Multifractal analysis of multiple ergodic averages

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Outline

Motivation and Problem

I. Multirecurrence and Multiple ergodic averages

Theorem (Furstenberg-Weiss, 1978) If

- (X, d) a compact metric space.
- $T_i : X \to T$ continuous, $T_i T_j = T_j T_i$ $(1 \le i, j \le d)$.

Then there exists $x \in X$ and $(n_k) \subset \mathbb{N}$ such that

$$\lim_{k\to\infty} T_i^{n_k} x = x, \quad \forall i = 1, 2, \cdots, \ell.$$

Applied to $X = \{0,1\}^{\mathbb{N}}$, $T_i = T^i$, T being the shift.

Theorem (Szemeredi, 1975) If $\Lambda \subset \mathbb{N}$ satisfies

$$\limsup_{N\to\infty} \frac{|\Lambda \cap [1, N]|}{N} > 0,$$

Then Λ contains arithmetic progressions of arbitrary length.

II. Multiple ergodic theorem

Multiple ergodic averages

$$\frac{1}{n}\sum_{k=1}^{n} f_1(T^k x) f_2(T^{2k}x)\cdots f_\ell(T^{\ell k}x)$$

- Furstenberg : when (X, T) is mixing, L^2-limit is $\prod_{j=1}^{d}\int f_j d\mu$.
- Host-Kra : L^2-convergence (von Neumann $\ell = 1$, Furstenberg $\ell = 2$, Conze-Lesigne $\ell = 3+$ total ergodicity).
- Bourgain : Almost everywhere convergence when $\ell = 2$.
- ...

N. B. (Question of limit) When $\ell = 2$, the limit depends on the Kronecker factor but may be not constant. When $\ell \geq 3$, the Kronecker factor can not capture relations among $x, T^n x, T^{2n} x, T^{3n} x$.

III. Setting of Multifractal Analysis

- $T : X \to X$ topological dynamical system
- $\Phi : X^\ell \to \mathbb{R}$ continuous function ($\ell \geq 1$)
- Denote, if the limit exists

$$A_\Phi(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Phi(T^k x, T^{2k} x, \cdots, T^{\ell k} x).$$

- For given α, denote

$$E(\alpha) = \{x \in X : A_\Phi(x) = \alpha\}.$$

Problem : What is the size of $E(\alpha)$?

N.B. The case $\ell = 1$ is classical. The case $\ell \geq 2$ is a challenging problem. Most interesting case is $\Phi = f_1 \otimes \cdots \otimes f_\ell$.

IV. Problem : different Spectra

Hausdorff Spectrum

$$F_{\text{hausdorff}}(\alpha) = \dim_H E(\alpha).$$

Invariance Spectrum

$$F_{\text{invariance}}(\alpha) = \sup\{\dim \mu : \mu \text{ invariant}, \mu(E(\alpha)) = 1\}.$$

Mixing Spectrum

$$F_{\text{mixing}}(\alpha) = \sup\{\dim \mu : \mu \text{ mixing}, \mu(E(\alpha)) = 1\}.$$

dimension of a measure :

$$\dim \mu = \inf\{\dim B : B \text{ Borel set }, \mu(B^c) = 0\}.$$

N.B. When $\ell = 1$, all these spectra are the same. But when case $\ell \geq 2$, they may be different ($E(\alpha)$ is no longer invariant).

V. V-statistics

- $T : X \to X$ topological dynamical system
- $\Phi : X^\ell \to \mathbb{R}$ continuous function ($\ell \geq 1$)
- Denote, if the limit exists

$$V_\Phi(x) = \lim_{n\to\infty} \frac{1}{n^\ell} \sum_{1\leq k_1,\cdots,k_\ell\leq d}^{n} \Phi(T^{k_1}x, T^{k_2}x, \cdots, T^{k_\ell}x).$$

- For given α, denote

$$V(\alpha) = \{x \in X : V_\Phi(x) = \alpha\}.$$

Problem : What is the size of $V(\alpha)$?

N. B. 1. A satisfactory result will be obtained for the entropy spectrum of $V(\alpha)$ when (X, T) has the specification property.
2. In general, there is no ergodic theorem (Aaronson et al, 1996).

