On pointwise dimensions and spectra of measures

(Sur les dimensions locales et les spectres des mesures)

Section : **Analyse mathématique**

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Abstract

We give a new definition of the lower pointwise dimension associated with a Borel probability measure with respect to a general Carathéodory-Pesin structure. Then we show that the spectrum of the measure coincides with the essential supremum of the lower pointwise dimension. We provide an example coming from dynamical systems.

Résumé

Nous donnons une nouvelle définition de la dimension locale inférieure associée à une mesure de probabilité borélienne par rapport à une structure de Carathéodory-Pesin générale. Nous prouvons ensuite que le spectre de la mesure est égal au supremum essentiel de la dimension locale inférieure. Un exemple tiré des systèmes dynamiques illustre cette approche.

Keywords: pointwise dimension, spectrum of a measure, dynamical systems.
Mots-clés : dimension locale, spectre d'une measure, système dynamique.

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Version française abrégée. Ya. Pesin a introduit et étudié une construction généralisant celle de Hausdorff et de Carathéodory permettant de définir de nombreuses quantités s'apparentant à une dimension. Cette approche s'est avérée particulièrement intéressante dans les systèmes dynamiques où de nombreux invariants sont en fait des dimensions de Carathéodory-Pesin [3]. Deux types de dimensions apparaissent naturellement : les dimensions des ensembles et les dimensions des mesures. Ces dernières sont calculables dans certaines conditions à partir de certaines dimensions locales calculées pour un point "typique" de la mesure. Mais alors que la dimension d'une mesure existe toujours, la dimension locale existe rarement en dehors de contextes très particuliers [3]. Dans cette note, nous poursuivons un objectif double : définir une dimension locale inférieure (qui existe toujours) et qui permet de calculer simplement la dimension de la mesure.

Commençons par la construction de Carathéodory-Pesin. Soit X un sous-ensemble mesurable de $\mathbb{R}^N$ ([1]). Soit τ une fonction définie sur les sous-ensembles de X à valeur dans $\mathbb{R}$. Soit $Q \subset (0, +\infty)$ un ensemble dénombrable dont la fermeture contient 0. Pour tout ensemble $A \subset X$, $q \in \mathbb{R}$ et $\alpha \in \mathbb{R}$, nous définissons la quantité suivante :

$$\mathcal{M}^\tau(A, q, \alpha, \varepsilon) \stackrel{\text{def}}{=} \inf_{\substack{(x_i,\varepsilon_i)\in X\times Q \\ \varepsilon_i \le \varepsilon}} \sum_i \exp\big[-q\tau(B(x_i, \varepsilon_i))\big]\varepsilon_i^\alpha ,$$

où l'infimum est pris sur toutes les collections finies ou dénombrables de boules $B(x_i, \varepsilon_i)$ recouvrant A. Notons que cette définition coïncide avec celle donnée dans [3] à ceci près que nous introduisons le paramètre q. Celui-ci permet d'ajuster la balance entre le diamètre d'une boule B et la valeur $\tau(B)$. De plus en posant $q = 0$, on retrouve la dimension de Hausdorff habituelle. Une autre différence mineure est que nous forçons la *suffisance* de la famille $(B(x, \varepsilon))_{(x,\varepsilon)\in X\times Q}$, en ne prenant l'infimum que sur les boules de rayon appartenant à Q (voir [3, p. 24]). La limite $\mathcal{M}^\tau(A, \alpha, q) \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0} \mathcal{M}^\tau(A, \alpha, q, \varepsilon)$ existe par monotonicité et on pose la définition suivante : pour tout ensemble non-vide $A \subset X$ et tout $q \in \mathbb{R}$,

$$\alpha_\tau(A, q) \stackrel{\text{def}}{=} \inf\{\alpha : \mathcal{M}^\tau(A, \alpha, q) = 0\} = \sup\{\alpha : \mathcal{M}^\tau(A, \alpha, q) = \infty\},$$

avec les conventions $\inf \emptyset = +\infty$ et $\sup \emptyset = -\infty$. Nous appelons *spectre de l'ensemble A par rapport à la fonction τ* la fonction $q \mapsto \alpha_\tau(A, q)$.

