When rare events dominate transport: stickiness in Hamiltonian systems

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Rare & Extreme
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Outline

Original Motivations

Main physics

- Understanding of transport properties from the tracers dynamics:
  - in two-dimensional flows
  - and three dimensional ones.

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Theory and Mathematical tools

- Hamiltonian Chaos:
  - $1 - \frac{1}{2}$ degrees of freedom
  - Phase space analysis with Poincaré sections
- Transport analysis
  - Poincaré recurrences etc..
  - Displacements PDF's and associated momenta analysis
  - Fractional derivatives?

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Anomalous transport

- anomalous = non-diffusive $\langle X^2 \rangle \sim t^{\mu},\ \mu \neq 1$
  - $\mu < 1$ Sub-diffusion
  - $\mu = 1$ Diffusion
  - $\mu > 1$ Super-diffusion
- Memory effects to generate Lévy type statistics.

Outline

Advection of passive tracers

Passive particles trajectories are obtained by:

$$\dot{\mathbf{r}} = \mathbf{v}(\mathbf{r}, t) \quad (1)$$

For a two dimensional flow one can rewrite Eq. (1) using an Hamiltonian formalism

$$\dot{x} = -\frac{\partial \Psi}{\partial y}, \qquad \dot{y} = \frac{\partial \Psi}{\partial x} \quad (2)$$

where (x, y) corresponds to the coordinates of the tracer as well as canonical conjugates for the stream function Ψ which acts as a Hamiltonian.

Outline

A system of point vortices

Consider the Euler equation in two-dimensions:

$$\frac{\partial \Omega}{\partial t} + [\Omega, \psi] = 0 \,, \; \Omega = -\nabla^2 \psi \,, \tag{3}$$

and vorticity distribution given by a superposition of Diracs:

$$\Omega(\mathbf{x}, t) = \sum_{i=1}^{N} k_i \delta \left(\mathbf{x} - \mathbf{x}_i(t) \right) \,, \tag{4}$$

Point vortex properties

- The system can be reduced to $N-$body Hamiltonian dynamics of point-vortices.
- Finite-time singularities exist.

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Hamiltonian system

For an unbounded plane the Hamiltonian writes:

$$H = \frac{1}{2\pi} \sum_{i>j} k_i k_j \ln |z_i - z_j| \,, \tag{5}$$

where z_i and $\bar{z}_i$ are canonically conjugate

Invariances

- By translation
- By rotation
  but only three integral in involution $\Rightarrow$ chaos for $N > 3$.
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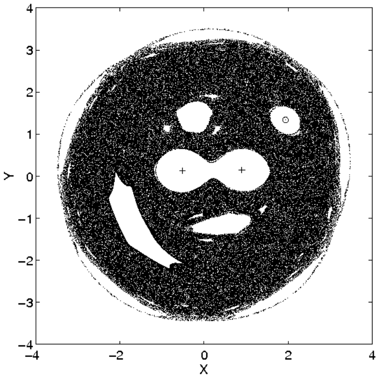
Poincaré map for 3 vortices

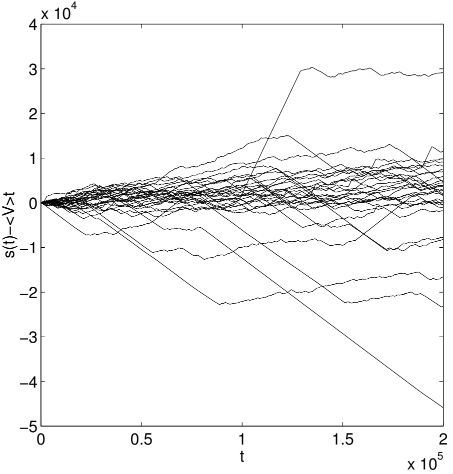


The Map is defined as:

$$z_{n+1} = \hat{P}(z_n) = e^{-i\Theta} z(T, z_n)$$

Phenomenon of chaotic advection

Lévy Flights



Evolution of $s_i(t)$ for an ensemble of 30 particles in the 3-point vortex system case

