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Abstract. Given a point S (the light position) in $\mathbb{P}^3$ and an algebraic surface $\mathcal{Z}$ (the mirror) of $\mathbb{P}^3$, the caustic by reflection $\Sigma_S(\mathcal{Z})$ of $\mathcal{Z}$ from S is the Zariski closure of the envelope of the reflected lines $\mathcal{R}_m$ got by reflection of (Sm) on $\mathcal{Z}$ at $m \in \mathcal{Z}$. We use the ramification method to identify $\Sigma_S(\mathcal{Z})$ with the Zariski closure of the image, by a rational map, of an algebraic 2-covering space of $\mathcal{Z}$. We also give a general formula for the degree (with multiplicity) of caustics (by reflection) of algebraic surfaces of $\mathbb{P}^3$.

Introduction

Let $S[x_0 : y_0 : z_0 : t_0] \in \mathbb{P}^3 := \mathbb{P}(\mathbf{W})$ (with $\mathbf{W}$ a 4-dimensional complex vector space) and let $\mathcal{Z} = V(F)$ be a surface of $\mathbb{P}^3$ given by some $F \in Sym^d(\mathbf{W}^\vee)$ (i.e. F corresponds to a polynomial of degree d in $\mathbb{C}[x, y, z, t]$). The **caustic by reflection** $\Sigma_S(\mathcal{Z})$ of $\mathcal{Z}$ from $S \in \mathbb{P}^3$ is the Zariski closure of the envelope of the reflected lines $\mathcal{R}_m$ of the lines (mS) after reflection at m on the mirror surface $\mathcal{Z}$.

Since the seminal work of von Tschirnhaus [14, 15], caustics by reflection of planar curves have been studied namely by Chasles [6], Quetelet [12] and Dandelin [7]. Let us also mention the work of Bruce, Giblin and Gibson [3, 1, 2] in the real case. A precise computation of the degree and class of caustics by reflection of planar algebraic curves has been done in [9, 10, 11]. The idea was based on the fact that the caustic by reflection of an irreducible algebraic curve $\mathcal{C}$ of $\mathbb{P}^2$ from source $S_0 \in \mathbb{P}^2$ is the Zariski closure of the image of $\mathcal{C}$ by a rational map. Moreover, in the planar case, the generic birationality of the caustic map has been established in [11, 4]. The study of caustics by reflection of algebraic surfaces is more delicate. We will see that a generic point m of $\mathcal{Z}$ is associated to two (instead of a single one) points on $\Sigma_S(\mathcal{Z})$.

A classical way to study envelopes is the ramification theory. Let us mention that this approach has been used namely by Trifogli in [16] and by Catanese and Trifogli in [5] for focal loci (which generalize the notion of evolute to higher dimension). We use here the ramification theory to construct the caustic by reflection $\Sigma_S(\mathcal{Z})$ and to identify it with the Zariski closure of the image by some rational map Φ, of an algebraic 2-covering space $\hat{Z}$ of $\mathcal{Z}$. We will see that, contrary to the case of caustics by reflection of planar curves, **the set of base points of $\Phi_{|\hat{Z}}$ is never empty**. We give a general formula expressing the degree (with multiplicity) mdeg of $\Sigma_S(\mathcal{Z})$ in terms of intersection numbers of $\mathcal{Z}$ with a particular curve (called reflected polar curve) computed at the projection on $\mathcal{Z}$ of the base points of $\Phi_{|\hat{Z}}$. As a consequence of our general result, we prove namely the following generic result (see Theorem 37 for precisions).

Theorem 1. *Let $d \geq 1$. For a generic irreducible surface $\mathcal{Z} \subset \mathbb{P}^3$ of degree d and for a generic $S \in \mathbb{P}^3$, we have*

$$\operatorname{mdeg} \Sigma_S(\mathcal{Z}) = d(d-1)(8d-7).$$

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We denote by $\mathcal{H}^\infty$ the plane at infinity of $\mathbb{P}^3$: $\mathcal{H}^\infty = \{[x:y:z:t] \in \mathbb{P}^3 \;:\; t=0\}$ and $\mathcal{Z}_\infty := \mathcal{Z} \cap \mathcal{H}^\infty$. In this study the **umbilical curve** $\mathcal{C}_\infty$ plays a particular role. Recall that $\mathcal{C}_\infty$ is the intersection of $\mathcal{H}^\infty$ with any sphere (see Section 1).

In practice, the degree of the caustic will namely depend on the position of S with respect to the surface $\mathcal{Z}$, to $\mathcal{H}^\infty$, to $\mathcal{C}_\infty$ and to the **isotropic** tangent planes to $\mathcal{Z}$ (see Section 1 for the notion of isotropic planes).

We illustrate this by a precise study of the degrees of caustics of a paraboloid $\mathcal{Z}$. In this case, $\mathcal{Z}_\infty$ is the union of two lines intersecting at the focal point at infinity. We will see that the caustic of the paraboloid is a surface if the light position is outside $\mathcal{Z}_\infty$ and outside the focal points of $\mathcal{Z}$.

Still in the case of the paraboloid, $\mathcal{Z} \cap \mathcal{C}_\infty$ is made of two points I and J and the tangent planes to $\mathcal{Z}$ at these two points are isotropic. Moreover the revolution axis $\mathcal{D}$ of the paraboloid is the intersection of these two tangent planes.

Proposition 2. *Let $\mathcal{Z}$ be the paraboloid $V(x^2+y^2-2zt) \subset \mathbb{P}^3$ of axis $\mathcal{D} = V(x,y)$ and let $S \in \mathbb{P}^3$.*

If S is a focal point of the paraboloid (either $F_1[0:0:1:0]$ or $F_2[0:0:1:2]$), then $\Sigma_S(\mathcal{Z})$ is reduced to the other focal point.

If $S \in \mathcal{Z}_\infty \setminus \{F_1\}$, then $\Sigma_S(\mathcal{Z})$ is a planar curve of degree 2.

If $S \in \mathcal{D}$, then the degree of $\operatorname{mdeg}\Sigma_S(\mathcal{Z}) = 4$ *if* $S[0:0:0:1]$ *and* $\operatorname{mdeg}\Sigma_S(\mathcal{Z}) = 6$ *elsewhere.*

Assume now that S is not on $\mathcal{D} \cup \mathcal{Z}_\infty \cup \{F_2\}$.

If S is neither at infinity nor on the paraboloid $\mathcal{Z}$, then:

- *If $x_0^2+y_0^2=0$ and $z_0 \neq t_0/2$ (i.e. if S is on $\mathcal{T}_I\mathcal{Z}$ or on $\mathcal{T}_J\mathcal{Z}$ and on two other isotropic tangent planes to $\mathcal{Z}$), then* $\operatorname{mdeg}\Sigma_S(\mathcal{Z}) = 14$.
- *If $x_0^2+y_0^2=0$ and $z_0 = t_0/2$ (i.e. if S is on $\mathcal{T}_I\mathcal{Z}$ or on $\mathcal{T}_J\mathcal{Z}$ and on another isotropic tangent plane to $\mathcal{Z}$), then* $\operatorname{mdeg}\Sigma_S(\mathcal{Z}) = 12$.
- *If $x_0^2+y_0^2 \neq 0$ and $x_0^2+y_0^2+(z_0-t_0/2)^2=0$ (i.e. if S is on three isotropic planes but neither on $\mathcal{T}_I\mathcal{Z}$ nor on $\mathcal{T}_J\mathcal{Z}$), then* $\operatorname{mdeg}\Sigma_S(\mathcal{Z}) = 17$.
- *Otherwise (generic case: S is on four isotropic tangent planes to $\mathcal{Z}$), then* $\operatorname{mdeg}\Sigma_S(\mathcal{Z}) = 18$.

If S is at infinity, then:

- *If $S \notin \mathcal{C}_\infty$, then* $\operatorname{mdeg}\Sigma_S(\mathcal{Z}) = 12$.
- *If $S \in \mathcal{C}_\infty$, then* $\operatorname{mdeg}\Sigma_S(\mathcal{Z}) = 6$.

If S is on $\mathcal{Z}$, then:

- *If $x_0^2+y_0^2+t_0^2=0$ (i.e. if the tangent plane to $\mathcal{Z}$ at S is isotropic), then* $\operatorname{mdeg}\Sigma_S(\mathcal{Z}) = 12$.
- *If $x_0^2+y_0^2=0$ (i.e. if S is on $\mathcal{T}_I\mathcal{Z}$ or on $\mathcal{T}_J\mathcal{Z}$), then* $\operatorname{mdeg}\Sigma_S(\mathcal{Z}) = 14$.
- *Otherwise* $\operatorname{mdeg}\Sigma_S(\mathcal{Z}) = 16$.

The paper is organized as follows. Section 1 is devoted to the (complex) projectivization of orthogonality in the real euclidean affine 3-space (which plays a crucial role in the present work) and its link with the umbilical curve. In Section 2, we construct the reflected lines. In Section 3, we use the reflected lines and the ramification method to define the caustic by reflection. In Section 4, we define the appropriate 2-covering $\hat{Z}$ of $\mathcal{Z}$ and the rational map Φ. In Section 5, we determine precisely the base points of $\Phi_{|\hat{Z}}$. We define the reflected polar in section 6 and use it in Section 7 to establish a formula for the degree of the caustic by reflection. In Section 8, we prove Theorem 1. In Section 9, we prove Proposition 2. In Section 10, we precise a significative difference between the caustic by reflection studied in this paper and the focal loci of generic varieties considered in [16, 5]. In appendix A, we study two families of caustics by reflection of surfaces which are related to caustics by reflection of planar curves.

1. Affine and projective perpendicularity, link with umbilical conjugation

Consider the real euclidean affine 3-space E_3 of direction the 3-vector space $\mathbf{E}_3$ (endowed with some fixed basis). Let $\mathbf{W} := (\mathbf{E}_3 \oplus \mathbb{R}) \otimes \mathbb{C}$ (endowed with the induced basis). Let $j : E_3 \hookrightarrow \mathbb{P}^3 := \mathbb{P}(\mathbf{W})$ be the natural map defined on coordinates by $j(x, y, z) := [x : y : z : 1]$ for every $\underline{m}(x, y, z) \in E_3$. We are interested in the interpretation in the plane at infinity of $\mathbb{P}^3$ of perpendicularity at a point of two affine subvarieties of E_3. Consider the two following quadratic forms

$$q(x, y, z) = x^2 + y^2 + z^2 \text{ on } \mathbf{E}_3 \otimes \mathbb{C} \quad \text{and} \quad Q(x, y, z, t) = x^2 + y^2 + z^2 \text{ on } \mathbf{W}.$$

Definition 3. *The* ***umbilical curve*** *of $\mathbb{P}^3$ is the irreducible conic $\mathcal{C}_\infty := V(Q_{|\mathcal{H}^\infty}) \cong V(q) \subset \mathbb{P}(\mathbf{E}_3 \otimes \mathbb{C})$. We call* ***cyclic point*** *any point of $\mathcal{C}_\infty$.*

We recall that every (complex projectivized) sphere contains $\mathcal{C}_\infty$. It is worth noting that, for every $\underline{m} \in E_3$, we have the following classical diagram

$$\begin{tikzcd} E_3 \arrow[r, hook, "j"] \arrow[dr, "\xi_{\underline{m}}"'] & \mathbb{P}(\mathbf{W}) & \mathbf{W}\backslash\{0\} \arrow[l, "\Pi"'] \\ & \mathcal{H}^\infty & \end{tikzcd}$$

where Π is the canonical projection and with $\xi_{\underline{m}}$ is defined on coordinates by $\xi_{\underline{m}}(\underline{m} + (x, y, z)) = [x : y : z : 0]$. Given any vector subspace $\mathbf{V} \subset \mathbf{E}_3$, the projective subspace $\mathcal{V} := \overline{j(\underline{m} + \mathbf{V})}$ of $\mathbb{P}^3$ (where $\overline{K}$ denotes the Zariski closure of K) is the complex projectivization of the affine subspace $V = \underline{m} + \mathbf{V}$ of E_3. We observe that $\xi_{\underline{m}}(V)$ is $\mathcal{V}_\infty := \mathcal{V} \cap \mathcal{H}^\infty$.

An affine line L (resp. an affine plane H) containing $\underline{m} \in E_3$ is defined by $\underline{m} + V_1$ (resp. $\underline{m} + V_2$) with V_i an i-dimensional subspace of $\mathbf{E}_3$. Recall that the (complex) projectivization $\mathcal{L}$ of L (resp. $\mathcal{H}$ of H) is the projective line (resp. plane) of $\mathbb{P}^3$ of equations obtained by homogeneization of the equations of L (resp. H).

Hence, two lines L, L' containing $\underline{m}$ are perpendicular at $\underline{m}$ if and only if their points at infinity are conjugated with respect the conic $\mathcal{C}_\infty$.

A line L and a plane H containing $\underline{m}$ are perpendicular if and only if $\mathcal{H}_\infty$ is the **polar** of ℓ_∞ with respect to the conic $\mathcal{C}_\infty$ in $\mathcal{H}^\infty \cong \mathbb{P}^2$. This leads to the following definition of projective normal lines to a plane.

Definition 4. *Let $\mathcal{H} = V(h) \subset \mathbb{P}^3$ (with $h \in \mathbf{W}^\vee \setminus \{\mathbf{0}\}$) be a projective plane and $m \in \mathcal{H} \setminus \mathcal{H}^\infty$. The normal line $\mathcal{N}_m(\mathcal{H})$ to $\mathcal{H}$ at m is the line containing m and $n_\infty(\mathcal{H}) := \Pi(\boldsymbol{\kappa}(\nabla h))$ with $\boldsymbol{\kappa} : \mathbf{W} \to \mathbf{W}$ defined on coordinates by $\boldsymbol{\kappa}(a, b, c, d) := (a, b, c, 0)$.*

Remark 5. *Given a projective plane $\mathcal{H} \subset \mathbb{P}^3$ ($\mathcal{H} \neq \mathcal{H}^\infty$), if $n_\infty(\mathcal{H}) = [u : v : w : 0]$ lies on the umbilical (i.e. (u, v, w) lies on the* ***isotropic cone*** *$V(q)$ in $\mathbf{E}_3 \otimes \mathbb{C}$), then the line $\mathcal{H}_\infty$ is* ***tangent*** *to $\mathcal{C}_\infty$ at $n_\infty(\mathcal{H})$ in $\mathcal{H}^\infty$. In this case we have $\mathcal{N}_m(\mathcal{H}) \subset \mathcal{H}$.*

Let $m = \Pi(\mathbf{m})$ be a non singular point of $\mathcal{Z} \setminus \mathcal{H}^\infty$. We write $\mathcal{T}_m(\mathcal{Z})$ for the **projective tangent plane** at m to $\mathcal{Z}$. We also define the **projective normal line** $\mathcal{N}_m(\mathcal{Z})$ at m to

$\mathcal{Z}$ is the projective normal line to $\mathcal{T}_m(\mathcal{Z})$ at m, i.e. $\mathcal{N}_m(\mathcal{Z})$ is the line containing m and $n_{\infty,m}(\mathcal{Z}) = \Pi(\boldsymbol{\kappa}(\nabla F(\mathbf{m})))$.

Observe that the line at infinity $\mathcal{T}_{\infty,m}(\mathcal{Z})$ of $\mathcal{T}_m(\mathcal{Z})$ is the **polar** of the point at infinity $n_{\infty,m}(\mathcal{Z})$ of $\mathcal{N}_m(\mathcal{Z})$ with respect the conic $\mathcal{C}_\infty$.

Later, we will see that the base points of the reflected map can be seen on the geometry on the normals at infinity with respect to the umbilical. In particular isotropic tangent plane to $\mathcal{Z}$ containing S will play some role.

Definition 6. *A plane $\mathcal{H} = V(h)$ (with $h \in \mathbf{W}^\vee \setminus \{\mathbf{0}\}$) is said to be* **isotropic** *if ∇h is an isotropic vector for Q.*

Remark 7. *A plane $\mathcal{H} \subset \mathbb{P}^3$ is isotropic if and only if either it is the plane at infinity $\mathcal{H}^\infty$ or if $n_\infty(\mathcal{H})$ is in $\mathcal{C}_\infty$ (i.e. $\mathcal{H}$ contains its normal lines).*

In particular, the surface $\mathcal{Z}$ admits an isotropic tangent plane at one of its nonsingular point $m[x : y : z : 1]$ if and only if m belongs to $V(Q(\nabla F), F)$. We note that the whole curve $\mathcal{C}_\infty$ is contained in every complex projectivized sphere $\mathcal{S}_r$ and that we have $\mathcal{N}_m(\mathcal{S}_r) \subset \mathcal{T}_m(\mathcal{S}_r)$ for all $m \in \mathcal{S}_r \setminus \mathcal{H}^\infty$. This is also true for tori.

Consider some particular points on $\mathcal{Z}$, playing a particular role in the construction of the caustic map. Let $\mathcal{B}_0 := V(F, \Delta_{\mathbf{S}} F, Q(\nabla F))$ in $\mathbb{P}^3$, the interpretation in the plane at infinity is the following one. Let m be a nonsingular point of $\mathcal{Z} \setminus \mathcal{H}^\infty$ then

$$m \in \mathcal{B}_0 \Longleftrightarrow \left\{ \begin{array}{c} S \in \mathcal{T}_m(\mathcal{Z}) \\ n_{\infty,m}(\mathcal{Z}) \in \mathcal{C}_\infty \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{c} (mS) \subset \mathcal{T}_m(\mathcal{Z}) \\ n_{\infty,m}(\mathcal{Z}) \in \mathcal{C}_\infty \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{c} (mS)_\infty \in \mathcal{T}_{\infty,m}(\mathcal{Z}) \\ \mathcal{T}_{\infty,m}(\mathcal{Z}) = \mathcal{T}_{n_{\infty,m}(\mathcal{Z})}(\mathcal{C}_\infty) \end{array} \right. . \tag{1}$$

We observe that $\mathcal{B}_0$ is in general a finite set, but that, for the unit sphere, $\mathcal{B}_0$ is a curve (the circle apparent contour of $\mathcal{Z}$ seen from S).

Let us now specify some additional notations used in this paper. We write $\mathbf{S}(x_0, y_0, z_0, t_0) \in \mathbf{W} \setminus \{0\}$. For any $m[x : y : z : t] \in \mathbb{P}^3$, we will write $\mathbf{m}(x, y, z, t) \in \mathbf{W} \setminus \{0\}$. For any $d' \geq 1$ and any $G \in Sym^{d'}(\mathbf{W}^\vee)$, we write as usual $G_x, G_y, G_z, G_t \in Sym^{d'-1}(\mathbf{W}^\vee)$ for the partial derivatives of in x, y, z and t respectively.

2. Reflected lines

The incident lines are the lines $(S\,m)$ with $m \in \mathcal{Z}$. We will define the reflected line $\mathcal{R}_m$ as the orthogonal symmetric of $(S\,m)$ with respect to the tangent plane to $\mathcal{Z}$ at m. To this end, we will define the orthogonal symmetric $\sigma(m)$ of S with respect to the tangent plane to $\mathcal{Z}$ at m. Let us first explain how one can give a sense to the notion of orthogonal symmetries in $\mathbb{P}^3$ by complex projectivization of the euclidean affine situation.

2.1. Orthogonal symmetric and map σ.

To every injective linear map $\mathbf{W} \overset{f}{\to} \mathbf{W}$, corresponds a unique morphism $\mathbb{P}(\mathbf{W}) \overset{\mathbb{P}(f)}{\to} \mathbb{P}(\mathbf{W})$. Therefore, to every injective affine map $E_3 \overset{g}{\to} E_3$, corresponds a unique algebraic map $\mathbb{P}(\mathbf{W}) \overset{\iota(g)}{\to} \mathbb{P}(\mathbf{W})$. This defines an injective groups homomorphism $\iota : Aff(E_3) \cong \mathbf{E}_3 \rtimes Gl(\mathbf{E}_3) \to \mathbb{P}(Gl(\mathbf{W}))$ such that $\iota(Is(E_3)) = \iota(\mathbf{E}_3 \rtimes O(\mathbf{E}_3)) \subset \mathbb{P}(O(\hat{Q}))$, with $\hat{Q} = x^2 + y^2 + z^2 + t^2$ on $\mathbf{W}$. We apply this to the orthogonal symmetry s_H with respect to some affine plane $H = V(\tilde{h}) \subseteq E_3$ with $\tilde{h} = ax + by + cz + d$. Recall that s_H is defined by

$s_H(P) = P - 2\,\tilde{h}(P)\frac{\nabla\tilde{h}}{q(\nabla\tilde{h}))}$. This leads to the morphism $s_{\mathcal{H}} := \iota(s_h) : \mathbb{P}^3 \to \mathbb{P}^3$ defined by $\mathbb{P}(\mathbf{s}_h)$ with

$$\forall \mathbf{P} \in \mathbf{W}, \quad \mathbf{s}_h(\mathbf{P}) := Q(\nabla h)\cdot \mathbf{P} - 2h(\mathbf{P})\cdot \boldsymbol{\kappa}(\nabla h) \in \mathbf{W},$$

with $\mathcal{H} = V(h) \subset \mathbb{P}^3$ and with $h = ax + by + cz + dt$ the homogeneized of $\tilde{h}$. Now we extend this definition to any projective plane $\mathcal{H} \subset \mathbb{P}^3$ as follows.

Definition 8. *Consider a plane $\mathcal{H} = V(h) \subseteq \mathbb{P}^3$ (with $h \in \mathbf{W}^\vee \setminus \{\mathbf{0}\}$). We define the orthogonal symmetry $s_{\mathcal{H}}$ with respect to $\mathcal{H}$ as the rational map given by $s_{\mathcal{H}} = \mathbb{P}(\mathbf{s}_h)$ with*

$$\forall \mathbf{P} \in \mathbf{W}, \quad \mathbf{s}_h(\mathbf{P}) := Q(\nabla h)\cdot \mathbf{P} - 2h(\mathbf{P})\cdot \boldsymbol{\kappa}(\nabla h) \in \mathbf{W}.$$

We can notice that, when $\mathcal{H} \neq \mathcal{H}^\infty$, $s_{\mathcal{H}}(P)$ is well defined in $\mathbb{P}^3$ except if $\mathcal{H}$ is an isotropic plane containing P (see Proposition 9). For any non singular $m[x : y : z : t] \in \mathcal{Z}$, we define $\sigma(m) := s_{\mathcal{T}_m\mathcal{Z}}(S) = \mathbb{P}(\boldsymbol{\sigma})(m)$ with

$$\boldsymbol{\sigma} := Q(\nabla F)\cdot \mathbf{S} - 2\Delta_{\mathbf{S}}F \cdot \boldsymbol{\kappa}(\nabla F) \in \mathbf{W} \tag{2}$$

on $\Pi^{-1}(\mathcal{Z})$ with $\Delta_{\mathbf{S}}F$ the equation of the polar hypersurface of $\mathcal{P}_S(\mathcal{Z})$ given by $\Delta_{\mathbf{S}}F := DF \cdot \mathbf{S}$ (where DF is the differential of F). We extend the definition of $\boldsymbol{\sigma}(\mathbf{m})$ to any $\mathbf{m} \in \mathbf{W} \setminus \{0\}$. Observe that $\boldsymbol{\sigma}$ defines a unique rational map $\sigma : \mathbb{P}^3 \to \mathbb{P}^3$.

