbb{C}[S_\sigma]$-module. In the other direction, let

$$E := \bigoplus_{m \in M} E_m$$

be an M-graded $\mathbb{C}[S_\sigma]$-module. The $\sim$-functor defines a quasicoherent sheaf of $\mathcal{O}_{U_\sigma}$-modules denoted $\mathcal{E}$ over U_σ such that

$$E = \Gamma(U_\sigma, \mathcal{E}).$$

To define the equivariant structure Φ on $\mathcal{E}$, consider the spaces of sections :

$$\Gamma(T \times U_\sigma, \alpha^*\mathcal{E}) = (\mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[S_\sigma]) \otimes_{\alpha^*} E$$

and

$$\Gamma(T \times U_\sigma, \pi_2^*\mathcal{E}) = (\mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[S_\sigma]) \otimes_{\pi_2^*} E.$$

We define an isomorphism of $\mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[S_\sigma]$-modules by setting

$$\begin{array}{cccc} \Phi : & \Gamma(T \times U_\sigma, \alpha^*\mathcal{E}) & \to & \Gamma(T \times U_\sigma, \pi_2^*\mathcal{E}) \\ & (\chi^m \otimes f_{m'}) \otimes e_{m''} & \mapsto & (\chi^{m+m''} \otimes f_{m'}) \otimes e_{m''}, \end{array}$$

where $(\chi^m, f_{m'}, e_{m''}) \in \mathbb{C}[M]_m \times \mathbb{C}[S_\sigma]_{m'} \times E_{m''}$ stand for weighted generators. To show that Φ fulfills the cocycle condition, consider the composition

$$E = \Gamma(U_\sigma, \mathcal{E}) \xrightarrow{\alpha^*} \Gamma(T \times U_\sigma, \alpha^*\mathcal{E}) \xrightarrow{\Phi} \Gamma(T \times U_\sigma, \pi_2^*\mathcal{E}) \xrightarrow{\simeq} \mathbb{C}[M] \otimes_{\mathbb{C}} E,$$

that we denote

$$\Phi^* : E \to \mathbb{C}[M] \otimes_{\mathbb{C}} E.$$

As for $e_m \in E_m$, $\alpha^*(e_m) = (1 \otimes 1) \otimes e_m$, we obtain $\Phi^*(e_m) = \chi^m \otimes e_m$. Then, we easily derive

$$(\mu^* \otimes 1_E) \circ \Phi^* = (1_{\mathbb{C}[M]} \otimes \Phi^*) \circ \Phi^*$$

as $\mathbb{C}$-linear maps from E to $\mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[M] \otimes_{\mathbb{C}} E$. This is equivalent to the cocycle condition (1.5), and $(\mathcal{E}, \Phi)$ is a T-equivariant sheaf.

Let $\theta : (\mathcal{E}, \Phi) \to (\mathcal{F}, \Psi)$ be a T-equivariant morphism between two quasicoherent T-sheaves over U_σ, and let $e_m \in \Gamma(U_\sigma, \mathcal{E})_m$. Then

$$\begin{aligned} \theta \circ \Phi_{t^{-1}}(\alpha^*_{t^{-1}} e_m) =& \theta(t \cdot e_m) \\ =& \theta\big(\chi^m(t)\, e_m\big) \\ =& \chi^m(t)\, \theta(e_m) \end{aligned}$$

while

$$\begin{aligned} \Psi_{t^{-1}} \circ (\alpha^*_{t^{-1}}\theta)(\alpha^*_{t^{-1}} e_m) =& \Psi_{t^{-1}} \circ \alpha^*_{t^{-1}}\big(\theta(e_m)\big) \\ =& t \cdot \big(\theta(e_m)\big). \end{aligned}$$

Then, from (1.9), we deduce that θ induces a degree zero morphism of M-graded modules. As the global section functor induces an equivalence of categories between quasicoherent sheaves of $\mathcal{O}_{U_\sigma}$-modules and $\mathbb{C}[S_\sigma]$-modules, we deduce that it induces an equivalence between $\mathrm{qCoh}^T(U_\sigma)$ and M-graded $\mathbb{C}[S_\sigma]$-modules by restriction. ∎

Examples 1.2.11. Here are some examples of quasicoherent toric sheaves obtained by Proposition 1.2.10 on the affine toric variety $X = U_\sigma = \mathbb{C}^2$, as in Example 1.1.2.(2).

(1) The ideal sheaf of the point $0 \in \mathbb{C}^2$, denoted $\mathcal{I}_0$, is a quasicoherent sheaf on $\mathbb{C}^2$ (it is in fact coherent, see below). Its space of global section is by definition the $\mathbb{C}[x, y]$-module of polynomials vanishing at 0 :

$$\begin{aligned} \Gamma(\mathbb{C}^2, \mathcal{I}_0) &= x \cdot \mathbb{C}[x, y] + y \cdot \mathbb{C}[x, y] \\ &= \bigoplus_{(p,q) \in \mathbb{N}^2 \setminus \{(0,0)\}} \mathbb{C} \cdot x^p y^q. \end{aligned}$$

This latter decomposition shows that $\Gamma(\mathbb{C}^2, \mathcal{I}_0)$ is $\mathbb{Z}^2$-graded, hence $\mathcal{I}_0$ is a quasicoherent toric sheaf on $\mathbb{C}^2$.

(2) Consider now the quotient sheaf $\mathcal{O}_0 = \mathcal{O}_X/\mathcal{I}_0$. This is the structure sheaf of the point $0 \in \mathbb{C}^2$. Its space of sections satisfies

$$\begin{array}{rcl} \Gamma(\mathbb{C}^2, \mathcal{O}_0) & = & \mathbb{C}[x, y]/(x, y) \\ & = & \mathbb{C} \cdot x^0 y^0 \end{array}$$

(we identify $x^p y^q$ with its class in the quotient to ease notations). This is again $\mathbb{Z}^2$-graded, by setting

$$\Gamma(\mathbb{C}^2, \mathcal{O}_0)_m = \begin{cases} \mathbb{C} \cdot 1 & \text{if} \quad m = (0, 0) \\ 0 & \text{if} \quad m \neq (0, 0), \end{cases}$$

and noting that for $m = (p, q) \neq (0, 0)$, the class of $x^p y^q \cdot 1$ in the quotient $\mathbb{C}[x, y]/(x, y)$ is zero.

(3) The ideal sheaf of any of the coordinate axes is also toric. For example, take $\mathcal{I} = (x)$. Then,

$$\begin{array}{rcl} \Gamma(\mathbb{C}^2, \mathcal{I}) & = & x \cdot \mathbb{C}[x, y] \\ & = & \displaystyle\bigoplus_{(p,q) \in \mathbb{N}^* \times \mathbb{N}} \mathbb{C} \cdot x^p y^q. \end{array}$$

This is again a $\mathbb{Z}^2$-graded $\mathbb{C}[x, y]$-module. The quotient $\mathcal{O}_X/\mathcal{I}$ is also a quasicoherent toric sheaf, with

$$\begin{array}{rcl} \Gamma(\mathbb{C}^2, \mathcal{O}_X/\mathcal{I}) & = & \mathbb{C}[x, y]/(x) \\ & = & \displaystyle\bigoplus_{q \in \mathbb{N}} \mathbb{C} \cdot y^q. \end{array}$$

(4) We can generalise those examples by considering ideal sheaves corresponding to torus invariant subschemes[1] in $\mathbb{C}^2$. Any ideal of the form

$$\mathcal{I}_S := \left(x^{p_i} y^{q_i},\ (p_i, q_i) \in S\right),$$

where $S \subset \mathbb{N}^2$ is a finite set, admits a $\mathbb{Z}^2$-graded $\mathbb{C}[x, y]$-module structure :

$$\Gamma(\mathbb{C}^2, \mathcal{I}_S) = \sum_{i \in S} (x^{p_i} y^{q_i}) \cdot \mathbb{C}[x, y].$$

Similarly, the quotient sheaf $\mathcal{O}_X/\mathcal{I}_S$ is toric.

[1] The reader unfamiliar with schemes may ignore the geometric interpretation of those ideals, or have a look at [25, Chapter II, Section 2]

Definition of σ-families

We will now translate the M-graded $\mathbb{C}[S_\sigma]$-module structure of $\Gamma(U_\sigma, \mathcal{E})$ into a family of vector spaces with morphisms. The starting point is the fact that for $m \in S_\sigma$, multiplication by the regular function χ^m defines a degree-m endomorphism of $\Gamma(U_\sigma, \mathcal{E})$ through the $\mathbb{C}[S_\sigma]$-module structure. Hence, for each pair $(m, m') \in M$, the following

$$\begin{array}{rccc} \chi^{m-m'}: & \Gamma(U_\sigma, \mathcal{E})_{m'} & \to & \Gamma(U_\sigma, \mathcal{E})_m \\ & f_{m'} & \mapsto & \chi^{m-m'} \cdot f_{m'} \end{array}$$

is well defined provided that $m - m' \in S_\sigma = \sigma^\vee \cap M$. This motivates the following definition:

Definition 1.2.12. For $(m, m') \in M^2$, we define the relation

$$m \leq_\sigma m' \iff m' - m \in \sigma^\vee.$$

We will write $m <_\sigma m'$ if $m \leq_\sigma m'$ but not $m' \leq_\sigma m$. We will also use the notation $m \nleq_\sigma m'$ when $m' - m \notin \sigma^\vee$.

We leave the proof of the following as an Exercise 1.2.29.

Lemma 1.2.13. *The following are satisfied, for $(m, m') \in M^2$:*

(i) *The relation $\leq_\sigma$ is reflexive and transitive,*

(ii) *$m \leq_\sigma m'$ and $m' \leq_\sigma m$ if and only if $m - m' \in \sigma^\perp$.*

In particular, $\leq_\sigma$ descends to a partial order on $M/(\sigma^\perp \cap M)$.

We can then define Perling's σ-*families*.

Definition 1.2.14. A σ-family (E^σ, χ^σ) is the data of a family of vector spaces $E^\sigma = (E^\sigma_m)_{m\in M}$ together with a family of morphisms $\chi^\sigma = (\chi^\sigma_{m,m'} : E^\sigma_m \to E^\sigma_{m'})_{m\leq_\sigma m'}$, satisfying :

(1) For each $m \in M$, $\chi^\sigma_{m,m} = 1_{E^\sigma_m}$,

(2) For each triple $m \leq_\sigma m' \leq_\sigma m''$, $\chi^\sigma_{m',m''} \circ \chi^\sigma_{m,m'} = \chi^\sigma_{m,m''}$.

When the situation is clear enough, we will simply drop χ^σ from the notations and refer to the σ-family as E^σ. Note that definition 1.2.14 implies the following :

Lemma 1.2.15. *Let (E^σ, χ^σ) be a σ-family. If $m - m' \in \sigma^\perp$, then $\chi^\sigma_{m,m'}$ and $\chi^\sigma_{m',m}$ are inverses of each other.*

We obtain a σ-family out of $\Gamma(U_\sigma, \mathcal{E})$ by setting

$$E^\sigma_m := \Gamma(U_\sigma, \mathcal{E})_m,$$

and, for $m \leq_\sigma m'$,

$$\chi^\sigma_{m,m'} := \chi^{m'-m}.$$

Example 1.2.16. The shifted module $\mathbb{C}[S_\sigma][m_0]$ defines the σ-family

$$E_m^\sigma \simeq \Gamma(U_\sigma, \mathcal{O}_X)[m_0]_m$$

that can be conveniently described as follows :

$$E_m^\sigma = \begin{cases} \mathbb{C} & \text{if} \quad m_0 \leq_\sigma m \\ \{0\} & \text{if} \quad m_0 \nleq_\sigma m, \end{cases} \tag{1.12}$$

while for $m \leq_\sigma m'$, the morphism $\chi^\sigma_{m,m'}$ is the obvious inclusion. Note in that case that all the $\chi^\sigma_{m,m'}$ are injective maps to $\mathbb{C}$. If we assume that X is the quadric surface of Example 1.1.5.(2), and $m_0 = (1,1) \in M \simeq \mathbb{Z}^2$, we can represent (E_m^σ) in Figure 1.9. The shaded zone corresponds to $m_0 + \sigma^\vee$, and to each dot is attached the vector space E_m^σ, with bigger dots corresponding to $E_m^\sigma = \mathbb{C}$. The arrows in the upper right quadrant represent the maps $\chi^\sigma_{m,m'}$ from E_m^σ to $E_{m'}^\sigma$, with initial $m = (2,3)$.