V-statistics

I. Topological entropy

s-**Hausdorff measure** : for $E \subset X$, $s > 0$,

$$\mathcal{H}^s(E) := \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : E \subset \cup_{i=1}^{\infty} U_i, |U_i| < \delta \right\}$$

Hausdorff dimension :

$$\dim_H(E) := \inf\{s > 0 : \mathcal{H}^s(E) = 0\} = \sup\{s > 0 : \mathcal{H}^s(E) = \infty\}$$

Bowen topological entropy :

$B_n(x, \epsilon) := \{y : d(T^j x, T^j y) < \epsilon, j = 0, 1, \cdots, n-1\}$ (Bowen ball).

$$H^s(E, \epsilon) := \lim_{n \to \infty} \inf \left\{ \sum_{i=1}^{\infty} |e^{-n_i}|^s : E \subset \cup_{i=1}^{\infty} B_{n_i}(x_i, \epsilon), n_i > n \right\}$$

$$h_{\text{top}}(E, \epsilon) := \inf\{s > 0 : H^s(E, \epsilon) = 0\} = \sup\{s > 0 : H^s(E, \epsilon) = \infty\}$$

$$h_{\text{top}}(E) := \lim_{\epsilon \to 0} h_{\text{top}}(E, \epsilon)$$

II. Specification property

Specification property of (X, T) : for any $\epsilon > 0$ there exists an integer $m(\epsilon) \geq 1$ having the property that for any integer $k \geq 2$, for any k points $x_1, \ldots, x_k$ in X, and for any integers

$$a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$$

with $a_i - b_{i-1} \geq m(\epsilon) \quad (\forall 2 \leq i \leq k)$, there exists a point $y \in X$ such that

$$d(T^{a_i+n}y, T^n x_i) < \epsilon \qquad (\forall\ 0 \leq n \leq b_i - a_i, \quad \forall 1 \leq i \leq k).$$

Examples :

topologically mixing Subshift of finite type.

topologically mixing continuous interval maps.

III. Bowen lemma

μ-generic points :

$$G_\mu := \left\{ x \in X : \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j x} \xrightarrow{w^*} \mu \right\},$$

Lemma (Bowen, 1973)

For any invariant measure μ, $h_{\text{top}}(G_\mu) \leq h_\mu$.

Lemma (Fan-Liao-Peyrière, 2008)

Suppose (X, T) has the specification property. We have $h_{\text{top}}(G_\mu) = h_\mu$ for any invariant measure μ.

IV. Topological spectrum of V-statistics

$$\mathcal{M}_\Phi(\alpha) := \left\{ \mu \in \mathcal{M}_{\text{inv}} : \int \Phi d\mu^{\otimes \ell} = \alpha \right\}.$$

Theorem

(a) If $\mathcal{M}_\Phi(\alpha) = \emptyset$, we have $V_\Phi(\alpha) = \emptyset$.
(b) If $\mathcal{M}_\Phi(\alpha) \neq \emptyset$, we have the conditional variational principle

$$h_{\text{top}}(V_\Phi(\alpha)) = \sup_{\mu \in \mathcal{M}_\Phi(\alpha)} h_\mu.$$

(b) $\alpha \mapsto h_{\text{top}}(V_\Phi(\alpha))$ is u.s.c.

N. B. The case $\ell = 1$ is classical and when Φ is "smooth", $\alpha \mapsto h_{\text{top}}(V_\Phi(\alpha))$ is analytic. But when $\ell \geq 3$, even when Φ is very "smooth", there may be discontinuity (phase transition).

IV.(Example) Product and quotient of Birkhoff averages

$f, g : X \to \mathbb{R}$ continuous functions.

Theorem

Assume $g(x) > 0$.

$$h_{\text{top}}\left\{x \in X : \lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} f(T^j x)}{\sum_{j=0}^{n-1} g(T^j x)} = \alpha\right\}$$
$$= \sup\{h_\mu : \mu \text{ invariant}, \frac{\mathbb{E}_\mu f}{\mathbb{E}_\mu g} = \alpha\}.$$

Theorem

$$h_{\text{top}}\left\{x \in X : \lim_{n\to\infty} \frac{1}{n^2} \sum_{i}^{n-1} \sum_{j}^{n-1} f(T^i x) g(T^j x) = \alpha\right\}$$
$$= \sup\{h_\mu : \mu \text{ invariant}, \mathbb{E}_\mu f \cdot \mathbb{E}_\mu g = \alpha\}.$$