Proposition 9. *The base points of the rational map $\sigma_{|\mathcal{Z}}$ are the singular points of $\mathcal{Z}$, the points of tangency of $\mathcal{Z}$ with $\mathcal{H}^\infty$ and the points at which $\mathcal{Z}$ has an isotropic tangent plane containing S.*

Proof. We prove that the base points of σ are the points of $\mathbb{P}^3$ such that $F_x = F_y = F_z = 0$ or such that $Q(\nabla F) = 0$ and $\Delta_{\mathbf{S}}F = 0$. It is easy to see that these points are base points of σ. Now let $m = [x : y : z : t]$ be a point of $\mathbb{P}^3$ such that $\sigma(m) = 0$.

- If $\Delta_{\mathbf{S}}(F) = 0$, then, since $\mathbf{S} \neq 0$, we get that $Q(\nabla F) = 0$.
- If $Q(\nabla F) = 0$, then either $\Delta_{\mathbf{S}}F = 0$ or $\boldsymbol{\kappa}(\nabla F) = 0$.
- Assume now that $Q(\nabla F) \neq 0$. We have $Q(\nabla F)\cdot \mathbf{S} = 2\Delta_{\mathbf{S}}F \cdot \boldsymbol{\kappa}(\nabla F)$. This implies that $\boldsymbol{\kappa}(\nabla F)$ is non zero and proportional to $\mathbf{S}$ (which is also non zero), so that $t_0 = 0$ and $0 = y_0F_x - x_0F_y = z_0F_y - y_0F_z = x_0F_z - z_0F_x$. Therefore, writing $\sigma^{(i)}$ for the ith coordinate of $\boldsymbol{\sigma}$, we have

$$\begin{aligned} 0 &= \sigma^{(1)} = Q(\nabla F)x_0 - 2(x_0F_x^2 + y_0F_xF_y + z_0F_xF_z) \\ &= Q(\nabla F)x_0 - 2(x_0F_x^2 + x_0F_y^2 + x_0F_z^2) = -Q(\nabla F)x_0. \end{aligned}$$

  In the same way, we get $0 = \sigma^{(2)} = -Q(\nabla F)y_0$ and $0 = \sigma^{(3)} = -Q(\nabla F)z_0$. This contradicts the fact that $Q(\nabla F) \neq 0$ (since $\mathbf{S} \neq 0$).

□

Remark 10. *Each $\sigma^{(i)}$ belongs to $Sym^{2(d-1)}(\mathbf{W}^\vee)$. Moreover, for a general $(\mathcal{Z}, S)$, the set $V(F, F_x, F_y, F_z)$ is empty and the base points of $\sigma_{|\mathcal{Z}}$ are the $2d(d-1)^2$ points of $V(F, Q(\nabla F), \Delta_{\mathbf{S}}F)$.*

2.2. **Reflected lines.**

Definition 11. *For any $m \in \mathcal{Z}$, the* **reflected line** *$\mathcal{R}_m$ on $\mathcal{Z}$ at m is the line $(m\sigma(m))$ when it is well defined.*

Definition 12. *We write $\mathcal{M}_{S,\mathcal{Z}}$ for the set of points $m \in \mathbb{P}^3$ such that $\mathbf{m}$ and $\boldsymbol{\sigma}(\mathbf{m})$ are proportional, i.e. $\mathcal{M}_{S,\mathcal{Z}} := \{m \in \mathbb{P}^3 \ : \ \exists[\lambda_0 : \lambda_1] \in \mathbb{P}^1, \quad \lambda_0 \cdot \mathbf{m} + \lambda_1 \cdot \boldsymbol{\sigma}(\mathbf{m}) = 0\}$.*

Observe that $\mathcal{R}_m$ is well defined if $m \in \mathcal{Z} \setminus \mathcal{M}_{S,\mathcal{Z}}$.

Proposition 13. *We have* $\mathcal{Z}\cap\mathcal{M}_{S,\mathcal{Z}} = \mathcal{Z}\cap(\mathrm{Base}(\sigma)\cup\{S\}\cup\mathcal{W})$, *with*

$$\mathcal{W} := \{m\in\mathcal{Z}\ :\ m = n_{\infty,m}(\mathcal{Z}),\quad \Delta_{\mathbf{S}}F(m)\neq 0,\ Q(m)=0\},$$

with $n_{\infty,m}(\mathcal{Z}) := \Pi(\boldsymbol{\kappa}(\nabla F(\mathbf{m})))$.

Proof. We prove $\mathcal{Z}\cap\mathcal{M}_{S,\mathcal{Z}}\subseteq\mathcal{Z}\cap(\mathrm{Base}(\sigma)\cup\{S\}\cup\mathcal{W})$, the inverse inclusion being clear. Let $m\in(\mathcal{Z}\cap\mathcal{M}_{S,\mathcal{Z}})\setminus\mathrm{Base}(\sigma)$. Observe that, due to the Euler identity, we have $0 = DF(\mathbf{m})\cdot\mathbf{m}$ and so $0 = DF\cdot\boldsymbol{\sigma} = -\Delta_{\mathbf{S}}F\cdot Q(\nabla F)$. If $\Delta_{\mathbf{S}}F = 0$, then $\boldsymbol{\sigma} = Q(\nabla F)\cdot\mathbf{S}$, so $m = \sigma(m) = S$. If $Q(\nabla F) = 0$, then $\boldsymbol{\sigma} = -2\Delta_{\mathbf{S}}F\cdot\boldsymbol{\kappa}(\nabla F)$. So $m = \sigma(m) = n_{\infty,m}(\mathcal{Z})$; moreover $\Delta_{\mathbf{S}}F\neq 0$ and $Q = 0$. □

Lemma 14. *If* $\dim\mathcal{M}_{S,\mathcal{Z}} = 3$, *then* $\mathcal{Z} = \mathcal{H}^\infty$ *or* $V(\Delta_{\mathbf{S}}F, Q(\nabla F)) = \mathbb{P}^3$.

Proof. Due to Proposition 13, we have $\mathcal{Z}\cap\mathcal{M}_{S,\mathcal{Z}}\subseteq\mathcal{Z}\cap(\mathrm{Base}(\sigma)\cup\{S\}\cup\mathcal{C}_\infty)$. Assume that $\dim\mathcal{M}_{S,\mathcal{Z}} = 3$. This implies that $\mathrm{Base}(\sigma) = \mathbb{P}^3$. So, due to the proof of Proposition 9, we conclude that $\mathbb{P}^3 = V(F_x, F_y, F_z)\cup V(\Delta_{\mathbf{S}}F, Q(\nabla F))$. So, either $\mathbb{P}^3 = V(F_x,F_y,F_z)$ (which implies $\mathcal{Z} = \mathcal{H}^\infty$) or $\mathbb{P}^3 = V(\Delta_{\mathbf{S}}F, Q(\nabla F))$. □

3. Caustic by reflection

Now, let us introduce some additional notations. We define $N_{\mathbf{S}}(\mathbf{m})$ as the complexified homogenized square euclidean norm of $\mathbf{Sm}$ by

$$N_{\mathbf{S}}(\mathbf{m}) := (xt_0 - x_0t)^2 + (yt_0 - y_0t)^2 + (zt_0 - z_0t)^2.$$

We will also consider the bilinear Hessian form Hess_F of F and its determinant H_F. Let us see how to construct two maps $\psi = \psi^\pm : \mathcal{Z}\to\mathbb{P}^3$ such that the surface $\psi(\mathcal{Z})$ is tangent to the reflected line $\mathcal{R}_m$ at $\psi(m)$, for a generic $m\in\mathcal{Z}$. Observe first that $\psi(m)$ is in $\mathcal{R}_m$ implies that $\psi(m)$ can be rewritten

$$\boldsymbol{\psi}(\mathbf{m}) = \lambda_0(\mathbf{m})\cdot\mathbf{m} + \lambda_1(\mathbf{m})\cdot\boldsymbol{\sigma}(\mathbf{m})\in\mathbf{W}\setminus\{0\},$$

with $[\lambda_0(\mathbf{m}):\lambda_1(\mathbf{m})]\in\mathbb{P}^1$ for every $m\in\mathcal{Z}$. The main result of this section is the next theorem specifying the form of λ_0 and λ_1 (belonging to an integral extension of the ring $Sym(\mathbf{W}^\vee)$) which ensures that, for a generic $m\in\mathcal{Z}$, $\mathcal{R}_m$ is tangent to $\psi(\mathcal{Z})$ at $\psi(m)$.

Theorem 15. *Let* $\psi : U\to\mathbb{P}^3$ *(with* $U\subseteq\mathcal{Z}$*) be given by*

$$\boldsymbol{\psi}(\mathbf{m}) = \lambda_0(\mathbf{m})\cdot\mathbf{m} + \lambda_1(\mathbf{m})\cdot\boldsymbol{\sigma}(\mathbf{m})\quad\in\mathbf{W},$$

with $\lambda_0(\cdot)$ *and* $\lambda_1(\cdot)$ *in an integral extension of* $Sym(\mathbf{W}^\vee)$ *such that*

$$\alpha(\mathbf{m})(\lambda_0(\mathbf{m}))^2 + \beta(\mathbf{m})\lambda_0(\mathbf{m})\lambda_1(\mathbf{m}) + \gamma(\mathbf{m})(\lambda_1(\mathbf{m}))^2 = 0 \tag{3}$$

with $\alpha,\beta,\gamma\in Sym(\mathbf{W}^\vee)$ *given by*

$$\alpha := \Delta_{\mathbf{S}}F\quad\in Sym^{d-1}(\mathbf{W}^\vee), \tag{4}$$

$$\beta := -2\left[\mathrm{Hess}\,F(\mathbf{S},\boldsymbol{\sigma}) + (\Delta_{\mathbf{S}}F)^2(F_{xx}+F_{yy}+F_{zz})\right]\quad\in Sym^{3d-4}(\mathbf{W}^\vee) \tag{5}$$

and

$$\gamma := -\frac{4\Delta_{\mathbf{S}}F}{(d-1)^2}N_{\mathbf{S}}\,H_F\quad\in Sym^{5d-7}(\mathbf{W}^\vee). \tag{6}$$

Then, for every $m\in\mathcal{Z}\setminus V(tQ(\nabla F))$, *the reflected line* $\mathcal{R}_m$ *is tangent to* $\psi(\mathcal{Z})$ *at* $\psi(m)$.

It will be useful to introduce

$$\forall(\mathbf{m},\lambda_0,\lambda_1)\in\mathbf{W}\times\mathbb{C}^2,\quad Q_{\mathbf{S},F}(\mathbf{m},\lambda_0,\lambda_1)=\alpha(\mathbf{m})\lambda_0^2+\beta(\mathbf{m})\lambda_0\lambda_1+\gamma(\mathbf{m})\lambda_1^2.$$

One may notice that, for a fixed $\mathbf{m}$, $Q_{\mathbf{S},F}(\mathbf{m},\lambda_0,\lambda_1)$ is a quadratic form in (λ_0,λ_1). Roughly speaking, Theorem 15 states that the image of $\mathcal{Z}$ by $\psi(\cdot)=\lambda_0(\cdot)\cdot\mathrm{Id}+\lambda_1(\cdot)\cdot\sigma(\cdot)$ (for some $\lambda_0,\lambda_1\in Sym(\mathbf{W}^\vee)[\sqrt{\beta^2-4\alpha\gamma}]$) corresponds to a part of the envelope of the reflected lines $\mathcal{R}_m$. More precisely:

Definition 16. *The* **caustic by reflection** $\Sigma_S(\mathcal{Z})$ *of* $\mathcal{Z}$ *from* S *is the Zariski closure of the following set*

$$\{P\in\mathbb{P}^3\ :\ \exists m\in\mathcal{Z},\ \exists[\lambda_0:\lambda_1]\in\mathbb{P}^1,\quad Q_{\mathbf{S},F}(\mathbf{m},\lambda_0,\lambda_1)=0\ \text{and}\ \mathbf{P}=\lambda_0\cdot\mathbf{m}+\lambda_1\cdot\boldsymbol{\sigma}(\mathbf{m})\}.$$

Remark 17. *If* $\mathcal{Z}\subseteq V(\Delta_{\mathbf{S}}F,(F_x^2+F_y^2+F_z^2)\,\mathrm{Hess}\,F(\mathbf{S},\mathbf{S}))$*, then (3) becomes* $0=0$ *on* $\mathcal{Z}$ *and* $\sigma_{S,\mathcal{Z}}(\mathcal{Z})$ *is either* $\{S\}$ *or empty. If it is* S *(i.e. if* $\Delta_{\mathbf{S}}F=0$ *in* $\mathbb{C}[x,y,z,t]$ *and if* $\mathcal{Z}\not\subseteq V(F_x^2+F_y^2+F_z^2)$*), we set* $\Sigma_S(\mathcal{Z})=\{S\}$*.*

Theorem 15 states that the points of the caustic $\Sigma_S(\mathcal{Z})$ corresponding to $m\in\mathcal{Z}$ are the points of coordinates $\boldsymbol{\psi}^{\pm}(\mathbf{m})$ with

$$\boldsymbol{\psi}^{\pm}(\mathbf{m})=\left(\tilde{\beta}(\mathbf{m})\pm\sqrt{\vartheta(\mathbf{m})}\right)\cdot\mathbf{m}+\Delta_{\mathbf{S}}F(\mathbf{m})\cdot\boldsymbol{\sigma}(\mathbf{m})\in\mathbb{C}^4,\tag{7}$$

with $\tilde{\beta}:=-\beta/2$ and $\vartheta:=\tilde{\beta}-\alpha\gamma$. Let us observe that if ϑ is a square in $\mathbb{C}[x,y,z,t]/(F)$, then, on $\mathcal{Z}$, (7) corresponds to two rational maps $\psi^{\pm}:\mathbb{P}^3\to\mathbb{P}^3$ and the caustic by reflection $\Sigma_S(\mathcal{Z})$ is the union of the Zariski closures of $\psi^+(\mathcal{Z})$ and of $\psi^-(\mathcal{Z})$. Let us give some examples.

Example 18 (A singular caustic of the saddle surface)**.** *Let us study the caustic by reflection of* $\mathcal{Z}=V(xy-zt)$ *from* $S=[0:0:1:0]$*. Observe that* $\alpha=-t$*,* $\beta=0$ *and* $\gamma=4t^3$*. So (3) becomes* $\lambda_0^2-4t^2\lambda_1^2=0$ *(if* $t\neq0$*). Hence* $\Sigma_S(\mathcal{Z})$ *is the union of the Zariski closure of the images of* $\mathcal{Z}$ *by the two rational maps* $\psi^{\pm}:\mathbb{P}^3\mapsto\mathbb{P}^3$ *defined on coordinates by* $\boldsymbol{\psi}^{\pm}(x,y,z,t)=(2ty\pm2tx,2tx\pm2ty,x^2+y^2-t^2\pm2tz,\pm2t^2)$*. Noting that on* $\mathcal{Z}$*,* $tz=xy$*, we obtain that the Zariski closure of* $\psi^{\pm}(\mathcal{Z})$ *is the parabola* $V(X\mp Y,\pm TZ-(X^2-T^2)/2)$ *and so that the caustic* $\Sigma_S(\mathcal{Z})$ *is the union of these two curves (which are parabolas contained in two orthogonal planes).*

Example 19 (The double-butterfly caustic of the saddle surface)**.** *We are interested in the caustic by reflection of* $\mathcal{Z}=V(xy-zt)$ *from* $S=[0:0:1:1]$*. We have* $\alpha=-z-t$*,* $\tilde{\beta}=-2(x^2+y^2-zt)$ *and* $\gamma=4(z+t)(x^2+y^2+z^2+t^2-2tz)$ *and so* $\vartheta=4(x^4+y^4+z^4+t^4-z^2t^2+2x^2y^2+x^2z^2+y^2z^2+x^2t^2+y^2t^2)$*. Since* $\vartheta(x,y,xy,1)$ *is not a square in* $\mathbb{C}[x,y]$*, we conclude that* ϑ *is not a square in* $\mathbb{C}[x,y,z,t]/(F)$*. In this case, the coordinates of* $\boldsymbol{\psi}^{\pm}$ *are in an extension of* $\mathbb{C}[x,y,z,t]$ *and do not corresponds to rational maps on* $\mathbb{P}^3$ *(see Figure 1 for a representation of this caustic).*

Example 20. *Let* $\mathcal{Z}$ *be the paraboloid* $V((x^2+y^2-2zt)/2)\subset\mathbb{P}^3$*.*

The caustic by reflection $\mathcal{Z}$ *from its focal point* $F_1=[0:0:1:0]$ *is its other focal point* $F_2=[0:0:1:2]$*. This can be quickly shown with our Theorem 15 (since* $\alpha=-t$*,* $\beta=-4t^2$ *and* $\gamma=4t^3$ *and so (3) admits a unique solution* $[\lambda_0:\lambda_1]\in\mathbb{P}^1$ *which is* $[-2t:1]$*. The unique ramification point* M_m *associated to* m *is then* $M_m=[0:0:-2zt+x^2+y^2-t^2:-2t^2]=F_2$ *(since* $x^2+y^2=2zt$*).*

If light position S *is another point of* $\mathcal{H}^\infty$*, then*

$$\vartheta=(x_0^2+y_0^2)[(x^2+y^2+t^2)^2(x_0^2+y_0^2)-4(x^2+y^2+t^2)(xx_0+yy_0-z_0t)z_0t-4(xx_0+yy_0-z_0t)^2t^2]$$

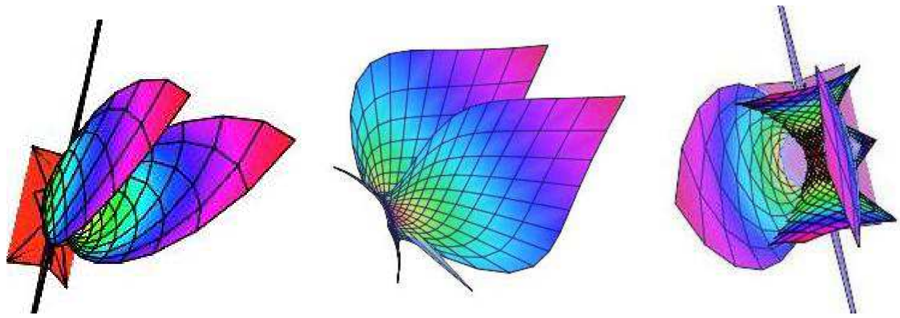

FIGURE 1. The picture on the right represents the caustic by reflection of $V(zt-xy)$ from $[0:0:1:0]$ (corresponding to the points of $V(zt-xy)$ in the chart $t=1$ for $x,y\in[-4,4]$). This caustic is obtained by gathering the two sheets given by (7) and represented in the two first pictures.

is not a square in $\mathbb{C}[x,y,z,t]/(F)$ unless if $x_0^2+y_0^2+z_0^2=0$ or $x_0^2+y_0^2=0$ (see Proposition 30 for the case when $x_0^2+y_0^2=0$ and $t_0=0$).[1] *The fact that ϑ is not a square means that the caustic map $\Phi_{S,F}$ cannot be decomposed in two rational maps on $\mathbb{P}^3$.*

The end of this section is devoted to the proof of Theorem 15.

Proof of Theorem 15. Let $m[x:y:z:t]\in\mathcal{Z}\setminus V(t(Q(\nabla F)))$. We will use several times the Euler identity ($xG_x+yG_y+zG_y+tG_t=d_1G$ if G is in $Sym^{d_1}(W^\vee)$). We use the idea of ramification (used for example in [16, 5]). The points of the caustic corresponding to m are the points $\Pi(\lambda_0\cdot\mathbf{m}+\lambda_1\cdot\boldsymbol{\sigma}(\mathbf{m}))$ with $[\lambda_0:\lambda_1]\in\mathbb{P}^1$ such that the rank of the Jacobian matrix J of

$$j:(\mathbf{m},\lambda_0,\lambda_1)\mapsto(\lambda_0\cdot\mathbf{m}+\lambda_1\cdot\boldsymbol{\sigma}(\mathbf{m}),F(\mathbf{m}))$$

is less than 5. We have

$$J:=\begin{pmatrix}\lambda_0+\lambda_1\sigma_x^{(1)} & \lambda_1\sigma_y^{(1)} & \lambda_1\sigma_z^{(1)} & \lambda_1\sigma_t^{(1)} & x & \sigma^{(1)}\\ \lambda_1\sigma_x^{(2)} & \lambda_0+\lambda_1\sigma_y^{(2)} & \lambda_1\sigma_z^{(2)} & \lambda_1\sigma_t^{(2)} & y & \sigma^{(2)}\\ \lambda_1\sigma_x^{(3)} & \lambda_1\sigma_y^{(3)} & \lambda_0+\lambda_1\sigma_z^{(3)} & \lambda_1\sigma_t^{(3)} & z & \sigma^{(3)}\\ \lambda_1\sigma_x^{(4)} & \lambda_1\sigma_y^{(4)} & \lambda_1\sigma_z^{(4)} & \lambda_0+\lambda_1\sigma_t^{(4)} & t & \sigma^{(4)}\\ F_x & F_y & F_z & F_t & 0 & 0\end{pmatrix},$$

with $\sigma^{(i)}$ the ith coordinates of $\boldsymbol{\sigma}$.

(1) Let us explain this briefly. Let $\psi(\cdot)$ of the form $\boldsymbol{\psi}(\mathbf{m}')=\lambda_0(\mathbf{m}')\cdot\mathbf{m}'+\lambda_1(\mathbf{m}')\cdot\boldsymbol{\sigma}(\mathbf{m}')$. We define the following property

$$\text{the line } (m\sigma(m)) \text{ is tangent to } \psi(\mathcal{Z}) \text{ at } \psi(m). \tag{8}$$

Recall that we have assumed $(Q(\nabla F))(\mathbf{m})\neq 0$. Assume for example $F_x(\mathbf{m})\neq 0$ (the proof is similar if we replace F_x by F_y or by F_z). Now, Property (8) means that there exists $A\in\mathbf{W}^\vee\setminus\{0\}$ such that

$$A(\mathbf{m})=0,\ A(\boldsymbol{\sigma}(\mathbf{m}))=0,\quad A((D\boldsymbol{\psi}(\mathbf{m})\cdot\begin{pmatrix}F_y(\mathbf{m})\\ -F_x(\mathbf{m})\\ 0\\ 0\end{pmatrix})=0,$$

[1]To prove that δ is not a square in $\mathbb{C}[x,y,z,t]/(F)$, it is enough to see that there exists no polynomial $P\in\mathbb{C}[x,y]$ such that $(P(x,y))^2=\delta(x,y,(x^2+y^2)/2,1)$.

$$A((D\boldsymbol{\psi}(\mathbf{m}))\cdot\begin{pmatrix}F_z(\mathbf{m})\\0\\-F_x(\mathbf{m})\\0\end{pmatrix})=0 \text{ and } A((D\boldsymbol{\psi}(\mathbf{m}))\cdot\begin{pmatrix}F_t(\mathbf{m})\\0\\0\\-F_x(\mathbf{m})\end{pmatrix})=0,$$

and so that

$$A(\mathbf{m})=0,\ A(\boldsymbol{\sigma}(\mathbf{m}))=0,\ \ A(\boldsymbol{\psi}_x(\mathbf{m}))F_y(\mathbf{m})=F_x(\mathbf{m})A(\boldsymbol{\psi}_y(\mathbf{m})),$$

$$A(\boldsymbol{\psi}_x(\mathbf{m}))F_z(\mathbf{m})=F_x(\mathbf{m})A(\boldsymbol{\psi}_z(\mathbf{m})) \text{ and } A(\boldsymbol{\psi}_x(\mathbf{m}))F_t(\mathbf{m})=F_x(\mathbf{m})A(\boldsymbol{\psi}_t(\mathbf{m})).$$

Therefore, by taking $b:=A(\boldsymbol{\psi}_x(\mathbf{m}))/F_x(\mathbf{m})$,

$$A(\mathbf{m})=0,\ \ A(\boldsymbol{\psi}_x(\mathbf{m}))=F_x(\mathbf{m})b,\ \ A(\boldsymbol{\psi}_y(\mathbf{m}))=F_y(\mathbf{m})b,$$

$$A(\boldsymbol{\psi}_z(\mathbf{m}))=F_z(\mathbf{m})b \text{ and } A(\boldsymbol{\psi}_t(\mathbf{m}))=F_t(\mathbf{m})b$$

and so that the rank of the following matrix is strictly less than 5

$$\hat{J}:=\begin{pmatrix}\boldsymbol{\psi}_x(\mathbf{m}) & \boldsymbol{\psi}_y(\mathbf{m}) & \boldsymbol{\psi}_z(\mathbf{m}) & \boldsymbol{\psi}_t(\mathbf{m}) & \mathbf{m} & \boldsymbol{\sigma}(\mathbf{m})\\ F_x & F_y & F_z & F_t & 0 & 0\end{pmatrix}\in Mat_{5,6}(\mathbb{C}).$$

Let us write C_i the i-th column of J. We observe that the four first columns of $\hat{J}$ are respectively equal to $C_1+(\lambda_1)_xC_6+(\lambda_0)_xC_5$, $C_2+(\lambda_1)_yC_6+(\lambda_0)_yC_5$, $C_3+(\lambda_1)_zC_6+(\lambda_0)_zC_5$ and $C_4+(\lambda_1)_tC_6+(\lambda_0)_tC_5$. Therefore the J and $\hat{J}$ have the same rank and so (8) means that $rank(J)<5$.