En suivant [3], nous définissons à partir du spectre précédent le spectre d'une mesure borélienne comme suit :

$$\alpha_\tau^\mu(q) \stackrel{\text{def}}{=} \inf\{\alpha_\tau(Y, q) : Y \text{ mesurable } \subset X,\ \mu(Y) = 1\} .$$

Introduisons maintenant la nouvelle quantité suivante, que nous appelons la *q-dimension locale inférieure* au point $x \in X$:

$$d^\tau_{\mu,q}(x) \stackrel{\text{def}}{=} \liminf_{Q\ni\varepsilon\to 0} \inf_{y\in B(x,\varepsilon)} \frac{\log \mu(B(y, \varepsilon)) + q\tau(B(y, \varepsilon))}{\log \varepsilon} \tag{1}$$

Nous renvoyons au lemme ci-dessous pour la preuve de la mesurabilité de cette quantité. Nous sommes maintenant en mesure de formuler notre théorème :

Théorème. Soit μ une mesure de probabilité borélienne sur X et $q \in \mathbb{R}$. Nous avons

$$\alpha_\tau^\mu(q) = \text{sup-essentiel } d^\tau_{\mu,q} .$$

[1]Nous pourrions considérer n'importe quel espace métrique séparable pourvu que la conclusion du Lemme de Recouvrement de Besicovitch soit vérifiée.

La preuve de ce résultat est donnée ci-dessous ainsi que quelques remarques et commentaires. Enfin, nous donnons un exemple récent tiré des systèmes dynamiques [1] où $\tau(A)$ est la récurrence de Poincaré de l'ensemble A : on suppose qu'agit sur l'ensemble X l'application mesurable T et on pose

$$\tau(A) \stackrel{\text{def}}{=} \inf\{k > 0 : T^{-k}A \cap A \neq \emptyset\}.$$

Introduction. The monography of Ya. Pesin [3] provides a state-of-the-art of the interrelation between dimension theory (originated by Carathéodory and Hausdorff) and the theory of dynamical systems. The general construction proposed therein leads to the definition of dimensions of sets and dimensions of measures. A central issue is to compute these characteristics and to establish their coincidence under appropriate assumptions. In the present work we focus on dimensions of measures that are shown to be accessible by using some lower pointwise dimensions we define which always does exist. After the definitions and the statement, we very briefly discuss the difference between our approach and the one presented in [3]. We end up with an example corresponding to a recently introduced dimension characteristic associated with Poincaré recurrence in dynamical systems.

Definitions and the result. Let us start with the general Carathéodory-Pesin construction [3]. Let X be a measurable subset of $\mathbb{R}^N$ ([2]) and τ a real-valued set function. Let $Q \subset (0, +\infty)$ be a countable set within 0 in its closure. For any $A \subset X$, any $\alpha \in \mathbb{R}$ and any $q \in \mathbb{R}$ we define

$$\mathcal{M}^\tau(A, \alpha, q, \varepsilon) \stackrel{\text{def}}{=} \inf_{\substack{(x_i,\varepsilon_i)\in X\times Q \\ \varepsilon_i \le \varepsilon}} \sum_i \exp\big[-q\tau(B(x_i, \varepsilon_i))\big]\varepsilon_i^\alpha\,, \tag{2}$$

where the infimum is taken over all finite or countable collections of balls $B(x_i, \varepsilon_i)$ such that $\bigcup_i B(x_i, \varepsilon_i) \supseteq A$. The limit $\mathcal{M}^\tau(A, \alpha, q) \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0} \mathcal{M}^\tau(A, \alpha, q, \varepsilon)$ exists by monotonicity and we give the following definition

Definition 1. *For any non-empty $A \subset X$ and any $q \in \mathbb{R}$,*

$$\alpha_\tau(A, q) \stackrel{\text{def}}{=} \inf\{\alpha : \mathcal{M}^\tau(A, \alpha, q) = 0\} = \sup\{\alpha : \mathcal{M}^\tau(A, \alpha, q) = \infty\} \tag{3}$$

is called the spectrum for the set-function τ of the set A, with the convention that $\inf\emptyset = +\infty$ and $\sup\emptyset = -\infty$.

The parameter q we introduce does not change anything with respect to the definition in [3]. This is just a very convenient way of allowing to "tune" the balance between the diameter of a ball B and the value $\tau(B)$. Another minor difference is that we have forced the *sufficiency* of the family $(B(x, \varepsilon))_{(x,\varepsilon)\in X\times Q}$ (see [3, p. 24]). Notice that $\alpha_\tau(A, 0)$ is nothing but the Hausdorff dimension of the set A.

Now we proceed to the definition of the spectra of measures, following [3].

[2] We could consider any Besicovitch metric space, i.e. any separable metric space for which the conclusion of the Besicovitch Covering Lemma holds.