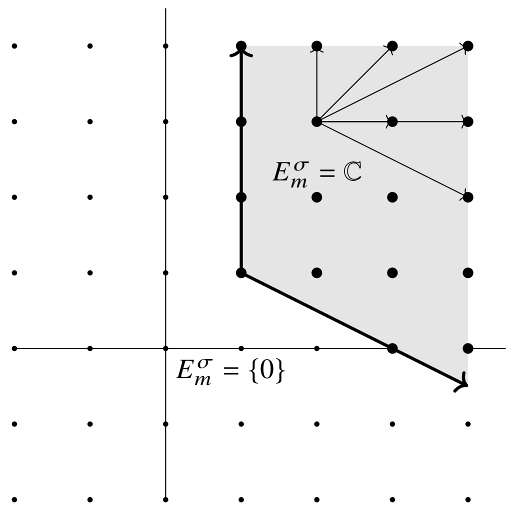

Figure 1.9. A σ-family.

Examples 1.2.17. We describe here the σ-families of the toric ideals in Examples 1.2.11. Let $S \subset \mathbb{N}^2$ be a finite set, and

$$\mathcal{I}_S := \left(x^p y^q,\ (p,q) \in S\right).$$

We define the subset of weights of $\mathbb{Z}^2$:

$$\mathbb{N}^2 \cdot S := \sum_{(p,q)\in S} (p,q) + \mathbb{N}^2 \subset \mathbb{Z}^2.$$

The σ-family

$$I_m^\sigma \simeq \Gamma(\mathbb{C}^2, \mathcal{I}_S)_m$$

satisfies

$$I_m^\sigma = \begin{cases} \mathbb{C} & \text{if} \quad m \in \mathbb{N}^2 \cdot S \\ \{0\} & \text{if} \quad m \notin \mathbb{N}^2 \cdot S \end{cases}$$

and the morphisms $(\chi_{m,m'}^\sigma)$ are again given by the obvious inclusions.

We wish to upgrade this to be functorial.

Definition 1.2.18. A morphism of σ-families

$$\phi^\sigma : (E^\sigma, \chi^\sigma) \to (F^\sigma, \psi^\sigma)$$

is a family of vector spaces morphisms $(\phi_m^\sigma : E_m^\sigma \to F_m^\sigma)_{m\in M}$ such that for any $m \leq_\sigma m'$,

$$\psi_{m,m'}^\sigma \circ \phi_m^\sigma = \phi_{m'}^\sigma \circ \chi_{m,m'}^\sigma.$$

Together with their morphisms, σ-families form a category, as introduced by Perling. He also proved the following [59].

Proposition 1.2.19. *The following categories are equivalent :*

(i) *The category of quasicoherent T-sheaves over U_σ,*

(ii) *The category of M-graded $\mathbb{C}[S_\sigma]$-modules with degree-0 morphisms,*

(iii) *The category of σ-families.*

Proof. By Proposition 1.2.10 we only have to prove the equivalence between the last two categories. The previous discussion shows that any M-graded $\mathbb{C}[S_\sigma]$-module naturally defines a σ-family, with degree-0 morphisms corresponding to morphisms of σ-families. In the other direction, given a σ-family (E^σ, χ^σ), we define an M-graded $\mathbb{C}[S_\sigma]$-module by forming the direct sum

$$E := \bigoplus_{m\in M} E_m^\sigma$$

and by setting, for $(m, m') \in S_\sigma \times M$, and $e_{m'} \in E_{m'}^\sigma$,

$$\chi^m \cdot e_{m'} = \chi_{m',m+m'}^\sigma(e_{m'}).$$

A morphism of σ-families

$$\phi^\sigma : (E^\sigma, \chi^\sigma) \to (F^\sigma, \psi^\sigma)$$

defines a degree-0 morphism

$$\begin{array}{rccc} \phi : & \bigoplus_{m\in M} E^\sigma & \to & \bigoplus_{m\in M} F_m^\sigma \\ & \sum e_m & \mapsto & \sum_m \phi_m^\sigma(e_m), \end{array}$$

clearly compatible with the $\mathbb{C}[S_\sigma]$-module structures. The fact that this functor is fully faithfull and essentially surjective is straightforward. ■

Gluing data and Σ-families

Consider now $\sigma \in \Sigma$ and $\tau \leq \sigma$. Recall that this induces an injection

$$\iota_{\tau\sigma} : U_\tau \to U_\sigma$$

with associated morphism of $\mathbb{C}$-algebras given by localisation

$$\iota_{\tau\sigma}^* : \mathbb{C}[S_\sigma] \to \mathbb{C}[S_\tau] \simeq \mathbb{C}[S_\sigma]_{\chi^m},$$

where $m \in \sigma^\vee$ is chosen so that $S_\tau = S_\sigma + \mathbb{N} \cdot (-m)$, cf Section 1.1. Let $(\mathcal{E}, \Phi)$ be a T-sheaf on U_σ. As the torus action commutes with the inclusion $\iota_{\tau\sigma}$, $\iota_{\tau\sigma}^*\mathcal{E}$ naturally inherits an equivariant structure $\iota_{\tau\sigma}^*\Phi := (1_T \times \iota_{\tau\sigma})^*\Phi$. When $\mathcal{E}$ is quasicoherent, if E^σ stands for the σ-family associated to $(\mathcal{E}, \Phi)$, we will denote $\iota_{\tau\sigma}^* E^\sigma$ the τ-family associated to $\iota_{\tau\sigma}^*(\mathcal{E}, \Phi)$ (we drop here the morphisms χ^σ to ease notations). A more explicit description is given in Exercise 1.2.30. In a similar way, any morphism $\mathcal{E} \to \mathcal{F}$ between two quasicoherent sheaves on U_σ yields a morphism $\phi^\sigma : E^\sigma \to F^\sigma$ between the associated σ-families that can be pulled back to a morphism $\iota_{\tau\sigma}^*\phi : \iota_{\tau\sigma}^* E^\sigma \to \iota_{\tau\sigma}^* F^\sigma$. We can now introduce Σ-families.

Definition 1.2.20. A Σ-*family* for a fan Σ is a pair (E^Σ, η^Σ) consisting of a collection $E^\Sigma = (E^\sigma)_{\sigma\in\Sigma}$ of σ-families and a collection $\eta^\Sigma = (\eta_{\tau\sigma})_{\tau\leq\sigma}$ of isomorphisms $\eta_{\tau\sigma} : \iota_{\tau\sigma}^* E^\sigma \to E^\tau$ such that

(1) for each $\sigma \in \Sigma$, $\eta_{\sigma\sigma} = 1_{E^\sigma}$,

(2) for each $\rho \leq \tau \leq \sigma$, $\eta_{\rho\sigma} = \eta_{\rho\tau} \circ \iota_{\rho\tau}^* \eta_{\tau\sigma}$.

A *morphism of* Σ-*families* $\phi^\Sigma : (E^\Sigma, \eta^\Sigma) \to (F^\Sigma, \nu^\Sigma)$ is a collection of morphisms $(\phi^\sigma : E^\sigma \to F^\sigma)_{\sigma\in\Sigma}$ of σ-families such that for all $\tau \leq \sigma$,

$$\nu_{\tau\sigma} \circ \iota_{\tau\sigma}^* \phi^\sigma = \phi^\tau \circ \eta_{\tau\sigma}.$$

The following was the goal of this section, and is due to Perling [59].

Theorem 1.2.21. *For a given fan* Σ*, the* Σ*-families together with their morphisms form a category that is equivalent to* $\mathrm{qCoh}^T(X_\Sigma)$.

Proof. By proposition 1.2.19, any T-equivariant quasicoherent sheaf on X_Σ defines σ-families $(E^\sigma)_{\sigma\in\Sigma}$. It is then clear by definition that $\iota_{\tau\sigma}^* E^\sigma \simeq E^\tau$ for any pair $\tau \leq \sigma$ and that those isomorphisms satisfy (1) and (2) of Definition 1.2.20. In the other direction, given a Σ-family (E^Σ, η^Σ), we obtain via Proposition 1.2.19 quasicoherent

T-sheaves $(\mathcal{E}^\sigma)_{\sigma\in\Sigma}$. Moreover, we can use the isomorphisms $(\eta_{\tau\sigma})_{\tau\leq\sigma}$ to define for each $\tau := \sigma\cap\sigma'$ an isomorphism of τ-families :

$$\phi_{\sigma\sigma'} = \eta_{\tau\sigma}^{-1}\circ\eta_{\tau\sigma'} : \iota_{\tau\sigma'}^* E^{\sigma'} \to \iota_{\tau\sigma}^* E^\sigma$$

and thus an isomorphism of T-sheaves

$$\tilde{\phi}_{\sigma\sigma'} : \iota_{\tau\sigma'}^* \mathcal{E}^{\sigma'} \to \iota_{\tau\sigma}^* \mathcal{E}^\sigma.$$

The collection $(\tilde{\phi}_{\sigma\sigma'})$ satisfy the cocycle conditions $\tilde{\phi}_{\sigma\sigma} = 1$ and $\tilde{\phi}_{\sigma'\sigma''}\circ\tilde{\phi}_{\sigma\sigma'} = \tilde{\phi}_{\sigma\sigma''}$ up to natural equivalence on overlaps $\sigma\cap\sigma'\cap\sigma''$, thanks to (1) and (2) in Definition 1.2.20. Hence, the sheaves $(\mathcal{E}^\sigma)_{\sigma\in\Sigma}$ glue to define a quasicoherent T-sheaf over X_Σ.

The facts that Σ-families form a category, and that the previously described correspondence is functorial, and an equivalence, are left as an exercise. ∎

Coherence

Recall that a quasicoherent sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ is coherent if and only if for any $x\in X$, there exists an affine chart $x\in U = \mathrm{Spec}(A)$ such that $\Gamma(U,\mathcal{E})$ is a finitely generated A-module. As this property is local, it is enough in our case to check it on the T-invariant affines U_σ, for $\sigma\in\Sigma$. The prototype example of a coherent T-sheaf is then given by $\mathcal{O}_X^{\oplus r}$, with a shifted equivariant structure. Consider r weights $(m_1,\dots,m_r)\in M^r$, and the equivariant sheaf $\mathcal{O}_X^{\oplus r}$ with equivariant structure

$$\chi^{m_1}\cdot\Phi_X\oplus\dots\oplus\chi^{m_r}\cdot\Phi_X.$$

Fix a cone $\sigma\in\Sigma$. Then the associated σ-family is given by the direct sum of the families defined in Example 1.2.16. In particular, it is a *finite* σ-family, in the following sense.

Definition 1.2.22. A σ-family (E^σ,χ^σ) is *finite* if it satisfies the following :

(1) For each $m\in M$, E_m^σ is finite dimensional,

(2) For each strictly increasing sequence $(m_i)_{i\in\mathbb{Z}}\in M^{\mathbb{Z}}$, i.e. such that

$$\dots <_\sigma m_{i-1} <_\sigma m_i <_\sigma m_{i+1} <_\sigma \dots,$$

there exists $i_0\in\mathbb{Z}$ such that $E_{m_i}^\sigma = \{0\}$ for $i\leq i_0$,

(3) For each $m\in M$, denote by C_m^σ the cokernel of the map

$$\sum_{m'<_\sigma m}\chi_{m',m}^\sigma : \bigoplus_{m'<_\sigma m} E_{m'}^\sigma \to E_m^\sigma.$$

Then the set of classes $[m]\in M/(\sigma^\perp\cap M)$ such $C_m^\sigma\neq\{0\}$ is finite.

A Σ-family (E^Σ,η^Σ) is *finite* if for each $\sigma\in\Sigma$, E^σ is finite.