(2) Now we observe that, on $\mathcal{Z}$, $xC_1+yC_2+zC_3+tC_4=\lambda_0C_5+\lambda_1C_6$. Since $t\neq 0$, C_4 is a linear combination of the other columns and so the rank of J is strictly less than 5 if and only if the following determinant is null:

$$D(\mathbf{m},\lambda_0,\lambda_1):=\begin{vmatrix}\lambda_0+\lambda_1\sigma_x^{(1)} & \lambda_1\sigma_y^{(1)} & \lambda_1\sigma_z^{(1)} & x & \sigma^{(1)}\\ \lambda_1\sigma_x^{(2)} & \lambda_0+\lambda_1\sigma_y^{(2)} & \lambda_1\sigma_z^{(2)} & y & \sigma^{(2)}\\ \lambda_1\sigma_x^{(3)} & \lambda_1\sigma_y^{(3)} & \lambda_0+\lambda_1\sigma_z^{(3)} & z & \sigma^{(3)}\\ \lambda_1\sigma_x^{(4)} & \lambda_1\sigma_y^{(4)} & \lambda_1\sigma_z^{(4)} & t & \sigma^{(4)}\\ F_x & F_y & F_z & 0 & 0\end{vmatrix}.$$

Now let us define

$$\boldsymbol{\tau}:=Q(\nabla F)\cdot\mathbf{S}+2\frac{(xt_0-x_0t)F_x+(yt_0-y_0t)F_y+(zt_0-z_0t)F_z}{t}\cdot\boldsymbol{\kappa}(\nabla F).$$

Observe that $\boldsymbol{\tau}=\boldsymbol{\sigma}+\frac{2t_0dF}{t}\boldsymbol{\kappa}(\nabla F)$ (due to the Euler identity). Therefore, on $\mathcal{Z}$, we have $\boldsymbol{\sigma}=\boldsymbol{\tau}$. Now we observe that, on $\mathcal{Z}$, we have

$$D(\mathbf{m},\lambda_0,\lambda_1)=\begin{vmatrix}\lambda_0+\lambda_1\tau_x^{(1)} & \lambda_1\tau_y^{(1)} & \lambda_1\tau_z^{(1)} & x & \tau^{(1)}\\ \lambda_1\tau_x^{(2)} & \lambda_0+\lambda_1\tau_y^{(2)} & \lambda_1\tau_z^{(2)} & y & \tau^{(2)}\\ \lambda_1\tau_x^{(3)} & \lambda_1\tau_y^{(3)} & \lambda_0+\lambda_1\tau_z^{(3)} & z & \tau^{(3)}\\ \lambda_1\tau_x^{(4)} & \lambda_1\tau_y^{(4)} & \lambda_1\tau_z^{(4)} & t & \tau^{(4)}\\ F_x & F_y & F_z & 0 & 0\end{vmatrix}, \tag{9}$$

with $\tau^{(i)}$ the ith coordinate of $\boldsymbol{\tau}$. Indeed, if we write L_i the i-th line of the matrix (with $\boldsymbol{\sigma}$) used in the definition of D and if we write $\tilde{L}_i$ the i-th line of the matrix (with $\boldsymbol{\tau}$) appearing in the above formula, we obtain (due to the Euler identity) that, on $\mathcal{Z}$, we have $\tilde{L}_4=L_4$, $\tilde{L}_5=L_5$ and $\tilde{L}_1=L_1+\lambda_1\frac{2t_0d}{t}F_xL_5$, $\tilde{L}_2=L_2+\lambda_1\frac{2t_0d}{t}F_yL_5$, $\tilde{L}_3=L_3+\lambda_1\frac{2t_0d}{t}F_zL_5$.

(3) On $\mathcal{Z}$, we have

$$D(\mathbf{m},\lambda_0,\lambda_1)=\alpha_1(\mathbf{m})\lambda_0^2+\beta_1(\mathbf{m})\lambda_0\lambda_1+\gamma_1(\mathbf{m})\lambda_1^2, \tag{10}$$

where α_1, β_1 and γ_1 can be expressed as follows (due to Euler's identity ensuring that $-xF_xt_0-yF_yt_0-zF_zt_0+tx_0F_x+ty_0F_y+tz_0F_z=t\Delta_{\mathbf{S}}F$ on $\mathcal{Z}$)

$$\alpha_1:=Q(\nabla F)t\Delta_{\mathbf{S}}F=tQ(\nabla F)\alpha \tag{11}$$

$$\beta_1:=-\frac{2}{t}Q(\nabla F)B \tag{12}$$

$$\gamma_1:=-4t^{-1}N_{\mathbf{S}}.Q(\nabla F).\Delta_{\mathbf{S}}F.h_F \tag{13}$$

with the following definitions of h_F and B. First, on $\mathcal{Z}$, we have

$$h_F:=\begin{vmatrix} F_{xx} & F_{xy} & F_{xz} & F_x\\ F_{xy} & F_{yy} & F_{yz} & F_y\\ F_{xz} & F_{yz} & F_{zz} & F_z\\ F_x & F_y & F_z & 0\end{vmatrix}=\frac{t^2}{(d-1)^2}H_F,$$

where H_F is the Hessian determinant of F [2]. Therefore

$$\gamma_1=-\frac{4t}{(d-1)^2}N_{\mathbf{S}}.Q(\nabla F).\Delta_{\mathbf{S}}F.H_F=tQ(\nabla F)\gamma. \tag{14}$$

Second $B:=\delta_xF_{xx}+\delta_yF_{yy}+\delta_zF_{zz}+2(\varepsilon_{x,y}F_{xy}+\varepsilon_{x,z}F_{xz}+\varepsilon_{y,z}F_{yz})$, with

$$\begin{aligned}
\delta_x &:= (x_0t-xt_0)^2(F_y^2+F_z^2)+((t_0y-ty_0)F_y+(t_0z-tz_0)F_z)^2\\
&= (x_0t-xt_0)^2(F_y^2+F_z^2)+(t_0(yF_y+zF_z)-t(y_0F_y+z_0F_z))^2\\
&= (x_0t-xt_0)^2(F_y^2+F_z^2)+(t_0(xF_x+tF_t)+t(y_0F_y+z_0F_z))^2\\
&= (x_0t-xt_0)^2(F_y^2+F_z^2)+((t_0x-x_0t)F_x+t\Delta_{\mathbf{S}}F)^2\\
&= x^2t_0^2(F_x^2+F_y^2+F_z^2)+2xt_0t[-x_0(F_x^2+F_y^2+F_z^2)+F_x\Delta_{\mathbf{S}}F]+\\
&\quad +x_0^2t^2(F_x^2+F_y^2+F_z^2)+t^2(\Delta_{\mathbf{S}}F)^2-2x_0t^2F_x\Delta_{\mathbf{S}}F\\
&= x^2t_0^2(F_x^2+F_y^2+F_z^2)+t\Delta_{\mathbf{S}}F(2xt_0F_x-2x_0tF_x)+\\
&\quad +t(F_x^2+F_y^2+F_z^2)(x_0^2t-2xx_0t_0)+t^2(\Delta_{\mathbf{S}}F)^2,
\end{aligned}$$

[2] Indeed, if we write $\hat{C}_i$ for the i-th column of $\operatorname{Hess} F$, due to the Euler formula, on $\mathcal{Z}$, we have $\hat{C}_4=\frac{d-1}{t}\nabla F-(x\hat{C}_1+y\hat{C}_2+z\hat{C}_3)/t$ (where ∇F is the gradient of F); therefore

$$H_F:=\begin{vmatrix} F_{xx} & F_{xy} & F_{xz} & F_{xt}\\ F_{xy} & F_{yy} & F_{yz} & F_{yt}\\ F_{xz} & F_{yz} & F_{zz} & F_{zt}\\ F_{xt} & F_{yt} & F_{zt} & F_{tt}\end{vmatrix}=\frac{d-1}{t}\begin{vmatrix} F_{xx} & F_{xy} & F_{xz} & F_x\\ F_{xy} & F_{yy} & F_{yz} & F_y\\ F_{xz} & F_{yz} & F_{zz} & F_z\\ F_{xt} & F_{yt} & F_{zt} & F_t\end{vmatrix}.$$

Now, if we write $\hat{L}_i$ the i-th line of the above matrix, using again the Euler identity, on $\mathcal{Z}$, we have $\hat{L}_4=\frac{d-1}{t}(F_x\ \ F_y\ \ F_z\ \ 0)-(x\hat{L}_1+y\hat{L}_2+z\hat{L}_3)/t$ and we get $H_F=(d-1)^2h_F/t^2$.

δ_y (resp. δ_z) being obtained from δ_x by interverting x and y (resp. x and z) and

$$\begin{aligned}
\varepsilon_{x,y} &:= -(t_0x-x_0t)F_y((t_0x-x_0t)F_x+(t_0z-z_0t)F_z)\\
&\quad -(t_0y-y_0t)F_x((t_0y-y_0t)F_y+(t_0z-z_0t)F_z)+(t_0x-x_0t)(t_0y-y_0t)F_z^2\\
&= (t_0x-x_0t)F_y[t_0yF_y+t_0tF_t+x_0tF_x+z_0tF_z]+\\
&\quad +(t_0y-y_0t)F_x[t_0xF_x+t_0tF_t+y_0tF_y+z_0tF_z]+\\
&\quad +(t_0x-x_0t)(t_0y-y_0t)F_z^2\\
&= (t_0x-x_0t)F_y[t_0yF_y+t\Delta_{\mathbf{S}}F-ty_0F_y]+(t_0y-y_0t)F_x[t_0xF_x+t\Delta_{\mathbf{S}}F-tx_0F_x]+\\
&\quad +(t_0x-x_0t)(t_0y-y_0t)F_z^2\\
&= t_0^2xy(F_x^2+F_y^2+F_z^2)+t\Delta_{\mathbf{S}}F((t_0x-x_0t)F_y+(t_0y-y_0t)F_x)\\
&\quad +t(F_x^2+F_y^2+F_z^2)(tx_0y_0-t_0(y_0x+yx_0))
\end{aligned}$$

$\varepsilon_{x,z}$ (resp. $\varepsilon_{y,z}$) being obtained from $\varepsilon_{x,y}$ by interverting y and z (resp. x and z). On $\mathcal{Z}$, we have

$$0=xF_x+yF_y+zF_z+tF_t \text{ and } (d-1)F_w=xF_{xw}+yF_{yw}+zF_{zw}+tF_{tw},\ \forall w\in\{x,y,z,t\}.$$

Therefore

$$0=x^2F_{xx}+y^2F_{yy}+z^2F_{zz}+t^2F_{tt}+2(xyF_{xy}+xzF_{xz}+xtF_{xt}+yzF_{yz}+ytF_{yt}+ztF_{zt})$$

and so

$$B=(F_x^2+F_y^2+F_z^2)(b_1+b_2+b_3)+2t\Delta_{\mathbf{S}}Fb_4+t^2(\Delta_{\mathbf{S}}F)^2(F_{xx}+F_{yy}+F_{zz}),$$

with

$$b_1=-t_0^2(t^2F_{tt}+2t(xF_{xt}+yF_{yt}+zF_{zt}))=-t_0^2t(2(d-1)F_t-tF_{tt}),$$

$$b_2=t^2(x_0^2F_{xx}+y_0^2F_{yy}+z_0^2F_{zz}+2x_0y_0F_{xy}+2x_0z_0F_{xz}+2y_0z_0F_{yz}),$$

$$\begin{aligned}
b_3 &= -2tt_0\sum_{w\in\{x,y,z\}}(w_0(xF_{xw}+yF_{yw}+zF_{zw}))\\
&= 2tt_0\sum_{w\in\{x,y,z\}}(w_0(tF_{tw}-(d-1)F_w)),
\end{aligned}$$

$$\begin{aligned}
b_4 &= \sum_{w\in\{x,y,z\}}F_w((t_0x-tx_0)F_{xw}+(t_0y-ty_0)F_{yw}+(t_0z-tz_0)F_{zw})\\
&= \sum_{w\in\{x,y,z\}}F_w(t_0(d-1)F_w-t(x_0F_{xw}+y_0F_{yw}+z_0F_{zw}+t_0F_{wt})).
\end{aligned}$$

Putting all these terms together, we get that B is equal to

$$t^2\left[Q(\nabla F)\cdot \mathrm{Hess}_F(\mathbf{S},\mathbf{S})-2\Delta_{\mathbf{S}}F\cdot\mathrm{Hess}_F(\mathbf{S},\boldsymbol{\kappa}(\nabla F))+(\Delta_{\mathbf{S}}F)^2(F_{xx}+F_{yy}+F_{zz})\right].$$

and so $B=t^2\left[\mathrm{Hess}_F(\mathbf{S},\boldsymbol{\sigma})+(\Delta_{\mathbf{S}}F)^2(F_{xx}+F_{yy}+F_{zz})\right]$, which leads to

$$\beta_1=tQ(\nabla F)\beta. \tag{15}$$

Hence the points of the caustic associated to m are the points $\Pi(\lambda_0\cdot\mathbf{m}+\lambda_1\cdot\boldsymbol{\sigma}(\mathbf{m}))$ where $[\lambda_0:\lambda_1]\in\mathbb{P}^1$ satisfies

$$\alpha_1(\mathbf{m})\lambda_0^2+\beta_1(\mathbf{m})\lambda_0\lambda_1+\gamma_1(\mathbf{m})\lambda_1^2=0, \tag{16}$$

with α_1, β_1 and γ_1 given by (11), (15) and (14). Now, since $tQ(\nabla F)\neq 0$, (16) means that $\alpha(\mathbf{m})\lambda_0^2+\beta(\mathbf{m})\lambda_0\lambda_1+\gamma(\mathbf{m})\lambda_1^2=0$.

□

4. Covering space $\hat{Z}$ and rational map Φ

We consider the algebraic covering space $\hat{Z}$ of $\mathcal{Z}$ given by

$$\hat{Z} := \{(m, [\lambda_0 : \lambda_1]) \in \mathcal{Z} \times \mathbb{P}^1 \ : \ Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0\}.$$

This set is a subvariety of a particular algebraic variety denoted $\mathbb{F}_{(3)}(-2d+3, 0)$ (by extending the notations used by Reid in [13, Chapter 2]) which corresponds to the cartesian product of sets $\mathbb{P}^3 \times \mathbb{P}^1$ endowed with an unusual structure of algebraic variety based on the following definition of multidegree multideg(P) for $P \in Sym(\mathbf{W}^\vee)[\lambda_0, \lambda_1] \cong \mathbb{C}[x, y, z, t, \lambda_0, \lambda_1]$:

$$\text{multideg}(x^{a'} y^{b'} z^{c'} t^{d'} \lambda_0^{e'} \lambda_1^{f'}) = (a' + b' + c' + d' + (2d - 3)e', e' + f').$$

With this notion of multidegree, we have $Sym(\mathbf{W}^\vee)[\lambda_0, \lambda_1] = \bigoplus_{k,\ell \geq 0} C_{k,\ell}$, where $C_{k,\ell}$ denotes the homogeneous component of multidegree (k, ℓ). Now, we define $\mathbb{F}_{(3)}(-2d+3, 0)$ as the quotient of $\mathbf{W} \times \mathbb{C}^2$ by the equivalence relation $\sim$ given by

$$(x, y, z, t, \lambda_0, \lambda_1) \sim (x', y', z', t', \lambda'_0, \lambda'_1)$$
$$\Leftrightarrow \ \exists \mu, \nu \in \mathbb{C}^*, \ (x', y', z', t', \lambda'_0, \lambda'_1) = (\mu x, \mu y, \mu z, \mu t, \mu^{2d-3} \nu \lambda_0, \nu \lambda_1).$$

We observe that $H^0(\mathbb{F}_{(3)}(-2d + 3, 0))$ corresponds to the set of $P \in \mathbb{C}[x, y, z, t, \lambda_0, \lambda_1]$ with homogeneous multidegree multideg defined above.

Now, since $F \in Sym^d(\mathbf{W}^\vee)$, $\alpha \in Sym^{d-1}(\mathbf{W}^\vee)$, $\beta \in Sym^{3d-4}(\mathbf{W}^\vee)$ and $\gamma \in Sym^{5d-7}(\mathbf{W}^\vee)$, we get that F and $Q_{\mathbf{S},F}$ are in $H^0(\mathbb{F}_{(3)}(-2d+3, 0))$. Therefore $\hat{Z}$ is a subvariety of $\mathbb{F}_{(3)}(-2d+3, 0)$ since it can be rewritten:

$$\hat{Z} = \{(m, [\lambda_0 : \lambda_1]) \in \mathbb{F}_{(3)}(-2d + 3, 0) \ : \ F(\mathbf{m}) = 0 \ \text{ and } \ Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0\}.$$

Since each coordinate of σ is in $Sym^{2d-2}(\mathbf{W}^\vee)$, the map $\mathbf{\Phi} : \mathbf{W} \times \mathbb{C}^2 \to \mathbf{W}$ given by

$$\mathbf{\Phi}(\mathbf{m}, \lambda_0, \lambda_1) := \lambda_0 \cdot \mathbf{m} + \lambda_1 \cdot \boldsymbol{\sigma}(\mathbf{m}) \quad \in (C_{2(d-1),1})^4$$

defines a rational map $\Phi : \mathcal{X} \to \mathbb{P}^3$ with

$$\mathcal{X} = \{(m, [\lambda_0 : \lambda_1]) \in \mathbb{F}_{(3)}(-2d + 3, 0) \ : \ Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0\}.$$

Let us denote by $B_{\Phi_{|\hat{Z}}}$ the set of base points of the map $\Phi_{|\hat{Z}}$, i.e.

$$B_{\Phi_{|\hat{Z}}} := \{(m, [\lambda_0 : \lambda_1]) \in \hat{Z} \ : \ \mathbf{\Phi}(\mathbf{m}, \lambda_0, \lambda_1) = 0\}.$$

We consider the canonical projection $\pi_1 : \mathbb{F}_{(3)}(-2d+3, 0) \to \mathbb{P}^3$ (given by $\pi_1(m, [\lambda_0 : \lambda_1]) = m$).

Notation 21. *We write $\mathcal{B} := \pi_1(B_{\Phi_{|\hat{Z}}})$.*

Observe that, for any $m \in \mathcal{B}$, there exists a unique $[\lambda_0 : \lambda_1] \in \mathbb{P}^1$ such that $\lambda_0 \cdot \mathbf{m} + \lambda_1 \cdot \boldsymbol{\sigma}(\mathbf{m}) = 0$. This gives the following scheme

$$\begin{array}{ccccccc}
B_{\Phi_{|\hat{Z}}} & \hookrightarrow & \hat{Z} & \hookrightarrow & \mathcal{X} & \xrightarrow{\Phi} & \mathbb{P}^3 \\
\pi_{1|B_\Phi} \downarrow 1:1 & & \pi_{1|\hat{Z}} \downarrow 2:1 & & \pi_{1|\mathcal{X}} \downarrow 2:1 & & \\
\mathcal{B} & \hookrightarrow & \mathcal{Z} & \hookrightarrow & \mathbb{P}^3 & &
\end{array}$$

Therefore $\#B_\phi = \#\mathcal{B}$.

Remark 22. *The caustic by reflection $\Sigma_S(\mathcal{Z})$ of $\mathcal{Z}$ from S satisfies $\Sigma_S(\mathcal{Z}) = \overline{\Phi(\hat{Z})} \subseteq \mathbb{P}^3$.*

Note that $\mathcal{B} \subseteq \mathcal{M}_{S,\mathcal{Z}}$ (with $\mathcal{M}_{S,Z}$ defined in Definition 12). Due to the classical blowing-up theorem, we obtain the following result valid in the general case.

Proposition 23. *Assume that the set $\mathcal{B}$ is finite and that* $\dim(\mathcal{Z}\cap\mathcal{M}_{S,\mathcal{Z}})\le 1$. *Then there exists $\delta\in\mathbb{N}^*\cup\{\infty\}$ such that, for a generic point $P\in\Sigma_S(\mathcal{Z})$, we have $\#[\pi_1(\Phi_{|\hat{Z}}^{-1}(\{P\}))\setminus\mathcal{B}]=\delta$.*

Proof. Observe that, by hypothesis, the set $B_{\Phi_{|\hat{Z}}}$ is finite. Now, applying the blowing-up result given in [8, Example II-7.17.3], we get the existence of a variety $\widetilde{\hat{Z}}$ and of two morphisms $\pi:\widetilde{\hat{Z}}\to\hat{Z}$ and $\widetilde{\Phi}:\widetilde{\hat{Z}}\to\mathbb{P}^3$ such that

- π defines an isomorphism from $\pi^{-1}(\hat{Z}\setminus B_\Phi)$ onto $\hat{Z}\setminus B_\Phi$,
- On $\pi^{-1}(\hat{Z}\setminus B_\Phi)$, we have $\tilde{\Phi}=\Phi\circ\pi$,
- $\widetilde{\Phi}(\widetilde{\hat{Z}})$ is the Zariski closure of $\Phi(\hat{Z}\setminus B_\Phi)$, i.e. $\widetilde{\Phi}(\widetilde{\hat{Z}})=\Sigma_S(\mathcal{Z})$,
- $\dim(\widetilde{\hat{Z}})=2$,
- $E:=\widetilde{\hat{Z}}\setminus\pi^{-1}(\hat{Z}\setminus B_\Phi)$ is a variety of dimension at most 1.