Definition 2 (Spectra of measures). *Let $\alpha_\tau(.,q)$ be the spectrum defined above. Let μ be a Borel probability measure on X.*

$$\alpha_\tau^\mu(q) \stackrel{\mathrm{def}}{=} \inf\{\alpha_\tau(Y,q) : Y \textit{ measurable } \subset X,\, \mu(Y)=1\}\ .$$

The next definition is a "local version" of these quantities, called q-pointwise dimension for the corresponding structure.

Definition 3 (Lower pointwise dimension). *The lower q-pointwise dimension of μ at the point x is defined by*

$$d^\tau_{\mu,q}(x) \stackrel{\mathrm{def}}{=} \liminf_{Q\ni\varepsilon\to 0}\ \inf_{y\in B(x,\varepsilon)} \frac{\log\mu(B(y,\varepsilon))+q\tau(B(y,\varepsilon))}{\log\varepsilon} \tag{4}$$

This definition is not exactly as in [3]. However, by adopting such a definition we may show directly that the spectrum of a measure coincides with the essential supremum of the lower pointwise dimension.

Lemma. The lower pointwise dimension is Borel measurable.

Proof. Let $\rho : X\times Q \to \mathbb{R}$ be any function. For any $\varepsilon \in Q$ the function $\psi_\varepsilon : X \to \mathbb{R}\cup\{-\infty\}$ defined by

$$\psi_\varepsilon(x) = \inf_{y\in B(x,\varepsilon)} \rho(y,\varepsilon)$$

is upper semi-continuous, hence measurable. In particular if $\rho(y,\varepsilon)$ denotes the ratio in (4) then the lower pointwise dimension is the pointwise limit, as $n\to\infty$, of the sequence of functions $(f_n)_n$ defined by

$$f_n(x) = \inf_{\varepsilon\in(0,1/n)\cap Q} \psi_\varepsilon(x). \tag{5}$$

Since Q is countable each f_n is measurable, and the conclusion follows. ■

We are now able to state the main result of this note.

Theorem. Let μ be a probability measure on X and $q\in\mathbb{R}$.

$$\alpha^\mu_\tau(q) = \text{ess-sup}\ d^\tau_{\mu,q}.$$

Note that this result is well known among specialists in the case of the Hausdorff dimension ($\tau\equiv 0$). We refer to [2] for further relations concerning dimensions of measures.

Proof. Set $\gamma = \text{ess-sup}\ d^\tau_{\mu,q}$ and let f_n be as in (5). Given $\delta>0$ we have

$$\mu(\{x : \lim_{n\to\infty} f_n(x) > \gamma-\delta\}) > 0,$$

hence there exists n such that $\Lambda_\delta = \{x : f_n(x) > \gamma - \delta\}$ has non-zero measure. Let $Y \subset X$ be such that $\mu(Y) = 1$. We obviously have

$$\alpha_\tau(Y, q) \geq \alpha_\tau(Y \cap \Lambda_\delta, q)\,.$$

Let $r \in (0, 1/n)$. For any ball $B = B(y, \varepsilon)$ with $\varepsilon \in (0, r) \cap Q$ such that $B \cap Y \cap \Lambda_\delta \neq \emptyset$ there exists $x \in \Lambda_\delta$ such that $y \in B(x, \varepsilon)$, which implies that

$$\mu(B)\, \mathrm{e}^{q\tau(B)} < \varepsilon^{\gamma-\delta}.$$

Thus remembering (2) we get

$$\mathcal{M}^\tau(Y \cap \Lambda_\delta, \gamma - \delta, q, r) \geq \inf_{\substack{(x_i,\varepsilon_i)\in X\times Q \\ \varepsilon_i \leq r}} \sum_i \mu(B(x_i, \varepsilon_i)) \geq \mu(Y \cap \Lambda_\delta),$$

where the infimum is taken according to the definition of $\mathcal{M}^\tau$. This yields

$$\mathcal{M}^\tau(Y \cap \Lambda_\delta, \gamma - \delta, q) \geq \mu(Y \cap \Lambda_\delta) > 0\,,$$

from which follows $\alpha_\tau(Y \cap \Lambda_\delta, q) \geq \gamma - \delta$. The arbitrariness of δ shows that $\alpha_\tau(Y, q) \geq \gamma$ for any set Y of full measure, therefore the inequality $\alpha_\tau^\mu(q) \geq \gamma$ follows.