Multiple ergodic averages

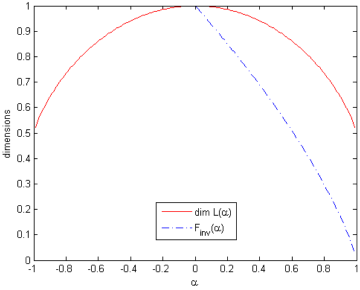
a striking new phenomenon

I. Example 1

On Σ_2. $f_1(x) = f_2(x) = 2x_1 - 1$ (valued $-1, 1$). We have

$$F_{\text{hausdorff}}(\alpha) = \frac{1}{2} + \frac{1}{2} H\left(\frac{1+\alpha}{2}\right), \quad F_{\text{invariant}}(\alpha) = H\left(\frac{1+\sqrt{\alpha}}{2}\right)$$

where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$.



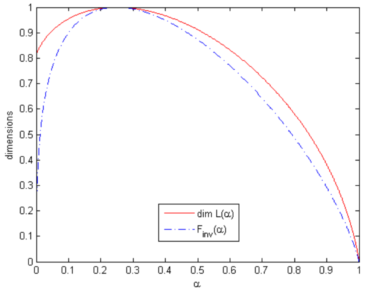
II. Example 2

On Σ_2. $f_1(x) = f_2(x) = x_1$ (valued $0, 1$).

$$F_{\text{invariant}}(\alpha) = H(\sqrt{\alpha})$$

$F_{\text{hausdorff}}(\alpha)$ is numerically computed : pressure $P(s) = 2\log t_0(s)$, $x = t_0(s) > 0$ is the solution of the third order equation

$$x^3 - (e^s + 1)x + (e^s - 1) = 0.$$



III. Remarks

- When $\ell = 1$, $F_{\text{hausdorff}} = F_{\text{invariant}}$. No longer the case when $\ell \geq 2$.
- It is possible that there is no invariant measure sitting on $E(\alpha)$. It is then necessary to construct non invariant measure for studying $E(\alpha)$.
- In general, $F_{\text{invariant}} \neq F_{\text{mixing}}$.
- $\alpha \mapsto F_{\text{mixing}}(\alpha)$ has discontinuity even for regular potentials.

Riesz product method

[A. H. Fan, L. M. Liao, J. H. Ma]

I. A special case on $X = \{-1, 1\}^{\mathbb{N}}$

- $X = \{-1, 1\}^{\mathbb{N}}$, T is the shift.
- $f_i(x) = x_1$ the projection on the first coordinates $(i = 1, 2, \cdots, \ell)$
- for $\theta \in \mathbb{R}$, denote

$$B_\theta := \left\{ x \in \{-1, 1\}^{\mathbb{N}} : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k x_{2k} \cdots x_{\ell k} = \theta \right\}.$$

Theorem (Fan-Liao-Ma, 2009)

For $\theta \notin [-1, 1]$, $B_\theta = \emptyset$. For any $\theta \in [-1, 1]$, we have

$$\dim_H(B_\theta) = 1 - \frac{1}{\ell} + \frac{1}{\ell} H\Big(\frac{1+\theta}{2}\Big),$$

where $H(t) = -t \log_2 t - (1-t) \log_2(1-t)$.

N.B. $\dim_H B_\theta \geq 1 - 1/\ell > 0$ if $\ell \geq 2$.

II. Proof using Riesz products

- Rademacher functions $r_n(x) = x_n$ are group characters
- Walsh functions

$$w_n = r_{n_1} \cdots r_{n_s}, \quad n = 2^{n_1 - 1} + 2^{n_2 - 1} + \cdots + 2^{n_s - 1}, \quad 1 \le n_1 < n_2 < \cdots$$

is a Hilbert basis in $L^2(\{-1, 1\}^{\mathbb{N}})$.

- The subsystem

$$\xi_k = r_k r_{2k} \cdots r_{\ell k} \quad (k \ge 1)$$

are dissociated in the sense of Hewitt-Zuckerman : different products of ξ_k give rise to different characters.