Let δ be the degree of the morphism $\widetilde{\Phi}$. If $\delta=\infty$, then $\dim(\Sigma_S(\mathcal{Z}))\le\dim(\widetilde{\Phi}(\widetilde{\hat{Z}}))<2$. Assume now that $\delta<\infty$. Since $\widetilde{\Phi}$ is a morphism, every point of $\widetilde{\Phi}(\widetilde{\hat{Z}})$ has δ preimages by $\widetilde{\Phi}$ in $\widetilde{\Phi}(\widetilde{\hat{Z}})$. Now, observe that $\dim(\Sigma_S(\mathcal{Z}))=2$ and that $\dim(\widetilde{\Phi}(E))<2$. Therefore, a generic point of $\Sigma_S(\mathcal{Z})$ is in $\Phi(\hat{Z})\setminus[(\mathcal{Z}\cap\mathcal{M}_{S,\mathcal{Z}})\cup\widetilde{\Phi}(E)]$. Let P in this set. We have

$$\begin{aligned}\delta &= \#\Phi_{|\mathcal{Z}}^{-1}(\{P\})\\ &= \#\{(m,[\lambda_0:\lambda_1])\in\mathcal{Z}\times\mathbb{P}^1\ :\ Q_{\mathbf{S},F}(\mathbf{m},\lambda_0,\lambda_1)=0\ \text{ and }\ \Pi(\lambda_0\cdot\mathbf{m}+\lambda_1\cdot\boldsymbol{\sigma}(\mathbf{m}))=P\}.\end{aligned}$$

Observe that, for $m\in\mathcal{Z}\cap\mathcal{M}_{S,\mathcal{Z}}$, $\Phi(\pi_1^{-1}(\{m\}))=\{m\}$ by Definition 12. Since $P\notin\mathcal{Z}\cap\mathcal{M}_{S,\mathcal{Z}}$, we know that $\Phi_{|\mathcal{Z}}^{-1}(\{P\})\cap\pi_1^{-1}(\mathcal{M}_{S,\mathcal{Z}})=\emptyset$. So, for any $m\in\pi_1(\Phi_{|\mathcal{Z}}^{-1}(\{P\}))$, there exists a unique $[\lambda_0:\lambda_1]\in\mathbb{P}^1$ such that $(m,[\lambda_0,\lambda_1])\in\Phi_{|\mathcal{Z}}^{-1}(\{P\})$. Therefore $\delta=\#\pi_1(\Phi_{|\mathcal{Z}}^{-1}(\{P\}))=\#[\pi_1(\Phi_{|\mathcal{Z}}^{-1}(\{P\}))\setminus\mathcal{B}]$ (since $\#[\pi_1(\Phi_{|\mathcal{Z}}^{-1}(\{P\}))\cap\mathcal{B}\subset\pi_1(\Phi_{|\mathcal{Z}}^{-1}(\{P\}))\cap\mathcal{M}_{S,\mathcal{Z}}=\emptyset$. □

Lemma 24. *If* $\dim(Z\cap\mathcal{M}_{S,\mathcal{Z}})=2$. *Then* $\Sigma_S(\mathcal{Z})=\overline{\mathcal{Z}\setminus(V(\beta,\gamma)\setminus V(\alpha))}$.

So if $\dim(Z\cap\mathcal{M}_{S,\mathcal{Z}})=2$, then $\Sigma_S(\mathcal{Z})=\mathcal{Z}$ except if $\mathcal{Z}\subset V(\beta,\gamma)\setminus V(\alpha)$ and in this last case $\Sigma_S(\mathcal{Z})=\emptyset$.

Proof of Lemma 24. Assume that $\dim(Z\cap\mathcal{M}_{S,\mathcal{Z}})=2$. Then $\mathcal{Z}\subset\mathcal{M}_{S,\mathcal{Z}}$. So, due to Proposition 13, we have $\mathcal{Z}\subset\mathrm{Base}(\sigma)$. Now, due to Definition 16, we have

$$\Sigma_S(\mathcal{Z})=\overline{\{m\in\mathcal{Z}\ :\ \exists[\lambda_0:\lambda_1]\in\mathbb{P}^1,\ Q_{\mathbf{S},F}(\mathbf{m},\lambda_0,\lambda_1)=0,\ \lambda_0\neq 0\}}$$

and the result follows. □

5. Base points of Φ

Proposition 25. *The base points of $\Phi_{|\hat{Z}}$ are the points $(m,[\lambda_0:\lambda_1])\in\hat{Z}$ satisfying one of the following conditions:*

1. *$m\in V(F,\Delta_{\mathbf{S}}F,Q(\nabla F))$ (i.e. $m\in Sing(\mathcal{Z})$ or m is a point of tangency of $\mathcal{Z}$ with an isotropic plane containing S) and $\lambda_0=0$,*
2. *$t=F_x=F_y=F_z=0$ and $x^2+y^2+z^2=0$ (i.e. m is a cyclic point with $\mathcal{T}_m\mathcal{Z}=\mathcal{H}^\infty$) and $\lambda_0=0$,*
3. *$t=F_x=F_y=F_z=0$ (i.e. $\mathcal{T}_m\mathcal{Z}=\mathcal{H}^\infty$) and $H_F=0$ and $\lambda_0=0$,*
4. *$m=S\in\mathcal{Z}$ and $[\lambda_0:\lambda_1]$ is the unique element of $\mathbb{P}^1$ such that $\lambda_0\cdot\mathbf{m}+\lambda_1 Q(\nabla F)\cdot\mathbf{S}=0$,*

(5) *m is a cyclic point (i.e. $m \in \mathcal{C}_\infty$), $[\lambda_0 : \lambda_1] = [2\Delta_{\mathbf{S}}F(F_{xx}+F_{yy}+F_{zz}) : 2d-1] \neq [0:1]$ and $(F_{xx}+F_{yy}+F_{zz})\cdot\mathbf{m} = (2d-1)\boldsymbol{\kappa}(\nabla F)$.*

Proof. Let us prove that any base point $(m,[\lambda_0 : \lambda_1])$ has one of the form announced in the statement of the proposition (the converse being direct). Let $(m,[\lambda_0:\lambda_1]) \in B_{\Phi_{|\hat{\mathcal{Z}}}}$. By definition of Φ, we have $0 = \lambda_0\cdot\mathbf{m} + \lambda_1\cdot\boldsymbol{\sigma}(\mathbf{m})$. So $m \in \mathcal{Z}\cap\mathcal{M}_{S,\mathcal{Z}}$ and these m have been determined in Proposition 13.

- Assume first that $\boldsymbol{\sigma}(\mathbf{m}) = 0$. Then the unique $[\lambda_0:\lambda_1]\in\mathbb{P}^1$ satisfying $\lambda_0\cdot\mathbf{m}+\lambda_1\cdot\boldsymbol{\sigma}(\mathbf{m}) = 0$ is $[\lambda_0:\lambda_1] = [0:1]$. Hence $\lambda_0 = 0$, $\lambda_1 \neq 0$ and so $Q_{\mathbf{S},F}(\mathbf{m},\lambda_0,\lambda_1) = \gamma(\mathbf{m})\lambda_1^2$.
  If $\Delta_{\mathbf{S}}F(\mathbf{m}) = 0$ and $Q(\nabla F(\mathbf{m})) = 0$, then $\gamma(\mathbf{m}) = 0$.
  Otherwise, according to Proposition 9, we have $F_x = F_y = F_z = 0$ and so $t = 0$. Now, we have $\gamma(\mathbf{m}) = 0$ if and only if $(x^2+y^2+z^2)t_0^2H_F = 0$.
- Assume now that $m = S$ and $\boldsymbol{\sigma}(\mathbf{m}) \neq 0$. Then $\Delta_{\mathbf{S}}F(\mathbf{m}) = 0$ and so $\boldsymbol{\sigma}(\mathbf{m}) = Q(\nabla F(\mathbf{m}))\cdot\mathbf{S}$. We consider the unique $[\lambda_0,\lambda_1]$ such that $\lambda_0\cdot\mathbf{m}+\lambda_1\cdot\boldsymbol{\sigma}(\mathbf{m}) = 0$. Observe that $\lambda_0 \neq 0$ and that $\lambda_1 \neq 0$. Since $\Delta_{\mathbf{S}}F(\mathbf{m}) = 0$, we have $Q_{\mathbf{S},F}(\mathbf{m},\lambda_0,\lambda_1) = \beta(\mathbf{m})\lambda_0\lambda_1$. But $\beta(\mathbf{m}) = -2\,\mathrm{Hess}_{F,\mathbf{m}}(\mathbf{S},\boldsymbol{\sigma}(\mathbf{m})) = 0$ due to $m = S = \sigma(m)$.
- Assume finally that $m \in \mathcal{W}$. We have $\Delta_{\mathbf{S}}F(\mathbf{m}) \neq 0$, $x^2+y^2+z^2 = 0$, $m = [F_x(\mathbf{m}) : F_y(\mathbf{m}) : F_z(\mathbf{m}) : 0]$, $t = 0$ and $\boldsymbol{\sigma}(\mathbf{m}) = -2\Delta_{\mathbf{S}}F(\mathbf{m})\boldsymbol{\kappa}(\nabla F(\mathbf{m}))$. Since $t = 0$, it follows that $N_{\mathbf{S}}(\mathbf{m}) = (x^2+y^2+z^2)t_0^2 = 0$ and so that $\gamma(\mathbf{m}) = 0$. Let $[\lambda_0,\lambda_1] \in \mathbb{P}^1$ be such that $\lambda_0\cdot\mathbf{m}+\lambda_1\cdot\boldsymbol{\sigma}(\mathbf{m}) = 0$. We observe that $\lambda_1 \neq 0$ and $\lambda_0 \neq 0$. We have $\alpha(\mathbf{m})\lambda_0^2 = \Delta_{\mathbf{S}}F(\mathbf{m})\lambda_0^2$ and

  $$-2\cdot\mathrm{Hess}_{F,\mathbf{m}}(\mathbf{S},\boldsymbol{\sigma}(\mathbf{m}))\lambda_0\lambda_1 = 2\,\mathrm{Hess}_{F,\mathbf{m}}(\mathbf{S},\mathbf{m})\lambda_0^2 = 2(d-1)\Delta_{\mathbf{S}}F(\mathbf{m}),$$

  since $(d-1)F_w = xF_{xw}+yF_{yw}+zF_{zw}+tF_{tw}$ for every $w\in\{x,y,z\}$. Hence we have

  $$Q_{\mathbf{S},F}(\mathbf{m},\lambda_0,\lambda_1) = (2d-1)\Delta_{\mathbf{S}}F(\mathbf{m})\lambda_0^2 - 2(\Delta_{\mathbf{S}}F(\mathbf{m}))^2(F_{xx}(\mathbf{m})+F_{yy}(\mathbf{m})+F_{zz}(\mathbf{m}))\lambda_0\lambda_1.$$

  Hence $Q_{\mathbf{S},F}(\mathbf{m},\lambda_0,\lambda_1) = 0$ if and only if $\lambda_0/\lambda_1 = 2\Delta_{\mathbf{S}}F(\mathbf{m})(F_{xx}(\mathbf{m})+F_{yy}(\mathbf{m})+F_{zz}(\mathbf{m}))/(2d-1)$. We conclude by using $\lambda_0\cdot\mathbf{m}+\lambda_1\cdot\boldsymbol{\sigma}(\mathbf{m}) = 0$ and the formula obtained for $\boldsymbol{\sigma}(\mathbf{m})$.

□

Corollary 26. *A point m is in $\mathcal{B}$ if and only if it satisfies one of the following conditions:*

(1) *$m \in \mathcal{B}_0 = V(F,\Delta_{\mathbf{S}}F,Q(\nabla F))$, i.e. m is a singular point of $\mathcal{Z}$ or m is a point of tangency of $\mathcal{Z}$ with an isotropic plane containing S (see also (1)),*
(2) *m is a point of tangency of $\mathcal{Z}$ with $\mathcal{H}^\infty$ and m lies on the umbilical curve $\mathcal{C}_\infty$,*
(3) *m is a point of tangency of $\mathcal{Z}$ with $\mathcal{H}^\infty$ and m lies in the hessian surface of $\mathcal{Z}$,*
(4) *$m = S \in \mathcal{Z}$,*
(5) *m lies on $\mathcal{C}_\infty$ and $(F_{xx}+F_{yy}+F_{zz})\cdot\mathbf{m} = (2d-1)\boldsymbol{\kappa}(\nabla F)$.*

This can be summarized in the following formula

$$\mathcal{B} = \mathcal{Z}\cap[V(\Delta_{\mathbf{S}}F,Q(\nabla F))\cup\{S\}\cup V(H_F\cdot Q,\boldsymbol{\kappa}(\nabla F))_\infty\cup\mathcal{G}_\infty],$$

with $\mathcal{G}_\infty = \{m\in\mathcal{C}_\infty \,:\, (F_{xx}+F_{yy}+F_{zz})\cdot\mathbf{m} = (2d-1)\boldsymbol{\kappa}(\nabla F)\}$.

Remark 27. *The set $\mathcal{B}$ is never empty. Except (iv), the forms of the base points are very similar to the base points of the caustic map of planar curves (see* [9]*).*

For a general $(\mathcal{Z},S)$, the set $\mathcal{B}$ consists of the points at which $\mathcal{Z}$ admits an isotropic tangent plane containing S, i.e. $\mathcal{B} = \mathcal{B}_0 = V(F,\Delta_{\mathbf{S}}F,Q(\nabla F))$, and in general $\mathcal{Z}$ has no singular point and $\mathcal{B}_0\cap\mathcal{H}^\infty = \emptyset$. In this case $\mathcal{B}$ is fully interpreted by (1).

Let us study the base points when $\mathcal{Z}$ is the paraboloid $V(x^2+y^2-2zt)$ (see Example 20).

Proposition 28 (Paraboloid)**.** *Let $\mathcal{Z}=V(x^2+y^2-2zt)$ and any $S\in\mathbb{P}^3\setminus\{F_1,F_2\}$. Then $\mathcal{B}=V(F,\Delta_{\mathbf{S}}P,Q(\nabla F))\cup(\{S\}\cap\mathcal{Z})$ and its points are the following ones:*

1. *the point S if S is in $\mathcal{Z}$,*
2. *the points $[1:\pm i:0:0]$ if $x_0=y_0=0$,*
3. *the point $[t_0:it_0:x_0+iy_0:0]$ and $[t_0:-it_0:x_0-iy_0:0]$ if $t_0\neq 0$,*
4. *the point $F_1[0:0:1:0]$ if $t_0=0$ and $x_0^2+y_0^2\neq 0$,*
5. *the points of the form $[ux_0:uy_0:z:0]$ (with $[u:z]\in\mathbb{P}^1$) if $x_0^2+y_0^2=0$, $t_0=0$,*
6. *the points m_1 and m_{-1} if $x_0^2+y_0^2\neq 0$, with*
$$m_\varepsilon:=[x_\varepsilon:y_\varepsilon:-\frac{1}{2}:1],$$
$$x_\varepsilon=\frac{x_0(z_0-\frac{t_0}{2})+i\varepsilon y_0\sqrt{x_0^2+y_0^2+(z_0-\frac{t_0}{2})^2}}{x_0^2+y_0^2}$$
and
$$y_\varepsilon=\frac{y_0(z_0-\frac{t_0}{2})-i\varepsilon x_0\sqrt{x_0^2+y_0^2+(z_0-\frac{t_0}{2})^2}}{x_0^2+y_0^2}.$$
7. *the point $\left[\frac{z_0-\frac{t_0}{2}}{2x_0}:\frac{z_0-\frac{t_0}{2}}{2y_0}:-\frac{1}{2}:1\right]$ if $x_0^2+y_0^2=0$, $x_0\neq 0$ and $z_0-\frac{t_0}{2}\neq 0$.*

Proof. We have $\alpha=\Delta_{\mathbf{S}}F=xx_0+yy_0-tz_0-zt_0$,
$$\tilde\beta=(x^2+y^2+t^2)(x_0^2+y_0^2-2t_0z_0)-2(tz_0+zt_0+t_0t)\Delta_{\mathbf{S}}F$$
and $\gamma=4((x_0t-xt_0)^2+(y_0t-yt_0)^2+(z_0t-zt_0)^2)\Delta_{\mathbf{S}}F$. First we observe that there is no point of $\mathcal{Z}$ satisfying (5) of Corollary 26. Assume that m is such a point. We have $t=0$, so $x^2+y^2=0$ and $z=0$ (since $x^2+y^2+z^2=0$). Now, using $(F_{xx}+F_{yy}+F_{zz})\mathbf{m}=(2d-1)\boldsymbol{\kappa}(\nabla F)$, we obtain $2x=3x$ and $2y=3y$, which implies $x=y=0$ which contradicts $z=t=0$.

Now we prove that $\mathcal{Z}\cap V(H_F\cdot Q,\boldsymbol{\kappa}(\nabla F))\cap\mathcal{H}^\infty=\emptyset$. Assume that $m\in\mathcal{Z}\cap V(H_F\cdot Q,\boldsymbol{\kappa}(\nabla F))\cap\mathcal{H}^\infty$. Due to $F_x(\mathbf{m})=F_y(\mathbf{m})=F_z(\mathbf{m})=0$, we obtain $x=y=t=0$. Since $H_F=-1$, we have $0=x^2+y^2+z^2$ and so $z=0$.

It remains to identify the points of $V(F,\Delta_SF,Q(\nabla F))$. Let $m[x:y:z:t]$ be a point of this set. Then $x^2+y^2=2zt$, $0=x_0x+y_0y-z_0t-t_0z$, $0=F_x^2+F_y^2+F_z^2=x^2+y^2+t^2$. So $-t^2=x^2+y^2=2zt$.

If $x_0=y_0=0$, since $t_0\neq 0$, we obtain $x^2+y^2+t^2=0=x^2+y^2-2zt$ and $z=-z_0t/t_0$. If $t=0$, we obtain $x^2+y^2=0$ and $z=0$. This gives (2). If $t=1$, from $t^2=-2zt$, it comes $z=-1/2$ and so $t_0=2z_0$, which contradicts $S\neq F_2$.

From now on, we assume that $(x_0,y_0)\neq 0$.

Assume first that $t=0$. Then we have $x^2+y^2=0$ and $0=\Delta_{\mathbf{S}}F=x_0x+y_0y-t_0z$. Let $\varepsilon\in\{\pm1\}$ such that $y=i\varepsilon x$. We have $0=x_0x+i\varepsilon y_0x-t_0z$. If $t_0=0$, this becomes $0=x(x_0+i\varepsilon y_0)$ and so either $x=0$ or $x_0+i\varepsilon y_0=0$. This gives (4) and (5). If $t_0\neq 0$ and if $x_0+i\varepsilon y_0=0$, we obtain $z=0$ and $y=i\varepsilon x$, so $m=|1:i\varepsilon:0:0]$. If $t_0\neq 0$ and if $x_0+i\varepsilon y_0\neq 0$, we obtain $x=t_0z/(x_0+i\varepsilon y_0)$, so $m=[t_0:i\varepsilon t_0:x_0+i\varepsilon y_0:0]$. This gives (3).

Assume now that $t=1$. We have $-1=x^2+y^2=2z$ and so $z=-1/2$. Since $x^2+y^2=-1$, we consider $\varepsilon\in\{\pm1\}$ be such that $y=\varepsilon i\sqrt{1+x^2}$. We have
$$0=x_0x+y_0y-z_0t-t_0z=x_0x+y_0\varepsilon i\sqrt{1+x^2}-z_0+\frac{t_0}{2}$$

and so $-y_0\varepsilon i\sqrt{1+x^2} = x_0x - z_0 + \frac{t_0}{2}$ and we obtain $-y_0^2(1+x^2) = (x_0x - z_0 + \frac{t_0}{2})^2$ and so

$$0 = (x_0^2+y_0^2)x^2 - 2xx_0\left(z_0 - \frac{t_0}{2}\right) + \left(z_0 - \frac{t_0}{2}\right)^2 + y_0^2.$$

This gives $m \in \{m_1, m_{-1}\}$ if $x_0^2+y_0^2 \neq 0$, with $m_\varepsilon := [x_\varepsilon : y_\varepsilon : -\frac{1}{2} : 1]$,

$$x_\varepsilon = \frac{x_0(z_0 - \frac{t_0}{2}) + \varepsilon\sqrt{(x_0(z_0-\frac{t_0}{2}))^2 - (x_0^2+y_0^2)(y_0^2 + (z_0 - \frac{t_0}{2})^2)}}{x_0^2+y_0^2}$$

and

$$y_\varepsilon = \frac{y_0(z_0 - \frac{t_0}{2}) - \varepsilon\sqrt{(y_0(z_0-\frac{t_0}{2}))^2 - (x_0^2+y_0^2)(x_0^2 + (z_0 - \frac{t_0}{2})^2)}}{x_0^2+y_0^2},$$

and so (6). Now, we assume moreover that $x_0^2+y_0^2 = 0$, we obtain

$$0 = -2xx_0\left(z_0 - \frac{t_0}{2}\right) + \left(z_0 - \frac{t_0}{2}\right)^2 + y_0^2.$$

If $x_0^2+y_0^2 = 0$ and $z_0 - \frac{t_0}{2} \neq 0$, then we obtain $x = \frac{z_0 - \frac{t_0}{2}}{2x_0}$ and $y = \frac{z_0 - \frac{t_0}{2}}{2y_0}$, since $x_0 \neq 0$ and $y_0 \neq 0$. This gives (7). If $x_0^2+y_0^2 = 0$ and $z_0 - \frac{t_0}{2} = 0$, we obtain $y_0^2 = 0$ and so $S = F_2$. □

Remark 29. *Let $\mathcal{Z} = V((x^2+y^2-2zt)/2)$ and $S \in \mathbb{P}^3 \setminus \{F_1, F_2\}$. Observe that $\#\mathcal{B} < \infty$ except if $S \in \mathcal{Z}_\infty$.*

In the particular case where $S \in \mathcal{Z}_\infty$, we have the following.

Proposition 30. *Let $\mathcal{Z} = V((x^2+y^2-2zt)/2)$ and $S = [1 : \varepsilon i : z_0 : 0]$ with $\varepsilon \in \{\pm 1\}$, then $\Sigma_S(\mathcal{Z})$ is the curve of equations $z_0 i\varepsilon x + z_0 y + i\varepsilon t, (z-t/2)^2 + x^2 + y^2$.*

Proof. In this case, we have $\alpha = x + i\varepsilon y - tz_0$, $\tilde{\beta} = -2\alpha t z_0$ and $\gamma = 4\alpha z_0^2 t^2$. So (3) becomes $\alpha(\lambda_0 + 2tz_0\lambda_1)^2 = 0$. Hence $\Sigma_S(\mathcal{Z})$ is the Zariski closure of the image of $\mathcal{Z}$ by the rational map given by $\boldsymbol{\psi}(\mathbf{m}) = -2tz_0 \cdot \mathbf{m} + \boldsymbol{\sigma}(\mathbf{m})$. On $\mathcal{Z}$, using the fact that $2tz = x^2+y^2$, this rational map can be rewritten $\boldsymbol{\psi}(\mathbf{m}) = \begin{pmatrix} -(x+i\varepsilon y)^2 + t^2 \\ i\varepsilon((x+i\varepsilon y)^2 + t^2) \\ -z_0t^2 + 2(x+i\varepsilon y)t \\ -2t^2 z_0 \end{pmatrix}$. To conclude observe $\boldsymbol{\psi}$ only depends on $(x+i\varepsilon y, t)$ and that if $(X,Y,Z,T) = \boldsymbol{\psi}(x,y,z,t)$, then $2t^2 = X - \varepsilon Y$ and $2(x+i\varepsilon y)^2 = -X - i\varepsilon Y$. □

6. Reflected Polar curves

Let $H \in Pic(\mathbb{P}^3)$ be the hyperplane class. We will identify $\pi_{1,*}(\Phi^* H^2) \in A_2(\mathbb{P}^3)$ with the class of sets $\mathcal{P}_{A,B} \subseteq \mathbb{P}^3$ defined as follows.