Transport properties

We measure the arc-length:

$$s_i(t) = \int_0^t |v(\tau)| d\tau \qquad (6)$$

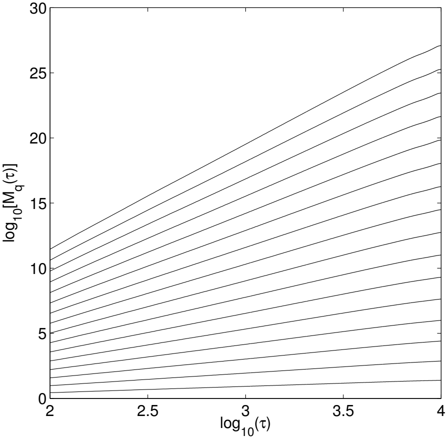
and compute the moments

$$M_q(t) \equiv \langle |s(t) - \langle s(t) \rangle|^q \rangle \ , \qquad (7)$$

for which the temporal behavior is fitted by a power law:

$$M_q(t) \sim t^{\mu(q)} \ . \qquad (8)$$

Moments evolution

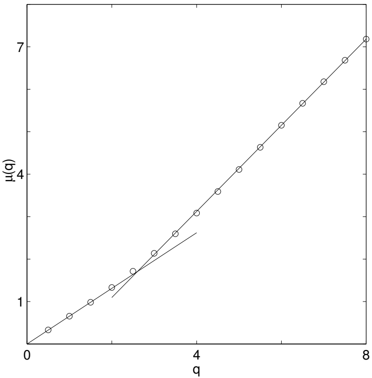


Typical evolution of the moments $M_q(t)$. The behavior is as expected:

$$M_q(t) \sim t^{\mu(q)} . \qquad (9)$$

$q = 0.5, 1, \cdots, 8$ are shown

Characterization of transport



3 point vortex system, transport is strongly anomalous $\mu(2) \approx 1.3$

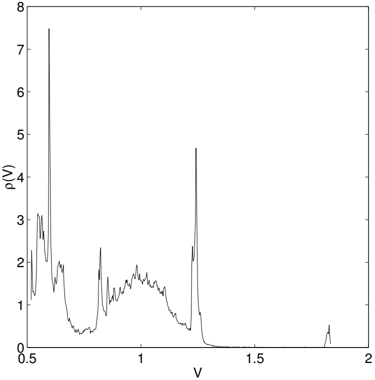
Behavior of $\mu(q)$ vs q:

- Gaussian Case: $\mu(q) = \lambda q$, $\lambda = \frac{1}{2}$
- Weakly anomalous (Self-similar) Case: $\mu(q) = \lambda q$, $\lambda \neq \frac{1}{2}$
- Strongly anomalous Case: $\mu(q) \neq \lambda q$

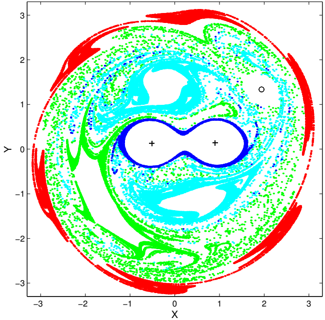
Outline

Origin of anomalous transport

Finite speed → Long time correlation are necessary to break the Central Limit Theorem



Origin of anomalous transport



Different peaks → Different sticky regions →strong anomalous (multi-fractal) transport

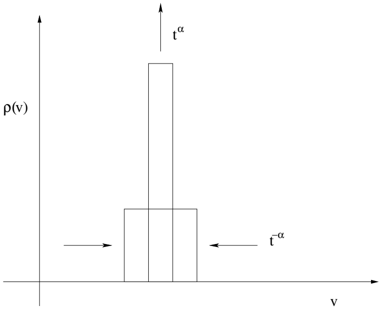
Distribution of average speeds

We consider a time-periodic Hamiltonian system $H(p, q, t)$ and given an initial condition i we consider the average speed:

$$\bar{v}_i(n) = \frac{1}{nT}\int_0^{nT}\sqrt{dq_i^2 + dp_i^2}\,. \qquad (10)$$

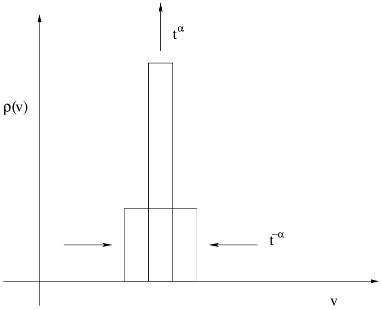
If the system is ergodic we expect that the distribution of $v_i(n)$, $\rho(n)$ to go towards a delta function as $n \to \infty$

Distribution of average speeds



A naive perspective....We check at which speed the maximum of $\rho(n)$ grows and we define an exponent α by $\rho_{max} \sim n^{\alpha}$
Note: if all is Gaussian we have $\alpha = 1/2$, et $\mu = 1$.