- The following Riesz product measure is well defined

$$d\mu_\theta = \prod_{k=1}^{\infty} (1 + \theta \xi_k(x)) dx.$$

II. Proof (continued)

Lemma 1 (Expectation)

If $f(x) = f(x_1, \cdots, x_n)$, we have

$$\mathbb{E}_{\mu_\theta}[f] = \int f(x) \prod_{k=1}^{\lfloor n/\ell \rfloor} \big(1 + \theta \xi_k(x)\big) dx.$$

Proof. Because r_n are Haar-independent. QED

II. Proof (continued)

Lemma 2 (Law of large numbers)

If $f(x) = \sum_{n=0}^{\infty} g_n x^n$ with $\sum_n |g_n| < \infty$, then for μ_θ-almost all x,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} g(\xi_k(x)) = \mathbb{E}_\theta[g(\xi_1)].$$

Proof. Apply Menchoff Theorem to $\sum_{k=0}^{\infty} \frac{1}{k}\Big(g(\xi_k) - \mathbb{E}_\theta[g(\xi_k)]\Big)$ and conclude by Kronecker theorem :

- $\xi_k^{2n}(x) = 1$, $\xi_k^{2n-1}(x) = \xi_k(x)$ $\forall n \geq 1$.
- $g(\xi_k) = \sum_{n=0}^{\infty} g_{2n} + \xi_k \sum_{n=1}^{\infty} g_{2n-1}$.
- $\mathbb{E}_\theta(\xi_k) = \theta, \mathbb{E}_\theta(\xi_j \xi_k) = \theta^2, \quad (j \neq k)$.
- $\mathbb{E}_\theta[g(\xi_k)] = \sum_{n=0}^{\infty} g_{2n} + \theta \sum_{n=1}^{\infty} g_{2n-1}$.
- $g(\xi_j) - \mathbb{E}_\theta g(\xi_k)$ are μ_θ-orthogonal.

QED

II. Proof (continued) : Proof of Theorem

$\mu_\theta(B_\theta) = 1$ (Lemma 2 applied to $g(x) = x$) :

$$\mu_\theta - a.e.\ x \quad \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \xi_k(x) = \mathbb{E}(\xi_1) = \theta.$$

By Lemma 1 (applied to 1_{I_n}) : $\forall x$, $\forall n \geq \ell$,

$$P_\theta(I_n(x)) = \frac{1}{2^n} \prod_{k=1}^{\lfloor n/\ell \rfloor} \big(1 + \theta \xi_k(x)\big).$$

Notice that $\log(1+\theta\xi_k(x)) = -\sum_{n=1}^{\infty} \frac{\theta^{2n}}{2n} + \sum_{n=1}^{\infty} \frac{\theta^{2n-1}}{2n-1}\xi_k(x)$. Then for all points $x \in B_\theta$,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \log(1+\theta\xi_k(x)) = -\sum_{n=1}^{\infty} \frac{\theta^{2n}}{2n} + \sum_{n=1}^{\infty} \frac{\theta^{2n-1}}{2n-1}\theta.$$

The right hand side can be written as

$$\theta \log(1+\theta) - \frac{\theta - 1}{2} \log(1-\theta^2) = \left[1 - H\left(\frac{1+\theta}{2}\right)\right] \log 2.$$

We conclude by Billingsley's theorem. QED

III. Riesz product : on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$

- F. Riesz (1918) : singular BV function

$$F(x) = \lim_{N\to\infty} \int_0^x \prod_{n=1}^{N} (1 + \cos 2\pi 4^n t) dt$$

- Zygmund (1932) : $a_n = r_n e^{2\pi i \phi_n} \in \Delta$, $3\lambda_n \leq \lambda_{n+1}$

$$F(x) = \lim_{N\to\infty} \int_0^x \prod_{n=1}^{N} (1 + r_n \cos 2\pi(\lambda_n t + \phi_n)) dt$$

- Notation

$$\mu_a := \prod_{n=1}^{\infty} (1 + r_n \cos 2\pi(\lambda_n t + \phi_n)) := \mu_F.$$

Riesz product : on a compact abelian G

- $\Gamma = \{\gamma_n\}(\subset \widehat{G})$ is dissociated if $\sharp W_n(\Gamma) = 3^n$

$$W_n := W_n(\Gamma) := \{\epsilon_1\gamma_1 + \cdots + \epsilon_n\gamma_n : \epsilon_j = -1, 0, 1\}$$

- Notation : $a = (a_n)_{n\geq 1} \subset \mathbb{C}, |a_n| \leq 1$

$$P_{a,n}(x) = \prod_{k=1}^{n}(1 + \mathrm{Re}\ a_k\gamma_k(x))$$

- Remarkable relation

$$W_{n+1} = W_n \sqcup (-\gamma_{n+1} + W_n) \sqcup (\gamma_{n+1} + W_n)$$

$$\widehat{P}_{a,n+1}(\gamma) = \widehat{P}_{a,n}(\gamma) \quad \forall \gamma \in W_n.$$