We compute now an upper bound for $\alpha_\tau^\mu(q)$. By assumption the set

$$Y = \{x \in X \colon \lim_{n\to\infty} f_n \leq \gamma\}$$

has full μ-measure.

Let $\delta > 0$. Observe that for any $x \in Y$ there exist a sequence $\varepsilon_k^x \in (0, 1) \cap Q$ converging to zero, and another sequence $y_k^x \in X$ such that $y_k^x \in B(x, \varepsilon_k^x)$ and

$$\frac{\log \mu(B(y_k^x, \varepsilon_k^x)) + q\tau(B(y_k^x, \varepsilon_k^x))}{\log \varepsilon_k^x} \leq \gamma + \delta\,, \tag{6}$$

for any integer k. Let $\varepsilon > 0$ and set $I_\varepsilon = \{(y_k^x, \varepsilon_k^x) : x \in Y,\ k \in \mathbb{N} \text{ and } \varepsilon_k^x < \varepsilon\}$. For any $x \in Y$ and $k \in \mathbb{N}$ we have $x \in B(y_k^x, \varepsilon_k^x)$, thus $\{B(y, r) \colon (y, r) \in I_\varepsilon\}$ is a Vitali cover of Y. The Besicovitch Covering Lemma implies that there exists a subcover with finite multiplicity. Namely, there exists a finite or countable set $J_\varepsilon \subset I_\varepsilon$ such that $Y \subset \cup_{(y,r)\in J_\varepsilon} B(y, r)$ and for any $x \in X$ the cardinal of $\{(y, r) \in J_\varepsilon \colon B(y, r) \ni x\}$ is bounded by the constant M (depending only on the ambient space $\mathbb{R}^N$). Using (6) we get

$$\begin{aligned}\mathcal{M}(Y,\gamma+\delta,q,\varepsilon) &\leq \sum_{(y,r)\in J_\varepsilon} \mathrm{e}^{-q\tau(B(y,r))} r^{\gamma+\delta} \\ &\leq \sum_{(y,r)\in J_\varepsilon} \mu(B(y,r)) \\ &= \sum_{(y,r)\in J_\varepsilon} \int_{B(y,r)} \mathbb{1}\, d\mu(x) \\ &= \int_X \sum_{(y,r)\in J_\varepsilon} \mathbb{1}_{B(y,r)}(x) d\mu(x) \\ &\leq M.\end{aligned}$$

The arbitrariness of ε gives us that $\mathcal{M}^\tau(Y,\gamma+\delta,q) \leq M < \infty$, which immediately implies that $\alpha_T(Y,q) \leq \gamma+\delta$. Since δ is arbitrary this proves the theorem. ■

Some comments and remarks. For any reasonable choice of set function τ one expects the sufficiency of the family $B(x,\varepsilon)_{(x,\varepsilon)\in X\times Q}$ to be true for some countable set Q. For example whenever the set function τ is monotone it can be proven that $Q = (\mathrm{e}^{-n})_n$ is sufficient.

A particular example is when the lower q-pointwise dimension is a constant, say $\gamma = \gamma(q)$, μ-almost everywhere. This arises in dynamical systems in many situations [3]. Then our theorem tells us that $\alpha^\mu_\tau(q) = \gamma$.

One could generalize the above Carathéodory-Pesin contruction (Definition 1) in the following way: Consider two positive set functions η, ψ such that η "dominates" ψ (that is, $\forall\delta > 0, \exists\epsilon > 0$ such that if $\eta(B) < \delta$ then $\psi(B) \leq \epsilon$). In Definition 1 we put $\eta(B) = \psi(B) = \mathrm{diam}(B)$. Our theorem remains valid in this situation modulo wide assumptions on η, ψ; see [3].

One should compare Definition 3 of lower pointwise dimension with the one in [3] p. 24, as well as our theorem with Theorem 4.2 p. 28 of the same book.

An example: dimension characteristics for Poincaré recurrence. Let $T : X \to X$ be a map preserving the Borel probability measure μ. For any Borel set U, let $\tau(U)$ be the smallest return time in U, that is $\tau(U) \stackrel{\text{def}}{=} \inf\{\tau_U(x) : x \in U\}$ where $\tau_U(x) \stackrel{\text{def}}{=} \inf\{k > 0 : T^k x \in U\}$. $\tau(\cdot)$ is a monotone set function. This situation is studied in [1] in the case when (X,T) is a weakly specified subshift and μ an ergodic measure of positive entropy. The case of special flows built over such systems is also considered.

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