Note that by Lemma 1.2.15, condition (3) in the previous definition makes sense as for $m - m' \in \sigma^\perp$, one has $C_m^\sigma \simeq C_{m'}^\sigma$ (cf the proof of next proposition). It is clear that the σ-families associated to the T-sheaves $(\mathcal{O}_X, \chi^{m_i} \cdot \Phi_X)$ as in Example 1.2.16 are finite. We easily deduce from this finiteness of the σ-families associated to $(\mathcal{O}_X^{\oplus r}, \chi^{m_1} \cdot \Phi_X \oplus \ldots \oplus \chi^{m_r} \cdot \Phi_X)$, cf Exercise 1.2.32.

Proposition 1.2.23. *A quasicoherent T-equivariant sheaf is coherent if and only if its associated Σ-family is finite.*

We will use the following lemma, whose proof is left as an exercise.

Lemma 1.2.24. *Let $\phi^\sigma : (E^\sigma, \chi^\sigma) \to (F^\sigma, \psi^\sigma)$ be a morphism of σ-families. Assume that ϕ^σ is surjective, i.e. for all $m \in M$, $\phi_m^\sigma : E_m^\sigma \to F_m^\sigma$ is surjective. If (E^σ, χ^σ) is finite, then (F^σ, ψ^σ) is finite.*

Proof of Proposition 1.2.23. Let $\mathcal{E}$ be a quasicoherent T-sheaf, with associated σ-families (E^σ, χ^σ), $\sigma \in \Sigma$.

If $\mathcal{E}$ is coherent, for any $\sigma \in \Sigma$, $\Gamma(U_\sigma, \mathcal{E})$ is a finitely generated $\mathbb{C}[S_\sigma]$-module. If $(s_i)_{1 \leq i \leq r}$ stands for a set of generators, we can decompose each s_i according to the M-grading of $\Gamma(U_\sigma, \mathcal{E})$ as

$$s_i = s_{m_{i1}} \oplus \ldots \oplus s_{m_{ip_i}}$$

with $s_{m_{ij}} \in \Gamma(U_\sigma, \mathcal{E})_{m_{ij}}$. Then, the sections $(s_{m_{ij}})$ generate $\Gamma(U_\sigma, \mathcal{E})$ as an M-graded module. That is, there is a surjective morphism of M-graded $\mathbb{C}[S_\sigma]$-modules

$$\begin{array}{ccc} \bigoplus_{ij} \mathbb{C}[S_\sigma][m_{ij}] & \to & \Gamma(U_\sigma, \mathcal{E}) \\ (f_{ij}) & \mapsto & \sum_{ij} f_{ij} \cdot s_{m_{ij}}. \end{array}$$

By the previous discussion, the module $\bigoplus_{ij} \mathbb{C}[S_\sigma][m_{ij}]$ corresponds to a finite σ-family. From Lemma 1.2.24, we deduce finiteness of E^σ.

In the other direction, assume that E^σ is a finite σ-family. We will show that $\Gamma(U_\sigma, \mathcal{E})$ is a finitely generated $\mathbb{C}[S_\sigma]$-module. For any $m \in M$, denote by $\chi_m^{<\sigma}$ the map

$$\chi_m^{<\sigma} := \sum_{m' <_\sigma m} \chi_{m',m}^\sigma : \bigoplus_{m' <_\sigma m} E_{m'}^\sigma \to E_m^\sigma$$

and by

$$\pi_m : E_m^\sigma \to C_m^\sigma$$

the projection to the cokernel of $\chi_m^{<\sigma}$. For any pair $m_1 \leq_\sigma m_2$, the following diagram is commutative

$$\begin{tikzcd} \bigoplus_{m_1' <_\sigma m_1} E^\sigma_{m_1'} \arrow[r, "\chi_{m_1}^{<\sigma}"] \arrow[d, "\chi^\sigma_{m_1',m_2'}"] & E^\sigma_{m_1} \arrow[r, "\pi_{m_1}"] \arrow[d, "\chi^\sigma_{m_1,m_2}"] & C^\sigma_{m_1} \arrow[d, "{[\chi^\sigma_{m_1,m_2}]}"] \\ \bigoplus_{m_2' <_\sigma m_2} E^\sigma_{m_2'} \arrow[r, "\chi_{m_2}^{<\sigma}"] & E^\sigma_{m_2} \arrow[r, "\pi_{m_2}"] & C^\sigma_{m_2} \end{tikzcd}$$

where the last vertical arrow is the map induced by $\chi^\sigma_{m_1,m_2}$ on the quotient $C^\sigma_{m_1}$. If $m_1 - m_2 \in \sigma^\perp$, by Lemma 1.2.15, the first two vertical arrows are isomorphisms, and so is the last one. Hence the set of classes in $M/(\sigma^\perp \cap M)$:

$$W_\sigma := \left\{[m] \in M/(\sigma^\perp \cap M) \mid C^\sigma_m \neq \{0\}\right\}$$

is well defined. By (3) of Definition 1.2.22, the set W_σ is finite. By Definition 1.2.22 (1), for any $[m] \in W_\sigma$, C^σ_m is finite dimensional. Consider, for each $[m] \in W_\sigma$, a splitting

$$E^\sigma_m = \mathrm{Im}(\chi_m^{<\sigma}) \oplus \hat{C}^\sigma_m \tag{1.13}$$

with $\hat{C}^\sigma_m \simeq C^\sigma_m$ through π_m. We may assume that for $m_1 - m_2 \in \sigma^\perp$,

$$\chi^\sigma_{m_1,m_2} : \hat{C}^\sigma_{m_1} \to \hat{C}^\sigma_{m_2}$$

is an isomorphism. Recall then that

$$\Gamma(U_\sigma, \mathcal{E}) = \bigoplus_{m \in M} E^\sigma_m$$

and that the $\mathbb{C}[S_\sigma]$-module structure is defined by setting, for $m \in S_\sigma$, and $e_{m'} \in E^\sigma_{m'}$,

$$\chi^m \cdot e_{m'} := \chi^\sigma_{m',m+m'}(e_{m'}).$$

Then, from the facts that W_σ is finite, $\dim(\hat{C}^\sigma_m)$ is finite for $[m] \in W_\sigma$, and $\hat{C}^\sigma_{m'} \simeq \hat{C}^\sigma_m$ for $m' \in [m]$, it is enough to conclude to show that for any $m' \in M$, any element $e_{m'} \in E^\sigma_{m'}$ admits a finite decomposition

$$e_{m'} = \sum_{[m] \in W_\sigma} \chi^\sigma_{m,m'}(c_m)$$

where $c_m \in \hat{C}_m$. Let then $e_{m_0} \in E^\sigma_{m_0}$. Consider the decomposition

$$e_{m_0} = \chi_{m_0}^{<\sigma}(e_{-1}) + c_{m_0} \in \mathrm{Im}(\chi_{m_0}^{<\sigma}) \oplus \hat{C}^\sigma_{m_0}$$

where

$$\chi_{m_0}^{<\sigma}(e_{-1}) = \sum_i \chi_{m_i,m_0}^{\sigma}(e_{m_i})$$

for a finite number of weights $m_i <_\sigma m_0$ and non-zero vectors $e_{m_i} \in E_{m_i}^{\sigma}$. If $e_{m_0} = c_{m_0}$, we end here. Otherwise, we decompose each e_{m_i} according to (1.13) again, and iterate this process. We claim that this terminates after a finite number of steps. If not, we could pick some $m_{-1} := m_i$ with $\chi_{m_i,m_0}^{\sigma}(e_{m_i}) \neq 0$ in the decomposition of e_{m_0}. Then, we could produce inductively sequences of weights

$$m_j <_\sigma m_{j+1} <_\sigma \ldots <_\sigma m_0$$

and non zero vectors $e_{m_j} \in E_{m_j}^{\sigma}$, in contradiction with Definition 1.2.22 (2). This process hence produces the desired decomposition for e_{m_0}, which concludes the proof. ■

Example 1.2.25. We will see in the next chapter, Section 2.1, that for any torus invariant subvariety $V(\tau) \subset X$, the ideal sheaf $\mathcal{I}_{V(\tau)}$ is naturally endowed with a torus equivariant structure. If $\iota : V(\tau) \longrightarrow X$ stands for the inclusion map, the structure sheaf $\iota_*\mathcal{O}_{V(\tau)}$ inherits a T-equivariant structure as well. The sheaves $\mathcal{I}_{V(\tau)}$ and $\iota_*\mathcal{O}_{V(\tau)}$ are examples of coherent toric sheaves.

1.2.1 Exercises for 1.2

Exercise 1.2.26 (The abelian categories of T-sheaves). Let $(\mathcal{E}, \Phi)$ and $(\mathcal{F}, \Psi)$ be two (quasi-)coherent T-sheaves.

(1) Show that $\mathrm{Hom}^T(\mathcal{E}, \mathcal{F})$ inherits an abelian group structure from $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$.

(2) Show that $\mathcal{E} \oplus \mathcal{F}$ admits a T-equivariant structure.

(3) If $\theta \in \mathrm{Hom}^T(\mathcal{E}, \mathcal{F})$, show that $\ker(\theta)$ and $\mathrm{coker}(\theta)$ admit T-equivariant structures.

Together with the facts that $\mathrm{qCoh}(X)$ and $\mathrm{Coh}(X)$ are abelian categories, this shows that $\mathrm{qCoh}^T(X)$ and $\mathrm{Coh}^T(X)$ are abelian.

Exercise 1.2.27 (Dual action and weight space decomposition). Let $(\mathcal{E}, \Phi)$ be a quasicoherent T-equivariant sheaf over some normal affine toric variety $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$. Set $E = \Gamma(U_\sigma, \mathcal{E})$ and consider the composition

$$\Phi^* := \Phi \circ \alpha^* : E \longrightarrow \mathbb{C}[M] \otimes_{\mathbb{C}} E,$$

where we used that $\Gamma(T \times U_\sigma, \pi_2^*\mathcal{E}) \simeq \mathbb{C}[M] \otimes_{\mathbb{C}} E$. This is called the *dual action*.

(1) Show that Φ^* satisfies the following :

(i) The $\mathbb{C}$-linear maps $(\mu^* \otimes 1_E) \circ \Phi^*$ and $(1_{\mathbb{C}[M]} \otimes \Phi^*) \circ \Phi^*$ from E to $\mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[M] \otimes_{\mathbb{C}} E$ are equal,

(ii) $1^* \circ \Phi^* = 1_E$, where $1^* : \mathbb{C}[M] \to \mathbb{C}$ is dual to the inclusion $1 \to T$.

(2) Show that the representation of T given by $t \mapsto \iota_{t^{-1}}^* \circ \Phi^*$ is regular.

(3) Show that this implies a weight space decomposition for $\Gamma(U_\sigma, \mathcal{E})$.

Exercise 1.2.28 (Equivariant structures). Let $\mathcal{E}$ be a quasicoherent sheaf of $\mathcal{O}_X$-modules on X. Assume we are given for each $t \in T$ an isomorphism $\Phi_t : t^*\mathcal{E} \to \mathcal{E}$ such that the cocycle condition (1.6) is satisfied, and for each $\sigma \in \Sigma$, the representation of T on $\Gamma(U_\sigma, \mathcal{E})$ given by $t \mapsto \Phi_{t^{-1}} \circ \alpha_{t^{-1}}^*$ is regular.

(1) Show that there exists a T-equivariant structure on $\mathcal{E}$ such that for all $t \in T$, $\Phi_t = \iota_t^* \Phi$. Hint : you may restrict to affines U_σ and define Φ as in the proof of Proposition 1.2.10.

(2) Consider two equivariant structures Φ and Ψ on $\mathcal{E}$. Assume that for all t, $\Phi_t = \Psi_t$. Show that $\Phi = \Psi$.

Exercise 1.2.29 (The partial orders $\leq_\sigma$). Prove Lemma 1.2.13.