Riesz product : some properties

- Zygmund dichotomy (1932)

$$F \text{ singular } \Leftrightarrow (a_n) \notin \ell^2; \quad F \text{ a.c. } \Leftrightarrow (a_n) \in \ell^2.$$

- Peyrière criterion (1973)

$$\sum |a_n - b_n|^2 = \infty \Rightarrow \mu_a \perp \mu_b;$$

$$\sum |a_n - b_n|^2 < \infty \Rightarrow \mu_a \ll \mu_b.$$

N. B. The second implication is proved under $\sup |a_n| < 1.$

- Parreau (1990) : $\sup |a_n| < 1$ replaced by $|a_n| = |b_n|$.
- Kilmer-Saeki (1988) : "$\sum |a_n - b_n|^2$" not "sufficient".
- Equivalence problem

Riesz product : randomization

- Random Riesz products of Rademacher type :

$$\prod_{n=1}^{\infty} (1 + \mathrm{Re} \pm \; a_n \gamma_n(x))$$

- Random Riesz products of Steinhaus type : $\forall \omega \in G^{\mathbb{N}}$

$$\mu_{a,\omega} := \prod_{n=1}^{\infty} (1 + \mathrm{Re}\; a_n \gamma_n(x + \omega_n))$$

- Homogeneous martingale (Kahane random multiplication) :

$$Q_n(x) := \prod_{k=1}^{n} (1 + \mathrm{Re}\; a_k \gamma_k(x + \omega_k)), \quad \forall x \in G.$$

Riesz product : two conjectures

- Conjecture 1 : $\forall \omega \in G^{\mathbb{N}}$

$$\mu_{a,\omega} \perp \mu_{b,\omega} \Leftrightarrow \mu_a \perp \mu_b; \quad \mu_{a,\omega} \ll \mu_{b,\omega} \Leftrightarrow \mu_a \ll \mu_b.$$

- Conjecture 2 :

$$\mu_a \perp \mu_b \Leftrightarrow \prod_{n=1}^{\infty} I(a_n, b_n) = 0.$$
$$\mu_a \ll \mu_b \Leftrightarrow \prod_{n=1}^{\infty} I(a_n, b_n) > 0.$$

$$I(a_n, b_n) := \mathbb{E}\sqrt{(1 + \mathrm{Re}\ a_k\gamma_k)(1 + \mathrm{Re}\ b_k\gamma_k)}.$$

Riesz product : return to $\mathbb{T}$

- A distance $d(\cdot,\cdot)$ on the unit disk :

$$ds^2 = d\theta^2 + \frac{dr^2}{\sqrt{1-r}}, \quad z = re^{2\pi i\theta}.$$

$$d(z_1, z_2)^2 \asymp |z_1 - z_2|^2 \left(1 + \frac{\cos^2(\phi - \psi)}{\sqrt{2 - |z_1 + z_2|}}\right)$$

$\phi = \arg(z_1 + z_2), \quad \psi = \arg(z_1 - z_2).$

- Conjecture 2 becomes

$$\sum d(a_n, b_n)^2 = \infty \Rightarrow \mu_a \perp \mu_b,$$

$$\sum d(a_n, b_n)^2 < \infty \Rightarrow \mu_a \ll \mu_b.$$

IV. A special case on $X = \{0,1\}^{\mathbb{N}}$

- $X = \{0,1\}^{\mathbb{N}}$, T is the shift.
- $f_i(x) = x_1$ the projection on the first coodinates $(i = 1, 2, \cdots, \ell)$
- for $\theta \in \mathbb{R}$, denote

$$A_\theta := \left\{ x \in \{0,1\}^{\mathbb{N}} : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k x_{2k} \cdots x_{\ell k} = \theta \right\}.$$

Remarks

- $f_i(T^i x) = x_i$ are not group characters.
- Riesz product method doesn't work and the study of A_θ is more difficult than B_θ.
- The study of A_θ was the motivation.

V. An attempt : a subset of A_0 For $\ell = 2$, define

$$X_0 := \left\{x \in \{0,1\}^{\mathbb{N}} : x_n x_{2n} = 0, \quad \text{for all } n\right\}.$$

Fibonacci sequence : $a_0 = 1,\ a_1 = 2, \quad a_n = a_{n-1} + a_{n-2}\ (n \geq 2)$.