Definition 31. *For any $A, B \in \mathbf{W}^\vee$, we define the set $\mathcal{D}_{A,B} := V(A,B) \subseteq \mathbb{P}^3$ and the* reflected polar *$\mathcal{P}_{A,B}$ by*

$$\mathcal{P}_{A,B} = \pi_1(\Phi^{-1}(\mathcal{D}_{A,B})) \cup \pi_1(Base(\Phi)),$$

i.e. $\mathcal{P}_{A,B}$ corresponds to the following set:

$$\{m \in \mathbb{P}^3 : \exists[\lambda_0, \lambda_1] \in \mathbb{P}^1,\ A(\lambda_0\mathbf{m} + \lambda_1\boldsymbol{\sigma}(\mathbf{m})) = 0,\ B(\lambda_0\mathbf{m} + \lambda_1\boldsymbol{\sigma}(\mathbf{m})) = 0,\ Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0\}.$$

Proposition 32. *For generic A, B in $\mathbf{W}^\vee$, $\mathcal{D}_{A,B}$ is a line and $\mathcal{P}_{A,B} = V(K_1, K_2, K_3)$, with*

$$K_1(\mathbf{m}) := A(\boldsymbol{\sigma}(\mathbf{m}))B(\mathbf{m}) - A(\mathbf{m})B(\boldsymbol{\sigma}(\mathbf{m})), \quad K_2(\mathbf{m}) := Q_{\mathbf{S},F}(\mathbf{m}, -A(\boldsymbol{\sigma}(\mathbf{m})), A(\mathbf{m})),$$
$$K_3(\mathbf{m}) := Q_{\mathbf{S},F}(\mathbf{m}, -B(\boldsymbol{\sigma}(\mathbf{m})), B(\mathbf{m})).$$

Proof. Recall that $\mathcal{M}_{S,\mathcal{Z}}$ has been defined in Definition 12. Assume that $\mathcal{D}_{A,B}$ is a line that does not correspond to any line $(m\,\sigma(m))$ for $m \in \mathbb{P}^3 \setminus \mathcal{M}_{S,\mathcal{Z}}$ (this is true for a generic (A, B) in $(\mathbf{W}^\vee)^2$). Hence

$$V(A, B, A \circ \boldsymbol{\sigma}, B \circ \boldsymbol{\sigma}) = \mathcal{M}_{S,\mathcal{Z}} \cap \mathcal{D}_{A,B}. \tag{17}$$

Let $m \in \mathcal{P}_{A,B}$ and $[\lambda_0 : \lambda_1] \in \mathbb{P}^1$ be such that $A(\boldsymbol{\Phi}(\mathbf{m}, \lambda_0, \lambda_1)) = 0 = B(\boldsymbol{\Phi}(\mathbf{m}, \lambda_0, \lambda_1)) = 0$ and $Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0$. This implies that

$$\lambda_0 \cdot A(\mathbf{m}) + \lambda_1 \cdot A(\boldsymbol{\sigma}(\mathbf{m})) = 0 = \lambda_0 \cdot B(\mathbf{m}) + \lambda_1 \cdot B(\boldsymbol{\sigma}(\mathbf{m})).$$

Therefore $(-A(\boldsymbol{\sigma}(\mathbf{m})), A(\mathbf{m}))$ and $(-B(\boldsymbol{\sigma}(\mathbf{m})), B(\mathbf{m}))$ are proportional to (λ_0, λ_1). But, since $Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0$, we conclude that m is in $V(K_1, K_2, K_3)$.

Conversely, assume now that m is a point of $V(K_1, K_2, K_3)$. Due to (17), we have

$$m \notin V(A, B, A \circ \boldsymbol{\sigma}, B \circ \boldsymbol{\sigma}) \quad \text{or} \quad m \in \mathcal{M}_{S,\mathcal{Z}} \cap \mathcal{D}_{A,B}.$$

Assume first that $m \notin V(A, B, A \circ \boldsymbol{\sigma}, B \circ \boldsymbol{\sigma})$, then $(-A(\boldsymbol{\sigma}(\mathbf{m})), A(\mathbf{m}))$ and $(-B(\boldsymbol{\sigma}(\mathbf{m})), B(\mathbf{m}))$ are proportional and at least one is non null. Let $[\lambda_0 : \lambda_1]$ be the corresponding point in $\mathbb{P}^1$. we have $A(\lambda_0 \cdot \mathbf{m} + \lambda_1 \cdot \boldsymbol{\sigma}(\mathbf{m})) = 0$, $B(\lambda_0 \cdot \mathbf{m} + \lambda_1 \cdot \boldsymbol{\sigma}(\mathbf{m})) = 0$, and $Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0$. So $m \in \mathcal{P}_{A,B}$.

Assume finally that $m \in \mathcal{M}_{S,\mathcal{Z}} \cap \mathcal{D}_{A,B}$, then there exists $[\lambda : \mu] \in \mathbb{P}^1$ such that $\lambda \cdot \mathbf{m} + \mu \cdot \boldsymbol{\sigma}(\mathbf{m}) = 0$, $A(\mathbf{m}) = B(\mathbf{m}) = 0$. We also have $A(\boldsymbol{\sigma}(\mathbf{m})) = B(\boldsymbol{\sigma}(\mathbf{m})) = 0$ and so $m \in \mathcal{P}_{A,B}$. □

Notation 33. *We write $B_{S,\mathcal{Z}}$ for the set of points $m \in \mathbb{P}^3$ for which $Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0$ in $\mathbb{C}[\lambda_0, \lambda_1]$.*

Observe that, for $m \in \mathbb{P}^3 \setminus B_{S,\mathcal{Z}}$, there are at most two $[\lambda_0 : \lambda_1] \in \mathbb{P}^1$ such that $Q_{\mathbf{S},F}(\mathbf{m}, \lambda_0, \lambda_1) = 0$, and so $\#(\mathcal{X} \cap \pi_1^{-1}(m)) \leq 2$.

Remark 34. *According to the expressions of σ, α, β and γ, we have*

$$B_{S,\mathcal{Z}} = V(\alpha, \beta, \gamma) = V(\Delta_{\mathbf{S}} F, \operatorname{Hess}_F(\mathbf{S}, \mathbf{S}) \cdot Q(\nabla F)).$$

We observe that $\dim B_{S,\mathcal{Z}} \geq 1$. Observe that $(\mathcal{M}_{S,\mathcal{Z}} \cup B_{S,\mathcal{Z}})$ is the set of $m \in \mathbb{P}^3$ such that $\Phi(\pi^{-1}(\{m\})) = \Pi(Vect(\mathbf{m}, \boldsymbol{\sigma}(\mathbf{m})))$. When $m \in \mathcal{M}_{S,\mathcal{Z}}$, $\Phi(\pi^{-1}(\{m\})) = \{m\}$ and when $m \in B_{S,\mathcal{Z}}$, $\Phi(\pi^{-1}(\{m\})) = \mathcal{R}_m$. Recall that $\mathcal{B} = \pi_1(B_{\Phi_{|\hat{\mathcal{Z}}}})$.

Proposition 35. *Assume that $\#\mathcal{B} < \infty$ and $\mathcal{Z} \neq \mathcal{H}^\infty$. Then, for generic $A, B \in \mathbf{W}^\vee$, $\dim \mathcal{P}_{A,B} = 1$ and $\deg \mathcal{P}_{A,B} = (d-1)(10d-9)$.*

Proof. As in the proof of the preceding proposition, we consider generic $(A, B) \in (\mathbf{W}^\vee)^2$ such that (17) holds. Recall that $\mathcal{P}_{A,B} = V(K_1, K_2, K_3)$. First, we observe that $K_1 \in Sym^{2d-1}(\mathbf{W}^\vee)$ whereas $K_2, K_3 \in Sym^{5(d-1)}(\mathbf{W}^\vee)$. Now, if m is a point of $\mathbb{P}^3 \setminus V(A, A \circ \boldsymbol{\sigma})$, then the following equivalence holds true $m \in \mathcal{P}_{A,B} \;\Leftrightarrow\; m \in V(K_1, K_2)$ and that, if m is a point of $\mathbb{P}^3 \setminus V(B, B \circ \boldsymbol{\sigma})$, then $m \in \mathcal{P}_{A,B} \Leftrightarrow m \in V(K_1, K_3)$. Therefore $\dim \mathcal{P}_{A,B} \in \{1, 2\}$.

- Let us prove that $\dim \mathcal{P}_{A,B} = 1$. Assume first that $\dim(\Sigma_S(\mathcal{Z})) \leq 1$. Then, for generic $(A, B) \in (\mathbf{W}^\vee)^2$, we have $\Sigma_{\mathcal{S}}(\mathcal{Z}) \cap \mathcal{D}_{A,B} = \emptyset$. Therefore, $\pi_1(\Phi_{|\hat{\mathcal{Z}}}^{-1}(\mathcal{D}_{A,B})) = \emptyset$ and so $\mathcal{Z} \cap \mathcal{P}_{A,B} = \mathcal{B}$ is a finite set, which implies that $\dim \mathcal{P}_{A,B} \leq 1$.

Assume now that $\dim(\Sigma_S(\mathcal{Z}))=2$. Let us consider a generic $(A,B)\in(\mathbf{W}^\vee)^2$ such that $\#(\Sigma_S(\mathcal{Z})\cap\mathcal{D}_{A,B})<\infty$ and such that, for every $P\in\Sigma_S(\mathcal{Z})\cap\mathcal{D}_{A,B}$, we have $\#[\pi_1(\Phi^{-1}_{|\hat{\mathcal{Z}}}(\{P\}))\setminus\mathcal{B}]=\delta$, (see Proposition 23). This implies that $\#\pi_1(\Phi^{-1}_{|\hat{\mathcal{Z}}}(\mathcal{D}_{A,B}))<\infty$ and so $\#(\mathcal{Z}\cap\mathcal{P}_{A,B})<\infty$ (since $\#\mathcal{B}<\infty$). Hence $\dim\mathcal{P}_{A,B}\le 1$ since $\dim\mathcal{Z}=2$.

- Let (A,B) as above. Since $\dim\mathcal{P}_{A,B}=1$, $\deg\mathcal{P}_{A,B}$ corresponds to $\#(\mathcal{P}_{A,B}\cap\mathcal{H})$ for a generic plane $\mathcal{H}$ in $\mathbb{P}^3$. Due to Corollary 26, we have $\#V(F,\Delta_{\mathbf{S}}F,Q(\nabla F))\le\#\mathcal{B}<\infty$. So $\dim V(\Delta_{\mathbf{S}}F,Q(\nabla F))=1$. Moreover we have $\mathcal{Z}\ne\mathcal{H}^\infty$. So, due to Lemma 14, we conclude that $\dim\mathcal{M}_{S,\mathcal{Z}}<3$ and we assume that $\#(\mathcal{M}_{S,\mathcal{Z}}\cap\mathcal{D}_{A,B})<\infty$. Now, since (17) holds, we conclude that $\dim V(A,A\circ\boldsymbol{\sigma})=1=\dim V(B,B\circ\boldsymbol{\sigma})$.

  Since $\dim\mathcal{P}_{A,B}=1$, we conclude that $\dim V(K_1,K_2)=\dim V(K_1,K_3)=1$. Moreover, for a generic $(A,B)\in(\mathbf{W}^\vee)^2$, we have

$$\#E<\infty,\ \text{with } E:=V(A,A\circ\boldsymbol{\sigma},K_3)\cup V(B,B\circ\boldsymbol{\sigma},K_2)<\infty.$$

  Let us explain how we get $\#V(A,A\circ\boldsymbol{\sigma},K_3)<\infty$. We recall that, due to (17), $\#V(A,A\circ\sigma,B,B\circ\sigma)<\infty$. We write $E_1:=V(A,A\circ\boldsymbol{\sigma},K_3)\setminus V(B,B\circ\sigma)$ to simplify notations.
  - First we observe that $E_1\cap\mathcal{M}_{S,\mathcal{Z}}\subset V(A)\cap\mathcal{M}_{S,\mathcal{Z}}$ which is finite for a generic $A\in\mathbf{W}^\vee$ since $\dim\mathcal{M}_{S,\mathcal{Z}}<3$.
  - Second we observe that $E_1\cap(B_{S,\mathcal{Z}}\setminus\mathcal{M}_{S,\mathcal{Z}})=\emptyset$ for a generic $A\in\mathbf{W}^\vee$. Indeed this set is contained in $V(A,A\circ\boldsymbol{\sigma},\Delta_{\mathbf{S}}F)\setminus V(Q(\nabla F))$. For m in this set, we have $\sigma(m)=S$. So, we just have to take A such that $A(\mathbf{S})\ne 0$.
  - Third we observe that $\#E_1\setminus B_{S,\mathcal{Z}}<\infty$ for generic $A,B\in\mathbf{W}^\vee$. Indeed, $\dim V(A,A\circ\sigma)=1$ (due to (17)) and, for any $m\in V(A,A\circ\boldsymbol{\sigma})\setminus B_{S,\mathcal{Z}}$, there are at most two $[\lambda_0:\lambda_1]\in\mathbb{P}^1$ such that $Q_{\mathbf{S},F}(m,\lambda_0,\lambda_1)=0$. So, for a generic $B\in\mathbf{W}^\vee$, we have

$$\#\{(m,[\lambda_0:\lambda_1])\in\mathcal{X}:\ m\in V(A,A\circ\boldsymbol{\sigma})\setminus B_{S,\mathcal{Z}},\ B(\Phi(m,[\lambda_0,\lambda_1]))=0\}<\infty.$$

    Now, if $m\in E_1\setminus B_{S,\mathcal{Z}}$, $[-B(\boldsymbol{\sigma}(\mathbf{m})):B(\mathbf{m})]\in\mathbb{P}^1$ and, due to $K_3(\mathbf{m})=0$, we observe that $M:=(m,[-B(\boldsymbol{\sigma}(\mathbf{m})):B(\mathbf{m})])$ is in $\mathcal{X}$ and $B(\Phi(M))=0$. So $\#E_1\setminus B_{S,\mathcal{Z}}<\infty$.

Let us prove that $\deg(\mathcal{P}_{S,\mathcal{Z},A,B})=(d-1)(10d-9)$ for a generic $(A,B)\in(\mathbf{W}^\vee)^2$. We consider generic A and B such that: $\deg A=1$, $\deg A\circ\sigma=2d-2$, $V(A,A\circ\sigma,B,B\circ\sigma)<\infty$ and $\#V(A,A\circ\sigma,K_3)<\infty$. We observe that the reflected polar curve corresponds to $V(K_1,K_2)$ outside $V(A,A\circ\sigma)$ and that the polar curve coincide with $V(K_3)$ on $V(A,A\circ\sigma)$ (since $V(A,A\circ\sigma)$ is contained in $V(K_1,K_2)$). We consider a generic plane $\mathcal{H}$ in $\mathbb{P}^3$, such that $\mathcal{H}\cap V(A,A\circ\sigma,B,B\circ\sigma)=\emptyset$ and $\mathcal{H}\cap V(A,A\circ\sigma,K_3)=\emptyset$. Since $\mathcal{H}\cap V(A,A\circ\sigma,K_3)=\emptyset$, we have

$$\begin{aligned}\deg(\mathcal{P}_{S,\mathcal{Z},A,B}) &= \deg V(K_1,K_2)-\sum_{m\in V(A,A\circ\sigma)\cap\mathcal{H}} i_m(\mathcal{H},V(K_1,K_2))\\ &= 5(d-1)(2d-1)-\sum_{m\in V(A,A\circ\sigma)\cap\mathcal{H}} i_m(\mathcal{H},V(K_1,K_2)).\end{aligned}$$

Now, we prove that

$$\forall m\in V(A,A\circ\sigma)\cap\mathcal{H},\quad i_m(\mathcal{H},V(K_1,K_2))=2i_m(\mathcal{H},V(A,A\circ\sigma)).\tag{18}$$

Let $m\in V(A,A\circ\sigma)\cap\mathcal{H}$. If $B(m)\ne 0$, then

$$\begin{aligned}i_m(\mathcal{H},V(K_1,K_2)) &= i_m(\mathcal{H},K_1,K_2)=i_m(\mathcal{H},K_1,B^2\cdot K_2)\\ &= i_m(\mathcal{H},K_1,\alpha(B\cdot A\circ\sigma)^2-\beta\cdot A\cdot B\cdot(B\cdot A\circ\sigma)+\gamma\cdot A^2\cdot B^2)\\ &= i_m(\mathcal{H},K_1,\alpha(A\cdot B\circ\sigma)^2-\beta\cdot A\cdot B\cdot(A\cdot B\circ\sigma)+\gamma\cdot A^2\cdot B^2),\end{aligned}$$

(since $B \cdot A \circ \sigma = A \cdot B \circ \sigma$ on $V(K_1)$) and so

$$\begin{aligned} i_m(\mathcal{H}, V(K_1, K_2)) &= i_m(\mathcal{H}, K_1, A^2 \cdot K_3) \\ &= i_m(\mathcal{H}, K_1, A^2) \quad \text{since } K_3(m) \neq 0 \\ &= i_m(\mathcal{H}, A \circ \sigma \cdot B - B \circ \sigma \cdot A, A^2) \\ &= 2\, i_m(\mathcal{H}, A \circ \sigma \cdot B - B \circ \sigma \cdot A, A) \\ &= 2\, i_m(\mathcal{H}, A \circ \sigma \cdot B, A) \\ &= 2\, i_m(\mathcal{H}, A \circ \sigma, A) \quad \text{since } B(m) \neq 0. \end{aligned}$$

Analogously, If $B(\sigma(m)) \neq 0$, then

$$\begin{aligned} i_m(\mathcal{H}, V(K_1, K_2)) &= i_m(\mathcal{H}, K_1, K_2) = i_m(\mathcal{H}, K_1, B^2 \circ \sigma \cdot K_2) \\ &= i_m(\mathcal{H}, K_1, \alpha \cdot B^2 \circ \sigma \cdot A^2 \circ \sigma - \beta \cdot A \circ \sigma . B \circ \sigma \cdot (A \cdot B \circ \sigma) + \gamma \cdot A^2 \cdot B^2 \circ \sigma) \\ &= i_m(\mathcal{H}, K_1, \alpha (A \circ \sigma \cdot B \circ \sigma)^2 - \beta \cdot A \circ \sigma \cdot B \circ \sigma \cdot (B \cdot A \circ \sigma) + \gamma \cdot B^2 \cdot A^2 \circ \sigma) \\ &= i_m(\mathcal{H}, K_1, A^2 \circ \sigma \cdot K_3) = i_m(\mathcal{H}, K_1, A^2 \circ \sigma) \\ &= i_m(\mathcal{H}, A \circ \sigma \cdot B - B \circ \sigma \cdot A, A^2 \circ \sigma) \\ &= 2\, i_m(\mathcal{H}, A \circ \sigma \cdot B - B \circ \sigma \cdot A, A \circ \sigma) = 2\, i_m(\mathcal{H}, A, A \circ \sigma). \end{aligned}$$

Hence we proved (18) and, for a generic plane $\mathcal{H}$, we have

$$\begin{aligned} \deg(\mathcal{P}_{S,\mathcal{Z},A,B}) &= 5(d-1)(2d-1) - \sum_{m \in V(A, A \circ \sigma) \cap \mathcal{H}} i_m(\mathcal{H}, V(K_1, K_2)) \\ &= 5(d-1)(2d-1) - 2 \sum_{m \in V(A, A \circ \sigma) \cap \mathcal{H}} i_m(\mathcal{H}, V(A, A \circ \sigma)) \\ &= 5(d-1)(2d-1) - 2 \deg(A) \deg(A \circ \sigma) \\ &= 5(d-1)(2d-1) - 4(d-1) = (d-1)(10d-9). \end{aligned}$$

□

7. A formula for the degree of the caustic

Recall that $\mathcal{B}$ has been completely described in Corollary 26 (see also Remark 27 for the general case). We refer to Definition 12 and Proposition 13 for $\mathcal{M}_{S,\mathcal{Z}}$ and to Notation 33 and Remark 34 for $\mathcal{B}_{S,\mathcal{Z}}$. Observe that $\dim \mathcal{M}_{S,\mathcal{Z}} \geq 1$ since $\mathrm{Base}(\sigma) \subseteq \mathcal{M}_{S,\mathcal{Z}}$.

Theorem 36. *We assume that $\#\mathcal{B} < \infty$.*

If $\dim(\Sigma_S(\mathcal{Z})) < 2$, then for a generic $(A, B) \in (\mathbf{W}^\vee)^2$, we have

$$0 = d(d-1)(10d-9) - \sum_{m \in \mathcal{B}} i_m(\mathcal{Z}, \mathcal{P}_{A,B}).$$

If $\dim(\Sigma_S(\mathcal{Z})) = 2$, $\dim(\mathcal{Z} \cap \mathcal{M}_{S,\mathcal{Z}}) \leq 1$ and $\#(\mathcal{Z} \cap B_{S,\mathcal{Z}} \setminus \mathcal{M}_{S,\mathcal{Z}}) < \infty$, then for a generic $(A, B) \in (\mathbf{W}^\vee)^2$, we have

$$\mathrm{mdeg}(\Sigma_S(\mathcal{Z})) = d(d-1)(10d-9) - \sum_{m \in \mathcal{B}} i_m(\mathcal{Z}, \mathcal{P}_{A,B}),$$

where $\mathrm{mdeg}(\Sigma_S(\mathcal{Z}))$ is the degree with multiplicity of $(\Sigma_S(\mathcal{Z}))$ ($\mathrm{mdeg}(\Sigma_S(\mathcal{Z})) = \delta \deg(\Sigma_S(\mathcal{Z}))$, see Proposition 23 for the property satisfied by δ), where d is the degree of $\mathcal{Z}$ and where $i_m(\mathcal{Z}, \mathcal{P}_{A,B})$ denotes the intersection number of $\mathcal{Z}$ with $\mathcal{P}_{A,B}$ at point m.

Let us notice, that in this formula, we can replace $i_m(\mathcal{Z},\mathcal{P}_{A,B})$ by $i_m(\mathcal{Z},V(K_1,K_2))$, with the notations of Proposition 32. Indeed, we can take A and B such that $\#V(A,A\circ\sigma,K_3)<\infty$ (see the proof of Proposition 35) and use $\mathcal{P}_{a,b}\setminus V(A,A\circ\sigma)=V(K_1,K_2)\setminus V(A,A\circ\sigma)$).

Let us recall that the case when $\dim(\mathcal{Z}\cap\mathcal{M}_{S,\mathcal{Z}})>1$ has been studied in Lemma 24.

Observe that, for $m\in\mathcal{Z}\cap B_{S,\mathcal{Z}}\setminus\mathcal{M}_{S,\mathcal{Z}}$, the reflected line $\mathcal{R}_m$ is well defined and contained in $\Sigma_S(\mathcal{Z})$. Therefore, in the degenerate case when $\dim(\mathcal{Z}\cap B_{S,\mathcal{Z}}\setminus\mathcal{M}_{S,\mathcal{Z}})\geq 1$, the surface constituted by the reflected lines $\mathcal{R}_m$ for $m\in\mathcal{Z}\cap B_{S,\mathcal{Z}}$ is contained in $\Sigma_S(\mathcal{Z})$.