Exercise 1.2.30 (Restriction of σ families to affine subsets). Let $\mathcal{E} \in \mathrm{qCoh}^T(U_\sigma)$ with associated σ-family (E^σ, χ^σ). Consider a face $\tau \leq \sigma$, with $\tau = a^\perp \cap M$ for some $a \in \sigma^\vee \cap M$, that we may assume to be primitive[2]. Denote the injection $\iota_{\tau\sigma} : U_\tau \to U_\sigma$. Set $(F^\tau, \psi^\tau) = \iota_{\tau\sigma}^*(E^\sigma, \chi^\sigma)$ to be the τ-family associated to the pulled back T-sheaf $\iota_{\tau\sigma}^*\mathcal{E}$. We introduce a relation $\sim$ on

$$\bigoplus_{p\in\mathbb{Z}} E^\sigma_{m+pa}$$

defined on generators $(e_{m+pa}, f_{m+qa}) \in E^\sigma_{m+pa} \times E^\sigma_{m+qa}$, by $e_{m+pa} \sim f_{m+qa}$ iff

$$\exists r \in \mathbb{N}\,,\ \chi^\sigma_{m+pa,m+(p+q+r)a}(e_{m+pa}) = \chi^\sigma_{m+qa,m+(p+q+r)a}(f_{m+qa}),$$

and then extending linearly.

(1) Show that $\sim$ defines an equivalence relation.

(2) Show that for each $m \in M$,

$$F^\tau_m = \Big(\bigoplus_{p\in\mathbb{Z}} E^\sigma_{m+pa}\Big)/\sim\,.$$

(3) Check that for $m - m' \in \tau^\perp \cap M$, $F^\tau_m \simeq F^\tau_{m'}$.

(4) Show that for $m \leq_\sigma m'$, the morphism

$$\bigoplus_{p\in\mathbb{Z}} \chi^\sigma_{m+pa,m'+pa}$$

descends modulo $\sim$ and gives $\psi^\tau_{m,m'} : F^\tau_m \to F^\tau_{m'}$.

[2] $m \in M$ is primitive if $\mathbb{Z}\cdot m = \mathbb{R}\cdot m \cap M$ in $M_\mathbb{R}$.

(5) Describe all morphisms $\psi^{\tau}_{m,m'}$, for $m \leq_{\tau} m'$.

Exercise 1.2.31 (The category of Σ-families).

(1) Show that Σ-families with their morphisms form a category.

(2) Conclude the proof of Theorem 1.2.21.

(3) Deduce that the category of Σ-families is abelian.

Exercise 1.2.32 (Direct sums of structure sheaves). We show in this exercise finiteness of the σ-families associated to $\mathcal{E} = (\mathcal{O}_X^{\oplus r}, \chi^{m_1} \cdot \Phi_X \oplus \ldots \oplus \chi^{m_r} \cdot \Phi_X)$.

(1) Show that $\chi^{m_1} \cdot \Phi_X \oplus \ldots \oplus \chi^{m_r} \cdot \Phi_X$ defines a T-equivariant structure on $(\mathcal{O}_X)^{\oplus r}$.

(2) Let $\sigma \in \Sigma$. Show that the σ-family $(E^{\sigma}, \chi^{\sigma})$ associated to $\mathcal{E}$ can be described as follows. Set $(e_1, \ldots, e_r)$ to be the canonical basis for $\mathbb{C}^r$. Then for each $m \in M$,

$$E^{\sigma}_m = \mathrm{Span}_{\mathbb{C}}\{e_i \mid m_i \leq_{\sigma} m\},$$

with morphisms $\chi^{\sigma}_{m,m'}$ given by the obvious inclusions.

(3) Conclude that it is a finite σ-family.

Exercise 1.2.33 (Surjective morphisms and finiteness). Give a proof of Lemma 1.2.24.

1.3 Torsion-free sheaves and Multifiltrations

We will specify our study to torsion-free sheaves, and characterize reflexivity and locally freeness. Recall that a coherent sheaf $\mathcal{E}$ is *torsion-free* if for any local sections $(s, e) \in \Gamma(U, \mathcal{O}_X) \times \Gamma(U, \mathcal{E})$ over some affine set $U = \mathrm{Spec}(A)$, $s \cdot e = 0$ implies $s = 0$ or $e = 0$. This is equivalent to saying that the A-module $\Gamma(U, \mathcal{E})$ is torsion-free. For any coherent-sheaf $\mathcal{E}$, the evaluation map yields a canonical morphism to its double dual $(\mathcal{E}^\vee)^\vee$, where the dual sheaf is by definition

$$\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X).$$

In the torsion-free case,

$$ev : \mathcal{E} \to (\mathcal{E}^\vee)^\vee$$

is an injection, and $(\mathcal{E}^\vee)^\vee$ is called the *reflexive hull* of $\mathcal{E}$. When ev is an isomorphism, we call $\mathcal{E}$ a *reflexive sheaf*. Finally, we recall that a *locally free sheaf* is a sheaf $\mathcal{E}$ such that at any point $x \in X$ there exists an affine chart $x \in U$ such that $\mathcal{E}_{|U}$ is isomorphic to $(\mathcal{O}_X^{\oplus r})_{|U}$. Locally free sheaves are reflexive, and reflexive ones are torsion-free, cf Exercise 1.3.30.

Torsion-free sheaves on the torus

A useful first result for this section is that a locally free T-sheaf over a torus is free, i.e. globally isomorphic to a direct sum of copies of the structure sheaf, with a shifted equivariant structure.

Proposition 1.3.1. *Let $\mathcal{E}$ be a locally-free T-sheaf over $T = \mathrm{Spec}(\mathbb{C}[M])$. Then there are weights $(m_1, \ldots, m_r) \in M^r$ such that*

$$\mathcal{E} \simeq (\mathcal{O}_T^{\oplus r}, \chi^{m_1} \cdot \Phi_T \oplus \ldots \oplus \chi^{m_r} \cdot \Phi_T)$$

as a T-equivariant sheaf.

Recall that the equivariant structure Φ_T, and its shifted versions, are defined in Examples 1.2.6 and 1.2.9. Later on we will extend this result to any affine toric varieties, following Klyachko [39, Proposition 2.1.1], or Kaneyama [37, Theorem 3.5]. We will use the following lemma in the proof :

Lemma 1.3.2. *Let $\mathcal{E}$ be a T-equivariant sheaf over the torus T. Consider two sections $(s_1, s_2) \in \Gamma(T, \mathcal{E})_m$ of given degree $m \in M$ and assume that there is a point $t_0 \in T$ such that $s_1(t_0) = s_2(t_0)$. Then $s_1 = s_2$.*

Proof. For all $t \in T$, we have

$$t \cdot s_i = \chi^{-m}(t) s_i,$$

hence, considering $t^{-1} \cdot s_i$ gives

$$\Phi_t(t^* s_1)(t_0) = \Phi_t(t^* s_2)(t_0).$$

As Φ_t is an isomorphism, we deduce that for all $t \in T$, $s_1(tt_0) = s_2(tt_0)$. The result follows. ∎

Proof of Proposition 1.3.1. Consider the distinguished point $\gamma_0 \in T = U_{\{0\}}$ (that is $(1, \ldots, 1) \in (\mathbb{C}^*)^n$), cf Exercise 1.1.15. The idea is to produce the equivariant trivialisation at γ_0, and then to extend it on T. By locally freeness, the stalk of $\mathcal{E}$ at γ_0 is

$$\mathcal{E}_{\gamma_0} \simeq \mathcal{O}_{T,\gamma_0}^{\oplus r}.$$

Let $\mathfrak{m}_{X,\gamma_0}$ be the maximal ideal of $\mathcal{O}_{T,\gamma_0}$. Using the isomorphism

$$(\mathcal{O}_{T,\gamma_0}/\mathfrak{m}_{T,\gamma_0})^{\oplus r} \simeq \mathbb{C}^r,$$

we define the following morphism of $\mathbb{C}$-vector spaces by evaluating sections at γ_0 :

$$\begin{array}{cccc} \phi : & \Gamma(T,\mathcal{E}) & \rightarrow & \mathbb{C}^r \\ & s & \mapsto & [s(\gamma_0)]. \end{array}$$

Note that if a section $s \in \Gamma(T,\mathcal{E})$ decomposes as $s = s_{m_1} + \ldots + s_{m_d}$ according to the M-module structure of $\Gamma(T,\mathcal{E})$, we have for all $t \in T$

$$\phi(t \cdot s) = \sum_{i=1}^{d} \chi^{m_i}(t)\, \phi(s_{m_i}). \tag{1.14}$$

We then look at a T-invariant subspace $E \subset \Gamma(T,\mathcal{E})$ such that $\phi_{|E}$ is an isomorphism. Then (1.14) will define a representation of the torus, that we may upgrade to an equivariant structure on $(\mathcal{O}_T^{\oplus r})$. Let then E be maximal for the inclusion amongst all T-invariant subspaces of $\Gamma(T,\mathcal{E})$ on which ϕ is injective. Then $\phi_{|E}$ is an isomorphism. Indeed, as ϕ is surjective, if $\phi(E) \subsetneq \mathbb{C}^r$, we may find $e_m \in \Gamma(T,\mathcal{E})_m$ such that $\phi(e_m) \notin \phi(E)$. But then $E \oplus \mathbb{C} \cdot e_m$ is a T-invariant subspace on which ϕ is injective, which contradicts maximality of E.

We then consider a basis $(e_i)_{1\le i\le r}$ of E which consists of T-eigenvectors, with $e_i \in \Gamma(T,\mathcal{E})_{m_i}$ for $1 \le i \le r$. We claim that

$$\begin{array}{cccc} \psi : & \bigoplus_{i=1}^{r} \mathbb{C}[M][m_i] & \rightarrow & \Gamma(T,\mathcal{E}) \\ & (s_i)_{1\le i\le r} & \mapsto & s_1 e_1 + \ldots + s_r e_r \end{array}$$

induces an isomorphism of coherent T-sheaves between $((\mathcal{O}_T)^{\oplus r}, \chi^{m_1} \cdot \Phi_T \oplus \ldots \oplus \chi^{m_r} \cdot \Phi_T)$ and $\mathcal{E}$. For this choice of shifted equivariant structure, ψ is a degree-0 morphism of $\mathbb{C}[M]$-modules. Hence it is enough to show that ψ is an isomorphism on each degree-m components. Consider then sections $s_i \in \mathbb{C}[M][m_i]_m$, that we can write explicitly as $s_i = c_i\,\chi^{m-m_i}$ for some $c_i \in \mathbb{C}$, satisfying

$$s_1 e_1 + \ldots + s_r e_r = 0.$$

We then have

$$0 = \phi\Big(\sum_{i=1}^{r} s_i e_i\Big) = \sum_{i=1}^{r} c_i\,\chi^{m-m_i}(\gamma_0)\,\phi(e_i) = \sum_{i=1}^{r} c_i\,\phi(e_i).$$

As $\phi_{|E}$ is injective, we deduce that $c_i = 0$ for all $1 \leq i \leq r$, and then ψ is injective. To prove surjectivity, let $e_m \in \Gamma(T, \mathcal{E})_m$, and evaluate at γ_0. Surjectivity of ϕ shows that there are constants $(c_i)_{1\leq i\leq r} \in \mathbb{C}^r$ such that:

$$\phi(e_m) = \sum_{i=1}^{r} c_i\,\phi(e_i).$$

Then, the sections e_m and $\sum_{i=1}^{r} c_i\,\chi^{m-m_i}\,e_i$ are of degree m, and agree at γ_0. By Lemma 1.3.2, we conclude that they agree on T. This concludes surjectivity of ψ, and the proof of the proposition. ∎

In the previous proof, we could choose a different space $E' \subset \Gamma(T, \mathcal{E})$ such that $\phi_{|E'}$ is an isomorphism. This would result in an isomorphism between $\mathcal{E}$ and another shifted free sheaf. In fact, it is easy to prove that a locally free sheaf $\mathcal{E}$ on T is isomorphic to $((\mathcal{O}_T)^{\oplus r}, \chi^{m_1}\cdot\Phi_T \oplus \ldots \oplus \chi^{m_r}\cdot\Phi_T)$ for *any* choice of weights $(m_1, \ldots, m_r)$. See Exercise 1.3.32 for a generalisation of this fact on any affine toric variety.