Theorem (Fan-Liao-Ma, 2009)

$$\dim_B(X_0) = \frac{1}{2\log 2}\sum_{n=1}^{\infty}\frac{\log a_n}{2^n} = 0.8242936\cdots$$

Theorem (Kenyon-Peres-Solomyak, 2011)

$$\dim_H(X_0) = -\log_2 p = 0.81137\cdots \quad (p^3 = (1-p)^2).$$

Remarks

- $\dim_H(X_0) < \dim_B(X_0)$.
- A class of sets like X_0 is studied by Kenyon-Peres-Solomyak.
- $\dim_H(X_0) = \dim_H(A_0)$.

VI. Combinatorial proof (of box dimension) Starting point

$$\dim_B X_0 = \lim_{n\to\infty} \frac{\log_2 N_n}{n}$$

where N_n is the cardinality of

$$\{(x_1 x_2 \cdots x_n) : x_k x_{2k} = 0 \;\; \text{for } k \geq 1 \text{ such that } \; 2k \leq n\}.$$

Let $\{1, \cdots, n\} = C_0 \sqcup C_1 \sqcup \cdots \sqcup C_m$ with

$$C_0 := \{1,\ 3,\ 5,\ \ldots,\ 2n_0 - 1\},$$
$$C_1 := \left\{2 \cdot 1,\ 2 \cdot 3,\ 2 \cdot 5,\ \ldots,\ 2 \cdot \left(2n_1 - 1\right)\right\},$$
$$\cdots$$
$$C_k := \left\{2^k \cdot 1,\ 2^k \cdot 3,\ 2^k \cdot 5,\ \ldots,\ 2^k \cdot (2n_k - 1)\right\},$$
$$\cdots$$
$$C_m := \{2^m \cdot 1\},$$

The conditions $x_k x_{2k} = 0$ with k in different columns in the above table are independent. On each column, (x_k, x_{2k}) is conditioned to be different from $(1, 1)$. Counting column by column, we get

$$N_n = a_{m+1}^{n_m} a_m^{n_{m-1} - n_m} a_{m-1}^{n_{m-2} - n_{m-1}} \cdots a_1^{n_0 - n_1}.$$

Mega-Gibbs measure and nonlinear transfer operator

[A. H. Fan, J. Schmeling, M. Wu]

I. Setting

- $X = \Sigma_m = \{0, 1, \cdots, m-1\}^{\mathbb{N}}$ $(m \geq 2)$, T is the shift.
- $\Phi : X \times X \to \mathbb{R}$ continuous ($\ell = 2$).

I. Invariance spectrum

Fiber of measures :

$$\mathcal{M}_\Phi(\alpha) := \{\mu : \mathbb{E}_{\mu\otimes\mu}\Phi = \alpha\}$$

Theorem (F-S-W)

If $\mathcal{M}_\Phi(\alpha) \neq \emptyset$, then

$$F_{\text{invariance}}(\alpha) = F_{\text{mixing}}(\alpha) = \sup_{\mu\in\mathcal{M}_\Phi(\alpha)} \dim \mu.$$

Remark 1 : It coincides with the spectrum of the V-statistics. But it is no longer the case for $\Phi : X \times X \times X \to \mathbb{R}$.

Remark 2 : If $\Phi = (\phi_1, \phi_2)$ with $\phi_1 = \phi_2$ taking negative values α, no invariant measure is supported by $E(\alpha)$ but $\dim E(\alpha) > 0$.

II. Hausdorff spectrum : partial result

Assumption : $\Phi(x,y) = \varphi(x_1, y_1)$ depend only on the first coordinates.

Nonlinear transfer equation :

$$t_s(x)^2 = \sum_{Ty=x} e^{s\Phi(x,y)} t_s(y), \qquad \forall s \in \mathbb{R}.$$

Fact : $t_s : \Sigma_m \to \mathbb{R}_+$ depends only on the first coordinate. $s \mapsto P(s)$ is strictly convex and analytic.