Proof of Theorem 36. Recall that for a generic (A,B) in $(\mathbf{W}^\vee)^2$, we have $\deg(\mathcal{P}_{A,B})=(d-1)(10d-9)$. We assume first that $\dim\Sigma_S(\mathcal{Z})<2$ (i.e. $\delta=\infty$) and that $\#\mathcal{B}<\infty$, Taking (A,B) such that $\deg(\mathcal{P}_{A,B})=(d-1)(10d-9)$ and $\mathcal{D}_{A,B}\cap\Sigma_S(\mathcal{Z})=\emptyset$, we have $\mathcal{P}_{A,B}\cap\mathcal{Z}=\mathcal{B}$ and so

$$\begin{aligned}d(d-1)(10d-9) &= \deg(\mathcal{Z})\deg(\mathcal{P}_{A,B})=\sum_{m\in\mathcal{Z}\cap\mathcal{P}_{A,B}} i_m(\mathcal{Z},\mathcal{P}_{A,B})\\ &= \sum_{m\in\mathcal{B}} i_m(\mathcal{Z},\mathcal{P}_{A,B})=\sum_{m\in\mathcal{B}} i_m(\mathcal{Z},\mathcal{P}_{A,B}).\end{aligned}$$

Assume now that $\dim\Sigma_S(\mathcal{Z})=2$ (i.e. that δ is finite), that $\dim(\mathcal{Z}\cap\mathcal{M}_{S,\mathcal{Z}})\leq 1$, that $\#(\mathcal{Z}\cap B_{S,\mathcal{Z}}\setminus\mathcal{M}_{S,\mathcal{Z}})<\infty$ and $\#\mathcal{B}<\infty$. We consider $(A,B)\in(\mathbf{W}^\vee)^2$ such that:

(a) $\mathcal{D}_{A,B}$ is a line containing no reflected line $\mathcal{R}_m=(m\,\sigma(m))$ $(m\in\mathcal{Z})$,
(b) $\deg(\mathcal{P}_{A,B})=(d-1)(10d-9)$ (this is generic due to Proposition 35),
(c) the points $P\in\mathcal{D}_{A,B}\cap\Sigma_S(\mathcal{Z})$ are such that $\#[\pi_1(\Phi^{-1}_{|\hat{\mathcal{Z}}}(\{P\}))\setminus\mathcal{B}]=\delta$ (this is generic due to Proposition 23),
(d) For any $P\in\mathcal{D}_{A,B}\cap\Sigma_S(\mathcal{Z})$, we have $i_P(\Sigma_S(\mathcal{Z}),\mathcal{D}_{A,B})=1$ (this is true for a generic (A,B) since $\Sigma_S(\mathcal{Z})$ is a surface),
(e) the line $\mathcal{D}_{A,B}$ intersects no reflected line $\mathcal{R}_m$ with $m\in B_{S,\mathcal{Z}}$ (this is generic since $\#(\mathcal{Z}\cap B_{S,\mathcal{Z}}\setminus\mathcal{M}_{S,\mathcal{Z}})<\infty$ and $\dim(\mathcal{Z}\cap\mathcal{M}_{S,\mathcal{Z}})\leq 1$),
(f) for any $m\in(\mathcal{P}_{A,B}\cap\mathcal{Z})\setminus\mathcal{B}$, we have $i_m(\mathcal{Z},\mathcal{P}_{A,B})=1$ (this is explained at the end of this proof),
(g) $\mathcal{D}_{A,B}$ does not intersect $\Phi(\pi^{-1}(\mathcal{Z}\cap V(\beta^2-4\alpha\gamma)))$ if $\dim(\mathcal{Z}\cap V(\beta^2-4\alpha\gamma))=1$.

Due to (b), we have

$$\begin{aligned}d(d-1)(10d-9) &= \deg(\mathcal{Z})\deg(\mathcal{P}_{A,B})=\sum_{m\in\mathcal{Z}\cap\mathcal{P}_{A,B}} i_m(\mathcal{Z},\mathcal{P}_{A,B})\\ &= \sum_{m\in\mathcal{B}} i_m(\mathcal{Z},\mathcal{P}_{A,B})+\sum_{m\in(\mathcal{Z}\cap\mathcal{P}_{A,B})\setminus\mathcal{B}} i_m(\mathcal{Z},\mathcal{P}_{A,B}).\end{aligned}$$

Now, we have

$$\begin{aligned}\sum_{m\in(\mathcal{Z}\cap\mathcal{P}_{A,B})\setminus\mathcal{B}} i_m(\mathcal{Z},\mathcal{P}_{A,B}) &= \#((\mathcal{Z}\cap\mathcal{P}_{A,B})\setminus\mathcal{B})\quad\text{due to } (f)\\ &= \#[\pi_1(\Phi^{-1}_{|\hat{\mathcal{Z}}}(\mathcal{D}_{A,B}))\setminus\mathcal{B}]\ \ \text{due to }\ \mathit{Definition}\ 31\\ &= \delta\#(\Sigma_S(\mathcal{Z})\cap\mathcal{D}_{A,B})\quad\text{due to } (c)\\ &= \delta\sum_P i_P(\Sigma_S(\mathcal{Z}),\mathcal{D}_{A,B})=\delta\deg(\Sigma_S(\mathcal{Z}))\quad\text{due to } (d).\end{aligned}$$

Let us now explain why (f) is true for a generic $(A,B)\in(\mathbf{W}^\vee)^2$. Let $m\in(\mathcal{P}_{A,B}\cap\mathcal{Z})\setminus\mathcal{B}$. Due to (e), $m\in\pi_1(\Phi^{-1}_{|\hat{\mathcal{Z}}}(\mathcal{D}_{A,B}))\setminus B_{S,\mathcal{Z}}$. We consider the cone hypersurface $\mathcal{K}_\mathcal{Z}$ of $\mathbf{W}$ associated to $\mathcal{Z}$. Since $m \in \mathcal{Z} \setminus B_{S,\mathcal{Z}}$, there exist two maps $\psi^{\pm} : U \to \mathbb{P}^3$ defined on a neighbourhood U of m in $\mathbb{P}^3$ such that, for any $m' \in U$, $\Phi(\pi_1^{-1}(\{m'\})) = \{\psi^-(m'), \psi^+(m')\}$. Let $\varepsilon \in \{+,-\}$ be such that $\Phi(\pi_1^{-1}(\{m\})) \cap \mathcal{D}_{A,B} = \{\psi^\varepsilon(m)\}$ (ψ^ε is unique for a generic $m \in \mathcal{Z}$ according to (a) and to (g)) and the tangent space to $\mathcal{P}_{A,B}$ at m is given by $V(A \circ D\psi^\varepsilon(\mathbf{m}), B \circ D\psi^\varepsilon(\mathbf{m}))$, where $D\psi^{\pm}(\mathbf{m})$ are the jacobian matrices of $\psi^{\pm}$ taken at $\mathbf{m}$. Now, with these notations, for a generic m in $\mathcal{Z}$, $D\psi^{\varepsilon'}(\mathbf{m})$ is invertible if $\dim \psi^{\varepsilon'}(\mathcal{Z}) = 2$ (if $\dim \psi^{\varepsilon'}(\mathcal{Z}) < 2$, take (A,B) such that $\mathcal{D}_{A,B} \cap \psi^{\varepsilon'}(\mathcal{Z}) = \emptyset$). This combined with (d) gives the result. □

8. Proof of Theorem 1

More precisely we prove the following (recall that $\#V(F, \Delta_{\mathbf{S}}F, Q(\nabla F))$ is the number of isotropic tangent planes to $\mathcal{Z}$ passing through S).

Theorem 37. *Let $\mathcal{Z} \subset \mathbb{P}^3$ be an irreducible smooth surface and $S \in \mathbb{P}^3 \setminus (\mathcal{Z} \cup \mathcal{H}^\infty)$ be such that $\mathcal{B} = V(F, \Delta_{\mathbf{S}}F, Q(\nabla F))$ (see Corollary 26) and such that this set contains $d(d-1)(2d-2)$ points. We assume moreover that $\mathcal{B} \cap V(H_F N_{\mathbf{S}}) = \emptyset$. Then* $\mathrm{mdeg}(\Sigma_S(\mathcal{Z})) = d(d-1)(8d-7)$.

Proof. Without any loss of generality, we assume that $S[0:0:0:1]$. Due to Theorem 36, we have $\mathrm{mdeg}(\Sigma_S(\mathcal{Z})) = d(d-1)(10d-9) - \sum_{P \in \mathcal{B}} i_P(\mathcal{Z}, V(K_1, K_2))$ for generic $A, B \in \mathbf{W}^\vee$. Since we know that $\mathcal{B} = V(F, \Delta_{\mathbf{S}}F, Q(\nabla F))$ contains $d(d-1)(2d-2)$ points, we just have to prove that $i_P(\mathcal{Z}, \mathcal{P}_{A,B}) = 1$ for any $P \in \mathcal{B}$ (for generic $A, B \in \mathbf{W}^\vee$).

Let such a point P. Assume that $A(P)B(P) \neq 0$ (this is true for generic $A, B \in \mathbf{W}^\vee$). Since F is smooth, either $F_x(\mathbf{P})F_y(\mathbf{P}) \neq 0$, or $F_x(\mathbf{P})F_z(\mathbf{P}) \neq 0$ or $F_y(\mathbf{P})F_z(\mathbf{P}) \neq 0$. Assume for example that $F_x(\mathbf{P})F_y(\mathbf{P}) \neq 0$, then there exists a local parametrization h of $\mathcal{Z}$ defined on an open neighbourhood of $(0,0)$ in $\mathbb{C}^2$ such that $h(0,0) = P$ and $[\partial h(u,v)/\partial u](0,0) \neq 0$. Since $\#V(F, \Delta_{\mathbf{S}}F, Q(\nabla F)) = d(d-1)(2d-2)$, we know that $i_P(V(F, \Delta_{\mathbf{S}}F, Q(\nabla F))) = 1$. Hence we assume that $\mathrm{val}_{u,v} A(\boldsymbol{\sigma}(h(u,v))) = \mathrm{val}_{u,v} B(\boldsymbol{\sigma}(h(u,v))) = 1$. (this is true for generic $A, B \in \mathbf{W}^\vee$ due to the formula of $\boldsymbol{\sigma}$). Moreover $\Delta_{\mathbf{S}}F(h(u,v))$, $\alpha(h(u,v))$, $\beta(h(u,v))$ and $\gamma(h(u,v))$ have valuation 1 in (u,v). Recall that K_1 and K_2 are given by $K_1(\mathbf{m}) = A(\boldsymbol{\sigma}(\mathbf{m}))B(\mathbf{m}) - B(\boldsymbol{\sigma}(\mathbf{m}))A(\mathbf{m})$ and $K_2(\mathbf{m}) = \alpha(\mathbf{m})(A(\boldsymbol{\sigma}(\mathbf{m})))^2 - \beta(\mathbf{m})A(\boldsymbol{\sigma}(\mathbf{m}))A(\mathbf{m}) + \gamma(\mathbf{m})(A(\mathbf{m}))^2$. On the one hand $K_2(h(u,v))$ has valuation 1 and its term of degree 1 is the term of degree 1 of

$$\gamma(h(u,v))(A(\mathbf{P}))^2 = -4(A(\boldsymbol{\sigma}(\mathbf{P})))^2 \Delta_{\mathbf{S}}F(h(u,v)) N_{\mathbf{S}}(P) H_F(\mathbf{P})/(d-1)^2.$$ On the other hand
$$K_1(h(u,v)) = \Delta_{\mathbf{S}}F(h(u,v))[\cdots] + Q(\nabla F(h(u,v)))[A(\mathbf{S})B(h(u,v)) - B(\mathbf{S})A(h(u,v))].$$
Hence, for generic $A, B \in \mathbf{W}^\vee$, $A(\mathbf{S})B(\mathbf{P}) \neq B(\mathbf{S})A(\mathbf{P})$ and so $i_P(V(F, K_1, K_2)) = 1$. □

9. Degree of caustics of a paraboloid

This section is devoted to the proof of Proposition 2. We consider again the case when $\mathcal{Z}$ is the paraboloid $V(F)$ with $F = (x^2 + y^2 - 2zt)/2$. We recall that the base points have been studied in Proposition 28 and that we have written $F_1[0:0:1:0]$ and $F_2[0:0:1:2]$ for the two focal points of this paraboloid. Let $S \in \mathbb{P}^3 \setminus \{F_1, F_2\}$. We have

$$F = (x^2 + y^2 - 2zt)/2, \quad \Delta_{\mathbf{S}}F = x_0 x + y_0 y - z_0 t - t_0 z,$$
$$\boldsymbol{\sigma} = (x^2 + y^2 + t^2) \cdot \mathbf{S} - 2\Delta_{\mathbf{S}}F \cdot \begin{pmatrix} x \\ y \\ -t \\ 0 \end{pmatrix},$$
$$K_1(\mathbf{m}) = A(\boldsymbol{\sigma}(\mathbf{m}))B(\mathbf{m}) - B(\boldsymbol{\sigma}(\mathbf{m}))A(\mathbf{m}),$$
$$K_2(\mathbf{m}) = \alpha(\mathbf{m})(A(\boldsymbol{\sigma}(\mathbf{m})))^2 - \beta(\mathbf{m})A(\boldsymbol{\sigma}(\mathbf{m}))A(\mathbf{m}) + \gamma(\mathbf{m})(A(\mathbf{m}))^2$$

$$\text{with}\quad \alpha=\Delta_{\mathbf{S}}F,\quad \gamma=-4\Delta_{\mathbf{S}}FN_{\mathbf{S}},$$
$$\beta=-2(x_0^2+y_0^2-2z_0t_0)(x^2+y^2+t^2)+4\Delta_{\mathbf{S}}F(t_0z+z_0t+t_0t),$$
$$\text{and}\quad N_{\mathbf{S}}=(x_0t-xt_0)^2+(y_0t-yt_0)^2+(z_0t-zt_0)^2.$$

It will be useful to observe that $\mathcal{Z}$ is invariant by composition by $[x:y:z:t]\mapsto[\bar{x}:\bar{y}:\bar{z}:\bar{t}]$ and by $[x:y:z:t]\mapsto[ax+by:-bx+ay:cz:t/c]$ with $a^2+b^2=c^2=1$.

- If $x_0=y_0=0$, then, due to Theorem 42, the degree of $\Sigma_S(\mathcal{Z})$ corresponds to the degree of a caustic of the parabola. Hence, we have $\operatorname{mdeg}\Sigma_S(\mathcal{Z})=4$ if $S[0:0:0:1]$ and $\operatorname{mdeg}\Sigma_S(\mathcal{Z})=6$ elsewhere (see [9]).

We assume now that $x_0\neq 0$.

- **[Generic case]** If $S\notin(\mathcal{Z}\cup\mathcal{H}^\infty)$, if $x_0^2+y_0^2\neq 0$ and if $x_0^2+y_0^2+(z_0-(t_0/2))^2\neq 0$, then Theorem 37 applies and $\operatorname{mdeg}\Sigma_S(\mathcal{Z})=18$. Indeed $\mathcal{B}=V(F,\Delta_{\mathbf{S}}F,Q(\nabla F))=\{C,D,m_1,m_{-1}\}$ with $C[t_0:it_0:x_0+iy_0:0]$, $C[t_0:-it_0:x_0-iy_0:0]$ and $m_\varepsilon[x_\varepsilon;y_\varepsilon:-1/2:1]$ with x_ε and y_ε defined in Proposition 28. Moreover we have $N_{\mathbf{S}}(\mathbf{C})=(x_0+iy_0)^2t_0^2\neq 0$, $N_{\mathbf{S}}(\mathbf{D})=(x_0-iy_0)^2t_0^2\neq 0$ and $N_{\mathbf{S}}(\mathbf{m}_\varepsilon)=(x_0^2+y_0^2)+(z_0-\frac{t_0}{2})^2\neq 0$.

In order to apply our Theorem 36, we will have to verify its assumptions on $\mathcal{B}_{S,\mathcal{Z}}$ and $\mathcal{M}_{S,\mathcal{Z}}$. This is the aim of the next proposition.

Proposition 38. *Let $S\in\mathbb{P}^3\setminus\{F_1,F_2\}$ be such that $\#\mathcal{B}<\infty$. Then $\#(\mathcal{Z}\cap\mathcal{M}_{S,\mathcal{Z}})<\infty$. Moreover if* $\operatorname{Hess}F(\mathbf{S},\mathbf{S})\neq 0$ *(i.e. if $S\notin\mathcal{Z}$), then $\#(\mathcal{Z}\cap\mathcal{B}_{S,\mathcal{Z}})<\infty$ and so Theorem 36 applies.*

Proof. Let us prove that $\#(\mathcal{Z}\cap\mathcal{M}_{S,\mathcal{Z}})<\infty$. Since $\#\mathcal{B}<\infty$, we already know that $\#V(F,\Delta_{\mathbf{S}}F,Q(\nabla F))<\infty$. Moreover we have $V(F,F_x,F_y,F_z)=\{[0:0:1:0]\}$ and $\mathcal{W}=\{[1:\pm i:0:0]\}$ (indeed for $m\in\mathcal{W}$, we have $(x,y)\neq 0$, so $z=t$ and $-z^2=x^2+y^2=2zt=2z^2$).

If $\operatorname{Hess}F(\mathbf{S},\mathbf{S})\neq 0$, then $\mathcal{Z}\cap\mathcal{B}_{S,\mathcal{Z}}=V(F,\Delta_{\mathbf{S}}F,Q(\nabla F))\subseteq\mathcal{B}$ which is finite. □

- Let $S\notin(\mathcal{Z}\cup\mathcal{H}^\infty)$ such that $x_0^2+y_0^2\neq 0$ and $x_0^2+y_0^2+(z_0-(t_0/2))^2=0$. Without loss of generality we assume that $x_0=0$ and $y_0=1$ and $z_0-(t_0/2)=i$. The fact that $S\notin\mathcal{Z}$ implies that $t_0\neq -i$. We have $\mathcal{B}=V(F,\Delta_{\mathbf{S}}F,Q(\nabla F))=\{C,D,E\}$ with $C[t_0:it_0:i:0]$, $D[t_0:-it_0:-i:0]$ and $E[0:i:-\frac{1}{2}:1]$.

  Around E, we parametrize $\mathcal{Z}$ by $h(x,y)=(x,i+y,\frac{x^2+(i+y)^2}{2},1)$. We have $\Delta_{\mathbf{S}}F\circ h(x,y)=y(1-it_0)-t_0\frac{x^2+y^2}{2}$ and $Q(\nabla F)\circ h(x,y)=2iy+x^2+y^2$. Hence $i_E(\mathcal{Z},V(\Delta_{\mathbf{S}}F,Q(\nabla F)))=2$ and so $i_C(\mathcal{Z},V(\Delta_{\mathbf{S}}F,Q(\nabla F)))=1$ and $i_D(\mathcal{Z},V(\Delta_{\mathbf{S}}F,Q(\nabla F)))=1$. Since moreover $N_{\mathbf{S}}(\mathbf{C})=(x_0+iy_0)^2t_0^2\neq 0$ and $N_{\mathbf{S}}(\mathbf{D})=(x_0-iy_0)^2t_0^2\neq 0$, due to the proof of Theorem 37, we have $i_C(\mathcal{Z},V(\Delta_{\mathbf{S}}F,Q(\nabla F)))=i_D(\mathcal{Z},V(\Delta_{\mathbf{S}}F,Q(\nabla F)))=1$. Observe that for a generic $A\in\mathbf{W}^\vee$, $A\circ\boldsymbol{\sigma}\circ h$ has valuation 1 with dominating term $2[a_2y(i-1+it_0)+a_3y(1-t_0i)+ia_4yt_0]$. Hence $\operatorname{val}_{x,y}K_1\circ h(x,y)=1$ and its dominating term is proportional to y. Using the fact that $z_0+\frac{t_0}{2}=i+t_0=i(1-it_0)$, we have $N_{\mathbf{S}}\circ h(x,y)=-it_0x^2-it_0(i+t_0)y^2+...$, $\operatorname{val}_{x,y}\alpha\circ h(x,y)=1$, $\operatorname{val}_{x,y}\gamma\circ h(x,y)=3$. Moreover $x_0^2+y_0^2-2z_0t_0=(1-it_0)^2$ so $\beta\circ h(x,y)=-2x^2(1-it_0)-2y^2(1-it_0)^2+....$ Therefore $\operatorname{val}_{x,y}K_2\circ h(x,y)=3$ and we conclude that $i_E(\mathcal{Z},V(K_1,K_2))=3$ and that $\operatorname{mdeg}\Sigma_S(\mathcal{Z})=22-1-1-3=17$.
- If $S\notin(\mathcal{Z}\cup\mathcal{H}^\infty)$, if $x_0^2+y_0^2=0$ ($x_0\neq 0$) and $z_0\neq t_0/2$. We assume without loss of generality that $x_0=1$ and $y_0=i$. $\mathcal{B}=V(F,\Delta_{\mathbf{S}}F,Q(\nabla F))=\{C,D,E\}$ with $C[1:i:0:0]$, $D[t_0:-it_0:2:0]$ and $E[\frac{z_0-(t_0/2)}{2x_0}:\frac{z_0-(t_0/2)}{2y_0}:-\frac{1}{2}:1]$.

  We use the parametrization $h(z,t)=(1,i\sqrt{1-2zt},z,t)$ at a neighbourhood of $\mathcal{Z}$ around C. We have $\Delta_{\mathbf{S}}F(h(z,t))=-z_0t-t_0z+(1-\sqrt{1-2zt})$ and $Q(\nabla F)(h(z,t))=$

$(2z+t)t$ and so $i_C(\mathcal{Z}, V(\Delta_{\mathbf{S}}F, Q(\nabla F))) = 2$ and so the intersection numbers of $\mathcal{Z}$ with $V(\Delta_{\mathbf{S}}F, Q(\nabla F))$ is equal to 1 at D and E. Since $N_{\mathbf{S}}(\mathbf{D}) = (x_0 - iy_0)^2 t_0^2 \neq 0$ and $N_{\mathbf{S}}(\mathbf{E}) = (z_0 - \frac{t_0}{2})^2 \neq 0$. Hence, due to the Proof of Theorem 37, we have $i_D(\mathcal{Z}, V(\Delta_{\mathbf{S}}F, Q(\nabla F))) = i_E(\mathcal{Z}, V(\Delta_{\mathbf{S}}F, Q(\nabla F))) = 1$. It remains to estimate $i_C(\mathcal{Z}, V(\Delta_{\mathbf{S}}F, Q(\nabla F)))$. We have

$$\boldsymbol{\sigma} \circ h(z,t) = \begin{pmatrix} t^2 + 2(z_0 t + t_0 z + \sqrt{1-2zt} - 1 - zt) \\ 2(z_0 t + t_0 z + \sqrt{1-2zt} - 1) i\sqrt{1-2zt} + 2izt + it^2 \\ 2(z_0+t_0)tz - z_0 t^2 - 2(\sqrt{1-2zt} - 1)t \\ (2z+t)t_0 t \end{pmatrix}.$$

Hence, for generic $A, B \in \mathbf{W}^\vee$, $K_1 \circ h(z,t)$ has valuation 2 with dominating terms

$$[z_0(2z+t)^2 + (t_0 - 2z_0)2z(z+t)]((a_1+ia_2)b_3 - (b_1+ib_2)a_3) +$$

$$+ t^2(t_0 - 2z_0)[(b_1+ib_2)a_4 - b_4(a_1+ia_2)].$$

Moreover $\mathrm{val}_{z,t}\, \alpha \circ h(z,t) = 1$, $\mathrm{val}_{z,t}\, \beta \circ h(z,t) = 2$ and $\mathrm{val}_{z,t}\, \gamma \circ h(z,t) = 2$. Hence $K_2 \circ h(z,t) = (4(z_0 t - t_0 z)^2 - 8(z_0 - 2t_0)zt)(z_0 t + t_0 z)(a_1+ia_2)^2 + \ldots$ for a generic $A \in \mathbf{W}^\vee$, so $i_C(\mathcal{Z}, V(K_1, K_2)) = 6$ and $\mathrm{mdeg}\, \Sigma_S(\mathcal{Z}) = 22 - 1 - 1 - 6 = 14$.