For a given coherent sheaf $\mathcal{E}$, the *singular locus* of $\mathcal{E}$ is the closed set of points $x \in X$ such that $\mathcal{E}$ is not locally free at x. If $\mathcal{E}$ is T-equivariant, and $\mathcal{E}$ is locally free on some affine set $U \subset X$, then it is locally free on $t^{-1}\cdot U$ for all $t \in T$. Indeed, if Φ denotes the T-equivariant structure, Φ_t defines an isomorphism from $t^*\mathcal{E}$ to $\mathcal{E}$. Hence, $\mathcal{E}$ is locally free on an open T-invariant set. Its singular locus is then a (finite) union of orbit closures (denoted $V(\tau)$ in the orbit-cone correspondence). In conclusion, we have:

Lemma 1.3.3. *Let $\mathcal{E}$ be a coherent T-equivariant sheaf on X. Then its singular locus is a finite union of orbit closures*

$$\mathrm{Sing}(\mathcal{E}) = \bigcup_{\tau\in\Sigma_s} V(\tau)$$

for some subset $\Sigma_s \subset \Sigma$ such that $\dim\tau \geq 1$ *for any $\tau \in \Sigma_s$. If $\mathcal{E}$ is torsion-free (resp. reflexive), the same conclusion holds with* $\dim\tau \geq 2$ *(resp.* $\dim\tau \geq 3$*) for all $\tau \in \Sigma_s$.*

Note that by definition, $\Sigma \setminus \Sigma_s$ is a fan, hence Σ_s cannot be any subset of Σ.

Proof. From the previous discussion, $\mathrm{Sing}(\mathcal{E})$ is closed and T-invariant. It is then a union of orbit closures. The statements about the dimension follows from the facts that $\dim(\tau) = n - \dim\big(V(\tau)\big)$ for each $\tau \in \Sigma$, and that for a torsion-free sheaf (resp. a reflexive sheaf), the singular locus is of codimension at least 2 (resp. at least 3), cf [55, Chapter 2, Section 1]. ∎

Combining Proposition 1.3.1 with Lemma 1.3.3, we obtain the following.

Corollary 1.3.4. *Let $\mathcal{E}$ be a coherent T-sheaf over the torus T. Then $\mathcal{E}$ is free, and isomorphic to $(\mathcal{O}_T^{\oplus r}, \chi^{m_1} \cdot \Phi_T \oplus \ldots \oplus \chi^{m_r} \cdot \Phi_T)$ for some weights $(m_1, \ldots, m_r)$.*

Torsion-free equivariant sheaves

We will now study in more details the Σ-families associated to torsion-free T-sheaves. We will need the following lemma, which together with Lemma 1.2.13, shows that $(M, \leq_\sigma)$ is a directed set.

Lemma 1.3.5. *Any finite set of weights $S \subset M$ admits an upper bound for $\leq_\sigma$. Hence, $(M, \leq_\sigma)$ is a directed set.*

The proof is left as an Exercise 1.3.33. Let $\mathcal{E}$ be a coherent T-equivariant sheaf on X with associated Σ-family (E^Σ, η^Σ). For each $\sigma \in \Sigma$, by Lemma 1.3.5, (E^σ, χ^σ) admits a direct limit, that we denote by

$$\mathbf{E}^\sigma := \varinjlim E^\sigma_m.$$

Recall that an element $[e_m] \in \mathbf{E}^\sigma$ is a class in

$$\bigcup_{m \in M} E^\sigma_m / \sim$$

where for $e_m \in E^\sigma_m$ and $f_p \in E^\sigma_p$, one has $e_m \sim f_p$ if there is $q \in M$ with $m \leq_\sigma q$ and $p \leq_\sigma q$ such that $\chi^\sigma_{m,q}(e_m) = \chi^\sigma_{p,q}(f_p)$. Then, for any pair $m \leq_\sigma m'$, we have a commutative diagram

$$\begin{tikzcd} E^\sigma_m \arrow[r, "\chi^\sigma_{m,m'}"] \arrow[dr] & E^\sigma_{m'} \arrow[d] \\ & \mathbf{E}^\sigma \end{tikzcd} \tag{1.15}$$

Proposition 1.3.6. *The following are equivalent:*

(i) *$\mathcal{E}_{|U_\sigma}$ is torsion-free,*

(ii) *For each $(m, m') \in S_\sigma \times M$, and each $e \in \Gamma(U_\sigma, \mathcal{E})_{m'}$, $\chi^m \cdot e = 0$ implies $e = 0$,*

(iii) *All the maps in the diagram (1.15) are injective.*

Proof. The fact that $\mathcal{E}_{U_\sigma}$ is torsion-free is equivalent to $\Gamma(U_\sigma, \mathcal{E})$ being a torsion-free $\mathbb{C}[S_\sigma]$-module. Then (i) implies (ii). Point (iii) is the transcription of (ii) in terms of σ-families, as the map $\chi^\sigma_{m,m'}$ is the multiplication by the character function $\chi^{m'-m}$. This shows the equivalence between (ii) and (iii). We now show that (ii) implies (i). Let

$$s = \sum_i c_i\, \chi^{m_i} \in \mathbb{C}[S_\sigma]$$

and

$$e = \sum_j e_{m_j} \in \Gamma(U_\sigma, \mathcal{E})$$

be sections such that $s \cdot e = 0$, that we decompose according to the M-grading. Fix the lexicographic order on $M \simeq \mathbb{Z}^n$. Let then m_{i_0} (resp. m_{j_0}) be minimal for this order such that $c_i \neq 0$ (resp. $e_{m_j} \neq 0$). Then, the minimal degree in the developed expression of $s \cdot e$ is $m_{i_0} + m_{j_0}$ with corresponding term $c_{i_0}\, \chi^{m_{i_0}} \cdot e_{m_{j_0}}$ (here we use translation invariance of the lexicographic order). As $s \cdot e = 0$, $c_{i_0}\, \chi^{m_{i_0}} \cdot e_{m_{j_0}} = 0$, and by assumption (ii), either $c_{i_0} = 0$ or $e_{m_{j_0}} = 0$, which yields a contradiction. We conclude that $s = 0$ or $e = 0$, and $\mathcal{E}_{|U_\sigma}$ is torsion-free. ∎

We now consider a face $\tau \leq \sigma$. We will use the following lemma, whose proof is straightforward :

Lemma 1.3.7. *If $\tau \leq \sigma$, then for any $(m, m') \in M^2$, $m \leq_\sigma m'$ implies $m \leq_\tau m'$.*

Assume that we have a family of linear maps $\varphi_m : E^\sigma_m \to E^\tau_m$ such that for any $m \leq_\sigma m'$, the following diagram is commutative :

$$\begin{tikzcd}
E^\sigma_m \arrow[r, "\varphi_m"] \arrow[d, "\chi^\sigma_{m,m'}"] & E^\tau_m \arrow[d, "\chi^\tau_{m,m'}"] \\
E^\sigma_{m'} \arrow[r, "\varphi_{m'}"] & E^\tau_{m'}
\end{tikzcd}$$

Then the collection $(\varphi_m)_{m \in M}$ induces a linear map $\varphi_{\tau\sigma} : \mathbf{E}^\sigma \to \mathbf{E}^\tau$. Assume from now on that $\mathcal{E}$ is torsion-free. Then the restrictions

$$\iota^*_{\tau\sigma} : \Gamma(U_\sigma, \mathcal{E})_m \to \Gamma(U_\tau, \mathcal{E})_m$$

are injective. Composing with the isomorphisms $\eta_{\tau\sigma}$, we obtain injective maps :

$$\varphi_m : E^\sigma_m \xrightarrow{\iota^*_{\tau\sigma}} \iota^*_{\tau\sigma} E^\sigma_m \xrightarrow{\eta_{\tau\sigma}} E^\tau_m.$$

Using the facts that $\tau^\vee = \sigma^\vee + \mathbb{N} \cdot (-m_0)$ for some $m_0 \in \sigma^\vee \cap \tau^\perp$, and that for weights (m, m') with $m - m' \in \tau^\perp$, $\chi^\tau_{m,m'}$ is an isomorphism, we deduce that the induced map $\varphi_{\tau\sigma} : \mathbf{E}^\sigma \to \mathbf{E}^\tau$ is injective. Consider now the directed set $(\Sigma, \leq)$. With the reverse order on faces, we obtain a direct system $(\mathbf{E}^\sigma, \varphi_{\tau\sigma})$, with direct limit $\mathbf{E}^0$, as can be seen from the diagrams

$$\begin{tikzcd} \mathbf{E}^{\sigma_1} \arrow[dr, "\varphi_{\tau\sigma_1}"] & & \\ & \mathbf{E}^\tau \arrow[r, "\varphi_{0\tau}"] & \mathbf{E}^0 \\ \mathbf{E}^{\sigma_2} \arrow[ur, "\varphi_{\tau\sigma_2}"] & & \end{tikzcd}$$

where $\tau = \sigma_1 \cap \sigma_2$ for a pair $(\sigma_1, \sigma_2) \in \Sigma^2$, and where 0 stands for the minimal cone in Σ. We will actually show that all the $\varphi_{\tau\sigma}$ are isomorphisms.

Proposition 1.3.8. *For any $\tau \leq \sigma$, $\varphi_{\tau\sigma} : \mathbf{E}^\sigma \to \mathbf{E}^\tau$ is an isomorphism.*

Before showing this, note that from Corollary 1.3.4, $\Gamma(U_0, \mathcal{E})$ is a free $\mathbb{C}[M]$-module of rank r. Then, all maps $\chi^0_{m,m'}$ are isomorphisms, and for all $m \in M$, $E^0_m \simeq \mathbf{E}^0 \simeq \mathbb{C}^r$. Together with Proposition 1.3.6 and Proposition 1.3.8, this shows that all the spaces E^σ_m admit injections to $\mathbf{E}^0$, in a compatible way with the morphisms $\chi^\sigma_{m,m'}$ and the restriction maps $\iota^*_{\tau\sigma}$. Proposition 1.3.8 is a corollary of the following lemma. Recall that $m_0 \in \sigma^\vee$ is such that $S_\tau = S_\sigma + \mathbb{N} \cdot (-m_0)$.

Lemma 1.3.9. *For any $m \in M$, there is $i_m \in \mathbb{Z}$ such that $E^\tau_m \simeq E^\sigma_{m+im_0}$ for all $i \geq i_m$.*

Proof. For each $m \in M$, consider the strictly increasing sequence

$$\ldots <_\sigma m + im_0 <_\sigma m + (i+1)m_0 <_\sigma \ldots$$

As $\mathbf{E}^\sigma$ injects into $\mathbf{E}^0$ which is finite dimensional, and by Proposition 1.3.6, the sequence of vector spaces $(E^\sigma_{m+im_0})_{i\in\mathbb{Z}}$ (with injections $\chi^\sigma_{m+im_0,m+(i+1)m_0}$) must be stationary after some $i_m \in \mathbb{Z}$. Using the fact that $\chi^\tau_{m+im_0,m+(i+1)m_0}$ is an isomorphism as $m_0 \in \tau^\perp$, we deduce that

$$(\iota^*_{\tau\sigma} E^\sigma)_m \simeq E^\sigma_{m+im_0}$$

for $i \geq i_m$ (compare with Exercise 1.2.30), hence the result. ∎

All these properties motivate the following definition introduced by Perling [58].

Definition 1.3.10. Let E be a finite dimensional vector space. A *family of multifiltrations* for E (indexed by Σ) is the data of vector subspaces $(E^\sigma_m)_{\sigma\in\Sigma, m\in M}$ of E satisfying the following, for each $\sigma \in \Sigma$:

(1) For any pair $m \leq_\sigma m'$, $E^\sigma_m \subset E^\sigma_{m'}$,

(2) E is the direct limit of the direct system (E^{σ}_m) over $(M, \leq_{\sigma})$:

$$E = \lim_{\rightarrow} E^{\sigma}_m,$$

(3) For each strictly increasing sequence $(m_i)_{i\in\mathbb{Z}} \in M^{\mathbb{Z}}$ with respect to $\leq_{\sigma}$,

$$\ldots <_{\sigma} m_{i-1} <_{\sigma} m_i <_{\sigma} m_{i+1} <_{\sigma} \ldots,$$

there exists $i_0 \in \mathbb{Z}$ such that $E^{\sigma}_{m_i} = \{0\}$ for $i \leq i_0$,

(4) There are finitely many classes $[m] \in M/(\sigma^{\perp} \cap M)$ such that

$$\mathrm{Span}_{\mathbb{C}}\{E^{\sigma}_{m'} \mid m' <_{\sigma} m\} \subsetneq E^{\sigma}_m,$$

(5) For any face $\tau \leq \sigma$, and any $m \in M$, $E^{\sigma}_m \subset E^{\tau}_m$. Moreover, if $\tau = m_0^{\perp} \cap \sigma$ for $m_0 \in \sigma^{\vee}$, there is $i_m \in \mathbb{Z}$ such that for all $i \geq i_m$, $E^{\tau}_m = E^{\sigma}_{m+im_0}$.