Distribution of average speeds



A naive perspective. Using this reasoning we end up with:

$$\alpha = 1 - \mu/2\,. \qquad (11)$$

Outline

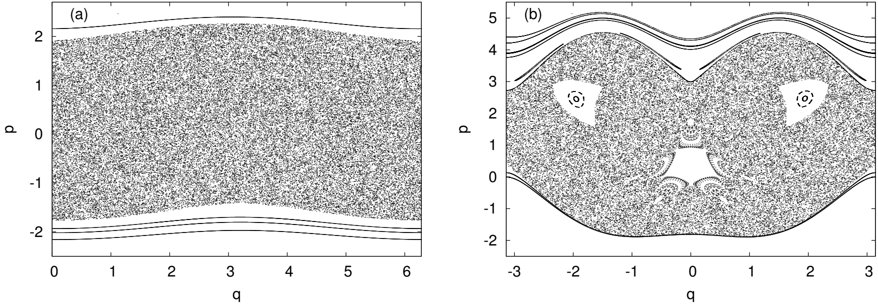
Motion of a charged particle in two waves

To test the ideas we shall consider a simpler Hamiltonian, describing for instance the motion of a charged particle in two electrostatic waves.

$$H = \frac{p^2}{2m} + A\left(\cos(k_1 q) + \varepsilon \cos(k_2 q - \omega t + \varphi)\right) , \qquad (12)$$

Motion of a charged particle in two waves

We consider two cases corresponding to different parameters:



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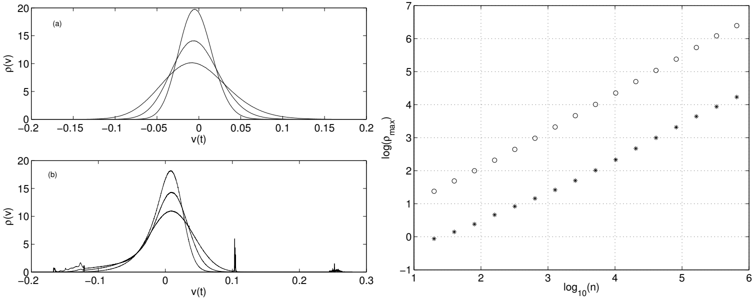


It works and when we compute transport we get $\alpha = 1 - \mu/2$.

Outline

The standard map

We test it further on the standard map (more data).

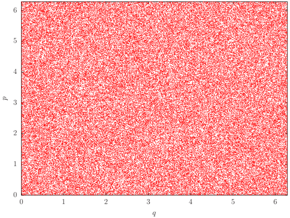
$$\begin{cases} p_{n+1} = p_n + K \sin q_n \; [2\pi] \\ q_{n+1} = q_n + p_{n+1} \; [2\pi], \end{cases} \tag{13}$$

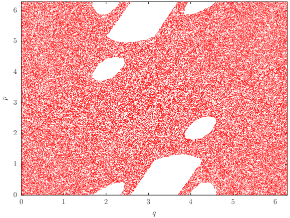
With the observable:

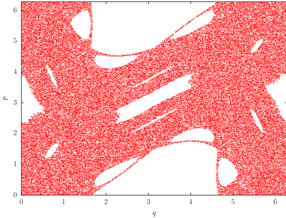
$$v = \sqrt{K^2 \sin^2 q + p^2} \, . \tag{14}$$

The standard map

We consider the following three cases:







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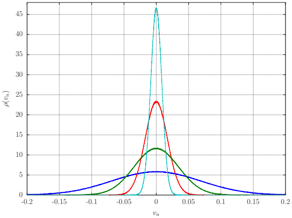