- If $S \notin (\mathcal{Z} \cup \mathcal{H}^\infty)$, if $x_0^2 + y_0^2 = 0$ ($x_0 \neq 0$) and $z_0 = t_0/2$. We assume without loss of generality that $x_0 = 1$ and $y_0 = i$. $\mathcal{B} = V(F, \Delta_{\mathbf{S}}F, Q(\nabla F)) = \{C, D\}$ with $C[1:i:0:0]$, $D[t_0 : -it_0 : 2 : 0]$. We use again the parametrization $h(z,t) = (1, i\sqrt{1-2zt}, z, t)$ at a neighbourhood of $\mathcal{Z}$ around C. We observe that $i_C(\mathcal{Z}, V(\Delta_{\mathbf{S}}F, Q(\nabla F))) = 3$ and so $i_D(\mathcal{Z}, V(\Delta_{\mathbf{S}}F, Q(\nabla F))) = 1$. Moreover $N_{\mathbf{S}}(\mathbf{D}) = (x_0 - iy_0)^2 t_0^2 \neq 0$. Hence, due to the proof of Theorem 37, we have $i_D(\mathcal{Z}, V(K_1, K_2)) = 1$. Moreover, we prove that $i_C(\mathcal{Z}, V(K_1, K_2)) = 9$ (probranches of $K_1 \circ h$ and $K_2 \circ h$ have intersection number 3/2) and so $\mathrm{mdeg}\, \Sigma_S(\mathcal{Z}) = 22 - 1 - 9 = 12$.

Proposition 39. *Let* $S \in \mathcal{Z} \setminus (\mathcal{D} \cup \mathcal{H}^\infty)$. *Then* $\mathrm{mdeg}\, \Sigma_S(\mathcal{Z}) = 12$ *if* $x_0^2 + y_0^2 + t_0^2 = 0$, $\mathrm{mdeg}\, \Sigma_S(\mathcal{Z}) = 14$ *if* $x_0^2 + y_0^2 = 0$ *and* $\mathrm{mdeg}\, \Sigma_S(\mathcal{Z}) = 16$ *otherwise.*

Proof. Observe that $\Delta_{\mathbf{S}}F$ divides α, β and γ. In this case we define $\Sigma_S(\mathcal{Z})$ by replacing $Q_{\mathbf{S},F}$ by $\tilde{Q}_{\mathbf{S},F} := Q/\Delta_{\mathbf{S}}F$, we define analogously $\tilde{\alpha} := 1$, $\tilde{\beta} := 4(t_0 z + z_0 t + t_0 t)$ and $\tilde{\gamma} := -4N_{\mathbf{S}}$. Following our argument above, we define $\tilde{\mathcal{B}}_{S,\mathcal{Z}} := V(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) = \emptyset$ and the new reflected polar curve $\tilde{\mathcal{P}}_{A,B} = V(K_1, \tilde{K}_2, \tilde{K}_3)$ by using $\tilde{Q}_{\mathbf{S},F}$ instead of $Q_{\mathbf{S},F}$. Following the proof of Proposition 35, we obtain that $\deg \mathcal{P}_{A,B} = \deg K_1 \deg \tilde{K}_2 - 2 \deg A \deg(A \circ \sigma) = 8$ and the corresponding set of base points $\tilde{\mathcal{B}}$ is contained in $(\{S\} \cap \mathcal{Z}) \cup V(F, \Delta_S F, Q(\nabla F), N_{\mathbf{S}}))$ (recall that $\mathcal{B} = (\{S\} \cap \mathcal{Z}) \cup V(F, \Delta_S F, Q(\nabla F)))$.

- If $S \in \mathcal{Z} \setminus \mathcal{H}^\infty$, $x_0^2 + y_0^2 \neq 0$ and $x_0^2 + y_0^2 + t_0^2 \neq 0$ (so $z_0 + \frac{t_0}{2} \neq 0$), then $V(F, \Delta_{\mathbf{S}}F, Q(\nabla F)) = \{C, D, m_1, m_{-1}\}$ with $C[t_0 : -it_0 : x_0 - iy_0 : 0]$, $D[t_0 : it_0 : x_0 + iy_0 : 0]$ and $m_\varepsilon[x_\varepsilon : y_\varepsilon : -\frac{1}{2} : 1]$ with $x_\varepsilon := \frac{x_0}{2z_0 t_0}(z_0 - \frac{t_0}{2}) + i\varepsilon \frac{y_0}{2t_0 z_0}(z_0 + \frac{t_0}{2})$ and $y_\varepsilon := \frac{y_0}{2z_0 t_0}(z_0 - \frac{t_0}{2}) - i\varepsilon \frac{x_0}{2t_0 z_0}(z_0 + \frac{t_0}{2})$. Hence the intersection number of $\mathcal{Z}$ with $V(\Delta_{\mathbf{S}}F, Q(\nabla F))$ is 1 at these four points. We observe that $N_{\mathbf{S}}(\mathbf{C}) = t_0^2(x_0 + iy_0)^2 \neq 0$, $N_{\mathbf{S}}(\mathbf{D}) = t_0^2(x_0 - iy_0)^2 \neq 0$ and $N_{\mathbf{S}}(\mathbf{m}_\varepsilon) = (z_0 + \frac{t_0}{2}) \neq 0$. Hence $i_C(V(F, K_1, \tilde{K}_2)) = i_D(V(F, K_1, \tilde{K}_2)) = i_{m_\varepsilon}(V(F, K_1, \tilde{K}_2)) = 0$.

  It remains to compute $i_S(V(F, K_1, \tilde{K}_2))$. We have $\sigma(S) = (x_0^2 + y_0^2 + t_0^2) \cdot S$ and $\tilde{K}_2(S) = -3(x_0^2 + y_0^2 + t_0^2)^2 (A(S))^2 \neq 0$. We conclude that $\mathrm{mdeg}\, \Sigma_S(\mathcal{Z}) = 2 \times 8 = 16$.
- If $S \in \mathcal{Z} \setminus \mathcal{H}^\infty$ and $x_0^2 + y_0^2 + t_0^2 = 0$, then $z_0 = -t_0/2$ and so $\mathcal{B} = V(F, \Delta_{\mathbf{S}}F, Q(\nabla F)) = \{S, C, D\}$ with $C[t_0 : it_0 : x_0 + iy_0 : 0]$ and $D[t_0 : -it_0 : x_0 - iy_0 : 0]$. Assume $t_0 = 1$.

  Observe that $N_{\mathbf{S}}(\mathbf{C}) \neq 0$ and so that $i_C(V(F, K_1, \tilde{K}_2)) = 0$. Analogously we have $i_D(V(F, K_1, \tilde{K}_2)) = 0$. Using the parametrization $h(x,y) = (x_0 + x, y_0 + y, \frac{(x_0+x)^2 + (y_0+y)^2}{2}, 1)$ of $\mathcal{Z}$ around S and the fact that $\tilde{\alpha} = 1$, $\tilde{\beta} \circ h(x,y) = 4(xx_0 + yy_0 + \frac{x^2+y^2}{2})$, that

$\tilde{\gamma}\circ h(x,y)=-4[x^2+y^2+(xx_0+yy_0+(x^2+y^2)/2)^2]$. Moreover

$$\boldsymbol{\sigma}\circ h(x,y)=(2xx_0+2yy_0+x^2+y^2)\cdot\mathbf{S}+\frac{x^2+y^2}{2}\cdot\begin{pmatrix}x_0+x\\ y_0+y\\ -1\\ 0\end{pmatrix}.$$

Hence $\tilde{K}_1\circ h(x,y)=\frac{x^2+y^2}{2}[(a_1x_0+a_2y_0-a_3)B(S)-(b_1x_0+b_2y_0-b_3)A(S)]$ and $\tilde{K}_2:=16(xx_0+yy_0)^2(A(S))^2$. So $i_S(V(F,\tilde{K}_1,\tilde{K}_2))=4$ and $\operatorname{mdeg}\Sigma_S(\mathcal{Z})=2\times 8-4=12$.

- If $S\in\mathcal{Z}\setminus\mathcal{H}^\infty$ and $x_0^2+y_0^2=0$, then we assume without loss of generality that $x_0=1$ and $y_0=i$ (so $z_0=0$). We have $\mathcal{B}=\{S,C,D,E\}$ with $C[1:i:0:0]$, $D[t_0:-it_0:2:0]$ and $E[-\frac{t_0}{4x_0}:-\frac{t_0}{4y_0}:-\frac{1}{2}:1]$. We have $N_{\mathbf{S}}(\mathbf{D})=4t_0^2$ and $N_{\mathbf{S}}(\mathbf{E})=5t_0^2/4$ and so $i_D(V(F,K_1,\tilde{K}_2))=i_E(V(F,K_1,\tilde{K}_2))=0$. Observe that $\Delta_{\mathbf{S}}F(\mathbf{S})=0$, $N_{\mathbf{S}}(\mathbf{S})=0$, $\boldsymbol{\sigma}(\mathbf{S})=t_0^2\cdot\mathbf{S}$ and $\tilde{\beta}(\mathbf{S})=4t_0^2\neq 0$. So $\tilde{K}_2(\mathbf{S})\neq 0$ and $i_S(\mathcal{Z},V(K_1,\tilde{K}_2))=0$.

  Around C, we parametrize $\mathcal{Z}$ by $h(z,t)=(1,i\sqrt{1-2zt},z,t)$. We have $\Delta_{\mathbf{S}}F\circ h(z,t)=1-\sqrt{1-2zt}-t_0z$ and

$$\boldsymbol{\sigma}\circ h(z,t)=\begin{pmatrix}2t_0z+2zt+t^2+2(\sqrt{1-2zt}-1)\\ 2it_0z\sqrt{1-2zt}+2izt+it^2+2i(\sqrt{1-2zt}-1)\sqrt{1-2zt}\\ 2t_0tz-2t(\sqrt{1-2zt}-1)\\ (2z+t)t_0t\end{pmatrix}.$$

  Hence $A\circ\boldsymbol{\sigma}\circ h(z,t)$ has valuation 1. Moreover $\tilde{\alpha}=1$, $\tilde{\beta}\circ h(z,t)=4(t_0z+t_0t)$ and $\tilde{\gamma}\circ h(z,t)=8t_0t+....$ Hence, for a generic $A\in\mathbf{W}^\vee$, $\tilde{K}_2\circ h(z,t)=8t_0tA(C)^2+....$ Moreover we have

$$K_1\circ h(z,t)=2t_0\{(a_3tz+a_4t^2-a_3z^2)(b_1+ib_2)-(b_3tz+b_4t^2-b_3z^2)(a_1+ia_2)\}+...$$

  Hence $i_C(\mathcal{Z},V(K_1,\tilde{K}_2))=2$ and so $\operatorname{mdeg}\Sigma_S(\mathcal{Z})=16-2=14$.

□

We assume now that S is at infinity. In this case $N_{\mathbf{S}}=(x_0^2+y_0^2+z_0^2)t^2$.

- If $S\in\mathcal{H}^\infty\setminus(\mathcal{Z}\cup\mathcal{C}_\infty)$, then $x_0^2+y_0^2\neq 0$ and $\mathcal{B}=\{F_1,m_1,m_{-1}\}$ with $m_\varepsilon[x_0z_0+i\varepsilon y_0\sqrt{x_0^2+y_0^2+z_0^2}:y_0z_0-i\varepsilon x_0\sqrt{x_0^2+y_0^2+z_0^2}:-\frac{x_0^2+y_0^2}{2}:x_0^2+y_0^2]$. Then we will prove that the intersection number of $V(F,K_1,K_2)$ is 8 at F_1 and 1 at the two other base points and so $\operatorname{mdeg}\Sigma_S(\mathcal{Z})=22-8-1-1=12$.

  Due to the proof of Theorem 37, since $N_{\mathbf{S}}(m_\varepsilon)\neq 0$, to prove that $i_{m_\varepsilon}(\mathcal{Z},V(K_1,K_2))=1$ it is enough to prove that $i_{m_\varepsilon}(V(F,\Delta_{\mathbf{S}}F,Q(\nabla F))=1$. To see this, we use the parametrization $h(x,y)=(x_1+x,y_1+y,((x_1+x)^2+(y_1+y)^2)/2,1)$ of $\mathcal{Z}$ around $m_\varepsilon=[x_1:y_1:-1/2:1]$. The terms of valuation 1 of $\Delta_{\mathbf{S}}F\circ h$ and $Q(\nabla F)\circ h$ are respectively x_0x+y_0y and $2(xx_1+yy_1)$ which are not proportional since $x_0^2+y_0^2\neq 0$.

  For F_1, we use the parametrization $h(x,y)=(x,y,1,(x^2+y^2)/2)$ of $\mathcal{Z}$ around F_1. We observe that, for generic $A,B\in\mathbf{W}^\vee$, $A\circ\boldsymbol{\sigma}\circ h$ and $B\circ\boldsymbol{\sigma}\circ h$ have valuation 2 with respective dominating terms:

$$\theta_A:=a_1[-x_0x^2+x_0y^2-2y_0xy]+a_2[-y_0y^2+y_0x^2-2x_0xy]+a_3(x^2+y^2)z_0,$$

$$\theta_B:=b_1[-x_0x^2+y_0y^2-2y_0xy]+b_2[-y_0y^2+x_0x^2-2x_0xy]+b_3(x^2+y^2)z_0.$$

  Therefore the lowest degree terms of $K_1\circ h$ are given by

$$(b_3a_1-a_3b_1)[-x_0x^2+x_0y^2-2y_0xy]+(b_3a_2-a_3b_2)[-y_0y^2+y_0x^2-2x_0xy].$$

Moreover the valuations of $\alpha\circ h$, $\beta\circ h$ and $\gamma\circ h$ are respectively 1, 2 and 5. Hence $K_2\circ h$ has valuation 4 and its dominating term is $a_3 2(x_0^2+y_0^2)(x^2+y^2)\theta_A$. Hence the curves of equations $K_1\circ h$ and $K_2\circ h$ are transverse and we conclude that $i_{F_1}(\mathcal{Z},V(K_1,K_2))=8$.

- If $S\in\mathcal{C}_\infty\setminus\mathcal{Z}$, then $x_0^2+y_0^2\neq 0$, $z_0\neq 0$ and $\mathcal{B}=\{F_1,m_1\}$ with $m_1[x_0z_0 : y_0z_0 : -\frac{x_0^2+y_0^2}{2} : x_0^2+y_0^2]$. Then $N_{\mathbf{S}}=0$ in $\mathbb{C}[x,y,z,t]$ (so $\gamma=0$ in $\mathbb{C}[x,y,z,t]$). We prove that $i_{F_1}(\mathcal{Z},V(K_1,K_2))=8$ as in the previous case. We compute $i_{m_1}(\mathcal{Z},V(K_1,K_2))$. We assume without loss of generality that $x_0=0$, $y_0=1$ and $z_0=i$. Around $m_1[0:z_0:-1/2:1]$, we parametrise $\mathcal{Z}$ by $h(x,y)=(x,z_0+y,((x_1+x)^2+(y_1+y)^2)/2,1)$. We have $\alpha\circ h(x,y)=\Delta_{\mathbf{S}}F\circ h(x,y)=y$, $Q(\nabla F)\circ h(x,y)=x^2+2iy+y^2$ and
$$\boldsymbol{\sigma}\circ h(x,y)=(x^2+2iy+y^2)\begin{pmatrix}0\\1\\i\\0\end{pmatrix}-2y\begin{pmatrix}x\\i+y\\-1\\0\end{pmatrix}=\begin{pmatrix}-2xy\\x^2-y^2\\z_0(x^2+y^2)\\0\end{pmatrix}.$$
Hence, for generic $A,B\in\mathbf{W}^\vee$, $K_1\circ h(x,y)$ has valuation 2. Moreover $\alpha(h(x,y))=y$, $\beta(h(x,y))=-2(x^2+y^2)$ and $\gamma(h(x,y))=0$. Hence $K_2\circ h(x,y)$ has valuation 4 and we have $i_{m_1}(\mathcal{Z},V(K_1,K_2))=8$, so $\operatorname{mdeg}\Sigma_S(\mathcal{Z})=22-8-8=6$.
- The case when $S\in\mathcal{Z}\cap\mathcal{H}^\infty$ has been studied in Proposition 30.

10. About a reflected bundle

Recall that $\mathcal{O}_{\mathcal{Z}}(-1)=\{(m,v)\in\mathcal{Z}\times\mathbf{W} \ :\ v\in m\}$. Observe that the set $\mathbf{R}(-1)$ of (m,v) in the trivial bundle $\mathcal{Z}\times\mathbf{W}$ such that v corresponds to a point of $\mathbb{P}^3$ on the reflected line $\mathcal{R}_m$ is:
$$\mathbf{R}(-1)=\mathcal{O}_{\mathcal{Z}}(-1)+\{(m,v)\in\mathcal{Z}\times\mathbf{W}\ :\ v\in\sigma(m)\}.$$
Observe that this sum is direct in the generic case (when $S\notin\mathcal{Z}$ and when $\mathcal{W}=\emptyset$, see Proposition 13). But, contrarily to the normal bundle considered in [16, 5] to study the evolute, $\mathbf{R}(-1)$ does not define a bundle since its rank is not constant. Indeed, the dimension of $Vect(\mathbf{m},\boldsymbol{\sigma}(\mathbf{m}))$ equals 2 in general but not at every point $m\in\mathcal{Z}$ (it is strictly less than 2 when m is a base point of $\sigma_{|\mathcal{Z}}$ and, as seen in Proposition 9, such points always exist).

Appendix A. Caustics of surfaces linked with caustics of curves

For the classes of examples studied in this section, caustics of surfaces are linked with of caustics of planar curves. We start with some facts on caustics of planar curves.

A.1. **Caustic of a planar curve.** Let $S_0[x_0 : y_0 : t_0]\in\mathbb{P}^2$ and an irreducible algebraic curve $\mathcal{C}=V(G)\subset\mathbb{P}^2$ with $G\in\mathbb{C}[x,y,t]$ homogeneous of degree $d\geq 2$. We write $\Delta_{\mathbf{S}_0}G:=x_0G_x+y_0G_y+z_0G_z$, $N_{\mathbf{S}_0}=(x_0t-xt_0)^2+(y_0t-yt_0)^2$, $\operatorname{Hess}G$ for the Hessian form of G and H_G for its determinant and $\sigma_{\mathbf{S}_0,G}=(G_x^2+G_y^2)\cdot\mathbf{S}-2\Delta_{\mathbf{S}_0}G\cdot(G_x,G_y,0)$.

Definition 40 ([9])**.** *The* **caustic map** *of $\mathcal{C}$ from S_0 is the rational map $\Phi_{S_0,\mathcal{C}}:\mathbb{P}^2\to\mathbb{P}^2$ corresponding to $\mathbf{\Phi}_{\mathbf{S}_0,G}:\mathbb{C}^3\to\mathbb{C}^3$ given by $\mathbf{\Phi}_{\mathbf{S}_0,G}=-\frac{2H_GN_{\mathbf{S}_0}}{(d-1)^2}\cdot\operatorname{Id}+\Delta_{\mathbf{S}_0}G\cdot\boldsymbol{\sigma}_{\mathbf{S}_0,G}$. The caustic by reflection $\Sigma_{S_0}(\mathcal{C})$ is the Zariski closure of $\Phi_{S_0,\mathcal{C}}(\mathcal{C})$.*

We start with a technical lemma making a link between the formulas involved in Theorems 40 and 15. Inspired by the three dimensional case, let us define the following quantities: $\alpha_{\mathbf{S}_0,G}:=\Delta_{\mathbf{S}_0}G$, $\beta_{\mathbf{S}_0,G}:=-2\left[\operatorname{Hess}G(\mathbf{S}_0,\boldsymbol{\sigma}_{\mathbf{S}_0,G})+(\Delta_{\mathbf{S}_0}G)^2(G_{xx}+G_{yy})\right]$ and $\gamma_{\mathbf{S}_0,G}:=-\frac{4\Delta_{\mathbf{S}_0}G}{(d-1)^2}N_{\mathbf{S}_0}\,H_G$.

Lemma 41. *Let $S_0 \in \mathbb{P}^2$ and $\mathcal{C} = V(G) \subset \mathbb{P}^2$ be an irreducible algebraic curve with $G \in \mathbb{C}[x,y,t]$ being homogeneous of degree $d \geq 2$. We have $\beta_{\mathbf{S}_0,G} = \frac{2}{(d-1)^2} N_{\mathbf{S}_0} H_G = \frac{\gamma_{\mathbf{S}_0,G}}{2\alpha_{\mathbf{S}_0,G}}$ and so $\Phi_{\mathbf{S}_0,G}(\mathbf{m}) = \lambda_0 \cdot \mathbf{m} + \lambda_1 \cdot \boldsymbol{\sigma}_{\mathbf{S}_0,G}(\mathbf{m})$, with $[\lambda_0 : \lambda_1] = [-\beta_{\mathbf{S}_0,G} : \alpha_{\mathbf{S}_0,G}](\mathbf{m})$ (i.e. $\alpha_{\mathbf{S}_0,G}(\mathbf{m})\lambda_0 + \beta_{\mathbf{S}_0,G}(\mathbf{m})\lambda_1 = 0$ if $(\alpha_{\mathbf{S}_0,G}, \beta_{\mathbf{S}_0,G})(\mathbf{m}) \neq \mathbf{0}$).*

We omit the straightforward proof of this lemma.

A.2. **Caustic of a surface from a light position on a revolution axis.** To simplify, we consider the case of a surface $\mathcal{Z} = V(F)$ with axis of revolution $V(x,y)$. We will use the fact that $F = G \circ h$ where $h(x,y,z,t) = (\sqrt{x^2+y^2}, z, t)$ for some homogeneous polynomial $G \in \mathbb{C}[r,z,t]$ with monomials of even degree in r. Such a surface $\mathcal{Z}$ is written $\mathcal{R}(G)$ and is called surface of revolution of axis $V(x,y)$ of the curve $V(G) \subset \mathbb{P}^2$.

Theorem 42. *Let $\mathcal{Z} = \mathcal{R}(G)$ with $G \in \mathbb{C}[r,z,t]$ irreducible homogeneous of degree $d \geq 2$ (the monomials of G being of even degree in r). Assume that $\mathcal{Z} \not\subseteq V(\Delta_{\mathbf{S}} F, (F_x^2 + F_y^2 + F_z^2) \operatorname{Hess} F(\mathbf{S},\mathbf{S}))$. Let $S[0:0:z_0:t_0] \in \mathbb{P}^3$ and $S_0[0:z_0:t_0] \in \mathbb{P}^2$.*

If $d = 2$ and if S is a focal point of $\mathcal{Z}$, then $\Sigma_S(\mathcal{Z})$ is reduced to another other focal point. Otherwise, we have $\Sigma_S(\mathcal{Z}) = V(x,y) \cup \mathcal{R}(\Sigma_{S_0}(V(G)))$.