Let $(E^{\sigma}_m)_{\sigma\in\Sigma,m\in M}$ (resp. $(F^{\sigma}_m)_{\sigma\in\Sigma,m\in M}$) be a family of multifiltrations for E (resp. for F). A *morphism* between these two families is a vector space homomorphism $\phi : E \to F$ such that for all $\sigma \in \Sigma$, and all $m \in M$, $\phi(E^{\sigma}_m) \subset F^{\sigma}_m$.

To ease notations, we will simply denote by (E^{σ}_m) a family of multifiltrations for a vector space E. The previous discussions leads to the following equivalence :

Theorem 1.3.11. *The category of torsion-free, coherent, T-equivariant sheaves on X is equivalent to the category of multifiltrations indexed by Σ.*

The previous propositions have shown that any torsion-free T-sheaf defines a family of multifiltrations. We leave to the reader the proof that this correspondence induces an equivalence of categories, cf Exercise 1.3.34.

Example 1.3.12. The structure sheaf $\mathcal{O}_X$ and the ideal sheaves $\mathcal{I}_V$ of torus invariant subvarieties are examples of torsion-free toric sheaves. For the latter, the families of multifiltrations will be given in Section 2.1.

Reflexivity

We now specify to reflexive sheaves. We recall an important characterisation of such sheaves.

Definition 1.3.13. A coherent sheaf $\mathcal{E}$ over X is *normal* if for any affine chart $U \subset X$ and any closed subset $Z \subset U$ with $\mathrm{codim}(Z) \geq 2$, the restriction map :

$$\Gamma(U, \mathcal{E}) \to \Gamma(U \setminus Z, \mathcal{E})$$

is an isomorphism.

By [27, Proposition 1.6], as X is a normal variety, we have :

Proposition 1.3.14. *The following are equivalent for a torsion-free sheaf $\mathcal{E}$ over X:*

(i) *$\mathcal{E}$ is reflexive,*

(ii) *$\mathcal{E}$ is normal,*

(iii) *For any affine chart $U \subset X$ and any closed subset $Z \subset U$ with* $\operatorname{codim}(Z) \geq 2$*, $\mathcal{F}_{|U} = j_*\mathcal{F}_{|U\setminus Z}$, where $j : U \setminus Z \to U$ is the inclusion.*

Consider now a torsion-free T-equivariant sheaf $\mathcal{E}$ over X. The normal variety X is covered by the affine chars $U_\sigma, \sigma \in \Sigma$. By the orbit-cone correspondence - Proposition 1.1.9 - for any $\sigma \in \Sigma$,

$$U_\sigma = \bigcup_{\tau \leq \sigma} O(\tau)$$

with $\dim(\tau) = \operatorname{codim}\big(O(\tau)\big)$. Moreover, for each $\tau \leq \sigma$,

$$\overline{O(\tau)} = \bigcup_{\tau \leq \tau'} O(\tau').$$

Hence, the set

$$Z_\sigma := \bigcup_{\tau \leq \sigma, \dim(\tau) \geq 2} O(\tau)$$

is closed in U_σ, with $\operatorname{codim}(Z_\sigma) \geq 2$. Denote by $j_\sigma : U_\sigma \setminus Z_\sigma \to U_\sigma$ the inclusion.

Proposition 1.3.15. *Let $\mathcal{E}$ be a T-equivariant torsion-free sheaf over X. Then $\mathcal{E}$ is reflexive if and only if for any $\sigma \in \Sigma$, the restriction map*

$$j_\sigma^* : \Gamma(U_\sigma, \mathcal{E}) \to \Gamma(U_\sigma \setminus Z_\sigma, \mathcal{E})$$

is an isomorphism.

Proof. If $\mathcal{E}$ is reflexive, by Proposition 1.3.14, we obtain that the restriction maps j_σ^* are isomorphisms. Conversely, assume that $\mathcal{E}$ is torsion-free, and that j_σ^* is an isomorphism for all $\sigma \in \Sigma$. By general theory, the dual of a coherent sheaf is reflexive, hence normal, cf [27, Corollary 1.2]. By Exercise 1.3.35, the dual of a T-equivariant sheaf is also T-equivariant. Hence $(\mathcal{E}^\vee)^\vee$ is T-equivariant, torsion-free and normal. To prove that $\mathcal{E} \simeq (\mathcal{E}^\vee)^\vee$, it is then enough by equivariance to show that for any $\sigma \in \Sigma$, the evaluation map induces an isomorphism

$$\Gamma(U_\sigma, \mathcal{E}) \simeq \Gamma(U_\sigma, (\mathcal{E}^\vee)^\vee).$$

Consider then the following diagram

$$\begin{tikzcd}
\Gamma(U_\sigma, \mathcal{E}) \arrow[r, "ev"] \arrow[d, "j_\sigma^*"] & \Gamma(U_\sigma, (\mathcal{E}^\vee)^\vee) \arrow[d, "j_\sigma^*"] \\
\Gamma(U_\sigma \setminus Z_\sigma, \mathcal{E}) \arrow[r, "ev"] & \Gamma(U_\sigma \setminus Z_\sigma, (\mathcal{E}^\vee)^\vee)
\end{tikzcd}$$

By assumption, and by normality of $(\mathcal{E}^\vee)^\vee$, the vertical arrows are isomorphisms. From Corollary 1.3.3, the singular loci of $\mathcal{E}_{|U_\sigma}$ and $(\mathcal{E}^\vee)^\vee_{|U_\sigma}$ are contained in Z_σ. Hence, they are locally free away from Z_σ, and the bottom horizontal arrow is an isomorphism. Then, the top horizontal arrow is an isomorphism as well, which concludes the proof. ∎

Using $U_{\{0\}} = T$, we can write

$$U_\sigma \setminus Z_\sigma = T \cup \Big(\bigcup_{\rho \in \sigma(1)} O(\rho) \Big) = \bigcup_{\rho \in \sigma(1)} U_\rho$$

where $\sigma(k)$ stands for the set of k-dimensional faces of σ. Hence, by Proposition 1.3.15, $\mathcal{E}$ is reflexive if and only if for any cone $\sigma \in \Sigma$, and any weight $m \in M$, the restriction j_σ^* induces an isomorphism

$$\Gamma(U_\sigma, \mathcal{E})_m \simeq \bigcap_{\rho \in \sigma(1)} \Gamma(U_\rho, \mathcal{E})_m.$$

This yields the following characterisation of reflexivity in terms of multifiltrations.

Proposition 1.3.16. *Let (E^σ_m) be a family of multifiltrations with associated torsion-free T-sheaf $\mathcal{E}$. Then the family of multifiltrations $(\hat{E}^\sigma_m)$ for $(\mathcal{E}^\vee)^\vee$ is given, for $\sigma \in \Sigma$ and $m \in M$ by*

$$\hat{E}^\sigma_m = \bigcap_{\rho \in \sigma(1)} E^\rho_m. \tag{1.16}$$

In the above proposition, the T-equivariant structure on the reflexive hull $(\mathcal{E}^\vee)^\vee$ is the one naturally induced by the T-equivariant structure on $\mathcal{E}$ that makes the injective map $ev : \mathcal{E} \to (\mathcal{E}^\vee)^\vee$ equivariant, cf Exercise 1.3.35. At the level of multifiltrations, this injection corresponds to the inclusions

$$E^\sigma_m \subset \bigcap_{\rho \in \sigma(1)} E^\rho_m.$$

Proof. From the proof of Proposition 1.3.15, we know that $\mathcal{E}$ agrees with its reflexive hull away from the sets Z_σ, and hence on all the affine charts U_ρ, for $\rho \in \Sigma(1)$. Then, $\hat{E}^\rho_m = E^\rho_m$ for each ray $\rho \in \Sigma(1)$ and each $m \in M$. As $(\mathcal{E}^\vee)^\vee$ is reflexive,

$$\hat{E}^\sigma_m = \bigcap_{\rho \in \sigma(1)} \hat{E}^\rho_m.$$

The result follows. ∎

Definition 1.3.17. A family of multifiltrations (E^σ_m) is *reflexive* if for any $\sigma \in \Sigma$ and any $m \in M$,

$$E^\sigma_m = \bigcap_{\rho \in \sigma(1)} E^\rho_m.$$

The family of multifiltrations defined by (1.16) is called the *reflexive hull* of (E^σ_m).

Corollary 1.3.18. *A T-equivariant torsion-free sheaf is reflexive if and only if its associated family of multifiltrations is.*

For a reflexive family of multifiltrations (E^{σ}_m), the data of the vector spaces E^{ρ}_m for $\rho \in \Sigma(1)$ and $m \in M$ is enough to recover the whole family. Moreover, for $\rho \in \Sigma(1)$ and $m \in M$, if $m' \in \rho^{\perp}$, then $E^{\rho}_{m+m'} = E^{\rho}_m$. Hence, the vector space $E^{\rho}_{[m]} := E^{\rho}_m$ is well defined, for any class $[m] \in M/(\rho^{\perp} \cap M)$. Fixing the unique primitive generator $u_{\rho} \in N$ of ρ such that

$$\rho = \mathbb{R}_+ \cdot u_{\rho}$$

induces a canonical isomorphism

$$\begin{array}{ccc} M/(\rho^{\perp} \cap M) & \to & \mathbb{Z} \\ [m] & \mapsto & \langle m, u_{\rho} \rangle \end{array}$$

This suggests to set for any $i \in \mathbb{Z}$:

$$E^{\rho}(i) := E^{\rho}_{[m]}$$

for any choice of m such that $\langle m, u_{\rho} \rangle = i$.

Definition 1.3.19. A *family of filtrations* (indexed by $\Sigma(1)$) for a finite dimensional vector space E is the data of vector subspaces $(E^{\rho}(i))_{\rho \in \Sigma(1), i \in \mathbb{Z}}$ of E satisfying the following, for each $\rho \in \Sigma(1)$:

(1) The sequence $i \mapsto E^{\rho}(i)$ is increasing,
(2) There is $i^+_{\rho} \in \mathbb{Z}$ such that for any $i \geq i^+_{\rho}$, $E^{\rho}(i) = E$,
(3) There exists $i^-_{\rho} \in \mathbb{Z}$ such that for any $i \leq i^-_{\rho}$, $E^{\rho}(i) = \{0\}$.

A *morphism of families of filtrations* between $(E^{\rho}(\bullet))_{\rho \in \Sigma(1)}$ and $(F^{\rho}(\bullet))_{\rho \in \Sigma(1)}$ is a morphism of vector spaces $\phi : E \to F$ such that for any $\rho \in \Sigma(1)$ and $i \in \mathbb{Z}$, $\phi(E^{\rho}(i)) \subset F^{\rho}(i)$.

We then obtained a new equivalence :

Theorem 1.3.20. *The category of T-equivariant reflexive sheaves, with T-equivariant morphisms of coherent sheaves, is equivalent to the category of families of filtrations with their morphisms.*

Once again, the details of the equivalence are left as an Exercise 1.3.36.

Example 1.3.21. We will study in the next chapter the tangent sheaf $\mathcal{T}_X$ of X, which is an example of reflexive toric sheaf. Its family of filtrations will be given in the next chapter, see Section 2.1.

Remark 1.3.22. The category of families of filtrations defined in 1.3.19 is *not* abelian. To obtain an abelian category, one needs to restrict to *saturated* morphisms. A morphism $\phi : (E^{\rho}(\bullet))_{\rho \in \Sigma(1)} \to (F^{\rho}(\bullet))_{\rho \in \Sigma(1)}$ between two families of filtrations is called *saturated* if for any $\rho \in \Sigma(1)$ and any $i \in \mathbb{Z}$, $\phi(E^\rho(i)) = \mathrm{Im}(\phi) \cap F^\rho(i)$, see Exercise 1.3.37. One should be careful though that the quotient of a reflexive sheaf by a saturated subsheaf is not reflexive in general.