Pressure :

$$P(s) = \log \int_{\Sigma_m} t_s(x)dx + \log m, \qquad \forall s \in \mathbb{R}.$$

Theorem (F-S-W)

For any $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, $P'(s) = \alpha$ has a unique solution s_α and we have

$$F_{\text{hausdorff}}(\alpha) = \frac{1}{2\log m}\left(P(s_\alpha) - s_\alpha P'(s_\alpha)\right)$$

III. Mega-Gibbs measures

Markov measure μ_s :

$$\pi(i) = \frac{t_s(i)}{\sum_{j=0}^{m-1} t_s(j)}, \qquad p_{i,j} = e^{s\varphi(i,j)} \frac{t_s(j)}{t_s(i)^2}.$$

Decomposition $\mathbb{N}^* = \bigsqcup_{i:2 \nmid i} \Lambda_i$ with $\Lambda_i = \{i2^k\}_{k\geq 0}$
Decomposition $\Sigma_m = \prod_{i:2\nmid i} \{0, 1, \cdots, m-1\}^{\Lambda_i}$.
Mega-Gibbs measure $\mathbb{P}_s$: Take a copy μ_s on each $\{0, 1, \cdots, m-1\}^{\Lambda_i}$ and then define

$$\mathbb{P}_s = \mu_s \times \cdots \times \mu_s \times \cdots .$$

IV. Gibbs measure For $n \geq 1$, μ_n is the probability measure uniformly distributed on each nq-cylinder and such that

$$\mu_n([x_1, ..., x_{2n}]) = \frac{1}{Z_n(t)} \exp(t \sum_{j=1}^{n} \varphi(x_j, x_{2j})).$$

Theorem (Existence of Gibbs measure)

For each t, the measures μ_n converge weakly to a probability measure μ_t, called Gibbs measure.

Theorem (Distribution of μ_t)

Let $N \geq 1$ and $F_1, ..., F_N$ be N arbitrary real functions defined on $S \times S$. We have

$$\lim_{n\to\infty} \int \prod_{j=1}^{N} F_j(x_j, x_{jq}) d\mu_n = \prod_{k=1}^{\lfloor \log_q N \rfloor} \prod_{\frac{N}{q^k} < i \leq \frac{N}{q^{k-1}}} \frac{1^t (\prod_{j=0}^{k-1} \Phi_{F_{iq^j}}(t)) w(t)}{\rho(t)^k}.$$

V. Sketched proof for Hausdorff spectrum

$\underline{D}(\mathbb{P}_s, x)$ (lower local dimension of $\mathbb{P}_s$ at x) : $\forall x \in E(\alpha)$, we have

$$\underline{D}(\mathbb{P}_s, x) \leq \frac{1}{2\log m}[P(s) - \alpha s].$$

$\mathbb{P}_{s_\alpha}(E(P'(s_\alpha))) = 1$: $\varphi(T^j x, T^{2j} x)$ is $\mathbb{P}_s$-mixing. So we have the law of large numbers :

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(T^j x, T^{2jx}) = P'(s) \quad \mathbb{P}_s\text{-a.e..}$$

VI. Study of transfer equation

$$t_i = \left(\sum_{j=0}^{m-1} A(i,j) t_j \right)^{1/2}, \qquad 0 \le i \le m-1.$$

Lemma

If the matrix A is positive, the above equation admits a unique positive solution.

The RHS of the equation defines a map $F : \mathbb{R}_+^{*m} \to \mathbb{R}_+^{*m}$ such that

$$F \uparrow, \quad F([a,b]^m) \subset [a,b]^m \qquad (a = \min A(i,j), b = \max A(i,j)).$$

$\lim F^n(a, \cdots, a)$ is a fixed point of F.

Lemma

If $A(i,j) = e^{s\varphi(i,j)}$, the solution $t(s)$ is analytic and $\log \sum_j t_j(s)$ is convex.

VI. Open questions

- Nearly nothing is known for

$$f_1(T^j x) f_2(T^{2j} x) f_3(T^{3j} x).$$

  [Riesz product method applicable to
  $f_1(x) = f_2(x) = f_3(x) = x_1 = \pm 1$.]
  If $f_i(x) = f_i(x_1)$ $(i = 1, 2, 3)$, the mixing spectrum = the spectrum of the V-statistics, but the mixing spectrum $\neq$ invariant spectrum. There is phase transition.

- The methods cannot be adapted to the case

$$f_1(x) = f_1(x_1, x_2), \quad f_2(x) = f_2(x_1, x_2).$$

  Because we lose the independence like : $x|_{\Lambda_i}$ and $x|_{\Lambda_j}$ $(i \neq j)$ are independent.