Proof. We have

$$\alpha = [z_0 F_z + t_0 F_t] = \Delta_{\mathbf{S}_0} G \circ h. \tag{19}$$

Now let us prove that

$$\gamma = \left[\frac{G_r}{r}\gamma_{\mathbf{S}_0,G}\right] \circ h, \tag{20}$$

Since $\Delta_{\mathbf{S}} F = \Delta_{\mathbf{S}_0} G \circ h$ and $N_{\mathbf{S}} = N_{\mathbf{S}_0} \circ h$, we just have to prove that $H_F = \left[\frac{G_r}{r} H_G\right] \circ h$ on $\mathcal{Z}$. Recall that $\frac{t^2}{(d-1)^2} H_F = h_F$ on $\mathcal{Z}$ and that $\frac{t^2}{(d-1)^2} H_G = h_G$ on $V(G)$ with

$$h_F := \begin{vmatrix} F_{xx} & F_{xy} & F_{xz} & F_x \\ F_{xy} & F_{yy} & F_{yz} & F_y \\ F_{xz} & F_{yz} & F_{zz} & F_z \\ F_x & F_y & F_z & 0 \end{vmatrix} \quad \text{and} \quad h_G := \begin{vmatrix} G_{rr} & G_{rz} & G_r \\ G_{rz} & G_{zz} & G_z \\ G_r & G_z & 0 \end{vmatrix}.$$

We will write as usual G_r, G_z, G_t for the first order derivatives of G and G_{rr}, G_{rz}, G_{rt}, G_{zt} and G_{tt} for the second order derivatives. We also write $F_r := G_r \circ h$ and we define analogously F_{xr}, F_{yr}, F_{zr} and F_{rr}. Due to the particular form of F, we immediately obtain that

$$F_x = \frac{x}{r}F_r, \quad F_y = \frac{y}{r}F_r, \quad F_{xt} = \frac{x}{r}F_{rt}, \quad F_{yt} = \frac{y}{r}F_{rt},$$
$$F_{xx} = \frac{F_r}{r} + \frac{x^2}{r^2}F_{rr} - \frac{x^2}{r^3}F_r, \quad F_{yy} = \frac{F_r}{r} + \frac{y^2}{r^2}F_{rr} - \frac{y^2}{r^3}F_r,$$
$$F_{xy} = \frac{xy}{r^2}F_{rr} - \frac{xy}{r^3}F_r, \quad F_{xz} = \frac{x}{r}F_{rz}, \quad F_{yz} = \frac{y}{r}F_{rz}.$$

with $r := \sqrt{x^2+y^2}$. Due to these relations and to the above formula of h_F, we have [3]

$$h_F = \frac{1}{ry^2}\begin{vmatrix} rF_r & -\frac{xF_r}{r} & 0 & 0 \\ 0 & \frac{y^2}{r}F_{rr} & yF_{rz} & yF_r \\ 0 & \frac{y}{r}F_{rz} & F_{zz} & F_z \\ 0 & \frac{y}{r}F_r & F_z & 0 \end{vmatrix} = \frac{F_r}{r}\begin{vmatrix} F_{rr} & F_{rz} & F_r \\ F_{rz} & F_{zz} & F_z \\ F_r & F_z & 0 \end{vmatrix}$$

[3] writing respectively L_i and C_i for the i-th line and for the i-th row, we make successively the following linear changes: $L_1 \leftarrow yL_1 - xL_2$, $C_1 \leftarrow yC_1 - xC_2$ and $L_2 \leftarrow rL_2 + \frac{x}{r}L_1$

and (20) follows. Now let us prove that

$$\beta = \left[\beta_{\mathbf{S}_0,G} - 2(\Delta_{\mathbf{S}_0} G)^2 \frac{G_r}{r}\right] \circ h. \tag{21}$$

Using the above formulas, we obtain $F_{xx} + F_{yy} + F_{zz} = F_{rr} + F_{zz} + \frac{F_r}{r}$. Now (21) comes from the definition of β, and from the above expressions of F_x, F_y, F_{xz}, F_{yz} F_{zz}, F_{xt}, F_{yt} and F_{tt}. Let $m[x : y : z : t] \in \mathcal{Z} \setminus V(\alpha, \beta)$. To prove the result, it is enough to prove that the two solutions $[\lambda_0 : \lambda_1]$ of (3) are $[\lambda_0^{(1)} : \lambda_1^{(1)}] = [2\Delta_{\mathbf{S}_0} G.G_r : r] \circ h$ and $[\lambda_0^{(2)} : \lambda_1^{(2)}] = [-2h_G N_{\mathbf{S}_0} : t^2 \Delta_{\mathbf{S}_0} G] \circ h = \Phi_{\mathbf{S}_0,G} \circ h$. Indeed, since $r = \sqrt{x^2+y^2}$, $x_0 = y_0 = 0$, $F_x = \frac{x}{r} F_r$ and $F_y = \frac{y}{r} F_r$, we have $\mathbf{M}_1 := \lambda_0^{(1)} \cdot \mathrm{Id} + \lambda_1^{(1)} \cdot \boldsymbol{\sigma} \in V(x,y)$ and $\mathbf{M}_2 := \lambda_0^{(2)} \cdot \mathrm{Id} + \lambda_1^{(2)} \cdot \boldsymbol{\sigma} = \left[x\frac{R_2}{r} : y\frac{R_2}{r} : Z_2 : T_2\right] \circ h$ (using Lemma 41) with $\boldsymbol{\Phi}_{\mathbf{S}_0,G} = (R_2, Z_2, T_2)$. Observe that $R_2/r \in \mathbb{C}[r,z,t]$. Due to Theorem 40, the Zariski closure of $\Phi_{S_0,V(G)}(V(G))$ is the caustic $\Sigma_{S_0}(V(G))$. Observe that, for every (r, z, t), the set $[x : y]$ goes along $\mathbb{P}^1$ when (x, y) moves in $\{(x,y) \in \mathbb{C}^2 \ : \ x^2 + y^2 = r^2\}$. So the Zariski closure of $M_2(\mathcal{Z})$ is the revolution surface $\mathcal{R}(\Sigma_{S_0}(V(G)))$. Observe moreover that the Zariski closure of $M_1(\mathcal{Z})$ is $V(x, y)$ unless it is a single point A of $V(x, y)$, which would mean that every reflected line contains this point A, this would imply that $\Sigma_{S_0}(V(G))$ is reduced to a point and so that $V(G)$ is a conic and S_0 one of its focal point (see e.g. [10]). To prove that $[\lambda_0^{(1)} : \lambda_1^{(1)}]$ and $[\lambda_0^{(2)} : \lambda_1^{(2)}]$ are the solutions of (3), it is enough to prove that

$$[\gamma : \alpha] = [\lambda_0^{(1)}\lambda_0^{(2)} : \lambda_1^{(1)}\lambda_1^{(2)}] \tag{22}$$

$$\text{and} \quad [-\beta : \alpha] = [\lambda_0^{(1)}\lambda_1^{(2)} + \lambda_0^{(2)}\lambda_1^{(1)} : \lambda_1^{(1)}\lambda_1^{(2)}]. \tag{23}$$

Now (22) comes from $[\lambda_0^{(1)}\lambda_0^{(2)} : \lambda_1^{(1)}\lambda_1^{(2)}] = [-4h_G N_{\mathbf{S}_0} G_r : rt^2] \circ h$, (19) and (20). Now (23) is equivalent to $\beta = \left[-2(\Delta_{\mathbf{S}_0} G)^2 \frac{G_r}{r} + \frac{2h_G N_{\mathbf{S}_0}}{t^2}\right] \circ h$. So (23) comes from (21) and Lemma 41. □

Remark 43. *Due to Theorem 42, the caustic by reflection of a sphere $\mathcal{S}$ of center A from $S \neq A$ is the union of the line $(A\ S)$ and of the revolution surface of axis $(A\ S)$ obtained from the caustic curve of the circle $\mathcal{S} \cap \mathcal{P}$ where $\mathcal{P}$ is any plane containing $(A\ S)$.*

We consider the case where $\mathcal{Z} = V(x^2 + y^2 - 2zt)$ with $S = [0 : 0 : z_0 : 1]$ (with $S_0[0 : z_0 : 1]$) with $z_0 \neq 1/2$ (i.e. $S \in V(x, y)$ and S is not a focal point of $\mathcal{P}$). Observe that $\mathcal{Z} = \mathcal{R}(V(G))$ with $G(r, z, t) = r^2/2 - zt$. Due to [9], $\Sigma_{S_0}(V(G))$ has degree 6 except if $z_0 = 0$ (corresponding to $S \in \mathcal{Z}$) and, in this last case, $\Sigma_{S_0}(V(G))$ has degree 4. More precisely:

Proposition 44. *Let $\mathcal{Z} = V(x^2 + y^2 - 2zt) \subset \mathbb{P}^3$ and $S[0 : 0 : z_0 : 1] \in \mathbb{P}^3$ with $z_0 \neq 1/2$. Then $\Sigma_S(\mathcal{Z}) = V(x, y) \cup \mathcal{R}(V(H))$, where*

- *if $z_0 \neq 0$, the curve $V(H)$ is the sextic given by*

$$\begin{aligned} H(r,z,t) := {} & 27\,r^4z^2 - 512\,z^3t^3 + 288\,z^2r^2t^2 + 108\,r^4t^2z_0^4 + \\ & (3072\,zt^5 - 24r^4zt - 512t^6 - 6144\,z^2t^4 + 4992\,zr^2t^3 + 4096\,z^3t^3 - 1536\,z^2r^2t^2 - 2112\,r^2t^4 - 1068\,r^4t^2 - 8\,r^6)z_0^3 \\ & + (-1536\,zt^5 - 10560\,zr^2t^3 - 6144\,z^3t^3 + 6144\,z^2t^4 + 288\,r^2t^4 + 108\,r^4z^2 - 168\,r^4zt + 3195\,r^4t^2 + 72\,r^6 + 2688\,z^2r^2t^2)z_0^2 \\ & + (-1536\,z^2t^4 + 3072\,z^3t^3 + 90\,r^4zt - 108\,r^4z^2 - 1728\,r^4t^2 - 1536\,z^2r^2t^2 + 4032\,zr^2t^3 - 162\,r^6)z_0. \end{aligned}$$

- *if $z_0 = 0$, the curve $V(H)$ is the cuartic given by $H(r, z, t) := 27\,r^4 - 512\,zt^3 + 288\,r^2t^2$.*

Proof. We have $\mathcal{Z} = \mathcal{R}(V(G))$ with $G(r, z, t) = (r^2 - 2zt)/2$. Due to Theorem 42, we know that $\Sigma_S(\mathcal{Z}) = V(x, y) \cup \mathcal{R}(\Sigma_{S_0}(\mathcal{C}))$ with $\mathcal{C} = V(G) \subseteq \mathbb{P}^2$ and $S_0[0 : z_0 : 1] \in \mathbb{P}^2$. Observe that $\mathcal{C}$

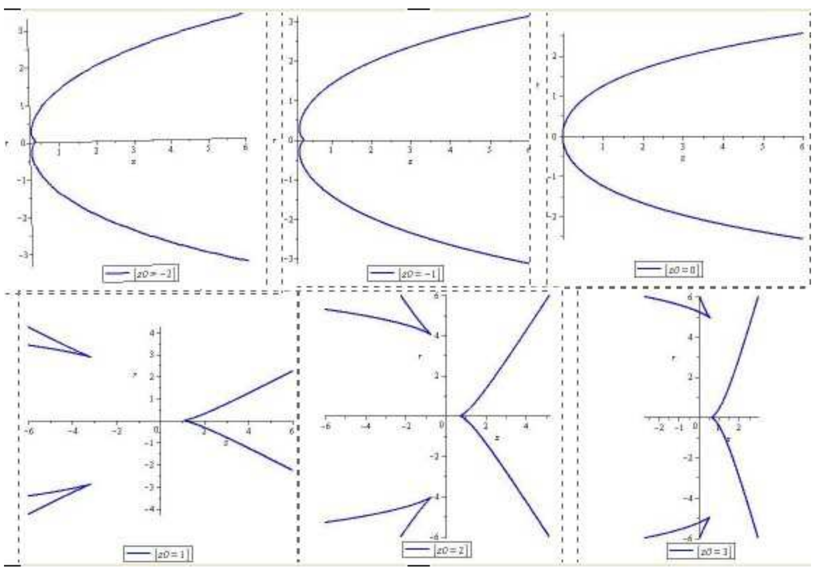

FIGURE 2. Caustics $\mathcal{C}'$ of $V(r^2 - 2zt)$ from $[0 : z_0 : 1]$ for $z_0 = -1, -2, 0, 1, 2, 3$. For every z_0, the two-dimensional part of the caustic by reflection of $V(x^2 + y^2 - 2zt)$ from $[0 : 0 : z_0 : 1]$ is the revolution surface of $\mathcal{C}'$ around $V(x, y)$.

admits the rational parametrization $(u, v) \mapsto [uv : (u^2/2) : v^2]$. Due to Theorem 40, $\Sigma_{S_0}(\mathcal{C})$ has the rational parametrization $(u, v) \mapsto \Phi_{\mathbf{S}_0,G}([uv : (u^2/2) : v^2])$ and

$$\Phi_{\mathbf{S}_0,G}\left(uv, \frac{u^2}{2}, v^2\right) = \left(2u^3v^3(1 - 2z_0), \frac{1}{4}(4z_0^2v^6 + 6z_0v^2u^2(v^2 - u^2) + u^4(u^2 + 6v^2)), \right.$$
$$\left. \frac{1}{2}(2z_0 - 1)v^4(2z_0v^2 - 3u^2)\right),$$

which parametrizes $\Sigma_{S_0}(\mathcal{C})$. So $\Sigma_{S_0}(\mathcal{C}) = V(H)$. □

A.3. **Caustic of a cylinder.** To simplify, we restrict ourselves to the study of a cylindrical surface $\mathcal{Z} = V(F)$ with axis $V(x, y)$. We will use the fact that $F(x, y, z, t) = G(x, y, t)$ for some homogeneous polynomial $G \in \mathbb{C}[x, y, t]$. Such a surface $\mathcal{Z}$ is called the cylinder of axis $V(x, y)$ and of basis $V(G) \subset \mathbb{P}^2$. We then write $\mathcal{Z} = Cyl(G)$. Observe that, in this particular case, the tangent plane to $\mathcal{Z}$ at $m = [x : y : z : t]$ does not depend on z.

Remark 45. *If $\mathcal{Z} = Cyl(G)$ (with G as above) and if $S[0 : 0 : 1 : 0] \in \mathbb{P}^3$, then $\mathcal{Z} \subseteq V(\Delta_{\mathbf{S}}F, (F_x^2 + F_y^2 + F_z^2)\operatorname{Hess} F(\mathbf{S}, \mathbf{S}))$.*

Theorem 46. *Let $S[x_0 : y_0 : z_0 : t_0] \in \mathbb{P}^3 \setminus \{[0 : 0 : 1 : 0]\}$ and let $\mathcal{Z} = Cyl(G)$ with $G \in \mathbb{C}[x, y, t]$ an irreducible homogeneous polynomial of degree $d \geq 2$. Assume that $\mathcal{Z} \not\subseteq V(\Delta_{\mathbf{S}}F, (F_x^2 + F_y^2 + F_z^2)\operatorname{Hess} F(\mathbf{S}, \mathbf{S}))$. We set $S_0[x_0 : y_0 : t_0] \in \mathbb{P}^2$.*

If $V(G) \not\subseteq V(H_G, N_{\mathbf{S}_0})$, then $\Sigma_S(\mathcal{Z}) = \overline{\sigma(\mathcal{Z})} \cup Cyl(\Sigma_{S_0}(V(G)))$, where $\overline{\sigma(\mathcal{Z})}$ is the algebraic curve corresponding to the Zariski closure of the sets of orthogonal symmetrics of S with respect to the tangent planes to $\mathcal{Z}$. Otherwise $\Sigma_S(\mathcal{Z}) = \overline{\sigma(\mathcal{Z})}$.

Proof. Observe that, since $F_{xz} = F_{yz} = F_{zz} = F_{zt} = 0$, we have $H_F = 0$ and so $\gamma = 0$. Let $m[x : y : z : t] \in \mathcal{Z} \setminus V(\alpha, \beta)$. We have $\alpha = \Delta_{\mathbf{S}}F = \Delta_{\mathbf{S}_0}G \circ h$ and $\beta = \beta_{\mathbf{S}_0,G} \circ h$ with $h(x, y, z, t) = (x, y, t)$. So (3) becomes $\lambda_0(\alpha(\mathbf{m})\lambda_0 + \beta(\mathbf{m})\lambda_1) = 0$ and its solutions $[\lambda_0 : \lambda_1] \in \mathbb{P}^1$ are $[0 : 1]$ and $[-\beta(\mathbf{m}) : \alpha(\mathbf{m})]$. The corresponding points on $\Sigma_S(\mathcal{Z})$ are $M_1(m) := \sigma(m)$ and

$M_2(m)[X_2(m):Y_2(m):Z_2(m):T_2(m)]$ with $[X_2:Y_2:T_2]=\Phi_{S_0,V(G)}$ (due to Lemma 41) and $Z_2(m)=\left[-\frac{2H_G(h(m))N_{\mathbf{S}_0}(h(m))}{(d-1)^2}z+\Delta_{\mathbf{S}_0}G(h(m))(G_x^2(h(m))+G_y^2(h(m))z_0\right]$. Due to Theorem 40, the Zariski closure of $\Phi_{S_0,V(G)}(V(G))$ is $\Sigma_{S_0}(V(G))$. If $V(G)\not\subseteq V(H_GN_{\mathbf{S}_0})$, then, for every $[x:y:t]\in V(G)$, $Z_2(x,y,z,t)$ goes all over $\mathbb{C}$ when z describes $\mathbb{C}$. If $V(G)\subseteq V(H_GN_{\mathbf{S}_0})$, then, due to Lemma 41, $\boldsymbol{\beta}=0$ on $\mathcal{Z}$, which implies that $M_2=M_1$ on $\mathcal{Z}$. □

Proposition 47 (parabolic cylinder with light at infinity)**.** *The caustic of reflection of* $\mathcal{Z}=V(y^2-2xt)\subset\mathbb{P}^3$ *from* $S[1-v^2:2v:z_0:0]$ *with* $v\neq 0$ *is* $\overline{\sigma(\mathcal{Z})}\cup V(H)$*, with*

$$\begin{aligned}H(x,y,z,t)=4y^3(1-v^6)+(-27t^3+108xt^2-72y^2t+24xy^2-108x^2t)(v+v^5)\\+12y(2x+3t+y)(2x+3t-y)(v^2-v^4)\\+2(16x^3+27t^3-36x^2t-24xy^2+216xt^2-144y^2t)v^3,\end{aligned}$$

(geometrically, v corresponds to the tangent of the half-angle of $(1,0)$ *with the direction of* S*).*

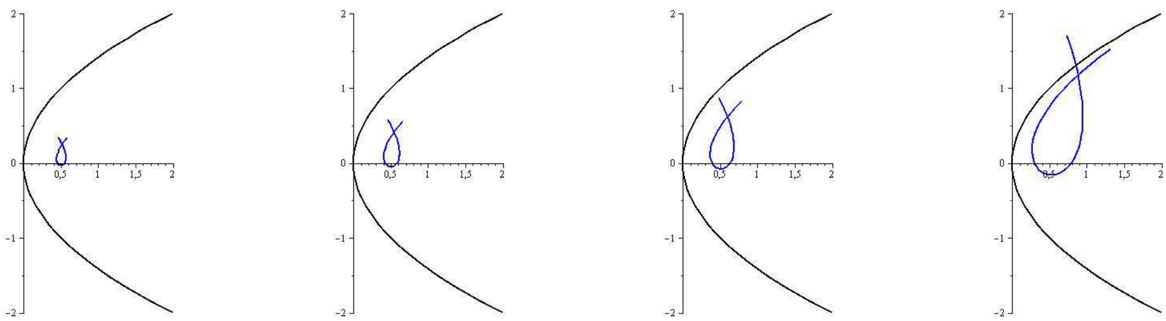

FIGURE 3. Caustics of $V(H)$ for $t_0=0$ and $x_0+iy_0=e^{i\theta}$ for $\theta=\frac{\pi}{50},\frac{\pi}{30},\frac{\pi}{20},\frac{\pi}{10}$.

Such surfaces are used in practice to concentrate sunrays on a tube (put along the line made of the focal points of the parabols) in order to heat the water circulating in it. Figure 3 is a transverse representation of this solar heater, the tube being at the focal point $(1/2,1)$.

Proof. Let $S_0[1-v^2:2v:0]$ and $\mathcal{C}:=V(G)\subset\mathbb{P}^2$ with $G(x,y):=(y^2-2xt)/2$. Due to Theorem 46, the caustic by reflection of $V(y^2-2xt)\subset\mathbb{P}^3$ from S is $V(x,y)\cup Cyl(\Sigma_{S_0}(\mathcal{C}))$. Moreover we know from [9] that $\deg\Sigma_{S_0}(\mathcal{C})=3$. Using the parametrization $\psi:(a,b)\mapsto\left[\frac{a^2}{2}:ab:b^2\right]$ of $\mathcal{C}$ together with Theorem 40, we conclude that $\Phi_{S_0,V(G)}\circ\psi(a,b)=[X_1(a,b):Y_1(a,b):Z_1(a,b)]$ is a parametrization of $\Sigma_{S'}(\mathcal{C})$. We obtain $\boldsymbol{\sigma}_{\mathbf{S}_0,G}=((y^2-t^2)x_0+2ty_0y,(t^2-y^2)y_0+2tx_0y$ and so

$$X_1=2v(1-v^2)a^3b^3+12v^2a^2b^4-6v(1-v^2)ab^5+(1-v^2)^2b^6$$

$$Y_1=-4v^2a^3b^3+6v(1-v^2)a^2b^4+12v^2ab^5-2v(1-v^2)b^6,\quad Z_1=2(1+v^2)^2b^6.$$

□

To complete the study of this example, let us specify $\overline{\sigma(\mathcal{Z})}$.

Proposition 48. *Under assumptions of the previous result, we have*

- *if* $v^2\neq -1$*, then* $\overline{\sigma(\mathcal{Z})}=\mathcal{H}^\infty\cap V((x^2+y^2)z_0^2-(v^2+1)^2z^2)$*;*
- *if* $z_0\neq 0$ *and* $v^2=-1$*, then* $\overline{\sigma(\mathcal{Z})}=\mathcal{H}^\infty\cap V(y+2x)$*,*
- *if* $z_0=0$ *and* $v^2=-1$*, then* $\overline{\sigma(\mathcal{Z})}=\{[2:-v:0:0]\}$*.*

Proof. We have $\boldsymbol{\sigma}(x,a,z,b) = \begin{pmatrix} a^2(1-v^2)+4vab-b^2(1-v^2) =: g_1(a,b) \\ a^2(-2v)+2(1-v^2)ab+b^2 2v =: g_2(a,b) \\ (a^2+b^2)z_0 =: g_3(a,b) \\ 0 \end{pmatrix}$.

First observe that $\overline{\sigma(\mathcal{Z})}$ is included in $\mathcal{H}^\infty$. Second, we compute the resultant in a of $(xg_3(a,1)-g_1(a,1), yg_3(a,1)-g_2(a,1))$ and obtain $4(1+v^2)^2((x^2+y^2)z_0-(v^2+1)^2)$. If $v^2 \neq -1$, this resultant gives the result (by homogeneization with z).

Assume now that $v^2=1$. We have $\boldsymbol{\sigma}(x,a,z,b) = (a+vb)\begin{pmatrix} 2(a+vb) \\ -v(a+vb) \\ (a-vb)z_0 \\ 0 \end{pmatrix}$.

If $z_0=0$, we have $\boldsymbol{\sigma}(x,a,z,b) = (a+vb)^2(2,-v,0,0)$ and so $\overline{\sigma(\mathcal{Z})} = \{[2:-v:0:0]\}$. Finally, if $z_0 \neq 0$ (still with $v^2=1$), using the fact that $[a-vb:a+vb]$ describes $\mathbb{P}^1$ when $[a:b]$ moves in $\mathbb{P}^1$, we obtain $\overline{\sigma(\mathcal{Z})} = \mathcal{H}^\infty \cap V(y+2x)$. □

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