We finish this section with the following beautiful formula that follows from Klyachko and Perling's work [39,59], which recovers a T-equivariant reflexive sheaf $\mathcal{E}$ on X from its family of filtrations $(E^\rho(\bullet))_{\rho\in\Sigma(1)}$. It is given by setting for each $\sigma \in \Sigma$:

$$\Gamma(U_\sigma, \mathcal{E}) := \bigoplus_{m\in M} \bigcap_{\rho\in\sigma(1)} E^\rho(\langle m, u_\rho\rangle) \otimes \chi^m. \tag{1.17}$$

Locally freeness

Recall that a locally free sheaf $\mathcal{E}$ on X corresponds to a *vector bundle*, denoted $p : \underline{\mathcal{E}} \to X$, such that sections of $\mathcal{E}$ coincide with sections of $\underline{\mathcal{E}}$, where the latter are defined on $U \subset X$ as regular maps $s : U \to \underline{\mathcal{E}}$ such that $p \circ s = 1_U$.

Definition 1.3.23. Let $p : \underline{\mathcal{E}} \to X$ be a vector bundle on X. A *T-equivariant structure*, or simply *T-structure*, on X, is a regular action

$$\underline{\Phi} : T \times \underline{\mathcal{E}} \to \underline{\mathcal{E}}$$

such that the following diagram is commutative:

$$\begin{tikzcd} T \times \underline{\mathcal{E}} \arrow[r, "\underline{\Phi}"] \arrow[d, "1_T \times p"] & \underline{\mathcal{E}} \arrow[d, "p"] \\ T \times X \arrow[r, "\alpha"] & X \end{tikzcd}$$

where recall α stands for the T-action on X. A pair $(\underline{\mathcal{E}}, \underline{\Phi})$ of a vector bundle together with a T-structure will be called a *T-vector bundle*.

A T-equivariant structure is also called *a linearisation* of T on $\underline{\mathcal{E}}$. Let $\mathcal{E}$ be a locally free sheaf on X with associated vector bundle $p : \underline{\mathcal{E}} \to X$. Assume that Φ is a T-equivariant structure on $\mathcal{E}$. It induces a T-equivariant structure $\underline{\Phi}$ on $\underline{\mathcal{E}}$ as follows. For each $t \in T$, the isomorphism

$$\Phi_t : t^*\mathcal{E} \to \mathcal{E}$$

induces an isomorphism on the fibers

$$\underline{\mathcal{E}}_{tx} = \mathcal{E}_{tx}/\big(\mathfrak{m}_{X,tx}\,\mathcal{E}_{tx}\big) \xrightarrow{\simeq} \mathcal{E}_x/\big(\mathfrak{m}_{X,x}\,\mathcal{E}_x\big) = \underline{\mathcal{E}}_x,$$

still denoted by Φ_t. Hence, $\underline{\Phi}_t := \Phi_{t^{-1}}$ sends the fiber of $\underline{\mathcal{E}}$ over x to the fiber over tx and produces a morphism $\underline{\mathcal{E}} \to \underline{\mathcal{E}}$. Then, $\underline{\Phi}(t, e) \mapsto \underline{\Phi}_t(e)$ is regular, and from (1.6), this defines a T-structure on $\underline{\mathcal{E}}$. This construction works both ways, and one can recover the T-equivariant structure on $\mathcal{E}$ out of the one on $\underline{\mathcal{E}}$. The details are left in Exercise 1.3.38.

Example 1.3.24. We will study in Chapter 2 many constructions of locally free T-equivariant sheaves. On a smooth toric variety, the cotangent sheaf, the tangent sheaf, invertible sheaves, their sums, duals, tensor products, are locally free T-sheaves.

An important result, due to Kaneyama [37] and Klyachko [39], is that a locally free T-sheaf over an affine toric variety is free. This generalizes previous Proposition 1.3.1.

Proposition 1.3.25. *Let $\mathcal{E}$ be a locally-free, T-equivariant sheaf over a normal affine toric variety $X = U_\sigma$. Then there are weights $(m_1, \ldots, m_r) \in M^r$ such that*

$$\mathcal{E} \simeq (\mathcal{O}_X^{\oplus r}, \chi^{m_1} \cdot \Phi_X \oplus \ldots \oplus \chi^{m_r} \cdot \Phi_X)$$

as a T-equivariant sheaf.

We will follow here the proof of Klyachko [39, Proposition 2.1.1], using the vector bundle formalism rather than the sheaf point of view. See also [37, Theorem 3.5] or [58, Proposition 5.20] for different proofs.

Proof. Let $\underline{\mathcal{E}}$ be the associated T-vector bundle. Consider the point $\gamma_\sigma \in U_\sigma$, such that the orbit of minimal dimension in U_σ is $O(\sigma) = T \cdot \gamma_\sigma$, cf Exercise 1.1.15. We first follow the proof of Proposition 1.3.1, and produce the equivariant trivialisation that we are looking for on $O(\sigma)$. We will then extend it on the whole U_σ by using the orbit-cone correspondence.

The fiber of $\underline{\mathcal{E}}$ at γ_σ is a $\mathbb{C}$-vector space :

$$\underline{\mathcal{E}}_{\gamma_\sigma} \simeq \mathbb{C}^r.$$

Precomposing with the evaluation map on sections, we define a morphism of $\mathbb{C}$-vector spaces:

$$\begin{array}{rccc} \phi : & \Gamma(U_\sigma, \underline{\mathcal{E}}) & \to & \mathbb{C}^r \\ & s & \mapsto & s(\gamma_\sigma). \end{array}$$

Let E be maximal amongst T-invariant subspaces of $\Gamma(U_\sigma, \underline{\mathcal{E}})$ on which ϕ is injective. Exactly as in the proof of Proposition 1.3.1, $\phi_{|E}$ is an isomorphism. Fix a basis $(e_i)_{1\leq i\leq r}$ of E consisting of T-eigenvectors. The sections $(e_i)_{1\leq i\leq r}$ are actually independent at any point $x \in U_\sigma$. Indeed, if the rank of $(e_i(x))_{1\leq i\leq r}$ is strictly less than r at some point $x \in U_\sigma$, then so it is on the whole orbit $T \cdot x$. But by Proposition 1.1.9, all the torus orbits contain the orbit of γ_σ in their closure. Semi-continuity of the rank then contradicts the fact that $(e_i(\gamma_\sigma))_{1\leq i\leq r}$ is free. We deduce that the sections

$(e_i)_{1\leq i\leq r}$ provide a global trivialisation

$$\begin{array}{rccc}\psi : & U_\sigma \times \mathbb{C}^r & \to & \underline{\mathcal{E}} \\ & (x,(c_i)_{1\leq i\leq r}) & \mapsto & \displaystyle\sum_{i=1}^{r} c_i\, e_i(x).\end{array}$$

If m_i denotes the weight of e_i, we can define a representation ϱ of T on $\mathbb{C}^r$ with weights $(m_i)_{1\leq i\leq r}$. Consider the action of T on $U_\sigma \times \mathbb{C}^r$ induced by α and ϱ. By construction, ψ is T-equivariant for this action, which concludes the proof, cf Exercise 1.3.38. ∎

Remark 1.3.26. The weights $(m_i)_{1\leq i\leq r}$ in the previous proof have a geometric interpretation. Let $T_\sigma \subset T$ be the stabiliser of the point γ_σ. Its character lattice is given by $M/(\sigma^\perp \cap M)$. The T-equivariant structure on $\underline{\mathcal{E}}$ induces a representation of T_σ on $\underline{\mathcal{E}}_{\gamma_\sigma}$, the fiber over γ_σ. The equivariance of the global trivialisation ψ precisely means that the weight space decomposition of $\underline{\mathcal{E}}_{\gamma_\sigma}$ for the T_γ-action is

$$\underline{\mathcal{E}}_{\gamma_\sigma} = \bigoplus_{i=1}^{r} \mathbb{C}\cdot e_i(\gamma_\sigma) \simeq \bigoplus_{i=1}^{r} E_{m_i}$$

where E_{m_i} is a one dimensional irreducible representation with weight $[m_i] \in M/(\sigma^\perp \cap M)$. Hence, only the classes of the weights $(m_i)_{1\leq i\leq r}$ are uniquely determined, which reflects the fact that χ^m acts as an isomorphism on $\Gamma(U_\sigma, \mathcal{E})$ for $m \in \sigma^\perp$. The classes $([m_i])_{1\leq i\leq r}$ are simply the weights of the induced representation

$$T_\sigma \to \mathrm{GL}(\underline{\mathcal{E}}_{\gamma_\sigma}).$$

We can actually say more. For each $\rho \in \sigma(1)$, define a filtration of the space $\underline{\mathcal{E}}_1 \simeq \mathbb{C}^r \simeq E$, the fiber over $1 \in T \subset U_\sigma$, in the following way. Consider $t = \lambda_{u_\rho}(s)$, for $s \in \mathbb{C}^*$, where $\lambda_{u_\rho} : \mathbb{C}^* \to T$ is the one-parameter subgroup associated to $u_\rho \in N$. Then define for each $m \in M$

$$E^\rho(m) = \{e \in \underline{\mathcal{E}}_1 \mid \exists \lim_{s\to 0} \chi^m(t)\cdot \underline{\Phi}_t \cdot e\}$$

where $\underline{\Phi} : T \times \underline{\mathcal{E}} \to \underline{\mathcal{E}}$ is the equivariant structure given by ψ. By construction of ψ, we see that $E^\rho(m)$ only depends on $\langle m, u_\rho\rangle$, and that $e_i(1) \in E^\rho(\langle m, u_\rho\rangle)$ if and only if $m_i \leq_\rho m$. Hence, this filtration of $\underline{\mathcal{E}}_1 \simeq E$ satisfies

$$E^\rho(\langle m, u_\rho\rangle) \simeq \bigoplus_{m_i \leq_\rho m} E_{m_i}.$$

Applying the proof of Proposition 1.3.25 to $U_\rho \subset U_\sigma$ shows that this filtration $(E^\rho(\bullet))$ corresponds to the filtration of the reflexive toric sheaf $\mathcal{E}_{|U_\rho}$. Finally, a direct consequence of Exercise 1.1.15 is that

$$\lim_{s\to 0} \lambda_{u_\rho}(s) = \gamma_\rho.$$

Hence, the classes of the weights $(m_i)_{1\leq i\leq r}$ in $M/(\rho^\perp \cap M)$ correspond to the weights of the induced reprensentation of T_ρ on $\underline{\mathcal{E}}_{\gamma_\rho}$.

We end this chapter with the characterisation of families of filtrations that correspond to locally free sheaves, following Klyachko [39, Theorem 2.2.1]. Let $\mathcal{E}$ be a reflexive T-sheaf with family of filtrations $(E^\rho(\bullet))_{\rho\in\Sigma(1)}$. If $\mathcal{E}$ is locally free on some affine chart $U_\sigma \subset X$, then by Proposition 1.3.25, there are some weights $(m_1, \dots, m_r)$ such that

$$\Gamma(U_\sigma, \mathcal{E}) \simeq \mathbb{C}[S_\sigma][m_1] \oplus \dots \mathbb{C}[S_\sigma][m_r].$$

Those weights come from a representation of T on E that splits as

$$E = \bigoplus_{i=1}^{r} E_{m_i}$$

where each E_{m_i} is a one dimensional representation with weight m_i. By Example 1.2.16, or Remark 1.3.26, we deduce that for each $\rho \in \sigma(1)$,

$$E^\rho(\langle m, u_\rho\rangle) = \bigoplus_{m_i \leq_\rho m} E_{m_i}.$$

This condition on the family of filtrations actually characterizes locally freeness.

Theorem 1.3.27. *Let $\mathcal{E}$ be a reflexive T-equivariant sheaf with family of filtrations $(E^\rho(\bullet))_{\rho\in\Sigma(1)}$. Then $\mathcal{E}$ is locally free if and only if for each $\sigma \in \Sigma$, there are weights $(m_1, \dots, m_r) \in M^r$ and subspaces $E_{m_i} \subset E$ such that*

$$E := \bigoplus_{i=1}^{r} E_{m_i}$$

and for each $\rho \in \sigma(1)$,

$$E^\rho(\langle m, u_\rho\rangle) = \bigoplus_{m_i \leq_\rho m} E_{m_i}.$$

We will refer to Theorem 1.3.27 as *Klyachko's compatibility condition*.