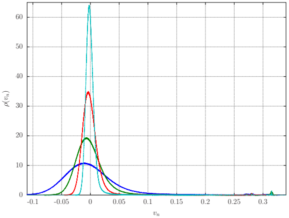

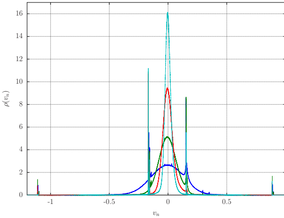

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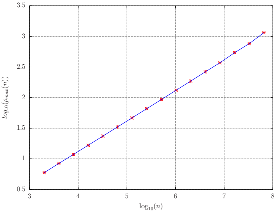

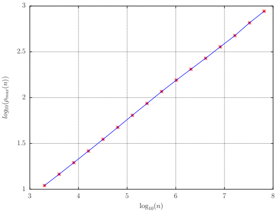

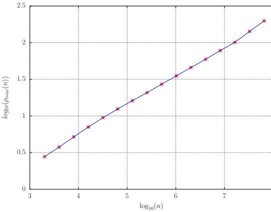

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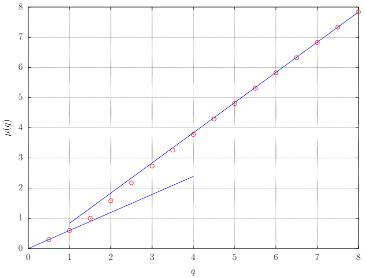


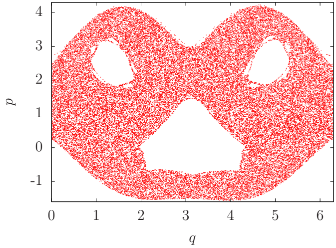
The naive perspective does not work, we use instead and it is ok

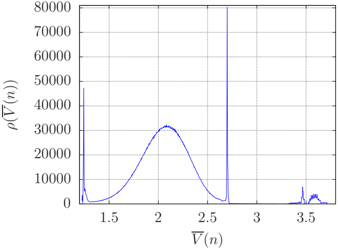
$$\frac{d\mu}{dq}(0) = 1 - \alpha\,. \tag{15}$$

Outline

Back to stickiness

We consider again the perturbed pendulum:





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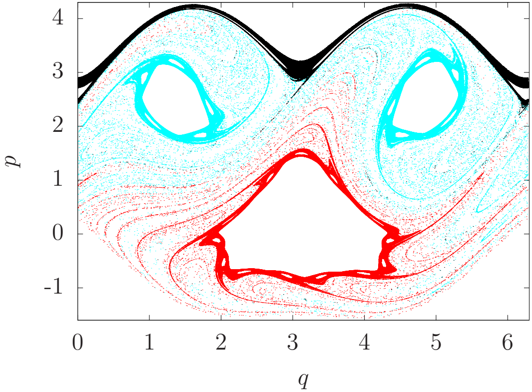
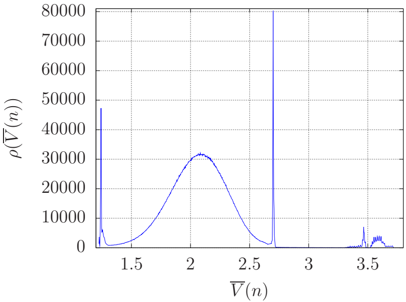


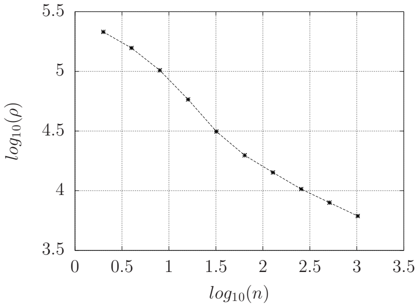
Back to stickiness



The area within a peak corresponds to the probability of sticking for at least n periods to the considered islands:

$$P_\nu(\tau) = \int_\tau^\infty \rho_\nu(t)\, dt \,, \tag{16}$$

Back to stickiness



But $P_\nu(\tau) \sim \tau^{1-\gamma}$ which gives $\gamma \approx 1.75 < 2$!
Potential problem with Kac's lemma unless the initial conditions are not typical.....

Back to stickiness

Thank you for your attention