Proof. We only have to show that this condition is sufficient. Let $\sigma \in \Sigma$ and assume that there exists $(m_1, \dots, m_r) \in M^r$ and subspaces $E_{m_i} \subset E$ such that

$$E := \bigoplus_{i=1}^{r} E_{m_i}$$

and for each $\rho \in \sigma(1)$,

$$E^\rho(\langle m, u_\rho\rangle) = \bigoplus_{m_i \leq_\rho m} E_{m_i}.$$

Then, the family of multifiltrations for $\mathcal{E}$ is given by

$$\begin{aligned} E_m^\sigma &= \bigcap_{\rho\in\sigma(1)} E^\rho(\langle m, u_\rho\rangle) \\ &= \bigcap_{\rho\in\sigma(1)} \bigoplus_{m_i\leq_\rho m} E_{m_i} \\ &= \bigoplus_{m_i\leq_\sigma m} E_{m_i} \end{aligned}$$

where the first equality follows by reflexivity, the second one by assumption, and the last one from the fact that the rays $\rho\in\sigma(1)$ span σ as a cone. From Example 1.2.16, this shows that

$$\Gamma(U_\sigma,\mathcal{E}) \simeq \mathbb{C}[S_\sigma][m_1]\oplus\ldots\oplus\mathbb{C}[S_\sigma][m_r].$$

Hence $\mathcal{E}$ is locally free on U_σ. The result follows. ∎

Remark 1.3.28. Note that the weights $(m_i)_{1\leq i\leq r}$ in the statement of Theorem 1.3.27 depend on the cone σ. From Exercise 1.3.32, even for a fixed σ, those weights are not uniquely determined. However, there classes in $M/(\sigma^\perp\cap M)$ are unique up to permutations. The one dimensional spaces E_{m_i} may not be uniquely determined either, if for example several weights amongst the m_i's are equal. When σ is n-dimensional and smooth, the primitive generators $(u_\rho)_{\rho\in\sigma(1)}$ of its rays form a basis of N. Then, Klyachko's compatibility condition determines the coordinate $\langle m_i, u_\rho\rangle$ in the dual basis of M as the least integer j such that the summand E_{m_i} appears in $E^\rho(j)$. We then see in that case that the spaces E_{m_i} are uniquely determined as soon as $m_i\neq m_j$ for all $i\neq j$.

Remark 1.3.29. Denote by (m_i^σ) the weights associated to σ in Theorem 1.3.27, for a given locally free toric sheaf $\mathcal{E}$. By the proof of Proposition 1.3.25 and Remark 1.3.26, they satisfy compatibility conditions. Indeed, if $\tau\leq\sigma$, then $\gamma_\sigma\in\overline{T\cdot\gamma_\tau}$, and then the projection of the classes $[m_i^\sigma]$ through

$$M/(\sigma^\perp\cap M)\to M/(\tau^\perp\cap M)$$

must agree with the classes $[m_i^\tau]$. This also shows that if Σ is complete, it is enough to know the (m_i^σ) for all maximal dimensional cones $\sigma\in\Sigma(n)$ to recover all the (classes of) weights (m_i^τ), $\tau\in\Sigma$. The collection of the classes $([m_i^\sigma])_{\sigma\in\Sigma}$ is called a collection of multisets of linear functions of size r in Payne's work [56]. Seen as $\mathbb{Z}$-linear maps, they satisfy the compatibilities

$$m(\sigma)_{|\tau} = m(\tau)$$

for $\tau\leq\sigma$, and form an important combinatorial object attached to $\mathcal{E}$.

1.3.1 Exercises for 1.3

Exercise 1.3.30 (Locally free, reflexive, torsion free). Let $\mathcal{E}$ be a coherent sheaf.

(1) Show that $\mathcal{E}^\vee$ is torsion-free.

(2) Deduce that a reflexive sheaf is torsion-free.

(3) Show that a locally-free sheaf is reflexive.

Exercise 1.3.31 (Torsion filtration). Let $\mathcal{E}$ be a coherent sheaf on X. The support of $\mathcal{E}$ is the closure of the set of points $x \in X$ with $\mathcal{E}_x \neq 0$. Define the dimension of $\mathcal{E}$ to be the dimension of its support:

$$d = \dim(\mathcal{E}) := \dim(\mathrm{Supp}(\mathcal{E})).$$

The *torsion filtration* of $\mathcal{E}$ is the unique filtration

$$0 \subset T_0(\mathcal{E}) \subset T_1(\mathcal{E}) \subset \ldots \subset T_d(\mathcal{E}) = \mathcal{E}$$

where for each $0 \leq i \leq d$, $T_i(\mathcal{E}) \subset \mathcal{E}$ is the maximal coherent subsheaf of dimension less or equal to i. Recall that a coherent sheaf of dimension d is *pure* if all its non-trivial coherent subsheaves have dimension d.

(1) Show existence and uniqueness of the torsion filtration of $\mathcal{E}$. Hint : $\dim(\mathcal{F}_1 \oplus \mathcal{F}_2) \leq \max \dim(\mathcal{F}_i)$ for two coherent sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$.

(2) Show that $\mathcal{E}$ is torsion-free if and only if $\dim(\mathcal{E}) = n$ and $T_{n-1}(\mathcal{E}) = 0$.

(3) Show that for each i, $T_i(\mathcal{E})/T_{i-1}(\mathcal{E})$ is either zero, or pure of dimension i.

(4) Assume that $\mathcal{E}$ is toric, with equivariant structure Φ.

   (a) Show that for any $t \in T$, $\dim(t^* T_i(\mathcal{E})) \leq i$ and thus $t^* T_i(\mathcal{E}) \subset T_i(\mathcal{E})$.

   (b) Conclude that Φ induces an equivariant structure on $T_i(\mathcal{E})$ for each $0 \leq i \leq d$. Hint : use Exercise 1.2.28.

Exercise 1.3.32 (Shifted equivariant structures on affine sets). Let $(\mathcal{E}, \Phi)$ be a torsion-free T-sheaf over a normal affine toric variety U_σ. Let $m \in \sigma^\perp \cap M$. Show that $(\mathcal{E}, \Phi)$ and $(\mathcal{E}, \chi^m \cdot \Phi)$ are isomorphic as T-sheaves.

Exercise 1.3.33 ($(M, \leq_\sigma)$ is a directed set). Give the proofs of Lemmas 1.3.5 and 1.3.7.

Exercise 1.3.34 (The category of families of multifiltrations). Conclude the proof of Theorem 1.3.11.

Exercise 1.3.35 (Duality and equivariance). Let $(\mathcal{E}, \Phi)$ be a T-equivariant sheaf.

(1) Let $(\mathcal{F}, \Psi)$ be another T-equivariant sheaf. Show that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ inherits a T-equivariant structure from Φ and Ψ.

(2) Show that the dual sheaf

$$\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$$

naturally admits a T-equivariant structure.

(3) Show that the T-equivariant structure on $(\mathcal{E}^\vee)^\vee$ can be chosen so that $ev : \mathcal{E} \to (\mathcal{E}^\vee)^\vee$ is T-equivariant.

Exercise 1.3.36 (The category of families of filtrations). Prove Theorem 1.3.20.

Exercise 1.3.37 (Saturated subsheaves). Let $\mathcal{E}$ be a T-equivariant reflexive sheaf. Consider a T-equivariant reflexive subsheaf $\mathcal{F} \subset \mathcal{E}$, cf Section 2.2. We say that $\mathcal{F}$ is a *saturated subsheaf* if the quotient $\mathcal{E}/\mathcal{F}$ is torsion-free.

(1) Show that the quotient sheaf $\mathcal{E}/\mathcal{F}$ naturally inherits the structure of a coherent T-equivariant sheaf.

(2) Show that $\mathcal{F}$ is saturated in $\mathcal{E}$ if and only if for all $\rho \in \Sigma(1)$ and all $i \in \mathbb{Z}$, $F^\rho(i) = F \cap E^\rho(i)$.

Exercise 1.3.38 (Equivariant vector bundles and sheaves). Let $\mathcal{E}$ be a locally free sheaf on X and let $p : \underline{\mathcal{E}} \to X$ be the associated vector bundle.

(1) Let Φ be a T-equivariant structure on $\mathcal{E}$. Show that it induces a T-equivariant structure on $\underline{\mathcal{E}}$.

(2) Assume that $\underline{\Phi}$ is an equivariant structure on $\underline{\mathcal{E}}$. Consider a toric affine chart $U_\sigma \subset X$. Define a representation of T on sections of $\underline{\mathcal{E}}_{|U_\sigma}$ as follows. For each $t \in T$, and $s \in \Gamma(U_\sigma, \underline{\mathcal{E}})$,

$$(t \cdot s)(x) = \underline{\Phi_t}^{-1}\big(s(tx)\big).$$

Show that this induces a T-equivariant structure on $\mathcal{E}$.

(3) Show that $(\mathcal{O}_X, \chi^m \cdot \Phi_X)$ corresponds to the trivial vector bundle $\mathbb{C} \times X \to X$ with T-structure given, for $(t, (c, x)) \in T \times \mathbb{C} \times X$, by

$$\underline{\Phi}\big(t, (c, x)\big) = \big(\chi^m(t)\, c,\ \alpha(t, x)\big) \in \mathbb{C} \times X.$$

(4) Conclude the proof of Proposition 1.3.25.

Notes for Chapter 1

Monomial ideals

The first wide class of examples of coherent toric sheaves is given by toric ideal sheaves on affine spaces, as in Examples 1.2.11.(4). Such sheaves correspond to *monomial ideals* in $\mathbb{C}[x_1, \dots, x_n]$. A monomial ideal is an ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$ generated by monomials of the form $x_1^{p_1} \dots x_n^{p_n}$. Equivalently, I is invariant under the torus action of $T = (\mathbb{C}^*)^n$ on $\mathbb{C}[x_1, \dots, x_n]$ induced by the standard action of T on $\mathbb{C}^n$ (cf Example 1.1.2). The study of monomial ideals is a branch of commutative algebra on its own, with connections to combinatorics, algorithmics, and geometry. We refer the interested reader to the textbook [49].

Cox ring and toric sheaves

Let X be a normal toric variety with fan Σ. The *Cox ring* of X, also called its *total coordinate ring*, is defined by

$$S := \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)],$$

where for each $\rho \in \Sigma(1)$, x_ρ stands for a degree one variable. The Cox ring is a $\mathrm{Cl}(X)$-graded polynomial ring, where $\mathrm{Cl}(X)$ denotes the *class group* of X. The group $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X), \mathbb{C}^*)$ naturally acts on $\mathbb{C}^{|\Sigma(1)|}$, and it was shown by Cox that the toric variety X could be recovered as a categorical quotient

$$X \simeq (\mathbb{C}^{|\Sigma(1)|} \setminus Z)/G,$$

where Z is the zero set of a monomial ideal $B \subset S$ encoded by the combinatorial structure of Σ, see [17] or [16, Chapter 5].

It was also shown in [17] that there is a correspondence between $\mathrm{Cl}(X)$-graded S-modules and quasicoherent sheaves on X. This correspondence sends finitely generated modules to coherent sheaves, and is surjective, when X has no torus factor, cf [16, Chapter 6, Appendix]. This very useful point of view on quasicoherent sheaves over toric varieties has seen many applications in algebraic geometry and commutative algebra, for instance to compute Hilbert series, cohomology groups, Euler characteristics, among others.

Later on, Batyrev and Cox identified the graded S-modules corresponding to toric sheaves on X. They proved that a graded S-module gives rise to a toric sheaf if it is endowed with an additional $\mathbb{Z}^{|\Sigma(1)|}$-grading [6, Proposition 4.17]. The relation between the Σ-family description of toric sheaves and Batyrev–Cox point of view has interesting applications in the study of graded algebras. As an example, we refer to [50] for results on the multigraded Mumford–Castelnuovo regularity. As [16, Chapters 5 and 6] already offers an excellent introduction to Cox's construction, we will focus in the rest of the text on Klyachko's approach to toric sheaves.